Working Paper 97-11
Statistics and Econometrics Series 05
February, 1997

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ON THE PROPERTIES OF THE DICKEY-PANTULA TEST AGAINST FRACTIONAL ALTERNATIVES

Juan José Dolado and Francesc Marmol*

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This version, January 1997

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1. Introduction

Although the stochastic behaviour of many economic time series has been reported in the literature to be well approximated by integrated processes of order one, denoted $I(1)$, there are some series, especially nominal ones (e.g., money holdings, prices, wages, etc.) which appear to be potentially better described as $I(2)$ processes. In this case, a shock to the series in one period will have a permanent influence on both future growth rates and the levels of the series so that it will appear having an extremely smooth path. Some recent contributions on this topic include Haldrup (1991, 1994a, 1994b), Haldrup and Simon (1995), Johansen (1992a, 1992b, 1995a, 1995b), Juselius (1993, 1994), King et al. (1991), Kitamura (1995) and Paruolo (1992, 1994).

Considering the problem of testing for a double unit root, Dickey and Pantula (1987) and Pantula (1989) have suggested a popular sequential testing procedure in applied work which takes the largest number of unit roots under consideration as the first maintained hypothesis and then decreases the order of differencing each time the current null hypothesis is rejected.

In general, $I(1)$ and $I(2)$ processes can be considered as particular cases of the more general family of fractionally integrated processes, denoted *FI*(d). As is well-known, a stochastic process y_t is called *FI*(d) if $\Delta^d y_t \sim I(0)$, where d is allowed to be a real number rather than just an integer one. When $d \geq \frac{1}{2}$, the series is nonstationary, denoted *NFI*(d). Sowell (1990) derived the limit distribution of the standard Dickey-Fuller (*DF*) test for a single unit root, based upon the t-statistic when the true process is *NFI*(d) with $d \in (\frac{1}{2}, \frac{3}{2})$, showing that it diverges to $-\infty$ $(+\infty)$ when $d < 1$ $(d > 1)$. Hence, the t-test is consistent for $d \in (\frac{1}{2}, 1)$ and has zero power for $d \in (1, \frac{3}{2})$.

In this paper we extend the previous arguments to the sequential testing procedure advocated by Dickey and Pantula (*DP*), allowing the analysis to cover the case of $I(2)$ processes. We find that Sowell's results can be generalized to these higher integrated processes. In fact, the properties of the testing sequence are derived for all values of d within the nonstationary range and thus can be extended to testing for three or more unit roots in the presence of *NFI*(d) alternatives.

2. The Model and the Dickey-Pantula Test

Throughout the paper, we shall assume that the true data generating process (*DGP*) of y_t is the following *NFI* process

$$\Delta^d y_t = \varepsilon_t, \tag{1}$$

where $d \geq \frac{1}{2}$, $\varepsilon_t \sim iid(0, \sigma^2)$ and $y_0 = 0$ for $t \leq 0$. Noticing that d can always be decomposed as $d = \alpha + \delta$, where $\alpha = 1,2,3,\ldots$ and $|\delta| < \frac{1}{2}$, (1) can be reparameterized as

$$\Delta^\alpha y_t = \eta_t, \Delta^\delta \eta_t = \varepsilon_t, \tag{2}$$

that is, any $NFI(d)$ process can be expressed as an integer $I(\alpha)$ process with stationary and ergodic fractionally integrated $SFI(\delta)$ innovations.

Let $\sigma_{\eta T}^2 = \mathrm{var}(S_T)$, where $S_t = \sum_{j=1}^{t} \eta_j$. The growth rate of this partial sums' variance was proved by Sowell (1990) to be equal to

$$T^{-1-2\delta}\sigma_{\eta T}^2 \xrightarrow{p} \frac{\sigma^2 \Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)} \equiv \theta_\eta^2, \tag{3}$$

say, where $\Gamma(\cdot)$ denotes the gamma or generalized factorial function. Furthermore, under the additional assumption that ε_t verifies $E|\varepsilon_t|^g < \infty$ for $g \geq max\{4, -8\delta/1+2\delta\}$, the following functional central limit theorem due to Davydov (1970) and Taqqu (1975) applies to this type of processes:

$$\sigma_{\eta T}^{-1} S_{[Tr]} \Rightarrow \frac{1}{\Gamma(1+\delta)} \int_0^r (r-s)^\delta dW(s), \quad (\equiv W_\delta(r)), \tag{4}$$

where $W(r)$ is a standard Brownian motion on [0,1] associated with the ε_t sequence and the symbols "$\Rightarrow$" and "$\xrightarrow{p}$" denote weak convergence and convergence in probability, respectively.

Consider now the two steps involved in the *DP* sequential tests. In the first stage, the *t*-ratio of $\hat{\beta}_1$ in the following regression

$$\Delta^2 y_t = \hat{\beta}_1 \Delta y_{t-1} + res., \tag{5}$$

is compared with the corresponding *DF* critical values in a one-sided lower-tail test in order to test the null hypothesis of two unit roots $(y_t \sim I(2))$ against the alternative of a single unit root $(y_t \sim I(1))$. Then, the following theorem holds.

Theorem 1. *Under DGP* (2), *the t-test of* $\hat{\beta}_1$ *in* (5) *verifies that*

(*i*) *if* $0.5 \le d < 1.5$, $t_{\beta_1} = O_p\left(T^{1/2}\right)$ *and* $t_{\beta_1} \xrightarrow{p} -\infty$,

(*ii*) *if* $1.5 \le d < 2$, $t_{\beta_1} = O_p\left(T^{2-d}\right)$ *and* $t_{\beta_1} \xrightarrow{p} -\infty$,

(*iii*) *if* $d = 2$, $t_{\beta_1} = O_p(1)$,

(*iv*) *if* $2 < d < 2.5$, $t_{\beta_1} = O_p\left(T^{d-2}\right)$ *and* $t_{\beta_1} \xrightarrow{p} \infty$, *and*

(*v*) *if* $d \ge 2.5$, $t_{\beta_1} = O_p\left(T^{1/2}\right)$ *and* $t_{\beta_1} \xrightarrow{p} \infty$.

Proof: See Appendix.

As expected, these properties mimic those obtained by Sowell (1990) in the test of the null of a single unit root versus the alternative of stationarity. Thus, the *t*-ratio only has a well-defined asymptotic distribution when $d = 2$, it is a consistent test for $d < 2$ and has zero power when $d > 2$.

Next, if the null hypothesis above is rejected, the second stage in the *DP* procedure proceeds to test the null of $y_t \sim I(1)$ against the alternative of $y_t \sim I(0)$ computing the *t*-ratio of $\hat{\beta}_2$ in the regression model

$$\Delta^2 y_t = \hat{\beta}_1 \Delta y_{t-1} + \hat{\beta}_2 y_{t-1} + res., \tag{6}$$

and comparing it with the *DF* critical value in a one-sided test. In this case, the following theorem applies.

Theorem 2. *Under DGP* (2), *the t-test of* $\hat{\beta}_2$ *in* (6) *verifies that*

(*i*) *if* $0.5 \le d < 1$, $t_{\beta_2} = O_p\left(T^{1-d}\right)$ *and* $t_{\beta_2} \xrightarrow{p} -\infty$,

(*ii*) *if* $d = 1$, $t_{\beta_2} = O_p(1)$,

(*iii*) *if* $1 < d < 1.5$, $t_{\beta_2} = O_p\left(T^{d-1}\right)$ *and* $t_{\beta_2} \xrightarrow{p} \infty$, *and*

(*v*) *if* $d \ge 1.5$, $t_{\beta_2} = O_p\left(T^{1/2}\right)$ *and* $t_{\beta_2} \xrightarrow{p} \infty$.

Proof: See Appendix.

The most remarkable feature of this result is that it mimics the findings in Theorem 1. The relevant t-statistic has only a well-defined limiting distribution when $d = 1$. If $d < 1$, the test will be consistent and if $d > 1$, it will have zero power. Finally, if $1 < d < 2$, the sequential testing procedure will classify the process as an $I(1)$ one. For instance, if $d = 1.8$, then, following Theorem 1, the null hypothesis of two unit roots is asymptotically rejected in a one-sided test whereas, following Theorem 2, the null of a single unit root is not rejected. Consequently, the *DP* test classifies a $NFI(1.8)$ as an $I(1)$ process as the sample size gets sufficiently large. More in general, Theorems 1 and 2 together conform a complete testing-decision sequence as the following figure illustrates:

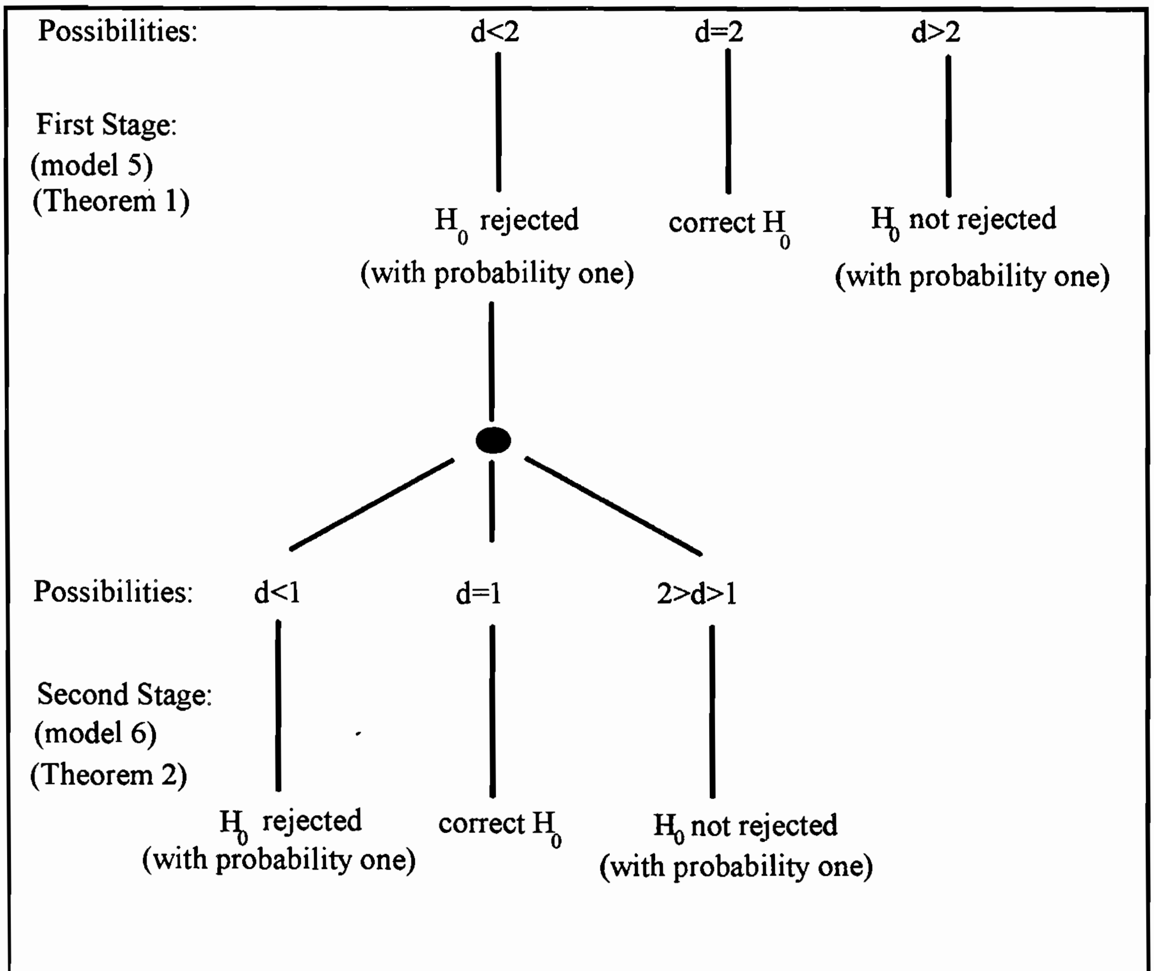

Figure: Testing sequence of the Dickey-Pantula test against *NFI* alternatives.

To check these results, we generated a $NFI(1.8)$ process, based on 5,000 replications with $T = 100$ and $T = 250$ observations, with standard Gaussian innovations. For $T = 100$ and considering one-sided tests with level 0.05[1], the percentage of rejections in the first stage of the *DP* procedure is 39.3 % whilst the corresponding proportion in the second stage is 17.7 %. For $T = 250$, the rejection rates are 68.1 % and 4.3 %, respectively.

In line with the Monte Carlo evidence reported in Diebold and Rudebusch (1991) and Hassler and Wolters (1994), the previous results point out that asymptotic considerations can be severely misleading in finite samples, since the power of unit root tests can be very low within the Euclidean interval $|d - 1/2|$ of the corresponding null hypothesis. As Sowell (1990) conjectures, this is because the limit distributions of the *t*-statistics depend upon two underlying random variables (expressions (A6)-(A8) in Appendix) with a slow rate of convergence to its asymptotic distribution for a plausible range of *d* values.

3. Concluding Remarks

Theorems 1 and 2 characterize the asymptotic behaviour of the *DP* testing sequence under fractional alternatives, extending previous results on the properties of *DF* unit root tests.

Our basic conclusion is that mechanical application of the *DP* procedure can lead to misleading results. Moreover, this conclusion can be extended to other testing approaches within the same family, such as those proposed by Hasza-Fuller (1979), Sen and Dickey (1987) and Haldrup (1994b).

Finally, although, for the sake of brevity, we have confined the results to a maximum of two unit roots, the above findings can be easily generalized to the more general sequential procedure consisting of testing a null hypothesis of *k* unit roots against an alternative of *k*-1 unit roots. Namely, as $T \uparrow \infty$, the sequence will stop when the true *d* verifies the inequality $k - 1 < d < k$.

[1] All computations were done in *S*-plus.

Appendix: Proofs of the theorems

In order to safe notation, stochastic processes such that $W(r)$ or $W_\delta(r)$ will be written as W and W_δ, until otherwise be stated. Similarly, we shall write integrals with respect to Lebesgue measure such as $\int_0^1 W(s)ds$ more simply as $\int W$ and the symbol $\sum_{t=1}^{T}$ will be denoted simply as $\sum$. Equally, $W_\delta^\alpha(r)$ stands for the $(\alpha-1)$-fold integral of $W_\delta(r)$ recursively defined as $W_\delta^\alpha(r)=\int_0^r W_\delta^{\alpha-1}(s)ds$, with $W_\delta^1(r)=W_\delta(r)$.

PROOF OF THEOREM 1. Under model (5), the least squares slope estimate and its corresponding t-ratio have the following expressions:

$$\hat{\beta}_1 = \frac{\sum(\Delta y_{t-1})(\Delta^2 y_t)}{\sum(\Delta y_{t-1})^2}, \tag{A1}$$

and

$$t_{\beta_1} = \frac{\sum(\Delta y_{t-1})(\Delta^2 y_t)}{\hat{\sigma}\left[\sum(\Delta y_{t-1})^2\right]^{1/2}}, \tag{A2}$$

where

$$\hat{\sigma}^2 = T^{-1}\sum\left(\Delta^2 y_t - \hat{\beta}_1 \Delta y_{t-1}\right)^2. \tag{A3}$$

When $d\in\left(\frac{3}{2},\frac{5}{2}\right)$, $d=2+\delta$ and $\Delta y_t \sim NFI(d-1)\equiv NFI(1+\delta)$, i.e., Δy_t satisfies the following difference equation:

$$\Delta y_t = \Delta y_{t-1} + \eta_t = S_t = S_{t-1} + \eta_t = S_{[Tr]} + \eta_t,$$

under the assumptions made on the initial terms, and where $\frac{t-1}{T}\le r<\frac{t}{T}$, for $t=1,2,\ldots,T$. Consequently, with respect to the denominator in (A1), using (3), (4) and the continuous mapping theorem (*CMT*), we get

$$\begin{aligned} T^{2-2d}\sum(\Delta y_{t-1})^2 &= T^{-2-2\delta}\sum(\Delta y_{t-1})^2 = T^{-1-2\delta}\sigma_{\eta T}^2 T^{-1}\sum\left(\sigma_{\eta T}^{-1}\Delta y_{t-1}\right)^2 \\ &= T^{-1-2\delta}\sigma_{\eta T}^2 T^{-1}\sum\left(\sigma_{\eta T}^{-1} S_{[Tr]}\right)^2 \Rightarrow \theta_\eta^2 \int (W_\delta)^2. \end{aligned} \tag{A4}$$

As regards the $\sum(\Delta y_{t-1})(\Delta^2 y_t)$ term, we have that

$$\sum(\Delta y_{t-1})(\Delta^2 y_t) = \sum(\Delta y_{t-1})\eta_t = \frac{1}{2}(\Delta y_T)^2 - \frac{1}{2}\sum \eta_t^2 .$$

The first term when multiplied by $T^{-1-2\delta}$ converges in distribution to $\theta_\eta^2 \frac{1}{2}[W_\delta(1)]^2$ because of the *CMT*, (3) and (4), whilst the limiting distribution of the second term follows by using the ergodic theorem

$$T^{-1}\sum \eta_t^2 \xrightarrow{p} \operatorname{var}(\eta_t) = \frac{\sigma^2\Gamma(1-2\delta)}{\Gamma^2(1-\delta)}, \tag{A5}$$

Therefore, when $\delta = 0$, i.e., when $d = 2$, $\theta_\eta^2 = \sigma^2$, $W_\delta(r) = W(r)$ for $\forall r \in [0,1]$ and

$$T^{-1}\sum(\Delta y_{t-1})(\Delta^2 y_t) \Rightarrow \frac{1}{2}\sigma^2[W(1)]^2 - \frac{1}{2}\sigma^2, \tag{A6}$$

whereas when $\delta > 0$, i.e., when $2 < d < \frac{5}{2}$,

$$T^{-1-2\delta}\sum(\Delta y_{t-1})(\Delta^2 y_t) \Rightarrow \frac{1}{2}\theta_\eta^2[W_\delta(1)]^2, \tag{A7}$$

and when $\delta < 0$, i.e., when $\frac{3}{2} < d < 2$,

$$T^{-1}\sum(\Delta y_{t-1})(\Delta^2 y_t) \xrightarrow{p} -\frac{\sigma^2\Gamma(1-2\delta)}{2\Gamma^2(1-\delta)}. \tag{A8}$$

Hence, using (A4), (A6)-(A8), and the *CMT*, we obtain that

$$T\hat{\beta}_1 = \frac{T^{-1}\sum(\Delta y_{t-1})(\Delta^2 y_t)}{T^{-2}\sum(\Delta y_{t-1})^2} \Rightarrow \frac{\frac{1}{2}[W^2(1)-1]}{\int W^2}, \tag{A9}$$

when $\delta = 0$,

$$T\hat{\beta}_1' = \frac{T^{-1-2\delta}\sum(\Delta y_{t-1})(\Delta^2 y_t)}{T^{-2-2\delta}\sum(\Delta y_{t-1})^2} \Rightarrow \frac{\frac{1}{2}[W_\delta(1)]^2}{\int (W_\delta)^2}, \tag{A10}$$

if $\delta > 0$, and

$$T^{1+2\delta}\hat{\beta}_1 = \frac{T^{-1}\sum(\Delta y_{t-1})(\Delta^2 y_t)}{T^{-2-2\delta}\sum(\Delta y_{t-1})^2} \Rightarrow -\frac{\sigma^2\Gamma(5-2d)}{2\theta_\eta^2\int(W_\delta)^2\Gamma^2(3-d)}, \tag{A11}$$

if $\delta < 0$.

Therefore, when $d \in (\frac{3}{2},\frac{5}{2})$ we observe that $\hat{\beta}_1 \xrightarrow{p} 0$ for all d, even that, when $d \in (\frac{3}{2},2)$, this convergence depends on d. Notice, as well, that if $d \in (2,\frac{5}{2})$, the limiting distribution of $\hat{\beta}_1$ has nonnegative support. Similarly, if $d \in (\frac{3}{2},2)$, the support is nonpositive. If $d = 2$, the support of the limiting distribution of $\hat{\beta}_1$ is the entire real line.

With regard the *t*-Student statistic, first notice that

$$\hat{\sigma}^2 = T^{-1}\sum\left(\Delta^2 y_t - \hat{\beta}_1 \Delta y_{t-1}\right)^2 = T^{-1}\sum \eta_t^2 + T^{-1}\hat{\beta}_1^2\sum(\Delta y_{t-1})^2 - 2T^{-1}\hat{\beta}\sum(\Delta y_{t-1})\eta_t.$$

Hence, by using (A4), (A5), (A9)-(A11) and the *CMT*, it follows that $\hat{\sigma}^2 \xrightarrow{p} \mathrm{var}(\eta_t)$ for all $\delta \in (-\frac{1}{2},\frac{1}{2})$, meaning that

$$t_{\beta_1} \Rightarrow \frac{\left[\frac{1}{2}\{W^2(1)-1\}\right]^{1/2}}{\sigma\left[\int W^2\right]^{1/2}},$$

when $\delta = 0$,

$$T^{2-d} t_{\beta_1} \Rightarrow \frac{\theta_\eta \frac{1}{4} W_\delta(1)\Gamma(3-d)}{\sigma\left[\int (W_\delta)^2\right]^{1/2} \Gamma^{1/2}(5-2d)},$$

when $\delta > 0$, and

$$T^{d-2} t_{\beta_1} \Rightarrow -\frac{\sigma\Gamma^{1/2}(5-2d)}{2\theta_\eta\left[\int (W_\delta)^2\right]^{1/2}\Gamma(3-d)},$$

in the case where $\delta < 0$.

When $d > \frac{5}{2}$, then $\alpha \geq 3$, $\Delta y_t \sim NFI(d-1)$ and $\Delta^2 y_t \sim NFI(d-2)$. In this case, if $\Delta^d y_t$ is decomposed as in (2), then

$$\Delta^{d-1}(\Delta y_t) = \varepsilon_t \Leftrightarrow \Delta^{\alpha-1}(\Delta y_t) = \eta_t \quad \Delta^\delta \eta_t = \varepsilon_t,$$

and

$$\Delta^{d-2}(\Delta^2 y_t) = \varepsilon_t \Leftrightarrow \Delta^{\alpha-2}(\Delta^2 y_t) = \eta_t \quad \Delta^\delta \eta_t = \varepsilon_t,$$

with $\alpha - 2 \geq 1$.

Using (3), (4) and the *CMT*, it is straightforward to prove that

$$T^{1-\alpha}\sigma_{\eta T}^{-1}S_{[Tr]} \Rightarrow W_\delta^\alpha(r), \quad \alpha \geq 1. \tag{A12}$$

Hence,

$$T^{2-2d}\sum(\Delta y_{t-1})^2 = T^{2-2\alpha-2\delta}\sum(\Delta y_{t-1})^2 = T^{-1-2\delta}\sigma_{\eta T}^2 T^{-1}\sum\left(T^{1-(\alpha-1)}\sigma_{\eta T}^{-1}\Delta y_{t-1}\right)^2$$

$$\Rightarrow \theta_\eta^2\int\left(W_\delta^{\alpha-1}\right)^2. \tag{A13}$$

In the same manner, after some manipulation, it follows that

$$\sum(\Delta y_{t-1})(\Delta^2 y_t) = \frac{1}{2}(\Delta y_T)^2 - \frac{1}{2}\sum(\Delta^2 y_t)^2$$

and then,

$$T^{3-2d}\sum(\Delta y_{t-1})(\Delta^2 y_t) = \frac{1}{2}T^{3-2d}(\Delta y_T)^2 + o_p(1)$$

$$= \frac{1}{2}T^{-1-2\delta}\sigma_{\eta T}^2\left[T^{1-(\alpha-1)}\sigma_{\eta T}^{-1}\Delta y_T\right]^2 + o_p(1) \Rightarrow \frac{1}{2}\theta_\eta^2\left[W_\delta^{\alpha-1}(1)\right]^2. \tag{A14}$$

Consequently, using (A1), (A13), (A14) and the *CMT* yields

$$T\hat{\beta}_1 = \frac{T^{3-2d}\sum(\Delta y_{t-1})(\Delta^2 y_t)}{T^{2-2d}\sum(\Delta y_{t-1})^2} \Rightarrow \frac{\left[W_\delta^{\alpha-1}(1)\right]^2}{2\int\left(W_\delta^{\alpha-1}\right)^2}. \tag{A15}$$

In the same manner, using (A13)-(A15) and the *CMT*, it follows that

$$T^{5-2d}\hat{\sigma}^2 = T^{4-2d}\sum(\Delta^2 y_t)^2 + T^2\hat{\beta}_1^2 T^{2-2d}\sum(\Delta y_{t-1})^2 - 2T\hat{\beta}T^{3-2d}\sum(\Delta y_{t-1})(\Delta^2 y_t)$$

$$\Rightarrow \theta_\eta^2\int\left(W_\delta^{\alpha-2}\right)^2 + \theta_\eta^2\frac{\left[W_\delta^{\alpha-1}(1)\right]^4}{4\int\left(W_\delta^{\alpha-1}\right)^2} - \theta_\eta^2\frac{\left[W_\delta^{\alpha-1}(1)\right]^4}{2\int\left(W_\delta^{\alpha-1}\right)^2}$$

$$= \theta_\eta^2\left[\int\left(W_\delta^{\alpha-2}\right)^2 - \frac{\left[W_\delta^{\alpha-1}(1)\right]^4}{4\int\left(W_\delta^{\alpha-1}\right)^2}\right] \equiv \theta_\eta^2\sigma_\infty^2, \tag{A16}$$

say, and then, from (A13), (A14), (A16) and the *CMT*, we get

$$T^{-1/2}t_{\beta_1} = \frac{T^{3-2d}\sum(\Delta y_{t-1})(\Delta^2 y_t)}{T^{5/2-d}\hat{\sigma}T^{1-d}\left[\sum(\Delta y_{t-1})^2\right]^{1/2}} \Rightarrow \frac{\left[W_\delta^{\alpha-1}(1)\right]^2}{2\sigma_\infty\left[\int\left(W_\delta^{\alpha-1}\right)^2\right]^{1/2}}.$$

Finally, when $d \in \left(\frac{1}{2},\frac{3}{2}\right)$, $\Delta y_t = \eta_t \sim SFI(\delta)$ and $\Delta^2 y_t = \Delta\eta_t = \eta_t - \eta_{t-1}$, a non-invertible (but stationary) fractionally integrated process of order $\bar{d} \in \left(-\frac{3}{2},-\frac{1}{2}\right)$, denoted $NIFI(\bar{d})$, with $\bar{d} = \delta - 1$. In this case, notice that

$$\sum(\Delta y_{t-1})(\Delta^2 y_t) = \sum \eta_{t-1}(\Delta\eta_t) = \sum \eta_t\eta_{t-1} - \sum \eta_{t-1}^2,$$

which implies

$$\hat{\beta}_1 = \frac{\sum \eta_t\eta_{t-1} - \sum \eta_{t-1}^2}{\sum \eta_{t-1}^2} = \frac{\sum \eta_t\eta_{t-1}}{\sum \eta_{t-1}^2} - 1. \tag{A17}$$

Therefore,

$$\hat{\beta}_1 = \frac{T^{-1}\sum \eta_t\eta_{t-1}}{T^{-1}\sum \eta_{t-1}^2} - 1 \xrightarrow{p} \frac{\gamma_\eta(1)}{\gamma_\eta(0)} - 1,$$

where

$$\gamma_\eta(j) = E\left(\eta_t\eta_{t-j}\right) = \sigma^2 \frac{\Gamma(j+\delta)\Gamma(1-2\delta)}{\Gamma(j+1-\delta)\Gamma(\delta)\Gamma(1-\delta)} \tag{A18}$$

and then,

$$\hat{\beta}_1 \xrightarrow{p} \frac{2\delta - 1}{1-\delta} < 0. \tag{A19}$$

With respect to the corresponding t-Student statistic, from the manipulation of its expression, we have that

$$t_{\beta_1} = \frac{\sum \eta_t\eta_{t-1}}{\hat{\sigma}\left(\sum \eta_{t-1}^2\right)^{1/2}} - \frac{\left(\sum \eta_{t-1}^2\right)^{1/2}}{\hat{\sigma}}. \tag{A20}$$

As regards the estimator of the variance of the perturbation terms, one gets

$$\hat{\sigma}^2 = T^{-1}\sum(\Delta\eta_t)^2 + \hat{\beta}_1^2 T^{-1}\sum \eta_{t-1}^2 - 2\hat{\beta}_1 T^{-1}\sum \eta_{t-1}\Delta\eta_t$$

$$= T^{-1}\sum \eta_t^2 + \left(\hat{\beta}_1 + 1\right)^2 T^{-1}\sum \eta_{t-1}^2 - 2\left(\hat{\beta}_1 + 1\right)T^{-1}\sum \eta_t\eta_{t-1}$$

which converges in probability to

$$\sigma^2 \frac{(1-2\delta)\Gamma(1-2\delta)}{(1-\delta)^2\Gamma^2(1-\delta)}, \tag{A21}$$

after using (A18), (A19) and the *CMT*. Consequently, from (A18), (A20), (A21) and the *CMT* we finally obtain that $T^{-1/2}t_{\beta_1} = O_p(1)$. Thus, when $d \in \left(\frac{1}{2}, \frac{3}{2}\right)$ the least squares estimator $\hat{\beta}_1$ is inconsistent but the *t*-Student test provides a consistent statistics. ■

PROOF OF THEOREM 2. Under model (6), the least squares slope estimate of y_{t-1} and its corresponding *t*-ratio have the following expressions:

$$\hat{\beta}_2 = \frac{\left[\sum y_{t-1}\Delta^2 y_t\right]\left[\sum (\Delta y_{t-1})^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]\left[\sum \Delta y_{t-1}\Delta^2 y_t\right]}{\left[\sum y_{t-1}^2\right]\left[\sum (\Delta y_{t-1})^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]^2}, \tag{A22}$$

and

$$t_{\beta_2} = \frac{\hat{\beta}_2}{\hat{\sigma}_{\beta_2}}, \tag{A23}$$

with

$$\hat{\sigma}^2_{\beta_2} = \hat{\sigma}^2 \frac{\left[\sum (\Delta y_{t-1})^2\right]}{\left[\sum y_{t-1}^2\right]\left[\sum (\Delta y_{t-1})^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]^2}. \tag{A24}$$

When $d > \frac{5}{2}$, by using (A12) and the *CMT* we get

$$T^{-2d}\sum y_{t-1}^2 = T^{-1-2\delta}\sigma^2_{\eta T} T^{-1}\sum \left(T^{1-\alpha}\sigma^{-1}_{\eta T} y_{t-1}\right)^2 \Rightarrow \theta^2_\eta \int \left(W^\alpha_\delta\right)^2, \tag{A25}$$

$$T^{1-2d}\sum y_{t-1}\Delta y_{t-1} = T^{-1-2\delta}\sigma^2_{\eta T} T^{-1}\sum \left(T^{1-\alpha}\sigma^{-1}_{\eta T} y_{t-1}\right)\left(T^{1-(\alpha-1)}\sigma^{-1}_{\eta T}\Delta y_{t-1}\right)$$
$$\Rightarrow \theta^2_\eta \int W^\alpha_\delta W^{\alpha-1}_\delta \tag{A26}$$

$$T^{1-2d}\sum y_{t-1}\Delta y_t = \frac{1}{2}T^{1-2d}y_T^2 + o_p(1) = \frac{1}{2}T^{-1-2\delta}\sigma^2_{\eta T}\left(T^{1-\alpha}\sigma^{-1}_{\eta T} y_T\right)^2 + o_p(1) \Rightarrow \frac{1}{2}\theta^2_\eta\left[W^\alpha_\delta(1)\right]^2, \tag{A27}$$

and

$$T^{1-2d}\sum y_{t-1}\Delta^2 y_t = T^{1-2d}\sum y_{t-1}\Delta y_t - T^{1-2d}\sum y_{t-1}\Delta y_{t-1} \Rightarrow \frac{1}{2}\theta^2_\eta\left[W^\alpha_\delta(1)\right]^2 - \theta^2_\eta\int W^\alpha_\delta W^{\alpha-1}_\delta. \tag{A28}$$

Then, using (A13), (A14), (A22), (A25)-(A28) and the *CMT*, we have that

$$T\hat{\beta}_2 = \frac{T^{1-2d}\left[\sum y_{t-1}\Delta^2 y_t\right]T^{2-2d}\left[\sum(\Delta y_{t-1})^2\right] - T^{-1}T^{1-2d}\left[\sum y_{t-1}\Delta y_{t-1}\right]T^{3-2d}\left[\sum \Delta y_{t-1}\Delta^2 y_t\right]}{T^{-2d}\left[\sum y_{t-1}^2\right]T^{2-2d}\left[\sum(\Delta y_{t-1})^2\right] - T^{2-4d}\left[\sum y_{t-1}\Delta y_{t-1}\right]^2}$$

$$\Rightarrow \frac{\left[\frac{1}{2}\left[W_\delta^\alpha(1)\right]^2 - \int W_\delta^\alpha W_\delta^{\alpha-1}\right]\left[\int (W_\delta^{\alpha-1})^2\right]}{\left[\int (W_\delta^\alpha)^2\right]\left[\int (W_\delta^{\alpha-1})^2\right] - \left[\int W_\delta^\alpha W_\delta^{\alpha-1}\right]^2} \equiv \beta_{2\infty}. \tag{A29}$$

Equally, given that

$$\hat{\beta}_1 = \frac{\left[\sum \Delta y_{t-1}\Delta^2 y_t\right]\left[\sum y_{t-1}^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]\left[\sum y_{t-1}\Delta^2 y_t\right]}{\left[\sum y_{t-1}^2\right]\left[\sum(\Delta y_{t-1})^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]^2}, \tag{A30}$$

it is straightforward to prove that

$$\hat{\beta}_1 \Rightarrow \frac{\left[\int W_\delta^\alpha W_\delta^{\alpha-1}\right]^2 - \frac{1}{2}\left[W_\delta^\alpha(1)\right]^2 \int W_\delta^\alpha W_\delta^{\alpha-1}}{\left[\int (W_\delta^\alpha)^2\right]\left[\int (W_\delta^{\alpha-1})^2\right] - \left[\int W_\delta^\alpha W_\delta^{\alpha-1}\right]^2} \equiv \beta_{1\infty}. \tag{A31}$$

In the same manner, given that

$$\begin{aligned}\hat{\sigma}^2 &= T^{-1}\sum\left(\Delta^2 y_t - \hat{\beta}_1\Delta y_{t-1} - \hat{\beta}_2 y_{t-1}\right)^2 \\ &= T^{-1}\sum(\Delta^2 y_t)^2 + \hat{\beta}_1^2 T^{-1}\sum(\Delta y_{t-1})^2 - 2\hat{\beta}_1 T^{-1}\sum \Delta^2 y_t \Delta y_{t-1} \\ &\quad + \hat{\beta}_2^2 T^{-1}\sum y_{t-1}^2 - 2\hat{\beta}_2 T^{-1}\sum y_{t-1}\Delta^2 y_t + 2\hat{\beta}_1\hat{\beta}_2 T^{-1}\sum y_{t-1}\Delta y_{t-1}\end{aligned}$$

it follows that

$$\begin{aligned}T^{3-2d}\hat{\sigma}^2 &\Rightarrow \theta_\eta^2\Big\{\beta_{1\infty}^2 \int (W_\delta^{\alpha-1})^2 + \beta_{2\infty}^2 \int (W_\delta^\alpha)^2 - \beta_{2\infty}\left[W_\delta^\alpha(1)\right]^2 \\ &\quad + 2\beta_{2\infty}\int W_\delta^\alpha W_\delta^{\alpha-1} + 2\beta_{1\infty}\beta_{2\infty}\int W_\delta^\alpha W_\delta^{\alpha-1}\Big\} \equiv \theta_\eta^2 \sigma_\infty^2,\end{aligned} \tag{A32}$$

which, in turn, from (A24) implies that

$$T^3\hat{\sigma}_{\hat{\beta}_2}^2 = T^{3-2d}\hat{\sigma}^2 \frac{T^{2-2d}\left[\sum(\Delta y_{t-1})^2\right]}{T^{2-4d}\left\{\left[\sum y_{t-1}^2\right]\left[\sum(\Delta y_{t-1})^2\right] - \left[\sum y_{t-1}\Delta y_{t-1}\right]^2\right\}}$$

$$\Rightarrow \sigma_\infty^2 \theta_\eta^2 \frac{\int \left(W_\delta^{\alpha-1}\right)^2}{\left[\int \left(W_\delta^\alpha\right)^2\right]\left[\int \left(W_\delta^{\alpha-1}\right)^2\right]-\left[\int W_\delta^\alpha W_\delta^{\alpha-1}\right]^2} \equiv \sigma_{\beta_2}^2 \theta_\eta^2$$

and hence,

$$T^{-1/2} t_{\beta_2} \Rightarrow \frac{\beta_{2\infty}}{\sigma_{\beta_2}^2 \theta_\eta^2}.$$

When $d \in \left(\frac{3}{2}, \frac{5}{2}\right)$, $y_t \sim NFI(d)$, $\Delta y_t \sim NFI(d-1)$, $\Delta^2 y_t \sim SFI(d-2) = SFI(\delta)$ and $\alpha = 2$. This, in turn, implies that

$$T^{-2d} \sum y_{t-1}^2 = T^{-1-2\delta} \sigma_{\eta T}^2 T^{-1} \sum \left(T^{-1} \sigma_{\eta T}^{-1} y_{t-1}\right)^2 \Rightarrow \theta_\eta^2 \int \left(W_\delta^2\right)^2, \tag{A33}$$

$$T^{1-2d} \sum y_{t-1} \Delta y_{t-1} = T^{-1-2\delta} \sigma_{\eta T}^2 T^{-1} \sum \left(T^{-1} \sigma_{\eta T}^{-1} y_{t-1}\right)\left(\sigma_{\eta T}^{-1} \Delta y_{t-1}\right)$$
$$\Rightarrow \theta_\eta^2 \int W_\delta^2 W_\delta, \tag{A34}$$

and

$$T^{2-2d} \sum y_{t-1} \eta_t \Rightarrow \theta_\eta^2 \left\{ W_\delta(1) \int W_\delta - \int W_\delta^2 \right\}. \tag{A35}$$

Collecting all the preceding results, it is rather direct to prove that, for $d \in \left(\frac{3}{2}, \frac{5}{2}\right)$, $T^2 \hat{\beta}_2 = O_p(1)$, $T^{3-2d} \hat{\sigma}^2 = O_p(1)$, $T^3 \hat{\sigma}_{\beta_2} = O_p(1)$ and $T^{-1/2} t_{\beta_2} = O_p(1)$.

Finally, in the case where $d \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $y_t \sim NFI(d)$, $\Delta y_t \sim SFI(d-1) = SFI(\delta)$, $\Delta^2 y_t \sim NIFI(d-2) = NIFI(\delta - 1)$, with $\alpha = 1$. This allows to rewrite $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}^2$ after some direct but tedious calculus as follows:

$$\hat{\beta}_2 = \frac{\left[\sum y_{t-1}\eta_t\right]\left[\sum \eta_{t-1}^2 - \sum \eta_t \eta_{t-1}\right] - \left[\sum \eta_t \eta_{t-1}\right]\left[\sum \eta_{t-1}^2\right]}{\left[\sum y_{t-1}^2\right]\left[\sum \eta_{t-1}^2\right] - \left[\sum y_{t-1}\eta_t\right]^2 - \left[\sum \eta_{t-1}^2\right]^2 - 2\left[\sum y_{t-1}\eta_t\right]\left[\sum \eta_{t-1}^2\right]}, \tag{A36}$$

$$\hat{\beta}_1 = \frac{\left[\sum y_{t-1}^2\right]\left[\sum \eta_t \eta_{t-1} - \sum \eta_{t-1}^2\right] - \left[\sum y_{t-1}\eta_t\right]^2 - \left[\sum y_{t-1}\eta_t\right]\left[\sum \eta_{t-1}^2\right]}{\left[\sum y_{t-1}^2\right]\left[\sum \eta_{t-1}^2\right] - \left[\sum y_{t-1}\eta_t\right]^2 - \left[\sum \eta_{t-1}^2\right]^2 - 2\left[\sum y_{t-1}\eta_t\right]\left[\sum \eta_{t-1}^2\right]}, \tag{A37}$$

and

$$\hat{\sigma}^2 = 2T^{-1} \sum \eta_{t-1}^2 - 2T^{-1} \sum \eta_t \eta_{t-1} + \hat{\beta}_1^2 T^{-1} \sum \eta_{t-1}^2$$
$$-2\hat{\beta}_1 T^{-1} \sum \eta_t \eta_{t-1} + 2\hat{\beta}_1 T^{-1} \sum \eta_{t-1}^2 + \hat{\beta}_2^2 T^{-1} \sum y_{t-1}^2 - 2\hat{\beta}_2 T^{-1} \sum \eta_t \eta_{t-1}$$
$$+2\hat{\beta}_2 T^{-1} \sum \eta_{t-1}^2 + 2\hat{\beta}_1 \hat{\beta}_2 T^{-1} \sum y_{t-1} \eta_t + 2\hat{\beta}_1 \hat{\beta}_2 T^{-1} \sum \eta_{t-1}^2 . \tag{A38}$$

Therefore, using (A4)-(A8), the weak law of large numbers and the *CMT*, it can be proved that when $d = 1$, $T\hat{\beta}_2 = O_p(1)$, $\hat{\beta}_1 = O_p(1)$, $\hat{\sigma}^2 = O_p(1)$ and $t_{\beta_2} = O_p(1)$, while that in the case where $1 < d < \frac{3}{2}$, $T\hat{\beta}_2 = O_p(1)$, $\hat{\beta}_1 = O_p(1)$, $\hat{\sigma}^2 = O_p(1)$, $T^{1-d} t_{\beta_2} = O_p(1)$ and $t_{\beta_2} \xrightarrow{p} \infty$. Conversely, in the case where $\frac{1}{2} < d < 1$, $\hat{\beta}_1 = O_p(1)$, $\hat{\sigma}^2 = O_p(1)$ and

$$T^{2d-1}\hat{\beta}_2 \Rightarrow -\frac{\gamma_\eta(0)}{\theta_\eta^2 \int (W_\delta)^2},$$

so that the limiting distribution of $\hat{\beta}_2$ has nonpositive support. This, in turn, induces the fact that its corresponding t-test be $T^{d-1} t_{\beta_2} = O_p(1)$ and $t_{\beta_2} \xrightarrow{p} -\infty$.

Lastly, it is also of interest to derive the asymptotic behaviour of t_{β_1} in model (6). In this case, and collecting all the above results, the following results can be proved: if $d \geq 2.5$, $t_{\beta_1} = O_p(T^{1/2})$ and $t_{\beta_1} \xrightarrow{p} \infty$; if $1.75 < d < 2.5$, $t_{\beta_1} = O_p(T^{7/2-2d})$ and $t_{\beta_1} \xrightarrow{p} 0$; if $d = 1.75$, $t_{\beta_1} = O_p(1)$; if $1.5 \leq d < 1.75$, $t_{\beta_1} = O_p(T^{7/2-2d})$ and $t_{\beta_1} \xrightarrow{p} \infty$ and if $0.5 \leq d < 1.5$, $t_{\beta_1} = O_p(T^{1/2})$ and $t_{\beta_1} \xrightarrow{p} \infty$. ■

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