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Towards Galois Connections over Positive Semifields

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Abstract. In this paper we try to extend the Galois connection construction of $\mathcal{K}$-Formal Concept Analysis to handle semifields which are not idempotent. Important examples of such algebras are the extended non-negative reals and the extended non-negative rationals, but we provide a construction that suggests that such semifields are much more abundant than suspected. This would broaden enormously the scope and applications of $\mathcal{K}$-Formal Concept Analysis.

Keywords: Formal Concept Analysis extensions; $\mathcal{K}$-Formal Concept Analysis; positive semifields; Galois Connection.

1 Introduction and Motivation

The most orthodox presentation of Standard FCA is still [1], whose Galois connection is interpreted as between the set of subsets of objects and attributes. But Standard FCA can also be understood in the context of the linear algebra of boolean spaces with sets substituted for characteristic functions, and other extensions, e.g. the $\mathcal{K}$-FCA [2, 3, 4], FCA in a fuzzy setting [5], etc., can also be considered in the light of linear algebra over a certain subclass of semirings. In this paper, we will understand a *semiring* [6] to be an algebra $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e\rangle$ for which

- the additive structure, $\langle S, \oplus, \epsilon\rangle$, is a commutative monoid,
- the multiplicative structure, $\langle S\backslash\{\epsilon\}, \otimes, e\rangle$, is a monoid,
- multiplication distributes over addition from right and left
- and the zero element is multiplicatively-absorbing i.e. $\forall a \in S,\ \epsilon \otimes a = \epsilon$.

We will only consider commutative semirings, those whose multiplicative structure is commutative.

Specifically, every commutative semiring accepts a canonical preorder, $a \leq b$ if and only if there exists $c \in D$ with $a \oplus c = b$. A *dioid* is a commutative semiring $\mathcal{D}$ where this relation is actually an order. And in fact the mentioned extensions

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to FCA all feature subtypes of dioids (e.g. $\mathcal{K}$-FCA or FCA in a fuzzy setting) or dioid-valued entries (e.g. interval-based FCA) in their formal contexts.

One of the most useful extensions to FCA uses $\mathcal{K}$-valued formal contexts where $\mathcal{K}$ is a complete idempotent semifield: this is the basis of $\mathcal{K}$-Formal Concept Analysis [2, 3, 4]. An idempotent semiring is one whose addition is idempotent, $u \oplus u = u$ while semifields are semirings whose multiplicative structure is a group. Idempotent semifields like $\overline{\mathbb{R}}_{\max,+}$ and $\overline{\mathbb{R}}_{\min,\times}$ are within this class, but also the semifields of non-negative rationals $\mathbb{Q}_0^+$ and that of completed nonnegative reals $\mathbb{R}_{\geq 0} = \langle [0, \infty], +, *, \bot = 0, e = 1, \top = \infty \rangle$ which is complete and totally ordered in its usual order.

It would be interesting to know whether dioids, in general, are FCA-generating, but for this paper we consider only a proper subclass of dioids: its intersection with the class of semifields, the "positive" semifields.

Regarding this extension we may wonder,

1. whether there are many positive semifields available that generate such extension of FCA.
2. whether the extension is useful, that is whether there are instances of FCA-related problems that are solved with positive semifields,

In this paper we try to address both these concerns. First, we review the theory of dioids and semimodules over them with an emphasis on semifields; next we present and approach to generating semifields and examples of their ubiquity. We also review the construction of Galois Connections from residuated semirings and finally provide a new application of this construction to matrix decomposition.

2 Positive Semifields and Semimodules

2.1 Dioids and positive semifields

Complete and positive dioids. Recall that a *dioid* is a commutative semiring $\mathcal{D}$ where the canonical preorder relation, $a \preccurlyeq b$ if and only if there exists $c \in D$ with $a \oplus c = b$ is actually an order $\langle D, \preccurlyeq \rangle$. For this order, the additive zero is always the bottom $\bot = \wedge D = \epsilon$.

In a dioid, the canonical orden relation is compatible with both $\oplus$ and $\otimes$ [7, Chap. 1, Prop. 6.1.7]. Dioids are all zero-sum free, that is, they have no non-null additive factors of zero: if $a, b \in D, a \oplus b = \epsilon$ then $a = \epsilon$ and $b = \epsilon$.

A dioid is *complete* if it is complete as an ordered set for the canonical order relation, and the following distributivity properties hold, for all $A \subseteq D, b \in D$,

$$\left(\bigoplus_{a \in A} a\right) \otimes b = \bigoplus_{a \in A} (a \otimes b) \qquad b \otimes \left(\bigoplus_{a \in A} a\right) = \bigoplus_{a \in A} (b \otimes a) \tag{1}$$

In complete dioids, there is already a top element $\top = \oplus_{a \in D} a$.

A semiring is e*ntire or zero-divisor free* if $a \otimes b = \epsilon$ implies $a = \epsilon$ or $b = \epsilon$. If the dioid is entire, its order properties justifies calling it a *positive dioid* or *information algebra* [7].

Positive semifields. A semifield, as mentioned in the introduction, is a semiring whose multiplicative structure $\langle K \setminus \{\epsilon\}, \otimes, e, \cdot^{-1} \rangle$ is a group, where $\cdot^{-1} : K \to K$ is the function to calculate the inverse such that $\forall u \in K, u \otimes u^{-1} = e$. Since all semifields are entire, dioids that are at the same time semifields are called *positive semifields*, of which the positive reals or rationals are a paragon.

Example 1 (Semifield of non-negative reals). Since we need the completion property to develop Galois connections, we concentrate on the completed nonnegative reals

$$\mathbb{R}_{\geq 0} = \langle [0, \infty], +, \times, \cdot^{-1}, \bot = 0, e = 1, \top = \infty \rangle$$

which is complete and totally ordered in its usual order $\langle \mathbb{R}_{\geq 0}, \leq \rangle$. Note that the multiplicative $\langle (0, \infty), \times \rangle$ structure must exclude also the infinity, since $0 \times \infty = 0$ does not have an inverse. □

Regarding the intrinsic usefulness of positive semifields that are not fields, and apart from the trivial case of $\mathbb{B}$ the booleans, there is not doubt of their usefulness: the best known semifield $\mathbb{R}_{\geq 0}$ is widely used, for instance, in Electrical Network theory, where the series or parallel addition of resistances and conductances is carried out entirely in it.

Pairs of mutually inverse semifields. In fact, this application provides a way forward regarding our first difficulty, to wit the fact that the idempotent semifields used in $\mathcal{K}$-FCA always comes in pairs: $\overline{\mathbb{R}}_{\max,+}$ and $\overline{\mathbb{R}}_{\min,+}$ or $\overline{\mathbb{R}}_{\max,\times}$ and $\overline{\mathbb{R}}_{\min,\times}$. Each member of these pairs appears as the dual of the other member by means of the $\cdot^{-1}$ involution so $\overline{\mathbb{R}}_{\max,+} = (\overline{\mathbb{R}}_{\min,+})^{-1}$ and $\overline{\mathbb{R}}_{\min,+} = (\overline{\mathbb{R}}_{\max,+})^{-1}$.

To settle notation straight, these semifields come in pairs $\left(\overline{\mathcal{K}}, (\overline{\mathcal{K}})^{-1}\right)$ with dual order structures $\overline{\mathcal{K}} = \langle K, \preccurlyeq \rangle$ and $(\overline{\mathcal{K}})^{-1} = \langle K, \succcurlyeq \equiv \preccurlyeq^{\delta} \rangle$, and dual algebraic structures: suppose that $\{\bot, \top\} \subseteq K$, then

$$\overline{\mathcal{K}} = \langle K, \underset{\cdot}{\oplus}, \underset{\cdot}{\otimes}, \cdot^{-1}, \bot, e, \top \rangle \qquad \overline{\mathcal{K}}^{-1} = \langle K, \dot{\oplus}, \dot{\otimes}, \cdot^{-1}, \top, e, \bot \rangle \tag{2}$$

On top of the individual laws as positive semifields, we have the modular laws:

$$(u \underset{\cdot}{\oplus} v) \underset{\cdot}{\otimes} (u \dot{\oplus} v) = u \underset{\cdot}{\otimes} v \qquad (u \underset{\cdot}{\oplus} v) \dot{\otimes} (u \dot{\oplus} v) = u \dot{\otimes} v$$

the analogues of the De Morgan laws:

$$u \underset{\cdot}{\oplus} v = (u^{-1} \dot{\oplus} v^{-1})^{-1} \qquad u \dot{\oplus} v = (u^{-1} \underset{\cdot}{\oplus} v^{-1})^{-1}$$

$$u \underset{\cdot}{\otimes} v = (u^{-1} \dot{\otimes} v^{-1})^{-1} \qquad u \dot{\otimes} v = (u^{-1} \underset{\cdot}{\otimes} v^{-1})^{-1}$$

and the self-dual inequality

$$(u \underset{\cdot}{\otimes} v) \dot{\otimes} w \succcurlyeq u \underset{\cdot}{\otimes} (v \dot{\otimes} w)$$

Note that:

- the notation to "speak" about these semirings tries to follow a convention reminiscent of that of boolean algebra, where the inversion is complement.
- the dot notation is a mnemonic for where do the multiplication of the bottom and top go:

$$\bot \underset{\cdot}{\otimes} \top = \bot \qquad\qquad \bot \dot{\otimes} \top = \top$$

  implying that the "lower" addition and multiplication are aligned with the usual order in the semiring while the "upper" addition and multiplication are aligned with its dual.

Example 2 (Dual semifields for the Non-negative Reals). The previous procedure shows that there are some problems with the notation of Example 1, and this led to the definition of the following signatures for this semifield and its inverse in convex analysis [8]:

$$\mathbb{R}_{\geq 0} = \langle [0, \infty], \underset{\cdot}{+}, \underset{\cdot}{\times}, \cdot^{-1}, 0, 1, \infty \rangle \qquad \mathbb{R}^{-1}_{\geq 0} = \langle [0, \infty], \dot{+}, \dot{\times}, \cdot^{-1}, \infty, 1, 0 \rangle \tag{3}$$

Both of these algebras are used, for instance, in (Electrical) Network Analysis: the algebra of complete positive reals to carry out the series sum of resistances, and its dual semifield to carry out parallel summation of resistances. With the convention that $\mathbb{R}_{\geq 0}$ semiring models *resistances*, it is easy to see that the bottom element, $\bot = 0$ models a shortcircuit, that the top element $\top = \infty$ models an open circuit (infinite resistance) and these conventions are swapped in the dual semifield of *conductances*. Interestingly, the required formulae for the multiplication of the extremes:

$$0 \underset{\cdot}{\otimes} \infty = 0 \qquad\qquad 0 \dot{\otimes} \infty = \infty \tag{4}$$

are a no-go for circuit analysis, which suggests that what is actually being operated with are the incomplete versions of these semifields, and the many problems that EE students have in learning how to properly deal with these values may stem from this fact. □

2.2 Semimodules over positive semifields

Let $\mathcal{D} = \langle D, +, \times, \epsilon_D, e_D \rangle$ be a commutative semiring. A $\mathcal{D}$-*semimodule* $\mathcal{X} = \langle X, \oplus, \odot, \epsilon_X \rangle$ is a commutative monoid $\langle X, \oplus, \epsilon_X \rangle$ endowed with a scalar action $(\lambda, x) \mapsto \lambda \odot x$ satisfying the following conditions for all $\lambda, \mu \in D$, $x, x' \in X$:

$$\begin{aligned} (\lambda \times \mu) \odot x &= \lambda \odot (\mu \odot x) & \lambda \odot (x \oplus x') &= \lambda \odot x \oplus \lambda \odot x' \\ (\lambda + \mu) \odot x &= \lambda \odot x \oplus \mu \odot x & \lambda \odot \epsilon_X &= \epsilon_X = \epsilon_D \otimes x \\ e_D \odot x &= x \end{aligned} \tag{5}$$

Matrices form a $\mathcal{D}$-semimodule $D^{g \times m}$ for given g, m. In this paper, we only use finite-dimensional semimodules where we can identify semimodules with column vectors, e.g. $\mathcal{X} \equiv \mathcal{D}^g$. If $\mathcal{D}$ is commutative, naturally-ordered or complete, then $\mathcal{X}$ is also commutative, naturally-ordered or complete [9]. If $\mathcal{K}$ is a semifield, we may also define an inverse for the semimodule by the coordinate-wise inversion, $(x^{-1})_i = (x_i)^{-1}$.

Similarly, the may define a *matrix conjugate* $(A^{\circledast})_{ij} = A^{-1}_{ji}$. For complete idempotent semifields, the following matrix algebra equations are proven in [10, Ch.8]:

Proposition 1. *Let $\mathcal{K}$ be an idempotent semifield, and $A \in \mathcal{K}^{m\times n}$. Then:*

1. $A \underset{\cdot}{\otimes} (A^{\circledast} \dot{\otimes} A) = A \dot{\otimes} (A^{\circledast} \underset{\cdot}{\otimes} A) = (A \dot{\otimes} A^{\circledast}) \underset{\cdot}{\otimes} A = (A \underset{\cdot}{\otimes} A^{\circledast}) \dot{\otimes} A = A$ *and* $A^{\circledast} \underset{\cdot}{\otimes} (A \dot{\otimes} A^{\circledast}) = A^{\circledast} \dot{\otimes} (A \underset{\cdot}{\otimes} A^{\circledast}) = (A^{\circledast} \dot{\otimes} A) \underset{\cdot}{\otimes} A^{\circledast} = (A^{\circledast} \underset{\cdot}{\otimes} A) \dot{\otimes} A^{\circledast} = A^{\circledast}$.
2. *Alternating $A - A^{\circledast}$ products of 4 matrices can be shortened as in:*
$$A^{\circledast} \dot{\otimes} (A \underset{\cdot}{\otimes} (A^{\circledast} \dot{\otimes} A)) = A^{\circledast} \dot{\otimes} A = (A^{\circledast} \dot{\otimes} A) \underset{\cdot}{\otimes} (A^{\circledast} \dot{\otimes} A)$$
3. *Alternating $A - A^{\circledast}$ products of 3 matrices and another terminal, arbitrary matrix can be shortened as in:*
$$A^{\circledast} \dot{\otimes} (A \underset{\cdot}{\otimes} (A^{\circledast} \dot{\otimes} M)) = A^{\circledast} \dot{\otimes} M = (A^{\circledast} \dot{\otimes} A) \underset{\cdot}{\otimes} (A^{\circledast} \dot{\otimes} M)$$
4. *The following inequalities apply:*
$$A^{\circledast} \dot{\otimes} (A \underset{\cdot}{\otimes} M) \geq M \qquad\qquad A^{\circledast} \underset{\cdot}{\otimes} (A \dot{\otimes} M) \leq M$$

2.3 Galois Connections over Idempotent Semifields

In this paper we presuppose the setting of [11]. When $\overline{\mathcal{K}}$ is a completed idempotent semifield and $\mathcal{X} \equiv \overline{\mathcal{K}}^g$ and $\mathcal{Y} \equiv \overline{\mathcal{K}}^m$ are idempotent vectors spaces or semimodules, the definition of the Galois connection involves the use of a scalar product $\langle \cdot \mid R \mid \cdot \rangle : X \times Y \to K$ and a scalar $\varphi \in \overline{\mathcal{K}}$ [4]:

$$x^{\uparrow}_{R,\varphi} = \vee\{y \in Y \mid \langle x \mid R \mid y\rangle \leq \varphi\} \qquad y^{\downarrow}_{R,\varphi} = \vee\{x \in X \mid \langle x \mid R \mid y\rangle \leq \varphi\}$$

This definition is quite general and might even be valid for any dioid, but we now want to use it when the semiring has the richer algebraic structure of a complete positive semifield. For simplicity's sake we will consider in this paper that $\varphi = e$. Generalizing it along the lines of [11, §3.1] is not difficult.

We consider the scalar product $\langle x \mid R \mid y\rangle = x^{\mathrm{T}} \underset{\cdot}{\otimes} R \underset{\cdot}{\otimes} y$, where $R \in \overline{\mathcal{K}}^{g\times m}$. Since $x^{\mathrm{T}} \underset{\cdot}{\otimes} R \underset{\cdot}{\otimes} y \leq e \Leftrightarrow y^{\mathrm{T}} \underset{\cdot}{\otimes} R^{\mathrm{T}} \underset{\cdot}{\otimes} x \leq e$, by using residuation we may write:

$$x^{\uparrow}_{R} = (x^{\mathrm{T}} \underset{\cdot}{\otimes} R) \underset{\cdot}{\backslash} e = R^{\circledast} \dot{\otimes} x^{-1} \qquad y^{\downarrow}_{R} = (y^{\mathrm{T}} \underset{\cdot}{\otimes} R^{\mathrm{T}}) \underset{\cdot}{\backslash} e = R^{-1} \dot{\otimes} y^{-1} \tag{6}$$

involving only transposition, inversion and operation in the dual semifield.

We recall the following proposition:

Proposition 2. $(\cdot^{\uparrow}_{R}, \cdot^{\downarrow}_{R}) : \mathcal{X} \rightharpoonup \mathcal{Y}$ *is a Galois connection between the semimodules* $\mathcal{X} \cong \overline{\mathcal{K}}^{g}$ *and* $\mathcal{Y} \cong \overline{\mathcal{K}}^{m}$*: for* $x \in X$*,* $y \in Y$*, we have* $y \leq x^{\uparrow}_{R} \Leftrightarrow x \leq y^{\downarrow}_{R}$ *.*

Proof. We need only prove in one sense, since the other is similar. If $y \leq x^{\uparrow}_{R} = R^{\circledast} \dot{\otimes} x^{-1}$, then by inversion, $R^{\mathrm{T}} \underset{\cdot}{\otimes} x \leq y^{-1}$ whence, by residuation $x \leq R^{\mathrm{T}} \underset{\cdot}{\setminus} y^{-1} = R^{-1} \dot{\otimes} y^{-1} = y^{\downarrow}_{R}$. □

The diagram in Fig. 1 summarizes this Galois connection [4] This immediately

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, bend left, "\cdot^{\uparrow}_{R}"] \arrow[d, bend right, "\pi_{R^{\mathrm{T}}}"'] \arrow[drr, "\cdot^{\uparrow}_{R}"] & & \mathcal{Y} \arrow[ll, bend left, "\cdot^{\downarrow}_{R}"] \arrow[d, bend right, "\pi_{R}"'] \arrow[dll, "\cdot^{\downarrow}_{R}"'] \\
\underline{\mathfrak{B}}^{\gamma}_{G}(G,M,R)_{\mathcal{K}} \arrow[u, bend right, "\hookrightarrow_{\mathcal{X}}"'] \arrow[rr, bend left, "\cdot^{\uparrow}_{R}"] & & \underline{\mathfrak{B}}^{\mu}_{M}(G,M,R)_{\mathcal{K}} \arrow[u, bend right, "\hookrightarrow_{\mathcal{Y}}"'] \arrow[ll, bend left, "\cdot^{\downarrow}_{R}"]
\end{tikzcd}$$

Fig. 1: $(\cdot^{\uparrow}_{R}, \cdot^{\downarrow}_{R}) : \mathcal{X} \rightharpoonup \mathcal{Y}$, the Galois connection between positive spaces.

puts at our disposal a number of results which we collect in the following proposition:

Proposition 3. *Consider the Galois connection* $(\cdot^{\uparrow}_{R}, \cdot^{\downarrow}_{R}) : \mathcal{X} \rightharpoonup \mathcal{Y}$ *. Then:*

1. *The polars are antitone, join-inverting functions:*

$$(x_1 \underset{\cdot}{\oplus} x_2)^{\uparrow}_{R} = x_1{}^{\uparrow}_{R} \dot{\oplus} x_2{}^{\uparrow}_{R} \qquad (y_1 \underset{\cdot}{\oplus} y_2)^{\downarrow}_{R} = y_1{}^{\downarrow}_{R} \dot{\oplus} y_2{}^{\downarrow}_{R} \,. \tag{7}$$

2. *The compositions of the polars:* $\pi_{R^{\mathrm{T}}} : X \to X$ *,* $\pi_{R} : Y \to Y$

$$\pi_{R^{\mathrm{T}}}(x) = (x^{\uparrow}_{R})^{\downarrow}_{R} = R^{-1} \dot{\otimes} (R^{\mathrm{T}} \underset{\cdot}{\otimes} x) \qquad \pi_{R}(y) = (y^{\downarrow}_{R})^{\uparrow}_{R} = R^{\circledast} \dot{\otimes} (R \underset{\cdot}{\otimes} y)$$

are closures, that is, extensive and idempotent operators.

$$\begin{aligned} \pi_{R^{\mathrm{T}}}(x) &\geq x & \qquad \pi_{R}(y) &\geq y \\ \pi_{R^{\mathrm{T}}}(\pi_{R^{\mathrm{T}}}(x)) &= \pi_{R^{\mathrm{T}}}(x) & \qquad \pi_{R}(\pi_{R}(y)) &= \pi_{R}(y) \end{aligned}$$

3. *The polars are mutual pseudo-inverses:*

$$(\cdot)^{\uparrow}_{R} \circ (\cdot)^{\downarrow}_{R} \circ (\cdot)^{\uparrow}_{R} = (\cdot)^{\uparrow}_{R} \qquad (\cdot)^{\downarrow}_{R} \circ (\cdot)^{\uparrow}_{R} \circ (\cdot)^{\downarrow}_{R} = (\cdot)^{\downarrow}_{R}$$

One of the advantages of working in idempotent semimodules is that we can strengthen statement 1 in Proposition 3 to reveal that the polars are idempotent semimodule morphisms:

Proposition 4. *The polar of intents of the Galois connection transforms a $\overline{\mathcal{K}}$-semimodule of extents into a $\overline{\mathcal{K}}^{-1}$-semimodule of intents, and dually for the polar of the extents.*

Proof. For linearity, consider $x_1{}^{\uparrow}_{R} = R^{\circledast} \dot{\otimes} x_1^{-1}$ and $x_2{}^{\uparrow}_{R} = R^{\circledast} \dot{\otimes} x_2^{-1}$.

$$\begin{aligned}
(\lambda_1 \underset{\cdot}{\otimes} x_1 \underset{\cdot}{\oplus} \lambda_2 \underset{\cdot}{\otimes} x_2)^{\uparrow}_{R} &= R^{\circledast} \dot{\otimes} (\lambda_1 \underset{\cdot}{\otimes} x_1 \underset{\cdot}{\oplus} \lambda_2 \underset{\cdot}{\otimes} x_2)^{-1} = \\
&= R^{\circledast} \dot{\otimes} (\lambda_1^{-1} \dot{\otimes} x_1^{-1} \dot{\oplus} \lambda_2^{-1} \dot{\otimes} x_2^{-1}) = \\
&= (\lambda_1^{-1} \dot{\otimes} R^{\circledast} \dot{\otimes} x_1^{-1}) \dot{\oplus} (\lambda_2^{-1} \dot{\otimes} R^{\circledast} \dot{\otimes} x_1^{-1}) = \\
&= (\lambda_1^{-1} \dot{\otimes} x_1{}^{\uparrow}_{R}) \dot{\oplus} (\lambda_2^{-1} \dot{\otimes} x_2{}^{\uparrow}_{R}) \,.
\end{aligned}$$

For the polar of extents the proof is similar. □

Note that this is the $\mathcal{K}$-FCA analogue of the fact that the polars are join-inverting. But the novelty is that the scalings for one semimodule and the other are inverted. We need two more results from [11]:

Lemma 1. *Let I_G and I_M be the identity matrices of dimension $g \times g$ and $m \times m$ in $\overline{\mathcal{K}}$. Then the object- and attribute-concepts of the Galois connection are:*

$$\gamma_R(I_G) = (R^{-1} \dot{\otimes} R^{\mathrm{T}}, R^{\circledast}) \qquad\qquad \mu_R(I_M) = (R^{-1}, R^{\circledast} \dot{\otimes} R)$$

taken as pairs of co-indexed vectors.

Corollary 1. *Consider the Galois connection $(\cdot^{\uparrow}_{R}, \cdot^{\downarrow}_{R}) : \widetilde{\mathcal{X}}^{\gamma} \rightharpoonup \widetilde{\mathcal{Y}}^{\mu}$. Then, its system of extents is $\overline{\mathcal{K}}^{-1}$-generated by the attribute-extents. Dually, its system of intents is $\overline{\mathcal{K}}^{-1}$-generated by the object-intents.*

3 A construction for positive semifields

There is a non-countable number of semifields obtainable from $\mathbb{R}_{\geq 0}$. Their discovery is probably due to Maslov, but we present here the generalized procedure introduced by Pap and collaborators that include Maslov results. The application to positive semifields is our own statement:

Construction 1 (Pap's dioids and semifields). Let $\mathbb{R}_{\geq 0}$ be the semiring of non-negative reals, and consider a strictly monotone *generator function* g on an interval $[a, b] \subseteq [-\infty, \infty]$ with values in $[0, \infty]$. Since g is strictly increasing it admits an inverse g^{-1}, so set

1. the pseudo-addition, $u \oplus v = g^{-1}(g(u) \underset{\cdot}{+} (g(v))$
2. the pseudo-multiplication, $u \otimes v = g^{-1}(g(u) \underset{\cdot}{\times} (g(v))$
3. neutral element, $e = g^{-1}(1)$
4. inverse, $x^{*} = g^{-1}(\frac{1}{g(x)})$,

Then,

1. if g is strictly increasing such that $g(a) = 0$ and $g(b) = \infty$, then a complete positive semifield whose order is aligned with that of $\mathbb{R}_0^+$ is:

$$\mathcal{K}_g = \langle [a, b], \underset{\cdot}{\oplus}, \underset{\cdot}{\otimes}, \cdot^*, \bot = a, e, \top = b\rangle \,.$$

2. if g is strictly decreasing such that $g(a) = \infty$ and $g(b) = 0$, then a complete positive semifield whose order is aligned with that of $(\mathbb{R}_{\geq 0})^{-1}$ is

$$(\mathcal{K}_g)^{-1} = \langle [a, b], \dot{\oplus}, \dot{\otimes}, \cdot^*, \bot^{-1} = b, e, \top^{-1} = a\rangle \,.$$

Proof. See [12, 13] for the basic dioid, and [7, p. 44] for the inverse operation and the fact that it is a semifield, hence a positive semifield. □

Our use of Construction 1 is to generate different kind of semifields by providing different generator functions:

Construction 2 (Multiplicative-cost semifields [14]). Consider a free parameter $\lambda \in [-\infty, 0) \bigcup (0, \infty]$ and the function $g(x) = x^\lambda$ in $[a, b] = [0, \infty]$ in Construction 1. For the operations we obtain:

$$u \oplus_\lambda v = \left(u^\lambda \underset{\cdot}{+} v^\lambda\right)^{\frac{1}{\lambda}} \quad u \otimes_\lambda v = \left(u^\lambda \underset{\cdot}{\times} v^\lambda\right)^{\frac{1}{\lambda}} = u \underset{\cdot}{\times} v \quad u^{\circledast} = \left(\frac{1}{x^\lambda}\right)^{\frac{1}{\lambda}} = x^{-1} \tag{8}$$

where the basic operations are to be interpreted in $\mathbb{R}_{\geq 0}$. Now,

– if $\lambda \in (0, \infty]$ then $g(x) = x^\lambda$ is strictly monotone increasing whence $\bot_\lambda = 0$, $e_\lambda = 1$, and $\top_\lambda = \infty$, and the complete positive semifield generated, order-aligned with $\mathbb{R}_{\geq 0}$, is:

$$\mathbb{R}_{\geq 0,\lambda} = \langle [0, \infty], \underset{\cdot}{\oplus}_\lambda, \underset{\cdot}{\times}, \cdot^{-1}, \bot_\lambda = 0, e, \top_\lambda = \infty\rangle \tag{9}$$

– if $\lambda \in [-\infty, 0)$ then $g(x) = x^\lambda$ is strictly monotone decreasing whence $\bot_\lambda = \infty$, $e_\lambda = 1$, and $\top_\lambda = 0$, and the complete positive semifield generated, order-aligned with $(\mathbb{R}_{\geq 0})^{-1}$, or dually aligned with $\mathbb{R}_{\geq 0}$, is:

$$\mathbb{R}_{\geq 0,-\lambda} = \mathbb{R}_{\geq 0,\lambda}^{-1} = \langle [0, \infty], \dot{\oplus}_\lambda, \dot{\times}, \cdot^{-1}, \bot_\lambda^{-1} = \infty, e, \top_\lambda^{-1} = 0\rangle \tag{10}$$

Proof. By instantiation of the basic case. □

In particular, consider the cases:

Proposition 5. *In the previous Construction 2, if* $\lambda \in \{\pm 1\}$ *then*

$$\mathbb{R}_{\geq 0,1} = \mathbb{R}_{\geq 0} \qquad (\mathbb{R}_{\geq 0,1})^{-1} = (\mathbb{R}_{\geq 0})^{-1} \tag{11}$$

and

$$\lim_{\lambda \to \infty} \mathbb{R}_{\geq 0,\lambda} = \overline{\mathbb{R}}_{\max,\times} \qquad \lim_{\lambda \to -\infty} \mathbb{R}_{\geq 0,\lambda}^{-1} = \overline{\mathbb{R}}_{\min,\times} \tag{12}$$

Proof. The proof of (11) by inspection. For (12) see [14]. □

This suggests the following corollary:

Corollary 2. $\mathbb{R}_{\geq 0,\lambda}$ *and* $\mathbb{R}_{\geq 0,\lambda}^{-1}$ *are inverse semifields.*

4 The Idempotent Singular Value Decomposition

This is our answer to the second question presented in the introduction as to the usefulness of the Galois Connection in the setting of semifields.

The singular value decomposition (SVD) is a well-known decomposition scheme for real- or complex-valued rectangular matrices [15].

Theorem 1. *Given a matrix $M \in \mathcal{M}_{m\times n}(\mathcal{K})$ where $\mathcal{K}$ is a field, there is a factorization $M = U\Sigma V^*$ where $\cdot^*$ stands for conjugation, given in term of three matrices*
- $U \in \mathcal{M}_{n\times n}(\mathcal{K})$ *is a unitary matrix of* left singular vectors*.*
- $\Sigma \in \mathcal{M}_{m\times n}(\mathcal{K})$ *is a diagonal matrix of non-negative real values called the* singular values*.*
- $V \in \mathcal{M}_{n\times n}(\mathcal{K})$ *is a unitary matrix of* right singular vectors*.*

Often the singular values are listed in descending order, and the left and right singular eigenvalues are re-ordered accordingly.

Note that M can also be written using outer products as:

$$M = \sum_{i=1}^{\min(m,n)} \sigma_i u_i v_i^* \tag{13}$$

hence, since the SVD is a costly procedure, it is also interesting to find $k < \min(m,n)$ such that using the k greatest singular values we may approximate:

$$M \approx \sum_{i=1}^{k} \sigma_i u_i v_i^* \,. \tag{14}$$

Note that, in particular, singular vectors of the null eigenvalue never contribute to the reconstruction so they may be discarded. This approximation procedure is particularly useful in applications like Latent Semantic Analysis [16].

Equation (13) suggests that the triples (σ_i, u_i, v_i) of a singular value and related left and right singular vectors have a special status in the theory. Indeed, [17] already suggested the use of formal concepts for this purpose. We are going next to introduce the idempotent Singular Value Decomposition (iSVD) for matrices over an idempotent semifield, in several flavours.

A full singular value decomposition is particularly easy in idempotent semifields. Recall from property 1 of Proposition 1 that we have no less than four decompositions of $A \in \mathcal{M}_{m\times n}(\mathcal{K})$ in terms of two other matrices:

$$A \underset{\cdot}{\otimes} (A^{\circledast} \overset{\cdot}{\otimes} A) = A \overset{\cdot}{\otimes} (A^{\circledast} \underset{\cdot}{\otimes} A) = (A \overset{\cdot}{\otimes} A^{\circledast}) \underset{\cdot}{\otimes} A = (A \underset{\cdot}{\otimes} A^{\circledast}) \overset{\cdot}{\otimes} A = A$$

Why these are SVDs is the answer produced by $\mathcal{K}$-Formal Concept Analysis.

Proposition 6. *The σ-concepts (u, v) of a matrix $M \in \mathcal{M}_{m\times n}(\mathcal{K})$ over a completed idempotent semifield $\mathcal{K}$ are (u, σ, v) triples in a singular value decomposition of M in the linear algebra of $\mathcal{K}$.*

Proof. Consider the massaged SVD equation in the dual semiring:

$$M = (U^{\mathrm{T}})^{\circledast} \mathbin{\dot{\otimes}} \varSigma \mathbin{\dot{\otimes}} V^{\circledast} \tag{15}$$

where the columns of U are left singular vectors, those of V are right singular vectors and $\varSigma$ is the (dual) diagonal matrix of singular values, whose off-diagonal entries are $\top$. We relax the equation $M \leq (U^{\mathrm{T}})^{\circledast} \mathbin{\dot{\otimes}} \varSigma \mathbin{\dot{\otimes}} V^{\circledast}$ to bring residuation in the dual semimodule into the picture and then use the residuation equalities to find:

$$(U^{\mathrm{T}})^{\circledast} \mathbin{\dot{\backslash}} M \mathbin{\dot{/}} V^{\circledast} \leq \varSigma \qquad\qquad \Leftrightarrow U^{\mathrm{T}} \mathbin{\underset{\cdot}{\otimes}} M \mathbin{\underset{\cdot}{\otimes}} V \leq \varSigma \tag{16}$$

These are actually $m \times n$ inequations, but those with $i \neq j$ for $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, are trivial $u_i^{\mathrm{T}} \mathbin{\underset{\cdot}{\otimes}} M \mathbin{\underset{\cdot}{\otimes}} v_j \leq \top$, so we concentrate on the ones involving the triples $(u_k, \sigma_k, v_k), k \in \{1, \ldots, \min(m, n)\}$ where $\sigma_k = \varSigma_{kk}$ and u_k, v_k, are, respectively the k-th columns of U, V so $u_k^{\mathrm{T}} \mathbin{\underset{\cdot}{\otimes}} M \mathbin{\underset{\cdot}{\otimes}} v_k \leq \sigma_k$

Now consider the definition of the σ-polars

$$u^{\uparrow}_{R,\sigma} = \{v \in K^n \mid u^{\mathrm{T}} \mathbin{\underset{\cdot}{\otimes}} M \mathbin{\underset{\cdot}{\otimes}} v \leq \sigma\} \quad u^{\uparrow}_{R,\sigma} = \{u \in K^m \mid u^{\mathrm{T}} \mathbin{\underset{\cdot}{\otimes}} M \mathbin{\underset{\cdot}{\otimes}} v \leq \sigma\} \tag{17}$$

So for a σ-concept (u, v) clearly $M \leq (u^{\mathrm{T}})^{\circledast} \mathbin{\dot{\otimes}} \sigma \mathbin{\dot{\otimes}} v^{\circledast}$ whence

$$M \leq \bigwedge_{(u,v) \in \underline{\mathfrak{B}}^{\sigma}(G,M,R)_{\mathcal{K}}} (u^{\mathrm{T}})^{\circledast} \mathbin{\dot{\otimes}} \sigma \mathbin{\dot{\otimes}} v^{\circledast} \tag{18}$$

□

Note that the number of SVD triples in (18) is enormous, hence the description is not practical. We can, of course, ignore in this description, all triples with the null singular value $\sigma = \bot$. But, more importantly, we can ignore collinear concepts, whereby we mean $\lambda \mathbin{\underset{\cdot}{\otimes}} (u, v) = (\lambda \mathbin{\underset{\cdot}{\otimes}} u, \lambda^{-1} \mathbin{\underset{\cdot}{\otimes}} v)$:

Lemma 2. *If (u_1, v_1) and (u_2, v_2) are collinear, one of them can be ignored in the reconstruction of the matrix.*

Proof. Mere algebra on the concepts as per Proposition 3. □

Another step would be to prove that a linear combination of concepts does not add to the individual generating concepts, but we can actually do much more than that.

Proposition 7. *Let $M \in \mathcal{M}_{m \times n}(\mathcal{K})$ be a matrix over a completed idempotent semifield $\mathcal{K}$. Then it can be synthesized from the join- and meet-irreducibles of the concept lattice $\underline{\mathfrak{B}}^{e}(G, M, R)_{\mathcal{K}}$ as:*

$$M = \left(M \mathbin{\underset{\cdot}{\otimes}} M^{\circledast}\right) \mathbin{\dot{\otimes}} M \qquad\qquad M = M \mathbin{\dot{\otimes}} \left(M^{\circledast} \mathbin{\underset{\cdot}{\otimes}} M\right) \tag{19}$$

Proof. The proof is easy: from Lemma 1 and Corollary 1 select the triples with $\sigma = e$ and write for the join-irreducible and meet irreducible concepts, respectively:

$$M = \left(M^{-1} \dot{\otimes} M^{\mathrm{T}}\right)^{-1} \dot{\otimes} (M^{\circledast})^{\circledast} \qquad M = (M^{-1})^{-1} \dot{\otimes} \left(M^{\circledast} \dot{\otimes} M\right)^{\circledast} \tag{20}$$

Then simplify algebraically. □

Note how the last result accounts for two of the previously introduced decompositions: the other two are their duals in the inverse semifield. Note also, how scalar multiples of any of the join- or meet-irreducibles may be further ignored.

5 Discussion and Further Work

We have presented two different contributions in this paper. First an analysis of the possibilities of positive semifields to generate Galois Connections, possibly by residuation. In this respect, we have mixed results:

– On the one hand, we have described Pap's construction which, instantiated with a suitable function, is able to generate dual pairs of idempotent semifields (e.g. max-times and min-times), among a plethora of other, commuative, complete non-idempotent semifields.
– On the other hand, commutative complete dioids are already complete residuated lattices, which make them good candidates to support Galois connections by residuation, although we have not been able to provide closed expressions for the polars.

Note that the Pap semifields are a side result of the lengthier process of defining a g-calculus [12, 13]. Also, in [18, §6.2,7] functions like g above a called *transforms*. This would seem to imply that we would have a non-standard calculus associated to semifields, as well as a non-standard algebra, e.g. concept lattices, in this algebraic setting.

The second contribution was a new application of Concept Lattices to decompose nonnegative matrices (NMF) by providing an analogue of Singular Value Decomposition. Since the procedure was developed in the completed idempotent semifield notation, this means there are different instantiations for semifields with different carrier sets, hence this is a *generic procedure.*

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