IMPLEMENTATION OF THE WALRASIAN CORRESPONDENCE BY MARKET GAMES*

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WP-AD 95-02

* This paper was presented in the seminar of professor William Thomson to whom we are very grateful for many suggestions. Also we would like to thank comments from J. Alcalde, R. Bhattacharya, C. Beviá, C. Herrero, P. Marhuenda, D. Pérez-Castrillo, T. Shinotsuka, T. Sjöström, and A. Villar. The responsibility of any remaining error is solely ours. The first author wishes to acknowledge financial help from the Generalitat Valenciana and DGCYT unde project PB91-0756 and the hospitality of the Royal Complutense College at Harvard University.

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A B S T R A C T

In this paper we present a set of axioms guaranteeing that, in exchange economies with or without indivisible goods, the set of Nash, Strong and active Walrasian Equilibria all coincide in the framework of market games.

Keywords: Mechanism, Market Games, Nash Equilibrium, Strong Equilibrium, Implementation.

1. INTRODUCTION

In this paper we present an unifying framework for the study of market games in exchange economies. In the framework of classical exchange economies when all goods are divisible particular market games have been considered by Shubik (1977), Schmeidler (1980) and Dubey (1982) (see also the related contributions of Shapley and Shubik (1977), Hahn (1978) and Dubey, Mas-Colell and Shubik (1980). The case of public goods is considered by Bagnoli and Lipman (1989) and Corchon and Wilkie (1990)). A general axiomatic framework for double auctions was introduced by Benassy (1986). In that paper he proposed a set of axioms that define a market game and was able to show that 1) the Walrasian allocation can be supported as a Nash Equilibrium (N.E. in the sequel) and 2) that when all markets are active, all N.E. yield Walrasian allocations. It was shown by Dubey (1982) in a specific model that N.E. with all markets being active and Strong Equilibrium coincide. Later on, Svensson (1991) studied the case of indivisible commodities, such as houses or jobs by means of a very specific model. He proposed a simple market game for which the set of Strong Equilibrium outcomes coincides with the set of Nash equilibrium outcomes when all the markets are open and with the set of Walrasian allocations (other models of indivisibilities include those of Demange and Gale (1985), Roth (1982), Kaneko (1983), Gale (1984), Quinzii (1984), Svensson (1984), Kaneko and Yamamoto (1986), Maskin (1987), Svensson (1988), Roth and Sotomayor (1988) and Tadenuma and Thomson (1990)).

A natural question is if the results explained above depend on either the special structure of the commodity space (as in Benassy) or to the particular game form used in Dubey or Svensson[1]. In this paper we address this question and show that neither is the case. We present a model where the commodity space has no special structure (and thus it could have both divisible and indivisible goods) and we propose a set of axioms that a market game should

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(1) It is easy to see that the proof of Proposition 1 in Benassy's paper (see pp. 103-104) does not use any convexity assumption. However, this is not true for other results in his paper. It must also be remarked that the results obtained by Schmeidler (1980) depend on the assumption that utility functions are differentiable.

satisfy. With these axioms in hand we prove that for any economy in the admissible domain, the set of Walrasian allocations, Strong Equilibrium outcomes and N.E. outcomes when all markets are open coincide.

Implementation by means of market games has intuitive appeal since the underlying mechanism is both natural and simple and implementation occurs in Nash and Strong equilibrium. There are, however, some costs associated with this approach. First, if a strong boundary condition is not assumed, there may exist no-trade, usually inefficient, N.E.. This kind of equilibria arise not only in the class of models analyzed in this paper (see Dubey (1982), Benassy (1986) and Svensson (1991)) but also in Cournot-type models of quantity competition (see Hart (1980) and Makowski (1980)). An interesting research would be to explore how a trembling hand argument might dispose of no-trade equilibria (see Bagnoli and Lipman (1989) for the public good case). In any case we can construct examples where a N.E. with some inactive markets is not dominated in the Pareto sense by the Walrasian Equilibria. This rules out focal point considerations. Second, the outcome function is discontinuous (see Benassy (1986) pp. 99-100 for a very convincing argument showing that discontinuity is an essential feature of efficient N.E. in market games. Certainly his game, as well as those of Dubey and Schmeidler, have discontinuous outcome functions (see Postlewaite and Wettstein (1989) for a continuous mechanism implementing the Walrasian correspondence). In our case two of our axioms (**VT** and **SBC**) imply discontinuities. Therefore small mistakes in the strategy space may translate into large discrepancies between intended and obtained allocations. However, we do not know if discontinuity is an issue here since our commodity space is not necessarily Euclidian and therefore for some weird topology anything is continuous.

Because of the above reasons, several authors have looked for continuous implementation of the the Walrasian correspondence in N.E. in games where strategies are prices and quantities and where the strong boundary condition is not postulated. The rules of the game -i.e. the outcome function- are, though, very different since it does not reflect market rules beyond the existence of a budget constraint. We will provide some axioms under which the Walrasian correspondence is implemented in N.E.. Some axioms are common to this approach and to market games. The reader is reminded that an alternative foundation to the idea of Walrasian equilibrium -not considered in this paper- is given by the consideration of arbitrarily large economies (i.e. core or Cournot limit theorems).

Our work can be regarded as an application of implementation theory to a very specific problem. The authors hope that it might modestly contribute to show that implementation is not only about abstruse mechanisms but it can shed some light on the functioning of real life institutions like double auction markets. It should be remarked though that market games mimic but are not markets (like an artificial heart mimic but is not a heart). In particular some tendencies that are spontaneous in a market are reflected here in the form of the outcome function.

The main insight of this paper is that from the point of view of implementation, convexity -or the lack of it- does not matter very much. Other insights gained by the study of market games are the following. 1) Individually feasibility cannot be guaranteed outside equilibrium[(2)]. 2) To sustain Walrasian allocations as equilibria (Nash or Strong) requires relatively harmless and simple assumptions. 3) However, to eliminate non-Walrasian equilibria requires very strong assumptions both on the mechanism (i.e. some kind of Bertrand competition) and on preferences (i.e. our strong boundary condition). And 4) the fact that coalitions can or can not be formed does not matter very much as long as markets are active. Possible extensions of our results include the consideration of production economies and the implementation of other social choice rules by market mechanisms.

The rest of the paper goes as follows. In the next Section we present our model of market games. Section 3 gathers our main results.

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(2) In this paper, we do not consider totally feasible implementation and thus for some messages, the resulting allocation may not be individually feasible. Given the informational requirements of totally feasible implementation when the feasible set is unknown to the designer (see e.g. Postlewaite and Wettstein (1989)), no market game can feasibly implement the Walrasian correspondence. If the feasible set is known to the designer and an extra condition on preferences is postulated, a simple modification of the original mechanism can be shown to implement the Walrasian correspondence (see Tian (1992)). We remark that in the Walrasian model with an auctioneer, feasibility is also not guaranteed outside equilibrium.

2. THE MODEL

This section has five parts. a) The description of the environment, b) the definition of a social choice correspondence, c) the description of a game form (also called mechanisms), d) the definition of the game-theoretical equilibrium concepts and e) the notion of implementation.

a) Environment

There are h consumers and $n + 1$ goods, where n and h are natural numbers. Goods 1, ..., n are (possibly) indivisible goods. Good 0 is a perfectly divisible good which can be loosely termed as money. A net trade for consumer i is an $n + 1$ dimensional vector $(x_{i1}, \ldots, x_{in}, m_i)$, sometimes also represented by (x_i, m_i), where x_{ig} (resp. m_i) is her net trade of commodity g (resp. money). The set of individually feasible net trades for consumer i is denoted by $X_i \equiv C_i \times M_i$ where $x_i \in C_i$ and $m_i \in M_i = [a_i, \infty)$. We do not impose any structure on C_i except that C_i is a subset or R^n.

The preferences of i over net trades in X_i are represented by a utility function $u_i\colon X_i \longrightarrow \mathbb{R}$ denoted by $u_i = u_i(x_i, m_i)$ which is assumed to be strictly increasing in money. We will also assume that any bundle inside X_i is preferred to any bundle outside X_i (3) and that $u_i()$ is never decreasing in any of its arguments, i.e. goods can be freely disposable by consumers. Let us denote by $(x, m) \in R^{(n+1).h}$ the h-tuple of net trades for consumers 1, ..., h also referred to as an *allocation*. An allocation (x, m) is said to be *balanced* if $\forall g = 1, \ldots, n$, $\sum_{i=1}^{h} x_{ig} \leq 0$ *and* $\sum_{i=1}^{h} m_i \leq 0$. We remark that a balanced allocation does not need to be totally feasible because agents may receive net trades which are not individually feasible. The set of balanced allocations is denoted by F. We will assume that the number of agents and the individually feasible net trade sets are fixed. Thus an *economy*, denoted either by e or by

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(3) A similar assumption is made by Hurwicz (1979 b), Schmeidler (1980) and Benassy (1986).

$(u_i)_{i=1,..h}$, is a list of utility functions, one for each agent. Let us denote by $\mathcal{E}$ the set of admissible economies.

b) Social Choice Correspondence

A *social choice correspondence* $\Phi : \mathcal{E} \longrightarrow F$ maps the set of economies into the set of balanced allocations. It is meant to represent the objectives of the society. In this paper Φ will be the Walrasian correspondence. In order to define this let us first define a Walrasian Equilibrium (W.E in the sequel) for a particular economy $e \in \mathcal{E}$.

DEFINITION 1. *A Walrasian Equilibrium for an economy* $(u_i)_{i=1,..h}$ *is a balanced allocation* (x^w, m^w) *and a vector of prices* $p^w \in \mathbb{R}^n_+$ *such that* $\forall i = 1, \ldots, h$, (x^w_i, m^w_i) *maximizes* $u_i(x_i, m_i)$ *subject to* $p^w \circ x_i + m_i = 0$

Notice that in the above definition money is the numeraire. The *Walrasian social choice correspondence* maps the set of economies in allocations that are Walrasian equilibria for these economies.

c) Game Forms and Market Games

A *Game Form* (or a *mechanism)* is a list (S_i, f), $i = 1,\ldots, h$, where S_i denotes the strategy set of consumer i and f is the outcome function.

In a market game a strategy for consumer (player) i is a $2n$ dimensional vector of real numbers $s_i = (\pi_i, q_i)$ where $\pi_i = (\pi_{i1}, \ldots, \pi_{in})$ (resp. $q_i = (q_{i1}, \ldots, q_{in})$) represents the vector of bids (resp. net trades) proposed by player i. Let us denote by π (resp. by q) the list of bids (resp. net trades) proposed by all players and by $s \equiv (\pi, q) \in \mathbb{R}^{2.n.h}$ the list of all strategies for all players. Let $S \equiv \prod_{i=1}^{h} S_i$.

The outcome function f maps S into the set of balanced allocations, i.e. $f : S \longrightarrow F$. Let f_i be the i^{th} component of f. It represents the allocation of goods and money obtained by i i.e. $f_i(s) \equiv (x_i(s), m_i(s))$. Let p be the

n-dimensional vector of trading prices as a function of s, i.e. $p : S \longrightarrow R^n_+$ or $p = p(s)$[4]. The outcome function f is designed to select allocations that satisfy the budget constraint of each consumer and thus the quantity of money allocated to consumer i, denoted by $m_i(s)$, is $m_i(s) \equiv - p(s) \circ x_i(s)$, where $x_i(s)$ is the net trade of goods allocated to i if the strategy s is played.

d) Equilibrium

Let $v_i(s) \equiv u_i(f_i(s))$ be the indirect utility function associated with the list of strategies $s \in S$. Also let us denote by s_{-i} the list of strategies played by all consumers except i. Then we have the following definition:

Definition 2. *A Nash Equilibrium (N.E.) of a game form (S, f) for an economy* $(u_i)_{i=1,..h}$ *is a list of strategies* $s^n \in S$ *such that* $\forall i = 1, ..., h$ *we have that* $v_i(s^n) \geq v_i(s_i, s^n_{-i})$ $\forall s_i \in S_i$.

A *coalition*, denoted generically by C, is a non empty subset of the set of all players. A list of strategies for all members of C is denoted by s_c and the corresponding strategy set by S_c. Similarly s_{-c} denotes the list of strategies for all players <u>not</u> in C.

Definition 3. *A Strong Equilibrium (S.E.) of a game form (S, f) for an economy* $(u_i)_{i=1,..h}$ *is a list of strategies* $s^s \in S$ *such that there is no coalition C, such that* $v_i(s_c, s^s_{-c}) \geq v_i(s^s)$, *some* $s_c \in S_c$, $\forall i \in C$ *and* $v_j(s_c, s^s_{-c}) > v_j(s^s)$ *for some* $j \in C$[5].

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(4) Benassy does not assume directly, as we do, that all traders buy at the same price (however at any N.E. their trading prices are identical). This possibility can be considered in our model at the cost of some complications.

(5) An alternative definition of strong equilibrium requires that the inequality be strict for all members of C. In our model both notions coincide.

e) **Implementation**

Let NE(S, f, e) be the set of Nash Equilibrium strategies for the mechanism (S, f) and the economy e. Let SE(S, f, e) be the set of Strong Equilibrium strategies for the mechanism (S, f) and the economy e. Then, we come to the main definitions of the paper.

Definition 4. *The Game form (S, f) implements a social choice correspondence Φ in the domain $\mathcal{E}$ in Nash Equilibrium if $\forall e \in \mathcal{E}$ we have that $NE(S, f, e) \neq \emptyset$ and $\Phi(e) = f(NE(S, f, e))$.*

Definition 5. *The Game form (S, f) implements a social choice correspondence Φ in the domain $\mathcal{E}$ in Strong Equilibrium if $\forall e \in \mathcal{E}$ we have that $SE(S, f, e) \neq \emptyset$ and $\Phi(e) = f(SE(S, f, e))$.*

Definitions 4 and 5 refer to the equilibrium concepts that we will use in the remainder of the paper. The notion of N.E. is relevant in a non cooperative framework, where agents can not engage in binding agreements. The notion of a S.E. is meant to capture the outcome of a game where cooperation is feasible, i.e. any binding agreement is possible. If the designer does not have a priori information about the feasibility of those agreements, it appears to be desirable that implementation occurs for both scenarios. This is the concept of double implementation, a term coined by Eric Maskin, and that we present formally in the next definition.

Definition 6. *The Game form (S, f) doubly implements in Nash and Strong equilibrium a social choice correspondence Φ in the domain $\mathcal{E}$ if $\forall e \in \mathcal{E}$ we have that $SE(S, f, e) \neq \emptyset$ and $\Phi(e) = f(SE(S, f, e)) = f(NE(S, f, e))$.*

3. RESULTS

In this Section we gather our main findings on market games. Let us first present our two first axioms:

Unanimity (U).- *If $s \equiv (\pi, q) \in S$ is such that for all active traders (i.e. those for which $x_i(s) \neq 0$) $\pi_i = \pi_j$, then $p(s) = \pi_i$. If there are no active traders but $\pi_i = \pi_j$, $\forall i, j = 1, ..., h$, then $p(s) = \pi_i$. Moreover, if q is a balanced allocation then $x(s) = q$.*

Voluntary Trade (VT).- *$\forall g = 1, ..., n$, $\forall i = 1, ..., h$, $\forall s \in S$, such that $x_{ig}(s) > 0$ (resp. < 0) then $p_g(s) \leq$ (resp. $\geq$) π_{ig}.*

Unanimity (**U**) roughly says that if all bids are the same, and the proposed net trade vector is balanced, trading prices must be equal to these bids and net trades allocated by the mechanism must be equal to the net trades asked by consumers. This is a very weak property which must be satisfied by any satisfactory model of resource allocation in market economies.

The Voluntary Trade (**VT**) property says that in order to get a positive (resp. negative) net trade of some commodity, say j, the trading price of j must be greater or equal (resp. lower or equal) than the minimum (resp. maximum) price at which any seller (resp. buyer) is willing to supply (resp. demand) this good. Under the **VT** property the interpretation of bids is that if consumer i is a net demander (resp. supplier) of good j, π_{ij} is the maximum (resp. minimum) price at which she is prepared to accept a net trade of q_{ij} units of good j. Special cases of this property have been proposed by Benassy ((1986) p. 100), Svensson (1991) and Silvestre (1985).

Now, we are prepared to prove our first result.

Proposition 1. *Let (S, f) be a game form in which* **VT** *and* **U** *hold. Then, if (p^w, x^w, m^w) is a W.E., $\exists$ s such that s is a S.E. and $f(s) = (x^w, m^w)$.*

Proof: *Let us first construct s. For a typical consumer i, let* $\pi_i = p^w$ *and* $q_i = x_i^w$. *Then, by* **U**, $f(s) = (x^w, m^w)$ *and* $p^w = p(s)$. *Therefore if the Proposition were not true there is a coalition C, and a* $s'_c \in S_c$ *such that* $v_i(s'_c, s_{-c}) \geq v_i(s)$, $\forall i \in C$ *and* $v_j(s'_c, s_{-c}) > v_j(s)$ *for some* $j \in C$. *Let* $(x'_i, m'_i) = f_i(s'_c, s_{-c})$, $\forall i \in C$. *Let also p' be the new vector of trading prices, i.e.* $p' = p(s'_c, s_{-c})$. *Then by revealed preference we have* $p^w \circ x'_i + m'_i \geq p^w \circ x^w + m^w$ $\forall i \in C$ *and* $\exists\ j \in C$ *for whom the inequality is strict.*

Since $p' \circ x'_i + m'_i = 0$, *it follows from that and the previous inequality that* $x'_i \circ (p' - p^w) \leq 0 \quad \forall\ i \in C$ *and* $\exists\ j \in C$ *for whom the inequality is strict. Adding over i we get:*

$$(p' - p^w) \circ \textstyle\sum_{i \in C} x'_i < 0 \qquad (1)$$

Since the Walrasian equilibrium is Pareto Efficient, C can not be composed of all agents. Also, if the sum of net demands of C of a good is negative (i.e. C is a net supplier of, say commodity g) we have two cases. If the complementary coalition is a net demander of g, by **VT** *it must be that* $p'_g \leq \pi_{rg} = p^w_g$ *some* $r \notin C$, *since all sellers outside the coalition are bidding* p^w. *If the complementary coalition is not a net demander of g by adding the budget constraints we get that* $p'_g = 0$ *since g( ) selects allocations in F. In any case, the corresponding term in (1) above is positive or zero. A similar (but simpler) argument can be used if some component of* x'_i *is positive. This shows that the inequality (1) above is impossible.*■

The logic behind Proposition 1 is the following: Suppose that market prices and the allocation are Walrasian but now a group of agents, say all the oil producing countries, collude and attempt to raise the price of oil. Since importing countries make no change in their bidding strategies, **VT** implies that oil producing countries cannot sell a single drop of oil and thus can not improve their welfare by colluding. Since a S.E. is a N.E., we have that:

Corollary 1. *Let (S, f) a game form in which* **VT** *and* **U** *hold. Then, if* (p^w, x^w, m^w) *is a W.E.,* $\exists s$ *such that s is a N.E. and* $f(s) = (x^w, m^w)$.

In order to obtain a converse to Proposition 1 we will impose additional properties on the outcome function. First we will assume that the utility function is continuous on money. We will also assume the following property:

Reactiveness (R).- *Let s a strategy profile such that in market g,* $\pi_{ig} > \pi_{jg}$ *with* $x_{ig}(s) > 0$ *and* $x_{jg}(s) < 0$. *Then* $\exists$ *s', identical to s except in component r (with r = i or r = j) such that:*

$$x_r(s') = x_r(s),\ (p_g(s') - p_g(s))\ x_{rg}(s) < 0 \text{ and } p_k(s) = p_k(s')\ \forall k \neq g.$$

This axiom means that, when the bid made by a buyer is greater than the bid made by a seller, there is a way for (at least) one of them to change the price without affecting either her consumption bundle or the other prices. Recall that we are interpreting π's as maximum buying (resp. minimum selling) bids. Thus, Axiom **R** says that any discrepancy between those bids can be eliminated by our market game with a minimal impact on allocations and other prices. In other words, the mechanism mimics what in this circumstances is a natural reaction of maximizing agents, namely to press the market price downwards (buyers) or upwards (sellers). We remark that this axiom does not imply that general equilibrium effects are somehow neglected. What it means is that when two agents want to change a particular price, the market games gives room to, at least, one of them to do so and to leave other prices unaffected. This axiom is automatically satisfied in general equilibrium models of price-making agents and in Svensson (1991). We will now postulate the following:

Possibility of Trade (PT). *$\forall i \in I$, $\forall s_{-i}$, $\exists s_i$ such that $x_i(s_i, s_{-i})$ has no zero component.*

This axiom can be understood as saying that all markets can be activated by an unilateral move of an agent. For instance, by offering a sufficiently low (resp. high) bid trader i can sell (resp. buy) some quantity of any good. Recall that bids are not required to be positive. Before we state a new axiom let us introduce a new piece of notation. $B_i(p)$ will denote the budget set of individual i at prices p, i.e., $B_i(p) = \{ (x_i, m_i) \in X_i / p \circ x_i + m_i \leq 0\}$.

Strong Bertrand Competition (SBC).- *Let s be a strategy profile where all active traders quote the same price p. Then if the number of active traders is greater than 2, we have the following:*

$$B_i(\pi_i) \subset B_i(p(s)) \text{ and } x_i \in B_i(\pi_i) \longrightarrow \exists\, q_i \text{ such that } x_i = f_i(\pi_i, q_i, s_{-i}).$$

The **SBC** Axiom says that a seller (resp. a buyer) by cutting (resp. increasing) the market price can transact as much as it wants. This is also satisfied in Svensson (1991). Notice that in the case of two active traders this axiom is problematic as shown by the following example:

Example. *Let $n = 2$, and suppose there are two units of a unique indivisible good owned by agent 2. Let: $s_1 = (150, 1)$, $s_2 = (150, -1)$, $\bar{s}_1 = (160, 2)$, $\bar{s}_2 = (150, -1)$ and $\hat{s}_1 = (160, 2)$, $\hat{s}_2 = (160, -2)$. By Unanimity $x(s_1, s_2) = (1, -1)$ and $x(\hat{s}_1, \hat{s}_2) = (2, -2)$. Applying the previous Axiom to (s_1, s_2) and $(\bar{s}_1, \bar{s}_2)$ we get that $x(\bar{s}_1, \bar{s}_2) = (2, -2)$. But applying this Axiom to $(s_1, \hat{s}_2)$ and $(\bar{s}_1, \bar{s}_2)$ we get that $x(\bar{s}_1, \bar{s}_2) = (1, -1)$.*

The reason why **SBC** does not work with 2 agents is that it is not possible to identify the deviant agent. This sort of problem is familiar to the theory of Nash implementation since Maskin (1977). Before we prove our next result, let us now introduce the following assumption:

Strong Boundary Assumption (SBA).- $C_i = \mathbb{R}^l_+$. *$\forall i = 1,\ldots, h$, any point interior of X_i is preferred to any point in the boundary of X_i.*

Proposition 2. *Under axioms* **U, VT, R, PT** *and* **SBC** *and assumption* **SBA** *if $h > 2$, any N.E. yields a Walrasian allocation.*

Proof: *First notice that* **SBA** *and* **PT** *imply that all markets are open. We will now prove that for any pair of active traders i and j, in a market, say g, we have that in any N.E. $\pi_{ig} = \pi_{jg}$.*

Because **VT** *if the market is active and, say $x_{ig}(s) > 0$ (resp. $<$), it must be that $p(s) \leq$ (resp. $\geq$) π_{ig}. Suppose that $\exists$ a pair of active traders i and j such that $\pi_{ig} > \pi_{jg}$ where i (resp. j) is the net buyer (resp. seller). Suppose that in Axiom* **R**, *$r = i$. Then, $\exists\, s'_i$ such that*

$x_i(s') = x_i(s)$, $p_g(s') < p_g(s))$ and $p_k(s) = p_k(s')$, $\forall k \neq g$.

But then, i is clearly better off by playing s'_i *because preferences are monotonic on money. This contradicts that we are at a N.E.. The same argument holds if* $r = j$. *Therefore, in any N.E., all active traders quote the same price, say* π_i. *By* **U**, *if s is a N.E. strategy profile,* $p(s) = \pi_i$.

Let us now assume that a trader, say i, is not obtaining her most preferred bundle in $B_i(p(s))$. *Thus* $\exists x'_i$ *such that* $u_i(x'_i, -p(s)\circ x'_i) > u_i(g_i(s), -p(s)\circ g_i(s))$. *Pick a* δ *such that* $u_i(x'_i, -p(s)\circ x'_i - \delta) \geq u_i(g_i(s), -p(s)\circ g_i(s))$.

Suppose that $x_{ig}(s) > 0$. *Let* π'_i *be such that* $\pi'_{ik} = \pi'_{ik}$ *for* $k \neq i$ *and* $\pi'_{ig} = p_g + \delta/2x'_{ig}$. *Note that* $\pi'_i \circ y = p(s)\circ y + \delta/2x'_{ig}$. *Thus if* $(y, m) \in B_i(\pi'_i)$, *then* $(y, m) \in B_i(p(s))$. *By Axiom* **SBC**, $\exists\, s'_i$ *such that* $x'_i = g_i(s'_i, s_{-i})$. *However by* **VT** *we have that* $p(s')\circ x'_i \leq \sum_{k\neq g} p_k(s)\circ x'_i + p_g(s)\circ x'_{ig} + \delta/2$. *Then, we have that* $u_i(x'_i, -p(s)\circ x'_i - \delta/2) > u_i(g_i(s), -p(s)\circ g_i(s))$ *contradicting that s was a N.E.*■

Thus, we have the following:

Corollary 2. *Under the conditions of Proposition 2, any S.E. yields Walrasian allocations.*

Theorem 1. *Any market game satisfying* **U, VT, R, PT, SBC** *and* **SBA** *with* $h > 2$ *doubly implements the Walrasian correspondence in Nash and Strong equilibrium.*

Notice that the failure of market games to implement the Walrasian correspondence is not related to the lack of Maskin monotonicity (see Maskin (1977)). In fact, it is not difficult to show that in our framework, the Walrasian correspondence is Maskin monotonic and implementable in N.E. (see e.g. Theorem 2 below). This is because we do not require the outcome function to select individually feasible net trades. In our case, what it might "trap" the economy in the wrong position is the combination of a violation of **SBC** and the **VT** axiom .**A** good example of it is provided by Svensson, p. 873. We now switch to the study of non market mechanisms. The first axiom is:

Avoidable Rationing (AR).- $\forall i = 1, \ldots, h$, $\forall s_{-i}$ *such that* $\pi_k = \pi_j$ $\forall k,j \neq i$, $\forall\, x'_i \in C_i$, $\exists q'_i$ *such that* $x'_i = x_i(\pi_k, q'_i, s_{-i})$.

This property means that if all agents submit identical bidding vectors, any consumer can achieve any bundle just asking for it. The difference of this axiom with **SBC** is that in order to obtain the desired bundle agents are not asked to beat the market price (as in **SBC**), merely to repeat it. Notice that this property is incompatible with totally feasible implementation (i.e. an outcome function which selects feasible allocations inside X_i for each agent, see Hurwicz, Maskin and Postlewaite (1982)), since for some strategies the resultant bundles may be individually unfeasible. A related property (i.e. that consumption bundles obtained by any agent are an increasing function of her proposals) has been termed *manipulable rationing* by Benassy (1982 p.24). A rule which comes very close to **AR** is Proportional Rationing (see Dubey (1982)). Similarly, many other manipulable rationing rules are in the spirit of **AR**. Two examples of **AR** are the following:

$$\text{Let } q_g = \sum q_{jg}. \text{ Then } \forall i = 1, \ldots, h, \ \forall g = 1, \ldots, n,$$
$$x_{ig} = q_{ig} - q_g/h \text{ or } x_{ig} = q_{ig}.h/\ (h-1) - q_g\ /(h-1)$$

(see Hurwicz (1979 b) and c) and Schmeidler (1980)).

In order to state our next axiom let us define the set $A_i(s_{-i},\pi_i)$, and the average bid of all agents except i, denoted by Π_i:

$$A_i(s_{-i},\ \pi_i) = \{\ (x_i,\ m_i) \in X_i/\ (x_i,\ m_i) = f_i(q_i,\ \pi_i,\ s_{-i}) \text{ some } q_i\}$$
$$\Pi_i = \sum_{j \neq i} \pi_j/\ (h-1).$$

$A_i(s_{-i},\pi_i)$ is the set of allocations that i can achieve by the choice of some proposed net trade, given her vector of bids and the list of strategies of all agents except i.

Aurea Mediocritas (AM).- $A_i(s_{-i},\ \pi_i) \subset A_i(s_{-i},\ \Pi_i),\ \forall\ \pi_i \neq \Pi_i.$

This axiom implies that agents whose bids are not equal to the average bid of the other agents are penalized. This is a generalization of an idea first introduced by Hurwicz (1979 b). It is clear that **AM** is compatible with a continuous outcome functions since the penalty to the deviating agent can be made continuous on the strategies. Then, we have the following:

Theorem 2: *Suppose that preferences are strictly monotonic. Then, any market game satisfying* **AR**, **U** *and* **AM** *implements in N.E. the Walrasian correspondence.*

Proof: *Let us first prove that the Walrasian allocation can be obtained as a N.E.. Give to the agents the Walrasian strategies as in Proposition 1. Thus, for a typical consumer, say i, let* $\pi_i = p^w$ *and* $q_i = x^w_i$. *Then, by* **U**, $f(s) = (x^w, m^w)$ *and* $p^w = p(s)$. *Therefore if the Proposition were not true there is an agent, say i, and a* $s'_i \in S_i$ *such that* $v_i(s'_i, s_{-i}) > v_i(s)$. *Let* $(x'_i, m'_i) = f_i(s'_i, s_{-i})$, $\forall i$. *Let also p' be the new vector of trading prices. Because axiom* **AM** *if i changes her strategy and gain it must be by repeating the Walrasian prices. But then* **AR** *implies that she can not do it better.*

Conversely in a N. E. **AM** *implies that all agents should send the same vector of bids. Therefore all bids are identical and by* **U** *trading prices equal these bids. But then* **AR** *implies that any trader can choose her most preferred bundle in her budget set and, thus, the allocation is Walrasian.* ■

The question about the axioms under which double implementation of the Walrasian correspondence in N.E. and S.E.occurs outside the realm of market mechanisms still open. See Schmeidler (1980) for an example of such a family of mechanisms.

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