A brief walk through Sampling Theory

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Contents

I. Starting point

Sampling Theory deals with the reconstruction of functions (signals) through their values (samples) on an appropriate sequence of points by means of sampling expansions involving these values. The most famous result in this direction goes by the name of Whittaker-Shannon-Kotel'nikov formula, which allows to reconstruct bandlimited signals i.e., signals containing no frequencies beyond a critical value ω_c from an equidistant sequence of samples whose spacing depends on ω_c.

Concerning the fatherhood of the above referred sampling formula there exists some historical controversy, which involves famous mathematicians as A. L. Cauchy or E. Borel, among others. The interested reader can find some historical sources in the references mentioned in the introduction of Section II. In any case, there is no doubt as to when did modern sampling theory start out: it was in 1949, when Shannon published his famous paper *Communication in the presence of noise* [113]. Although it is almost certain that Shannon was not the discoverer of "his formula", his paper triggered an avalanche of works, which have eventually produced a flourishing body of results on sampling methods and their applications.

What started as a theorem for reconstructing bandlimited signals from uniform samples has now become, from a mathematical point of view, a whole branch of applied mathematics, known as Sampling Theory. This new field turned out to be very useful in many mathematical areas, such as Approximation Theory, Harmonic Analysis, Theory of Entire Functions, Theory of Distributions, and Stochastic Processes, among others. The efforts in extending Shannon's fundamental result point in various directions: nonuniform samples, other discrete data taken from the signal, multidimensional signals, and more. This, together with the technological impact of sampling in Communication Theory and Signal Processing, can provide a clearer idea of the importance this topic has nowadays.

As a consequence, a tremendous amount of material can be gathered under the title Sampling Theory, and therefore, any survey in this topic, in particular this one, must necessarily be summarized. In answer to questions about the title of these notes, we have chosen the expression "brief walk" to indicate a personal choice of summarizing an introduction to Sampling Theory. The main aim in writing this paper is to serve as an introduction to Sampling Theory for the interested non-specialist reader. Despite the

introductory level, some hints and motivations are given on more advanced problems in Sampling Theory. The presentation of the work is self-contained and mostly elementary. The only prerequisites are a good understanding of the fundamentals of Hilbert spaces and Harmonic Analysis, although a mastery on those theories is by no means required. We have stressed motivations and ideas at the expense of a formal mathematical presentation. As a result, the reader will not find the customary sequences of definitions, theorems and corollaries, although the author has striven to keep the mathematical rigor in all arguments.

A few words about the structure of this work are in order. In Section II a survey about orthogonal sampling formulas is given. The classical Whittaker-Shannon-Kotel'nikov one is the *leitmotiv* to introduce a general theory for orthogonal sampling formulas in the framework of orthonormal bases in a Hilbert space. Most of this section stems from the reference [48]. The procedure, which is illustrated with a number of examples, closely parallels the theory of orthonormal bases in a Hilbert space and allows a quick immersion into orthogonal sampling results. Section III is devoted to a deeper study of the spaces of classical bandlimited functions, i.e., the classical Paley-Wiener spaces. It includes sampling formulas which use other types of samples, like derivatives or the Hilbert transform of a given signal, an idea already proposed in Shannon's paper. We also deal, at an introductory level, with nonuniform sampling involving Riesz bases or frames. For the sake of completeness, we also include an introductory theory of these mathematical concepts. In Section IV, a flavour on sampling bandlimited stationary stochastic processes is given from an abstract point of view. Finally, Section V covers a rapid overview on important sampling topics not included or mentioned in previous sections. They are accompanied with a suitable list of references for further reading. The main aim of this closing section is to address the interested reader to the appropriate references to be acquainted with more advanced topics on sampling.

Finally, most of the results stated throughout the work are well-known, and the author only claims for originality in the way of setting them out. He will be satisfied if this contributes to make Sampling Theory better known to the scientific community.

II. Orthogonal Sampling Formulas

In 1949 Claude Shannon [113] published a remarkable result:

If a signal $f(t)$ (with finite energy) contains no frequencies higher than w cycles per second, then $f(t)$ is completely determined by its values $f(n/2w)$ at a discrete set of points with spacing $1/2w$, and can be reconstructed from these values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2w}\right) \frac{\sin \pi(2wt - n)}{\pi(2wt - n)}. \tag{1}$$

In engineering-mathematical terminology, the signal f is *bandlimited to* $[-2\pi w, 2\pi w]$, meaning that $f(t)$ contains no frequencies beyond w cycles per second. Equivalently,

its Fourier transform F is zero outside this interval:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi w}^{2\pi w} F(x) e^{ixt} dx \,. \tag{2}$$

The engineering principle underlying (1) is that all the information contained in $f(t)$ is stored in its samples $\{f(n/2w)\}$. The cut-off frequency determines the so-called *Nyquist rate*, the minimum rate at which the signal needs to be sampled in order to recover it at all intermediate times t. In the case above, $2w = 4\pi w/2\pi$ is the sampling frequency and $1/2w$ is the sampling period. This rate is named after the engineer H. Nyquist, who was the first to point out its importance in connection with telegraph transmission [92].

The sampling functions used in the reconstruction (1) are

$$S_n(t) = \frac{\sin \pi(2wt - n)}{\pi(2wt - n)} \,.$$

They satisfy the interpolatory property $S_n(t_k) = \delta_{n,k}$ at $t_k = \frac{k}{2w}$, $k \in \mathbb{Z}$, where $\delta_{n,k}$ equals 1 if $n = k$, and 0 if $n \neq k$. A series as in (1) is known as a *cardinal series* because the sampling functions involve the *cardinal sine* function (or *sinc function*)

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin \pi t}{\pi t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

These series owe their name to J. M. Whittaker [129], a reference cited by Shannon in [113]. To be precise, J. M. Whittaker's work was a refinement of his father's, the eminent British mathematician E. T. Whittaker [128]. However, it is not clear whether or not they were the first mathematicians to introduce these kinds of expansions. Another famous mathematicians like E. Borel, A. L. Cauchy, W. L. Ferrar or K. Ogura are attributed its fatherhood. Some interesting historical notes concerning this controversy can be found in [28, 63, 64, 74, 135]. See also the master references [19, 33, 47, 93].

The Shannon sampling theorem provides the theoretical foundation for modern pulse code modulation communication systems, which were introduced, independently, by V. Kotel'nikov [72] in 1933 (an English translation from the original Russian manuscript can be found in [14]) and by Shannon in 1949. This sampling theorem is presently known in the mathematical literature as the *Whittaker-Shannon-Kotel'nikov Theorem* or WSK sampling theorem.

In general, the problem of *sampling and reconstruction* can be stated as follows: Given a set H of functions defined on a common domain Ω, is there a discrete set $D = \{t_n\} \subset \Omega$ such that every $f \in H$ is uniquely determined by its values on D? And if this is the case, how can we recover such a function? Moreover, is there a sampling series of the form

$$f(t) = \sum_n f(t_n) S_n(t) \tag{3}$$

valid for every f in H, where the convergence of the series is at least absolute and uniform on closed bounded intervals?

In many cases of practical interest, the set H is related to some integral transform as in (2), and the sampling functions satisfy an interpolatory property. All this leads us to propose a general method to obtain some sampling theorems in a unified way. In Section A we obtain orthogonal sampling theorems by following these steps:

1. Take a set of functions $\{S_n(t)\}$ interpolating at a sequence of points $\{t_n\}$.
2. Choose an orthonormal basis for an L^2 space.
3. Define an integral kernel involving $\{S_n(t)\}$ and the orthonormal basis. Consider the corresponding integral transform in the L^2 space.
4. Endow the range space of this integral transform with a norm which provides an isometric isomorphism between the range space and the L^2 space via the integral transform.
5. Thus, any Fourier expansion in the L^2 space is transformed into a Fourier expansion in the range space whose coefficients are the samples of the corresponding function, computed at the sequence $\{t_n\}$.
6. Convergence in this norm of the range space implies pointwise convergence and, as a consequence, we obtain a *sampling expansion* which holds for all functions in the range space. The idea underlying the whole procedure is borrowed from Hardy [61], who first noticed that (1) is an orthogonal expansion.

This methodology is put to use in Section B, where several well-known sampling formulas are derived in this way. Thus the main features of our approach are the following:

I. The fact of placing the problem in a functional framework, common to many diverse situations, allows us to introduce Sampling Theory through the well-developed theory of orthonormal bases in a Hilbert space. A number of well-known sampling formulas are obtained in this unified way.

II. The functional setting we have chosen only permits us, in principle, to derive *orthogonal* sampling expansions. However, it can be enlarged to more general settings including Riesz bases or frames as will be pointed out in Section III.E.1.

A. A unified approach

We begin this section with a brief reminder of *orthonormal bases* in a separable Hilbert space $\mathbb{H}$, i.e., a Hilbert space containing a countably dense set. This well-known concept will be a basic tool along this section, and it will allow us to draw nontrivial consequences in sampling.

An orthonormal basis for $\mathbb{H}$ *is a complete and orthonormal sequence* $\{e_n\}_{n=1}^{\infty}$ *in* $\mathbb{H}$*, i.e.,* $\langle e_n, e_m\rangle = \delta_{n,m}$ *(orthonormality) and the zero vector is the only vector orthogonal to every* e_n *(completeness).*

Given an orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ in $\mathbb{H}$, the following statements are equivalent [91, p. 307]

(1) For every $x \in \mathbb{H}$ we have the *Fourier series expansion*

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n \,, \tag{4}$$

in the $\mathbb{H}$-norm sense.

(2) For every x and y in $\mathbb{H}$ we have

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle y, e_n\rangle} \,. \tag{5}$$

(3) For every $x \in \mathbb{H}$, the *Parseval formula*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \tag{6}$$

holds.

In this section we will deal with $L^2(I)$ spaces, i.e.,

$$L^2(I) = \left\{ F : I \to \mathbb{C} \text{ measurable and } \int_I |F(x)|^2 \, dx < \infty \right\},$$

where I is an interval in $\mathbb{R}$, bounded or not. As usual, the inner product in $L^2(I)$ is given by $\langle F, G\rangle_{L^2(I)} = \int_I F(x)\overline{G(x)}dx$. All these spaces are separable and, consequently, possess a countable orthonormal basis [91, p. 314]. Throughout this section, $\{\phi_n(x)\}_{n=1}^{\infty}$ will denote an orthonormal basis for a fixed $L^2(I)$ space.

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of functions $S_n : \Omega \subset \mathbb{R} \longrightarrow \mathbb{C}$, defined for all $t \in \Omega$, and let $\{t_n\}_{n=1}^{\infty}$ be a sequence in Ω satisfying conditions C1 and C2:

C1. $S_n(t_k) = a_n\delta_{n,k}$ where $\delta_{n,k}$ denotes the Kronecker delta and $a_n \neq 0$,

C2. $\displaystyle\sum_{n=1}^{\infty} |S_n(t)|^2 < \infty$ for each $t \in \Omega$.

Define the function $K(x,t)$ as

$$K(x,t) = \sum_{n=1}^{\infty} S_n(t)\overline{\phi_n}(x)\,, \qquad (x,t) \in I \times \Omega \tag{7}$$

Note that, as a function of x, $K(\cdot,t)$ belongs to $L^2(I)$ since $\{\overline{\phi_n}\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(I)$ as well.

Now, consider $K(x,t)$ as an integral kernel and define on $L^2(I)$ the linear integral transformation which assigns

$$f(t) := \int_I F(x)K(x,t)dx \tag{8}$$

to each $F \in L^2(I)$.

The integral transform (8) is well defined because both F and $K(\cdot,t)$ belong to $L^2(I)$ and the Cauchy-Schwarz inequality implies that $f(t)$ is defined for each $t \in \Omega$. Also, this transformation is one-to-one, since $\{K(x,t_k) = a_k\overline{\phi_k}(x)\}_{k=1}^{\infty}$ is a complete sequence for $L^2(I)$, i.e., the only function orthogonal to every $K(x,t_k)$ is the zero function. Actually, if two functions f and g are equal on the sequence $\{t_k\}_{k=1}^{\infty}$, they necessarily coincide on the whole set Ω. Indeed, suppose that $f(t) = \int_I F(x)K(x,t)dx$ and $g(t) = \int_I G(x)K(x,t)dx$; then, $f(t_k) = g(t_k)$ for every k can be written as

$$\int_I [F(x) - G(x)]K(x,t_k)dx = 0\,,$$

and this implies $F - G = 0$ in $L^2(I)$. Hence, $f(t) = g(t)$ for each $t \in \Omega$.

Now, define $\mathcal{H}$ as the range of the integral transform (8)

$$\mathcal{H} = \left\{ f : \Omega \longrightarrow \mathbb{C} \;\middle|\; f(t) = \int_I F(x)K(x,t)dx\,, F \in L^2(I) \right\}$$

endowed with the norm $\|f\|_{\mathcal{H}} := \|F\|_{L^2(I)}$. Recall that, in a Hilbert space $\mathbb{H}$, the polarization identity [91, p. 276] allows us to recover the inner product from the norm by

$$\langle x,y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}\,, \quad x,y \in \mathbb{H},$$

in the case of a real vector space, or by

$$\langle x,y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\}\,, \quad x,y \in \mathbb{H},$$

in the case of a complex vector space. Using the polarization identity, we have a first result

• $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *is a Hilbert space, isometrically isomorphic to* $L^2(I)$. *For each* $f, g \in \mathcal{H}$

$$\langle f, g \rangle_{\mathcal{H}} = \langle F, G \rangle_{L^2(I)}\,, \tag{9}$$

where $f(t) = \int_I F(x)K(x,t)dx$ *and* $g(t) = \int_I G(x)K(x,t)dx$.

Since an isometric isomorphism transforms orthonormal bases into orthonormal bases, we derive the following important property for $\mathcal{H}$ by applying the integral transform (8) to the orthonormal basis $\{\phi_n(x)\}_{n=1}^{\infty}$

• $\{S_n(t)\}_{n=1}^{\infty}$ *is an orthonormal basis for* $\mathcal{H}$.

Now, we will see that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a *reproducing kernel Hilbert space*, a crucial step for our sampling purposes. For more details on this topic, see Aronszajn's seminal paper [7] or references [64, 109, 132, 135]. We recall that

A Hilbert space $\mathbb{H}$ *of functions on* Ω *is said to be a reproducing kernel Hilbert space, hereafter RKHS, if all the evaluation functionals* $E_t(f) := f(t)$, $f \in \mathbb{H}$, *are continuous for each fixed* $t \in \Omega$ *(or equivalently bounded since they are linear).*

Then, by the Riesz representation theorem [91, p. 345], for each $t \in \Omega$ there exists a unique element $k_t \in \mathbb{H}$ such that $f(t) = \langle f, k_t \rangle$, $f \in \mathbb{H}$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{H}$. Let $k(t,s) = \langle k_s, k_t \rangle = k_s(t)$ for $s, t \in \Omega$. Then,

$$\langle f(\cdot), k(\cdot, s) \rangle = \langle f, k_s \rangle = f(s)\,, \quad \text{for every } s \in \Omega\,. \tag{10}$$

The function $k(t,s)$ is called the *reproducing kernel* of $\mathbb{H}$. Equivalently, a RKHS can be defined through the function $k(t,s)$ instead of the continuity of the evaluation functionals. Namely,

A functional Hilbert space $\mathbb{H}$ *is a RKHS if there exists a function* $k : \Omega \times \Omega \longrightarrow \mathbb{C}$ *such that for each fixed* $s \in \Omega$, *the function* $k(\cdot, s)$ *belongs to* $\mathbb{H}$, *and the reproducing property* (10) *holds for every* $f \in \mathbb{H}$ *and* $s \in \Omega$.

In this case, the continuity of E_t follows from the Cauchy-Schwarz inequality. The reproducing property (10) looks somewhat strange since the knowledge of f at a point $s \in \Omega$ requires the inner product $\langle f, k(\cdot, s) \rangle$ which involves the whole f. However, this property has far-reaching consequences from a theoretical point of view as we will see later on.

One can easily prove that the reproducing kernel in a RKHS is unique. Indeed, let $k'(t,s)$ be another reproducing kernel for $\mathbb{H}$. For a fixed $s \in \Omega$, consider $k'_s(t) = k'(t,s)$. Then, for $t \in \Omega$ we have

$$\begin{aligned} k'_s(t) &= \langle k'_s, k_t \rangle = \overline{\langle k_t, k'_s \rangle} \\ &= \overline{k_t(s)} = \overline{\langle k_t, k_s \rangle} = k_s(t)\,, \end{aligned}$$

and hence, $k(s,t) = k'(s,t)$ for all $t, s \in \Omega$.

Finally, if $\{e_n(t)\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathbb{H}$, then the reproducing kernel can be expressed as $k(t,s) = \sum_{n=1}^{\infty} e_n(t)\overline{e_n(s)}$. Indeed, expanding k_t in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ we have

$$k_t = \sum_{n=1}^{\infty} \langle k_t, e_n \rangle e_n = \sum_{n=1}^{\infty} \overline{e_n(t)} e_n \,,$$

and by using (5),

$$k(t,s) = \langle k_s, k_t \rangle = \sum_{n=1}^{\infty} \overline{e_n(s)} e_n(t) \,. \tag{11}$$

As a consequence of the above discussion about RKHS we obtain

• *$(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a RKHS whose reproducing kernel is given by*

$$k(t,s) = \sum_{n=1}^{\infty} \overline{S_n(s)} S_n(t) = \langle K(\cdot,t), K(\cdot,s) \rangle_{L^2(I)} \,. \tag{12}$$

To prove it, we use the Cauchy-Schwarz inequality in (8), obtaining for each fixed $t \in \Omega$

$$|E_t(f)| = |f(t)| \leq \|F\|_{L^2(I)} \|K(\cdot,t)\|_{L^2(I)} = \|f\|_{\mathcal{H}} \|K(\cdot,t)\|_{L^2(I)} \tag{13}$$

for every $f \in \mathcal{H}$.

As to the reproducing kernel formula (12), due to (11) we only need to prove the second equality. To this end, consider

$$k'(t,s) = \langle K(\cdot,t), K(\cdot,s) \rangle_{L^2(I)} = \int_I K(x,t) \overline{K(x,s)} dx \,.$$

Then, for a fixed $s \in \Omega$, $k'(t,s)$ is the transform of $\overline{K(x,s)}$ by (8). Using the isometry (9) we have

$$\langle f, k'(\cdot,s) \rangle_{\mathcal{H}} = \langle F, \overline{K(x,s)} \rangle_{L^2(I)} = \int_I F(x) K(x,s) dx = f(s) \,.$$

The uniqueness of the reproducing kernel leads to the desired result.

It is worth pointing out that inequality (13) has important consequences for the convergence in $\mathcal{H}$. More precisely

• *Convergence in the norm $\|\cdot\|_{\mathcal{H}}$ implies pointwise convergence and uniform convergence on subsets of Ω where $\|K(\cdot,t)\|_{L^2(I)} = \sqrt{k(t,t)}$ is bounded.*

At this point we have all the ingredients to obtain a sampling formula for all the functions in $\mathcal{H}$. Indeed, expanding an arbitrary function $f \in \mathcal{H}$ in the orthonormal basis $\{S_n(t)\}_{n=1}^{\infty}$, we have

$$f(t) = \sum_{n=1}^{\infty} \langle f, S_n \rangle_{\mathcal{H}} S_n(t) \,,$$

where the convergence is in the $\mathcal{H}$-norm sense and hence pointwise in Ω. Taking into account the isometry between $\mathcal{H}$ and $L^2(I)$, we have that

$$\langle f, S_n\rangle_{\mathcal{H}} = \langle F, \phi_n\rangle_{L^2(I)} = \frac{f(t_n)}{a_n}$$

for each $n \in \mathbb{N}$. Hence, we obtain the following *sampling formula* for $\mathcal{H}$

• *Each function f in $\mathcal{H}$ can be recovered from its samples at the sequence $\{t_n\}_{n=1}^{\infty}$ through the formula*

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{S_n(t)}{a_n}\,. \tag{14}$$

The convergence of the series in (14) is absolute, and uniform on subsets of Ω where $\|K(\cdot,t)\|_{L^2(I)} = \sqrt{k(t,t)}$ is bounded.

Note that an orthonormal basis is an unconditional basis in the sense that, due to Parseval's identity (6), any of its reorderings is again an orthonormal basis. Therefore, the sampling series (14) is pointwise unconditionally convergent for each $t \in \Omega$ and hence, absolutely convergent. The uniform convergence follows from inequality (13).

Note that we could also have obtained the formula (14) by applying the integral transform (8) to the Fourier series expansion $F(x) = \sum_{n=1}^{\infty}\langle F, \phi_n\rangle_{L^2(I)}\phi_n(x)$ of a function F in $L^2(I)$.

A comment about the functional space $\mathcal{H}$ is in order. Any $f \in \mathcal{H}$ can be described using the sequence of its values $\{f(t_n)\}_{n=1}^{\infty}$ by means of formula (14). In particular, the inner product and the norm in $\mathcal{H}$ can be expressed as

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \frac{f(t_n)\overline{g(t_n)}}{|a_n|^2}\,, \qquad \|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{|f(t_n)|^2}{|a_n|^2}\,.$$

Some properties for the functional space $\mathcal{H}$ can be easily obtained by using the reproducing property (10). Namely

• *In the case when $\mathcal{H}$ is a closed subspace of a larger Hilbert space $\mathbb{H}$, the reproducing formula* (10) *applied to any $f \in \mathbb{H}$ gives its orthogonal projection, $P_{\mathcal{H}}f$, onto $\mathcal{H}$, i.e.,*

$$P_{\mathcal{H}}f(s) = \langle f, k(\cdot, s)\rangle_{\mathbb{H}}\,, \quad f \in \mathbb{H} \text{ and } s \in \Omega\,. \tag{15}$$

Indeed, let $f = f_1 + f_2$ be the orthogonal decomposition of $f \in \mathbb{H}$ with $f_1 \in \mathcal{H}$, i.e., $f_1 = P_{\mathcal{H}}f$. Then,

$$\langle f, k(\cdot, s)\rangle_{\mathbb{H}} = \langle f_1 + f_2, k(\cdot, s)\rangle_{\mathbb{H}} = \langle f_1, k(\cdot, s)\rangle_{\mathcal{H}} = f_1(s)\,,$$

since f_2 is orthogonal to any $k(\cdot, s)$, $s \in \Omega$.

Next we solve, in an easy way, some extremal problems in a RKHS. We refer the interested reader to [69, 109, 131] for deeper results.

• *Fixing $t_0 \in \Omega$, $E > 0$ and $M \in \mathbb{C}$, we have in $\mathcal{H}$ the following relations*

$$\max_{\|f\|^2 \leq E} |f(t_0)|^2 = E\, k(t_0, t_0)\,, \quad \textit{reached for} \quad f^*(s) = \pm\sqrt{E}\, \frac{k(s, t_0)}{\sqrt{k(t_0, t_0)}}\,,$$

and

$$\min_{f(t_0)=M} \|f\|^2 = \frac{M^2}{k(t_0, t_0)}\,, \quad \textit{reached for} \quad f^*(s) = M\, \frac{k(s, t_0)}{k(t_0, t_0)}\,.$$

In fact, both results come out from the inequality

$$|f(s)|^2 = |\langle f, k(\cdot, s)\rangle|^2 \leq \|f\|^2 k(s, s)\,, \quad s \in \Omega\,,$$

where we have used the reproducing property and the Cauchy-Schwarz inequality.

We close this section with two approaches to orthogonal sampling formulas which are readily seen to be related with the one proposed in this section:

(a) Note that given an integral kernel $K(x, t)$, conditions C1 and C2 can be read as the existence of a sequence $\{t_n\}_{n=1}^{\infty} \subset \Omega$, such that $\{K(x, t_n)\}_{n=1}^{\infty}$ is an orthogonal basis for $L^2(I)$. This was the way originally suggested by Kramer in [73] to obtain orthogonal sampling theorems. Kramer's result reads as follows

• *Let $K(x, t)$ be a kernel belonging to $L^2(I)$, I being an interval of the real line, for each fixed $t \in \Omega \subset \mathbb{R}$. Assume that there exists a sequence of real numbers $\{t_n\}_{n\in\mathbb{Z}}$ such that $\{K(x, t_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form*

$$f(t) = \int_I F(x) K(x, t)\, dx\,,$$

where $F \in L^2(I)$, we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n(t)\,, \tag{16}$$

with

$$S_n(t) = \frac{\int_I K(x, t)\overline{K(x, t_n)}\, dx}{\int_I |K(x, t_n)|^2\, dx}\,.$$

The series (16) *converges absolutely and uniformly wherever $\|K(\cdot, t)\|_{L^2(I)}$ is bounded.*

One of the richest sources of Kramer kernels is in the subject of self-adjoint boundary value problems. See [41, 64, 142, 134, 135] for more details and references.

By using orthonormal bases in ℓ^2 spaces to define the kernel (7), one can easily arrive to sampling expansions associated with discrete transforms of the type

$$f(t) = \sum_n F(n) K(n, t)\,, \quad \{F(n)\} \in \ell^2\,.$$

This leads to the discrete version of Kramer's result. See [6, 49] for a more specific account of the theory and examples.

(b) Another similar formulation is the one given in [90, 109]:

• *Let $\mathbb{H}$ be a RKHS of functions defined on a subset Ω of $\mathbb{R}$ with reproducing kernel k. Assume there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset \Omega$ such that $\{k(\cdot, t_n)\}_{n=1}^{\infty}$ is an orthogonal basis for $\mathbb{H}$. Then, any $f \in \mathbb{H}$ can be expanded as*

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)} ,$$

with convergence absolute and uniform on subsets of Ω where $k(t, t)$ is bounded.

This result follows from the expansion of f in the orthonormal basis $\left\{ \frac{k(\cdot, t_n)}{\sqrt{k(t_n, t_n)}} \right\}_{n=1}^{\infty}$. Note that, in our construction,

$$k(t, t_n) = \langle K(\cdot, t), K(\cdot, t_n) \rangle_{L^2(I)} = \overline{a}_n S_n(t) ,$$

and $k(t_n, t_n) = |a_n|^2$. We will use this approach in the Finite Sampling Section II.C.

B. Putting to work the theory

Our main aim in this section is to derive some of the well-known sampling formulas by following the method exposed in the previous section. All the examples in this section are based on the knowledge of specific orthonormal bases for some L^2-spaces (see [91, pp. 322-329] and [136] for an account of bases and integral transforms, respectively).

1. Classical bandlimited functions

The set of functions $\{e^{-inx}/\sqrt{2\pi}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi, \pi]$. We consider the Fourier integral kernel $K(x, t) = e^{itx}/\sqrt{2\pi}$. For a fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{e^{itx}}{\sqrt{2\pi}} &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \langle e^{itx}, e^{inx} \rangle_{L^2[-\pi,\pi]} \frac{e^{inx}}{\sqrt{2\pi}} \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} \frac{e^{inx}}{\sqrt{2\pi}} \quad \text{in} \quad L^2[-\pi, \pi] . \end{aligned}$$

Therefore, taking $S_n(t) = \dfrac{\sin \pi(t-n)}{\pi(t-n)}$ and $t_n = n$, $n \in \mathbb{Z}$, we obtain the WSK sampling theorem

• *Any function of the form*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx , \qquad \textit{with } F \in L^2[-\pi, \pi] ,$$

i.e., bandlimited to $[-\pi,\pi]$ in the classical sense, can be recovered from its samples at the integers by means of the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} . \tag{17}$$

The series converges absolutely, and uniformly on $\mathbb{R}$ because, in this case,

$$\|K(\cdot,t)\|^2_{L^2[-\pi,\pi]} = 1 \quad \text{for all} \quad t \in \mathbb{R}.$$

For the moment, we denote as $\mathcal{H}_\pi$ the corresponding $\mathcal{H}$ space. We will come back to this space, the so-called Paley-Wiener space, in a subsequent section. The reproducing kernel in $\mathcal{H}_\pi$ space is given by

$$\begin{aligned} k_\pi(t,s) &= \frac{1}{2\pi} \langle e^{itx}, e^{isx} \rangle_{L^2[-\pi,\pi]} = \frac{\sin \pi(t-s)}{\pi(t-s)} \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} \frac{\sin \pi(s-n)}{\pi(s-n)} , \end{aligned}$$

where we have used (12) and (11) respectively.

Actually, the sampling points need not be taken at the integers in order to recover functions in $\mathcal{H}_\pi$. For a fixed real number α, one can easily check that the sequence of functions $\{e^{-i(n+\alpha)x}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is also an orthonormal basis for $L^2[-\pi,\pi]$. For a fixed $t \in \mathbb{R}$, we have the expansion

$$\frac{e^{itx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n-\alpha)}{\pi(t-n-\alpha)} \frac{e^{i(n+\alpha)x}}{\sqrt{2\pi}} \quad \text{in} \quad L^2[-\pi,\pi] .$$

Taking $S_n(t) = \dfrac{\sin \pi(t-n-\alpha)}{\pi(t-n-\alpha)}$ and $t_n = n + \alpha$, $n \in \mathbb{Z}$, we obtain that

• *Any function in $\mathcal{H}_\pi$ can be recovered from its samples at the integers shifted by a real constant α by means of the cardinal series*

$$f(t) = \sum_{n=-\infty}^{\infty} f(n+\alpha) \frac{\sin \pi(t-n-\alpha)}{\pi(t-n-\alpha)} . \tag{18}$$

The above result shows that, in regular sampling, the significance relies on the spacing of the sampling points and not on the sampling points themselves.

Note that $\{e^{-inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is also an orthonormal basis for any $L^2[\omega_0-\pi,\omega_0+\pi]$, with ω_0 a fixed real number. We then obtain that

$$\begin{aligned} \frac{e^{itx}}{\sqrt{2\pi}} &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \langle e^{itx}, e^{inx} \rangle_{L^2[\omega_0-\pi,\omega_0+\pi]} \frac{e^{inx}}{\sqrt{2\pi}} \\ &= \sum_{n=-\infty}^{\infty} e^{i\omega_0(t-n)} \frac{\sin \pi(t-n)}{\pi(t-n)} \frac{e^{inx}}{\sqrt{2\pi}} \quad \text{in} \quad L^2[\omega_0-\pi,\omega_0+\pi] . \end{aligned}$$

As a consequence, the following sampling result for signals with non-symmetrical band of frequencies with respect to the origin arises

• *Any function of the form*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0-\pi}^{\omega_0+\pi} F(x) e^{itx} dx\,, \qquad \textit{with } F \in L^2[\omega_0 - \pi, \omega_0 + \pi]\,,$$

can be recovered by means of the series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) e^{i\omega_0(t-n)} \frac{\sin \pi(t-n)}{\pi(t-n)}\,. \tag{19}$$

It is worth pointing out the following result concerning the band of frequencies of a bandlimited real-valued signal f: if the Fourier transform F of a real-valued function f is zero outside an interval, then it must be symmetrical with respect to the origin. Indeed,

$$|F(x)|^2 = F(x)\overline{F(x)} = F(x)F(-x)\,,$$

is an even function.

The choice of the interval $[-\pi, \pi]$ is arbitrary. The same result applies to any compact interval $[-\pi\sigma, \pi\sigma]$ taking the samples $\{f(n/\sigma)\}_{n\in\mathbb{Z}}$ and replacing t by σt in the cardinal series (17). Indeed, $\{e^{-inx/\sigma}/\sqrt{2\pi\sigma}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi\sigma, \pi\sigma]$. For a fixed $t \in \mathbb{R}$, we have the expansion

$$\begin{aligned} \frac{e^{itx}}{\sqrt{2\pi}} &= \frac{1}{2\pi\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \langle e^{itx}, e^{inx/\sigma} \rangle_{L^2[-\pi\sigma,\pi\sigma]} \frac{e^{inx/\sigma}}{\sqrt{2\pi\sigma}} \\ &= \sqrt{\sigma} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)} \frac{e^{inx/\sigma}}{\sqrt{2\pi\sigma}} \quad \text{in } L^2[-\pi\sigma, \pi\sigma]\,. \end{aligned}$$

Therefore, taking $S_n(t) = \sqrt{\sigma}\, \dfrac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)}$, $t_n = \dfrac{n}{\sigma}$, $n \in \mathbb{Z}$ and $a_n = \sqrt{\sigma}$ we obtain that

• *Any function of the form*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma}^{\pi\sigma} F(x) e^{itx} dx\,, \qquad \textit{with } F \in L^2[-\pi\sigma, \pi\sigma]\,,$$

can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f\big(\frac{n}{\sigma}\big)\, \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)}\,. \tag{20}$$

We have the same convergence properties like in (17) since $\|K(\cdot, t)\|^2_{L^2[-\pi\sigma,\pi\sigma]} = \sigma$. Moreover, the reproducing kernel for the corresponding space $\mathcal{H}_{\pi\sigma}$ is

$$k_{\pi\sigma}(t, s) = \frac{\sin \pi\sigma(t-s)}{\pi(t-s)} = \sigma \operatorname{sinc} \sigma(t-s)\,. \tag{21}$$

2. Bandlimited functions in the fractional Fourier transform sense

The sequence $\left\{\frac{1}{\sqrt{2\sigma}}e^{-i\pi nx/\sigma}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\sigma,\sigma]$. It is easy to prove that $\left\{\frac{1}{\sqrt{2\sigma}}e^{-i\pi nx/\sigma}e^{iax^2}\right\}_{n\in\mathbb{Z}}$, with $a\in\mathbb{R}$, is also an orthonormal basis for $L^2[-\sigma,\sigma]$. Let a and b be two nonzero real constants. For notational ease we denote $2ab=\dfrac{1}{c}$. We will see later the meaning of these constants. Direct calculations show that the expansion

$$\begin{aligned} e^{-ia(t^2+x^2-2bxt)} &= \sum_{n=-\infty}^{\infty}\left\langle e^{-ia(t^2+x^2-2bxt)},\frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2}\right\rangle_{L^2[-\sigma,\sigma]}\frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2}\\ &= \sum_{n=-\infty}^{\infty}\sqrt{2\sigma}e^{-iat^2}\frac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}\frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2}\end{aligned}$$

holds in the $L^2[-\sigma,\sigma]$ sense. Set $S_n(t)=\sqrt{2\sigma}e^{-iat^2}\dfrac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}$ and $t_n=\dfrac{n\pi c}{\sigma}$, $n\in\mathbb{Z}$. Since $S_n(t_k)=\sqrt{2\sigma}e^{-iat_n^2}\delta_{n,k}$, we obtain the result

• *For any function f of the form*

$$f(t)=\int_{-\sigma}^{\sigma}F(x)e^{-ia(t^2+x^2-2bxt)}dx\,,\qquad \textit{with } F\in L^2[-\sigma,\sigma]\,, \tag{22}$$

the following sampling formula

$$f(t)=\sum_{n=-\infty}^{\infty}f(t_n)e^{-ia(t^2-t_n^2)}\frac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})} \tag{23}$$

holds.

Here, the reproducing kernel obtained from (12) is

$$k_\sigma(t,s)=2\sigma e^{-ia(t^2-s^2)}\frac{\sin\frac{\sigma}{c}(t-s)}{\frac{\sigma}{c}(t-s)}\,.$$

Since $k_\sigma(t,t)=2\sigma$, the series in (23) converges uniformly in $\mathbb{R}$.

Our next purpose is to see how formula (22) and the *fractional Fourier transform* (FRFT) are related. Recall that the FRFT with angle $\alpha\notin\{0,\pi\}$ of a function $f(t)$ is defined as

$$\mathcal{F}_\alpha\left[f\right](x)=\int_{-\infty}^{\infty}f(t)K_\alpha(x,t)\,dt\,,$$

where, apart from a normalization constant, the integral kernel $K_\alpha(x,t)$ is given by

$$e^{i\frac{\cot\alpha}{2}(t^2+x^2)-i\frac{xt}{\sin\alpha}}\,. \tag{24}$$

For $\alpha = 0$ the FRFT is defined by $\mathcal{F}_0[f](x) = f(x)$, and for $\alpha = \pi$ by $\mathcal{F}_\pi[f](x) = f(-x)$. Whenever $\alpha = \pi/2$, the kernel (24) coincides with the Fourier kernel. Otherwise, (24) can be rewritten as

$$e^{ia(\alpha)[t^2+x^2-2b(\alpha)xt]},$$

where $a(\alpha) = \dfrac{\cot\alpha}{2}$ and $b(\alpha) = \sec\alpha$. The inversion formula of the FRFT (see [138]) is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\alpha(x) K_{-\alpha}(x,t)\, dx\,.$$

Consequently, formula (23) is just the sampling expansion for a function bandlimited to $[-\sigma,\sigma]$ in the FRFT sense (22). Note that $2a(\alpha)b(\alpha) = \dfrac{1}{\sin\alpha}$, and $c = \sin\alpha$ in the sampling expansion (23).

The fractional Fourier transform has many applications in several areas including quantum mechanics, optics and signal processing [5, 89, 94, 95]. In particular, the propagation of light can be viewed as a process of continual fractional Fourier transform. This allows to pose the FRFT as a tool for analyzing and describing some optical systems [95]. For the FRFT properties and its relationship to sampling see [130, 138, 139, 140, 141].

3. Finite sine and cosine transforms

In this section we deal with two transforms closely related with the Fourier one.

a. Finite cosine transform

Let us consider the orthogonal basis $\{\cos nx\}_{n=0}^{\infty}$ in $L^2[0,\pi]$. Note that $\|\cos nx\|^2_{L^2[0,\pi]}$ equals $\pi/2$ for $n \geq 1$, and π for $n = 0$. For $t \in \mathbb{R}$ fixed, we expand the function $\cos tx$ in this basis obtaining

$$\begin{aligned}\cos tx &= \sum_{n=0}^{\infty} \left\langle \cos tx, \frac{\cos nx}{\|\cos nx\|} \right\rangle_{L^2[0,\pi]} \frac{\cos nx}{\|\cos nx\|} \\ &= \frac{\sin \pi t}{\pi t} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2t \sin \pi t}{\pi(t^2-n^2)} \cos nx\,, \qquad \text{in } L^2[0,\pi]\,.\end{aligned}$$

Therefore, choosing $S_0(t) = \dfrac{\sin\pi t}{\pi t}$, $S_n(t) = \dfrac{(-1)^n\, 2t\sin\pi t}{\pi(t^2-n^2)}$ and $t_n = n$, $n \in \mathbb{N}\cup\{0\}$, we have that

- *Any function of the form*

$$f(t) = \int_0^\pi F(x)\cos tx\, dx\,, \qquad \textit{with } F \in L^2[0,\pi]$$

can be expanded as

$$f(t) = f(0)\frac{\sin \pi t}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} f(n)\frac{(-1)^n\, t \sin \pi t}{t^2 - n^2}\,.$$

The convergence of the series is absolute and uniform on $\mathbb{R}$ since

$$\|K(\cdot, t)\|^2_{L^2[0,\pi]} = \frac{\pi}{2} + \frac{\sin 2t\pi}{4t}$$

is bounded for all $t \in \mathbb{R}$. The reproducing kernel for the corresponding $\mathcal{H}_{\cos}$ space is given by

$$\begin{aligned} k_{\cos}(t, s) &= \int_0^{\pi} \cos tx \cos sx\, dx = \frac{1}{2}\left[\frac{\sin \pi(t-s)}{t-s} + \frac{\sin \pi(t+s)}{t+s}\right] \\ &= \frac{1}{t^2 - s^2}[t \sin t\pi \cos s\pi - s \cos t\pi \sin s\pi]\,. \end{aligned}$$

b. Finite sine transform

In a similar way, let us consider the orthonormal basis $\left\{\sqrt{\frac{2}{\pi}}\sin nx\right\}_{n=1}^{\infty}$ in $L^2[0, \pi]$. For a fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \sin tx &= \frac{2}{\pi}\sum_{n=1}^{\infty}\langle \sin tx, \sin nx\rangle_{L^2[0,\pi]} \sin nx \\ &= \sum_{n=1}^{\infty}\frac{2(-1)^n\, n \sin \pi t}{\pi(t^2 - n^2)}\sin nx\,, \qquad \text{in } L^2[0, \pi]\,. \end{aligned}$$

Taking $S_n(t) = \dfrac{2(-1)^n\, n \sin \pi t}{\pi(t^2 - n^2)}$ and $t_n = n$, $n \in \mathbb{N}$, we obtain that

• *Any function of the form*

$$f(t) = \int_0^{\pi} F(x) \sin txdx\,, \qquad \textit{with } F \in L^2[0, \pi]$$

can be expanded as

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} f(n)\frac{(-1)^n\, n \sin \pi t}{t^2 - n^2}\,.$$

The convergence of the series is absolute and uniform on $\mathbb{R}$, since in this case

$$\|K(\cdot, t)\|^2_{L^2[0,\pi]} = \frac{\pi}{2} - \frac{\sin 2t\pi}{4t}$$

is bounded for all $t \in \mathbb{R}$. The reproducing kernel for the corresponding $\mathcal{H}_{\sin}$ space is given by

$$k_{\sin}(t,s) = \int_0^\pi \sin tx \sin sx \, dx = \frac{1}{2}\left[\frac{\sin \pi(t-s)}{t-s} - \frac{\sin \pi(t+s)}{t+s}\right]$$
$$= \frac{1}{t^2-s^2}[-t\cos t\pi \sin s\pi + s \sin t\pi \cos s\pi] \,.$$

The cardinal series (17) is absolutely convergent and hence unconditionally convergent. Therefore, it can be written, gathering terms, in the equivalent form

$$f(t) = \frac{\sin \pi t}{\pi}\left\{\frac{f(0)}{t} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{f(n)}{t-n} + \frac{f(-n)}{t+n}\right)\right\} \,.$$

As a consequence, the sampling expansion associated with the finite cosine transform (finite sine transform) is nothing more than the cardinal series (17) for an even (odd) function. Moreover, it is easy to prove that the orthogonal sum

$$\mathcal{H}_\pi = \mathcal{H}_{\sin} \oplus \mathcal{H}_{\cos}$$

holds. In fact, using Euler formulas

$$\sin tx = \frac{e^{itx} - e^{-itx}}{2i} \quad \text{and} \quad \cos tx = \frac{e^{itx} + e^{-itx}}{2} \,,$$

we obtain that $\mathcal{H}_{\sin} \subset \mathcal{H}_\pi$ and $\mathcal{H}_{\cos} \subset \mathcal{H}_\pi$ as sets, and $\langle f,g\rangle_{\mathcal{H}_{\sin}} = \frac{1}{\pi}\langle f,g\rangle_{\mathcal{H}_\pi}$ for $f,g \in \mathcal{H}_{\sin}$ (the same occurs for $f,g \in \mathcal{H}_{\cos}$). Then, having in mind the reproducing property (10) and equation (15), for $s \in \mathbb{R}$ and $f \in \mathcal{H}_\pi$ we have

$$f(s) = \langle f, k_\pi(\cdot,s)\rangle_{\mathcal{H}_\pi} = \frac{1}{\pi}\langle f, (k_{\sin} + k_{\cos})(\cdot,s)\rangle_{\mathcal{H}_\pi}$$
$$= \langle f, k_{\sin}(\cdot,s)\rangle_{\mathcal{H}_{\sin}} + \langle f, k_{\cos}(\cdot,s)\rangle_{\mathcal{H}_{\cos}}$$
$$= \frac{f(s)-f(-s)}{2} + \frac{f(s)+f(-s)}{2} \,.$$

Using an appropriate normalization one could avoid the factor $1/\pi$.

4. Classical bandlimited functions again

Consider the product Hilbert space $\mathbf{H} = L^2[0,\pi] \times L^2[0,\pi]$ endowed with the norm $\|F\|^2_{\mathbf{H}} = \|F_1\|^2_{L^2[0,\pi]} + \|F_2\|^2_{L^2[0,\pi]}$ for every $F = (F_1,F_2) \in \mathbf{H}$. The system of functions $\left\{\frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathbf{H}$. For a fixed $t \in \mathbb{R}$ we have

$$(\cos tx, \sin tx) = \sum_{n=-\infty}^{\infty}\left\langle (\cos tx, \sin tx), \frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\right\rangle_{\mathbf{H}} \frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)$$
$$= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\sqrt{\pi}(t-n)} \frac{1}{\sqrt{\pi}}(\cos nx, \sin nx) \,,$$

in the **H** sense. Taking $S_n(t) = \dfrac{\sin \pi(t-n)}{\sqrt{\pi}(t-n)}$ and $t_n = n \in \mathbb{Z}$ we have that $S_n(t_k) = \sqrt{\pi}\delta_{n,k}$. As a consequence

• *Any function of the form*

$$f(t) = \int_0^\pi \{F_1(x)\cos tx + F_2(x)\sin tx\}dx\,, \qquad \textit{with } F_1, F_2 \in L^2[0,\pi]$$

can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}\,.$$

The corresponding $\mathcal{H}$ space is the space $\mathcal{H}_\pi$ in section II.B.1. Indeed, for $f \in \mathcal{H}_\pi$ we have

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}}\left\{\int_{-\pi}^{0} F(x)e^{itx}dx + \int_0^\pi F(x)e^{itx}dx\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\pi}^{0} F(x)(\cos tx + i\sin tx)dx + \int_0^\pi F(x)(\cos tx + i\sin tx)dx\right\} \\ &= \int_0^\pi \left\{\frac{1}{\sqrt{2\pi}}[F(x)+F(-x)]\cos tx + \frac{i}{\sqrt{2\pi}}[F(x)-F(-x)]\sin tx\right\}dx \\ &= \int_0^\pi \{F_1(x)\cos tx + F_2(x)\sin tx\}dx\,, \end{aligned}$$

where $F_1(x) = \frac{1}{\sqrt{2\pi}}[F(x)+F(-x)]$ and $F_2(x) = \frac{i}{\sqrt{2\pi}}[F(x)-F(-x)]$ belong to $L^2[0,\pi]$.

In particular, taking $F_1 = F_2 = F \in L^2[0,\pi]$ we obtain the sampling expansion for a function f bandlimited to $[0,\pi]$ in the sense of the *Hartley transform*. To be more precise

• *Any function of the form*

$$f(t) = \int_0^\pi F(x)[\cos tx + \sin tx]dx\,, \qquad \textit{with } F \in L^2[0,\pi]\,,$$

can be expanded as a cardinal series (17).

Recall that the Hartley transform of a function F, defined as

$$f(t) = \int_0^\infty F(x)[\cos tx + \sin tx]dx\,,$$

was introduced by R. V. L. Hartley, an electric engineer, as a way to overcome what he considered a drawback of the Fourier transform, namely, representing a real-valued function $F(x)$ by a complex-valued one

$$g(t) = \int_{-\infty}^\infty F(x)[\cos tx - i\sin tx]dx\,.$$

For more information about the Hartley transform see, for instance, [136, p. 265].

5. The ν-Bessel-Hankel space

The Fourier-Bessel set $\{\sqrt{x}J_\nu(x\lambda_n)\}_{n=1}^\infty$ is known to be an orthogonal basis for $L^2[0,1]$, where λ_n is the nth positive zero of the Bessel function $J_\nu(t)$, $\nu > -1/2$ [127, p. 580]. The Bessel function of order ν is given by

$$J_\nu(t) = \frac{t^\nu}{2^\nu \Gamma(\nu+1)} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1+\nu)\cdots(n+\nu)} \left(\frac{t}{2}\right)^{2n} \right] .$$

J_ν satisfies the Bessel differential equation

$$t^2 y'' + ty' + (t^2 - \nu^2)y = 0 .$$

Using special function formulas [1, 11.3.29], for a fixed $t > 0$, we have

$$\sqrt{xt}J_\nu(xt) = \sum_{n=1}^{\infty} \frac{2\sqrt{t\lambda_n}J_\nu(t)}{J'_\nu(\lambda_n)(t^2-\lambda_n^2)} \sqrt{x}J_\nu(x\lambda_n) , \qquad \text{in } L^2[0,1] .$$

Hence

- *The range of the integral transform*

$$f(t) = \int_0^1 F(x)\sqrt{xt}J_\nu(xt)dx , \quad F \in L^2[0,1] \tag{25}$$

is a RKHS $\mathcal{H}_\nu$ and the sampling expansion

$$f(t) = \sum_{n=1}^{\infty} f(\lambda_n) \frac{2\sqrt{t\lambda_n}J_\nu(t)}{J'_\nu(\lambda_n)(t^2-\lambda_n^2)} \tag{26}$$

holds for $f \in \mathcal{H}_\nu$.

Using a well-known integral [127, p. 134], the reproducing kernel is

$$k_\nu(s,t) = \frac{\sqrt{st}}{t^2-s^2}\{tJ_{\nu+1}(t)J_\nu(s) - sJ_{\nu+1}(s)J_\nu(t)\} .$$

Furthermore,

$$\begin{aligned} \|K(\cdot,t)\|^2_{L^2(0,1)} = k_\nu(t,t) &= t\int_0^1 x|J_\nu(xt)|^2\,dx \\ &= \frac{t}{2}\left\{ [J'_\nu(t)]^2 + (1-\frac{\nu^2}{t^2})[J_\nu(t)]^2 \right\} = t\,O(\frac{1}{t}) \end{aligned}$$

as t goes to ∞ [127]. As a consequence, the convergence of the series in (26) is absolute and uniform in any interval $[t_0,\infty)$ with $t_0 > 0$.

Note that the integral kernel in (25) is that of *Hankel transform*. Recall that the Hankel transform of a function F is defined as

$$f(t) = \int_0^\infty F(x)\sqrt{xt}J_\nu(xt)dx\,, \quad t > 0\,, \quad \nu > -1/2\,.$$

It defines an unitary, i.e., a bijective isometry, operator $L^2[0,\infty) \longrightarrow L^2[0,\infty)$ which is self-inverse [91, p. 366]. Therefore, functions in $\mathcal{H}_\nu$ are those functions in $L^2[0,\infty)$, bandlimited to $[0,1]$ in the Hankel transform sense, and (26) is the associated sampling formula. See [62] and [136, p. 371] for more details about Hankel transform and its associated sampling series.

6. The continuous Laguerre transform

The sequence $\{e^{-x/2}L_n(x)\}_{n=0}^\infty$ is an orthonormal basis for $L^2[0,\infty)$, where $L_n(x) = \sum_{k=0}^n (-1)^k \frac{1}{k!}\binom{n}{k}x^k$ is the nth Laguerre polynomial. A continuous extension $L_t(x)$ of the Laguerre polynomials can be found in [135, p. 144]. It is given by

$$L_t(x) = \sum_{n=0}^\infty L_n(x)\frac{\sin\pi(t-n)}{\pi(t-n)}\,.$$

$L_t(x)$ is a $\mathcal{C}^\infty$-function that satisfies the Laguerre differential equation

$$xy'' + (1-x)y' + ty = 0\,,$$

which is the same differential equation satisfied by $L_n(x)$ when t is replaced by n. For our sampling purposes, the most important feature is that the expansion

$$e^{-x/2}L_t(x) = \sum_{n=0}^\infty \frac{\sin\pi(t-n)}{\pi(t-n)}e^{-x/2}L_n(x)$$

holds in $L^2[0,\infty)$. Therefore

• *Any function of the form*

$$f(t) = \int_0^\infty F(x)e^{-x/2}L_t(x)dx\,, \qquad \textit{with } F \in L^2[0,\infty)$$

can be expanded as the sampling series

$$f(t) = \sum_{n=0}^\infty f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}\,.$$

In a similar way, one can consider other families of special functions defining integral transforms and seek the associated sampling expansion. This is the case, for instance, of the *continuous Legendre transform* involving the Legendre function, the *finite continuous Jacobi transform* involving the Jacobi function or more general versions of the continuous Laguerre transform considered in this example. See [135, Ch. 4] for a complete discussion of this topic.

7. The multidimensional WSK theorem

The general theory in Section A can be easily adapted to higher dimensions. For simplicity we consider the bidimensional case.

The sequence $\{e^{-inx}e^{-imy}/2\pi\}$ is an orthonormal basis for $L^2(\mathbf{R})$, where $\mathbf{R}$ denotes the square $[-\pi,\pi]\times[-\pi,\pi]$. For a fixed $(t,s)\in\mathbb{R}^2$, we have

$$\frac{1}{2\pi}e^{itx}e^{isy}=\sum_{n,m}\frac{\sin\pi(t-n)}{\pi(t-n)}\frac{\sin\pi(s-m)}{\pi(s-m)}\frac{1}{2\pi}e^{inx}e^{imy}\quad\text{in } L^2(\mathbf{R})\,.$$

The functions $S_{nm}(t,s)=\dfrac{\sin\pi(t-n)}{\pi(t-n)}\dfrac{\sin\pi(s-m)}{\pi(s-m)}$ and the sequence $\{t_{nm}=(n,m)\}$, $n,m\in\mathbb{Z}$, satisfy conditions C1 and C2 in Section 2. Therefore

- *Any function of the form*

$$f(t,s)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F(x,y)e^{itx}e^{isy}dxdy\qquad \textit{with } F\in L^2(\mathbf{R})\,,$$

can be recovered by means of the double series

$$f(t,s)=\sum_{n,m}f(n,m)\frac{\sin\pi(t-n)}{\pi(t-n)}\frac{\sin\pi(s-m)}{\pi(s-m)}\,.$$

The series converges absolutely, and uniformly on $\mathbb{R}^2$.

Similarly, one can get bidimensional versions of sampling formulas like (18) or (19) by considering orthonormal bases in $L^2(\mathbf{R})$ obtained from orthonormal bases in each separate variable.

Certainly, one can always find a rectangle enclosing the bounded support B of the bidimensional Fourier transform of a bidimensional bandlimited signal f. Thus, we can use the bidimensional WSK formula to reconstruct f. However, this is clearly inefficient from a practical point of view, since we are using more information than strictly needed. In general, the support of the Fourier transform B will be an irregularly shaped set. So, obtaining more efficient reconstruction procedures depends largely on the particular geometry of B. See [64, Ch. 14] for a more specific account.

On the other hand, regular multidimensional sampling corresponds to a cartesian uniform sampling grid which is used in signal and image processing whenever possible. However, the practice imposes other sampling grids, like the polar grid used in computerized tomography or the spiral grid used for fast magnetic resonance (see for example [20, 118]). Consequently, in general, irregular sampling is more suitable than regular sampling for multidimensional signals.

8. The Mellin-Kramer sampling result

First, we introduce the necessary ingredients to understand the subsequent development. They are taken, besides the main result, from reference [24]. A function

$f : \mathbb{R}^+ \longrightarrow \mathbb{C}$ is called c-recurrent for $c \in \mathbb{R}$, if $f(x) = e^{2\pi c} f(e^{2\pi} x)$ for all $x \in \mathbb{R}^+$ where $\mathbb{R}^+$ stands for $(0, +\infty)$. The functional space

$$Y_c^2 := \left\{ f : \mathbb{R}^+ \longrightarrow \mathbb{C};\ f \in L^1_{loc}(\mathbb{R}^+),\ \text{c-recurrent, and } \int_{e^{-\pi}}^{e^{\pi}} |f(x)x^c|^2 \, \frac{dx}{x} < \infty \right\},$$

is a Hilbert space under the inner product

$$\langle f, g \rangle_{\mathcal{M}^c} = \int_{e^{-\pi}}^{e^{\pi}} f(x)\overline{g(x)} x^{2c} \, \frac{dx}{x} \, .$$

It is known that the sequence $\{e^{-ck} x^{-c-ik}\}_{k \in \mathbb{Z}}$ forms an orthogonal basis for Y_c^2. The same occurs for its conjugate sequence. Next, we consider the kernel

$$K_c(t, x) = t^{-c} x^{-c-i \log t} \, , \quad t \in \mathbb{R}^+ \, , \quad x \in [e^{-\pi}, e^{\pi}] \, .$$

For a fixed $t \in \mathbb{R}^+$, we have the expansion

$$K_c(t, x) = \sum_{k \in \mathbb{Z}} S_{c,k}(t) e^{-ck} \, x^{-c-ik}$$

in Y_c^2, where

$$S_{c,k}(t) = \frac{e^{2ck}}{2\pi} \langle K_c(t, \cdot), K_c(e^k, \cdot) \rangle_{\mathcal{M}^c} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(e^{-k} t \right)^{-c-iu} du \, .$$

For the proofs and more details, see the mentioned reference [24]. Therefore, taken $t_k = e^k$, $k \in \mathbb{Z}$, as sampling points and taking into account that $S_{c,k}(t_m) = \delta_{k,m}$ we obtain the following exponential sampling result

• *If f can be represented in the form*

$$f(t) = \int_{e^{-\pi}}^{e^{\pi}} F(x) K_c(t, x) \, x^{2c} \, \frac{dx}{x} \, , \quad t \in \mathbb{R}^+ \, ,$$

for some $c \in \mathbb{R}$ and some $F \in Y_c^2$, then f can be reconstructed by means of the exponential sampling

$$f(t) = \sum_{k=-\infty}^{\infty} f(e^k) S_{c,k}(t) \, .$$

This sampling result is valid for Mellin-bandlimited functions, i.e., functions f represented as

$$f(t) = \frac{1}{2\pi} \int_{-1}^{1} \mathcal{M}[f](c + iu) \, t^{-c-iu} \, du \, , \quad t \in \mathbb{R}^+ \, .$$

Recall that Mellin transform is defined by

$$\mathcal{M}[f](s) := \int_0^{\infty} f(u) \, u^{s-1} \, du \, , \quad s = c + it \in \mathbb{C}$$

whenever the integral exists. Again, we address the interested reader to [24] to complete the details. An application of exponential sampling in optics can be found in [55].

Finally we can add that, generally speaking, one can easily construct spaces $\mathcal{H}$ as in Section II.A having a sampling property at a sequence $\{t_n\}_{n=1}^{\infty}$ as in formula (14). To this end, let $t_1, t_2, \cdots$ be distinct real numbers such that $\sum_n 1/|t_n|^2 < \infty$. There exists an analytic function $P(t)$ with simple zeros at the sequence $\{t_n\}_{n=1}^{\infty}$ [85, p. 457]. Specifically, the function $P(t)$ is given by the canonical product

$$P(t) = \begin{cases} \prod_{n=1}^{\infty}(1 - \frac{t}{t_n})\exp(t/t_n) & \text{if } \sum_{n=1}^{\infty}|t_n|^{-1} = \infty \\ \prod_{n=1}^{\infty}(1 - \frac{t}{t_n}) & \text{if } \sum_{n=1}^{\infty}|t_n|^{-1} < \infty \end{cases}$$

whenever $t_n \neq 0$ for all $n \in \mathbb{N}$, or by

$$P(t) = \begin{cases} t\prod_{n=2}^{\infty}(1 - \frac{t}{t_n})\exp(t/t_n) & \text{if } \sum_{n=2}^{\infty}|t_n|^{-1} = \infty \\ t\prod_{n=2}^{\infty}(1 - \frac{t}{t_n}) & \text{if } \sum_{n=2}^{\infty}|t_n|^{-1} < \infty \end{cases}$$

in the case when, for instance, $t_1 = 0$ (see [132, p. 55] for the details).

Taking $S_n(t) = \dfrac{P(t)}{t - t_n}$ and any orthonormal basis $\{\phi_n(x)\}_{n=1}^{\infty}$ for an $L^2(I)$ space, we can follow the steps in Section A in order to construct a RKHS $\mathcal{H}$ with the sampling property at the given sequence $\{t_n\}_{n=1}^{\infty}$. Thus, taking into account the fact that $S_n(t_k) = P'(t_n)\delta_{n,k}$, formula (14) ensures that any function of the form (8) can be expanded as the Lagrange type interpolation series

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{P(t)}{(t - t_n)P'(t_n)}\,.$$

This result was introduced for the first time, in connection with an inverse sampling problem, in [137].

C. Finite sampling

Consider $(\mathcal{H}_N, \langle\cdot,\cdot\rangle_{\mathcal{H}_N})$ an euclidean finite dimensional functional space comprising functions defined on $\Omega \subset \mathbb{R}$. Let N be its dimension and let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be an orthonormal basis for $\mathcal{H}_N$. Since in a finite dimensional space every linear functional is bounded, $\mathcal{H}_N$ is a RKHS whose reproducing kernel, given by (11), is

$$k_N(t, s) = \sum_{i=1}^{N} \varphi_i(t)\overline{\varphi_i(s)}\,.$$

One can easily check that if $f(t) = \sum_{i=1}^{N} a_i \varphi_i(t) \in \mathcal{H}_N$, where $a_i \in \mathbb{C}$, then

$$\langle f, \sum_{i=1}^{N} \varphi_i(t)\overline{\varphi_i(s)}\rangle_{\mathcal{H}_N} = \langle \sum_{i=1}^{N} a_i\varphi_i(t), \sum_{i=1}^{N} \varphi_i(t)\overline{\varphi_i(s)}\rangle_{\mathcal{H}_N} = \sum_{i=1}^{N} a_i\varphi_i(s) = f(s)\,.$$

In the case where $\mathcal{H}_N$ is a subspace of a larger Hilbert space $\mathcal{H}$ (for instance, when $\{\varphi_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$), by applying property (15), for every $f \in \mathcal{H}$ we obtain that

$$\langle f, k_N(\cdot, s)\rangle_{\mathcal{H}_N} = \sum_{i=1}^{N} \langle f, \varphi_i\rangle \varphi_i(s)\,, \tag{27}$$

i.e., its orthogonal projection onto $\mathcal{H}_N$. The reproducing formula for $\mathcal{H}_N$ is a useful tool to prove pointwise convergence of the generalized Fourier expansion $\sum_{i=1}^{\infty} a_i\varphi_i(t)$ whenever it holds (see in this direction reference [126]).

We can derive a finite sampling expansion for $\mathcal{H}_N$ in the following way. Assume that there exists a finite sequence of points $\{s_n\}_{n=1}^{N}$ in Ω such that $\{k_N(t, s_n)\}_{n=1}^{N}$ are orthogonal in $\mathcal{H}_N$, i.e.,

$$\langle k_N(\cdot, s_m), k_N(\cdot, s_n)\rangle_{\mathcal{H}_N} = k_N(s_n, s_m)\delta_{nm}\,.$$

Then, expanding any function $f \in \mathcal{H}_N$ in the orthogonal basis $\{k_N(t, s_n)\}_{n=1}^{N}$ we obtain the following finite sampling expansion

$$f(t) = \sum_{n=1}^{N} f(s_n)\,\frac{k_N(t, s_n)}{k_N(s_n, s_n)}\,. \tag{28}$$

In this context, two examples are of particular interest

1. Trigonometric polynomials

Consider $\mathcal{H}_N$ the space of trigonometric polynomials of degree $\leq N$ and period 2π. $\mathcal{H}_N$ is a subspace of $L^2[-\pi, \pi]$ endowed with the usual inner product. An orthonormal basis for $\mathcal{H}_N$ is given by the set of exponential complex $\{e^{ikt}/\sqrt{2\pi}\}_{k=-N}^{N}$. Therefore, the reproducing kernel for $\mathcal{H}_N$ is

$$k_N(t, s) = \frac{1}{2\pi}\sum_{k=-N}^{N} e^{ik(t-s)} = \frac{1}{2\pi}D_N(t-s)\,,$$

where D_N denotes the N-th Dirichlet kernel defined as

$$D_N(t) = \sum_{k=-N}^{N} e^{ikt} = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}\,.$$

As a consequence of (27), we obtain the following well-known result for $f \in L^2[-\pi,\pi]$

$$\langle f, k_N(\cdot,t)\rangle_{L^2[-\pi,\pi]} = \frac{1}{2\pi}\sum_{k=-N}^{N}\langle f, e^{ikt}\rangle e^{ikt}\,,$$

i.e., its N-th Fourier partial sum.

Now we can obtain a sampling formula for the space $\mathcal{H}_N$ of the trigonometric polynomials of degree $\leq N$. To this end, consider the points $s_n = \frac{2\pi n}{2N+1} \in [-\pi,\pi]$, $n = -N,\cdots,0,\cdots,N$. Since

$$k_N(s_m,s_n) = \frac{1}{2\pi}\frac{\sin\pi(m-n)}{\sin\frac{\pi(m-n)}{2N+1}} = \frac{2N+1}{2\pi}\,\delta_{mn}\,,$$

a direct application of the sampling formula (28) gives

$$p(t) = \frac{1}{2N+1}\sum_{n=-N}^{N} p\Big(\frac{2\pi n}{2N+1}\Big)\frac{\sin\big(\frac{2N+1}{2}\big)\big(t-\frac{2\pi n}{2N+1}\big)}{\sin\frac{1}{2}\big(t-\frac{2\pi n}{2N+1}\big)}\,,$$

for every trigonometric polynomial $p(t) = \sum_{k=-N}^{N} c_k e^{ikt}$ in $\mathcal{H}_N$. This interpolation formula, due to Cauchy [34], goes back to 1841 and is related to the finite version of Shannon's sampling theorem.

2. Orthogonal polynomials

Another important class of examples is given by finite families of orthogonal polynomials on an interval of the real line. As an illustration, we restrict ourselves to the particular case of the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ defined, for instance, by means of their Rodrigues formula

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\big[(t^2-1)^n\big]\,.$$

It is known that they form an orthogonal basis for $L^2[-1,1]$ and that $\|P_n\|^2 = (n+\frac{1}{2})^{-1}$.

Consider $\mathcal{H}_N$ the finite subspace of $L^2[-1,1]$ spanned by $\{P_0,P_1,\ldots,P_N\}$. For this space we have that

$$k_N(t,s) = \sum_{n=0}^{N} P_n(t)P_n(s) = \frac{N+1}{2}\,\frac{P_{N+1}(t)P_N(s) - P_N(t)P_{N+1}(s)}{t-s}\,,$$

where we have used the Christoffel-Darboux formula for Legendre polynomials. Note that

$$k_N(t,t) = \frac{N+1}{2}[P'_{N+1}(t)P_N(t) - P'_N(t)P_{N+1}(t)]\,.$$

We seek points $\{s_n\}_{n=0}^N$ in $[-1,1]$ such that $k_N(s_m,s_n)=0$ for $m\neq n$, i.e.,

$$\frac{P_{N+1}(s_m)}{P_N(s_m)}=\frac{P_{N+1}(s_n)}{P_N(s_n)}\,.$$

In particular we can take for $\{s_n\}_{n=0}^N$ the $N+1$ simple roots of P_{N+1} in $(-1,1)$. Thus, we obtain the finite sampling formula

$$f(t)=\sum_{n=0}^{N} f(s_n)\frac{P_{N+1}(t)}{(t-s_n)P'_{N+1}(s_n)}\,,$$

for every $f(t)=\sum_{k=0}^N c_k\sqrt{(k+\frac{1}{2})}P_k(t)$. Note that this formula is nothing but Lagrange interpolation formula for the samples $\{f(s_n)\}_{n=0}^N$. In general, we can take as sampling points $\{s_n\}_{n=0}^N$ the $N+1$ simple roots of the polynomial $P_{N+1}(t)-cP_N(t)$ in $(-1,1)$, where $c\in\mathbb{R}$. The sampling formula in this general case reads

$$f(t)=\sum_{n=0}^{N} f(s_n)\frac{P_{N+1}(t)P_N(s_n)-P_N(t)P_{N+1}(s_n)}{(t-s_n)P_N(t)[P'_{N+1}(s_n)-cP'_N(s_n)]}\,.$$

General results about families of orthogonal polynomials can be found, for instance, in classical references [110, 121] or in [126]. More examples and applications can be found in [6, 64, 102].

III. Classical Paley–Wiener spaces revisited

In this section we will extend the important example of the *classical bandlimited functions* given in section II.B.1. by digging a little more in the space $(\mathcal{H}_\pi,\|\cdot\|_{\mathcal{H}_\pi})$ and in the isometry between $\mathcal{H}_\pi$ and $L^2[-\pi,\pi]$ obtained there.

It is well-known that the Fourier transform $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$, i.e.,

$$\begin{aligned}\mathcal{F}: L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R})\\ f &\longrightarrow \mathcal{F}(f)=\widehat{f}\,,\end{aligned}$$

is a linear, bijective transform satisfying $\|f\|_{L^2(\mathbb{R})}=\|\widehat{f}\|_{L^2(\mathbb{R})}$ for every $f\in L^2(\mathbb{R})$ [91, p. 362]. Whenever f or $\widehat{f}$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, the Fourier or inverse Fourier transform coincide with the parametric integrals

$$\widehat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-it\omega}dt \quad\text{or}\quad f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{it\omega}d\omega\,,$$

respectively [91, p. 335]. For functions f, $\widehat{f}$ in $L^2(\mathbb{R})$, the integrals must be understood as limits in the mean. Thus, for $\widehat{f}$ we have

$$\int_{-\infty}^{\infty}\left|\widehat{f}(\omega)-\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} f(t)e^{-it\omega}dt\right|^2 d\omega\to 0\,,$$

as $N \to \infty$ [91, p. 362].

As a consequence of this discussion, the space

$$\mathcal{H}_\pi = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \;\middle|\; f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx \,,\; F \in L^2[-\pi, \pi] \right\}$$

coincides with the closed subspace of $L^2(\mathbb{R})$ given by $\mathcal{F}^{-1}\big(L^2[-\pi, \pi]\big)$, i.e., the classical Paley-Wiener space given by

$$PW_\pi := \big\{ f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq L^2[-\pi, \pi] \big\},$$

where supp $\widehat{f}$ denotes the support of the Fourier transform of f. Hence, $\widehat{f}$ is zero outside $[-\pi, \pi]$ for any $f \in PW_\pi$. The isometry between $\mathcal{H}_\pi$ and $L^2[-\pi, \pi]$ is nothing more that the restriction of the Fourier transform to PW_π, and the inner product is given by

$$\langle f, g \rangle_{\mathcal{H}_\pi} = \int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx = \langle f, g \rangle_{L^2(\mathbb{R})} \,.$$

A. Fourier duality

The Paley-Wiener space PW_π can be expressed without resorting to the Fourier transform. Namely, any function $f \in PW_\pi$ can be extended to any $z \in \mathbb{C}$ as

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(\omega) e^{iz\omega} d\omega \,. \tag{29}$$

Thus we get a holomorphic (or analytic) function on $\mathbb{C}$, i.e., an *entire function*. To this end, first we prove that f is a continuous function on $\mathbb{C}$ by using a standard argument allowing interchange the limit with the integral. After, we apply Morera's theorem [85, p. 173]: whenever $\gamma : [a, b] \longrightarrow \mathbb{C}$ is a closed curve in $\mathbb{C}$, the integral

$$\int_\gamma f(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_a^b \left(\int_{-\pi}^{\pi} \widehat{f}(\omega) e^{i\gamma(t)\omega} d\omega \right) \gamma'(t) dt$$

is shown to be zero by interchanging the order of the integrals.

Moreover, f is a function of *exponential type at most* π, i.e., satisfies an inequality $|f(z)| \leq A e^{\pi |z|}$ for all $z \in \mathbb{C}$ and some positive constant A. It follows from (29) by using the Cauchy-Schwarz inequality. Indeed, for $z = x + iy \in \mathbb{C}$ we have

$$|f(x+iy)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |\widehat{f}(\omega)|\, e^{-y\omega}\, d\omega \leq \frac{e^{\pi|y|}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |\widehat{f}(\omega)|\, d\omega \leq e^{\pi|z|} \|f\|_{PW_\pi} \,. \tag{30}$$

Conversely, the classical Paley-Wiener theorem [132, p. 100] shows that PW_π coincides with the space of entire functions of exponential type at most π with square integrable restriction to the real axis, i.e.,

$$PW_\pi = \big\{ f \in \mathcal{H}(\mathbb{C}) \;:\; |f(z)| \leq A e^{\pi|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R}) \big\} \,.$$

The isometric isomorphism given by the Fourier transform

$$\begin{array}{ccc} PW_\pi & \xrightarrow{\mathcal{F}} & L^2[-\pi,\pi] \\ f & \longrightarrow & \widehat{f} \end{array} , \qquad f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(\omega)\, e^{iz\omega} d\omega ,$$

is called the *Fourier duality* between the spaces PW_π and $L^2[-\pi,\pi]$, and it has far-reaching consequences. Any expansion converging in $L^2[-\pi,\pi]$ is transformed by $\mathcal{F}^{-1}$ into another expansion which converges in the topology of PW_π. This implies, by the reproducing kernel property, that it converges uniformly on $\mathbb{R}$ as we showed in Section II.B.1.

The following nontrivial properties of PW_π can be easily established by using the Fourier duality:

(a) *The energy of $f \in PW_\pi$, i.e., its L^2 norm, is contained in that of its samples $\{f(n)\}_{n\in\mathbb{Z}}$.*

Indeed, since $\{f(n)\}_{n\in\mathbb{Z}}$ are the coefficients of the Fourier expansion of $\widehat{f}$ in the orthonormal basis $\{e^{-inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$, the Parseval formula (6) gives

$$\|f\|^2_{PW_\pi} = \|\widehat{f}\|^2_{L^2[-\pi,\pi]} = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \|\{f(n)\}\|^2_{\ell^2(\mathbb{Z})} .$$

(b) *The sequence $\{\mathrm{sinc}\,(z-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis in PW_π. Expanding $f \in PW_\pi$ in this basis we obtain its WKS expansion* (17). *Also, for each fixed $\alpha \in \mathbb{R}$, $\{\mathrm{sinc}\,(z-n-\alpha)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for PW_π giving the sampling expansion* (18).

(c) *PW_π is a RKHS whose reproducing kernel is given by $k_\pi(z,w) = \mathrm{sinc}(z-\overline{w})$, whenever z, w in $\mathbb{C}$.*

Recall that for real variables t, s we obtained $k_\pi(t,s) = \mathrm{sinc}(t-s)$. For complex variables is necessary to conjugate the second variable in the cardinal sine function. Indeed,

$$f(w) = \frac{1}{\sqrt{2\pi}} \langle \widehat{f}(x), e^{-i\overline{w}x} \rangle_{L^2[-\pi,\pi]} = \langle f(z), \mathrm{sinc}(z-\overline{w}) \rangle_{PW_\pi} , \quad w \in \mathbb{C} ,$$

by using the Fourier duality.

(d) *Convergence in the norm of $f \in PW_\pi$ implies uniform convergence in horizontal strips of $\mathbb{C}$.*

It is a consequence of the inequality $|f(z)| \le e^{\pi|y|}\|f\|_{PW_\pi}$, $z = x+iy \in \mathbb{C}$, in (30).

(e) *For any $f \in PW_\pi$, $|f(x)|$ goes to zero as to $|x| \to \infty$, $x \in \mathbb{R}$.*

This is a straightforward consequence of the Riemann-Lebesgue lemma [8, p. 170]. Furthermore, using the properties of the inverse Fourier transform with respect to the derivation, we see that the smoother $\widehat{f}$ is, the faster the decay of f is [8, p. 334].

(f) *PW_π is closed under derivation, and for every $f \in PW_\pi$ the following Bernstein-type inequality holds:*

$$\|f'\|_{PW_\pi} \leq \pi \|f\|_{PW_\pi} .$$

This follows from

$$f'(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} i\omega \widehat{f}(\omega)\, e^{iz\omega} d\omega ,$$

applying the Cauchy-Schwarz inequality.

The classical Bernstein's inequality also holds: for $f \in PW_\pi$, $\|f'\|_\infty \leq \pi \|f\|_\infty$, where we are using the supremun norm $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$ [99, p. 209].

(g) *The orthogonal projection $P_{PW_\pi} f$ of $f \in L^2(\mathbb{R})$ onto PW_π is given by*

$$P_{PW_\pi} f(t) = \mathcal{F}^{-1}(\chi_{[-\pi,\pi]} \mathcal{F} f)(t) = \langle f, \operatorname{sinc}(\cdot - t) \rangle_{PW_\pi} = (f * \operatorname{sinc})(t) ,$$

for $t \in \mathbb{R}$, where $\chi_{[-\pi,\pi]}$ denotes the characteristic function of the interval $[-\pi, \pi]$ and $*$ the convolution operator.

The first equality comes from the minimum norm property of the orthogonal projection [91, p. 302]. Indeed, for $f \in L^2(\mathbb{R})$ and $g \in PW_\pi$ we have

$$\|f - g\|^2_{L^2(\mathbb{R})} = \|\widehat{f} - \widehat{g}\|^2_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widehat{f}(\omega) - \widehat{g}(\omega)|^2 d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-\pi} |\widehat{f}(\omega)|^2 d\omega + \int_{-\pi}^{\pi} |\widehat{f}(\omega) - \widehat{g}(\omega)|^2 d\omega + \int_{\pi}^{\infty} |\widehat{f}(\omega)|^2 d\omega \right] ,$$

which is mimimun for $\widehat{g} = \widehat{f}\, \chi_{[-\pi,\pi]}$, when the second summand equals zero. The other equalities come from (15) and the definition of the convolution.

(h) *In PW_π, fixing $t_0 \in \mathbb{R}$, $E > 0$ and $M \in \mathbb{C}$, we have*

$$\max_{\|f\|^2 \leq E} |f(t_0)|^2 = E , \quad \textit{reached for} \quad f^*(s) = \pm\sqrt{E}\, \frac{\sin \pi(s - t_0)}{\pi(s - t_0)} .$$

Similarly,

$$\min_{f(t_0) = M} \|f\|^2 = M^2 , \quad \textit{reached for} \quad f^*(s) = M\, \frac{\sin \pi(s - t_0)}{\pi(s - t_0)} .$$

$$\begin{array}{ccc} f \in PW_\pi & \xrightarrow{\mathcal{F}} & \widehat{f} \in L^2[-\pi,\pi] \\ \mathcal{S}\downarrow & & \downarrow\mathcal{P} \\ \{f(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) & \xrightarrow[\mathcal{F}]{} & \widehat{f}_p \in L^2_p[-\pi,\pi] \end{array}$$

Figure 1: Fourier duality and sampling.

Using Fourier duality allows to derive other expansions, not necessarily a sampling one. For instance, the use of the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ leads to the so-called Bessel-Neumann expansion in

$$PW_1 = \left\{ f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq [-1,1] \right\}.$$

It is well-known that $\left\{ \sqrt{n+\frac{1}{2}} P_n(x) \right\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[-1,1]$ and that

$$\mathcal{F}\Big[\frac{i^n}{\sqrt{2\pi}} t^{-1/2} J_{n+\frac{1}{2}}(t) \Big](x) = P_n(x)\chi_{[-1,1]}(x)$$

for any $n \in \mathbb{N} \cup \{0\}$, where $J_{n+\frac{1}{2}}(t)$ is the Bessel function of half odd integer order. For any $f \in PW_1$, we expand its Fourier transform $\widehat{f}$ as

$$\widehat{f}(x) = \sum_{n=0}^{\infty} a_n P_n(x)\,, \quad a_n = (n+\frac{1}{2})\langle \widehat{f}, P_n \rangle_{L^2[-1,1]}\,.$$

As a consequence of the inverse Fourier transform, we get

- *Any $f \in PW_1$ can be expanded as*

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{i^n}{\sqrt{2\pi}} z^{-1/2} J_{n+\frac{1}{2}}(z)\,.$$

The convergence is absolute and uniform on horizontal strips of $\mathbb{C}$.

Next, we are going to explore the meaning of the Fourier duality in terms of Signal Theory by using the commutative diagram showed in figure 1.

All mappings included in this diagram are bijective isometries: the signal energy is preserved. $\mathcal{S}$ denotes the *sampling mapping* with sampling period $T_s = 1$. $\mathcal{P}$ is the *2π-periodization mapping* which extends a function $\widehat{f}$ in $[-\pi,\pi]$ to the whole $\mathbb{R}$ with period 2π. The other two mappings are, respectively, the functional Fourier transform and the Fourier transform in $\ell^2(\mathbb{Z})$, defined as

$$\mathcal{F}(\{a_n\})(\omega) := \sum_{n=-\infty}^{\infty} a_n \frac{e^{-in\omega}}{\sqrt{2\pi}}\,, \quad \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\,.$$

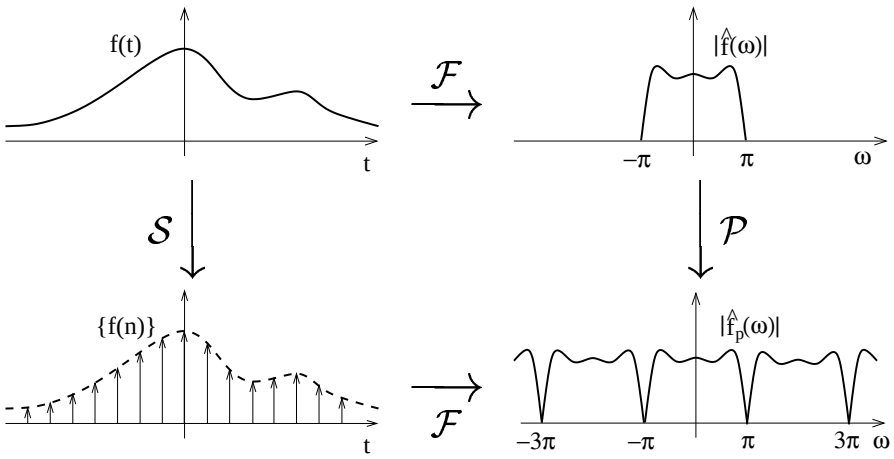

Figure 2: Time-frequency interpretation of the WSK sampling theorem.

Thus, we obtain the very well-known result by signal processing engineers, which states that sampling a signal (with a sampling period $T_s = 1$ in this case), matches a periodization of its spectrum (with a period 2π in this case). The situation described by the diagram in figure 1 is illustrated in figure 2. In the next section we will deal with the general case, i.e., when we sample a signal in PW_π with a sampling period $T_s > 0$.

Finally to say that, under minor changes, all the results in this section apply for any general Paley-Wiener space $PW_{\pi\sigma}$ defined by

$$PW_{\pi\sigma} := \left\{ f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq L^2[-\pi\sigma, \pi\sigma] \right\},$$

or expressed in the form

$$PW_{\pi\sigma} = \left\{ f \in \mathcal{H}(\mathbb{C}) \ : \ |f(z)| \leq Ae^{\pi\sigma|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R}) \right\},$$

by using the classical Paley-Wiener theorem.

B. Undersampling and oversampling

As it was mentioned in the preceding section, if we sample a signal f in PW_π with a general sampling period $T_s > 0$, the question arises whether it is possible to reconstruct it from its samples $\{f(nT_s)\}$. We will see that it is indeed possible in the case where $0 < T_s \leq 1$, i.e., sampling the signal at a frequency higher than that given by its bandwidth. For sampling periods $T_s > 1$, we cannot reconstruct the signal due to the *aliasing phenomenon*, which will be explained later.

In the next section, by using a version of the Poisson summation formula, it will be shown that sampling a signal with a sampling period T_s is equivalent to periodize its spectrum with a period $2\pi/T_s$.

1. Poisson summation formula

Consider the sequence of samples $\{f(nT_s)\}_{n\in\mathbb{Z}}$ taken from a signal $f \in PW_\pi$ with a sampling period $T_s > 0$. Let $\widehat{f_p}$ be the $\frac{2\pi}{T_s}$-periodized version of $\widehat{f}$, i.e.,

$$\widehat{f_p}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{f}(\omega + \frac{2\pi}{T_s}n)\,.$$

Obviously, $\widehat{f_p}$ is a $\frac{2\pi}{T_s}$ periodic function. Now, we calculate its Fourier expansion with respect to the orthonormal basis $\left\{\sqrt{\frac{T_s}{2\pi}}\,e^{-imT_s\omega}\right\}_{m\in\mathbb{Z}}$ of $L^2[0, \frac{2\pi}{T_s}]$. The Fourier coefficient c_m is calculated as

$$\begin{aligned} c_m &= \sqrt{\frac{T_s}{2\pi}}\int_0^{\frac{2\pi}{T_s}} \widehat{f_p}(\omega)e^{imT_s\omega}d\omega = \sqrt{\frac{T_s}{2\pi}}\int_0^{\frac{2\pi}{T_s}} \sum_{n=-\infty}^{\infty} \widehat{f}\big(\omega + \frac{2\pi}{T_s}n\big)e^{imT_s\omega}d\omega \\ &= \sqrt{\frac{T_s}{2\pi}}\sum_{n=-\infty}^{\infty}\int_0^{\frac{2\pi}{T_s}} \widehat{f}\big(\omega + \frac{2\pi}{T_s}n\big)e^{imT_s\omega}d\omega \end{aligned}$$

The change of variable $\omega + \frac{2\pi}{T_s}n = x$ allows us to obtain

$$\begin{aligned} c_m &= \sqrt{\frac{T_s}{2\pi}}\sum_{n=-\infty}^{\infty}\int_{\frac{2\pi}{T_s}n}^{\frac{2\pi}{T_s}(n+1)} \widehat{f}(x)e^{imT_sx}dx = \sqrt{\frac{T_s}{2\pi}}\int_{-\pi}^{\pi} \widehat{f}(x)e^{imT_sx}dx \\ &= \sqrt{T_s}f(mT_s) \end{aligned}$$

Therefore, the Fourier expansion for $\widehat{f_p}$ is

$$\widehat{f_p}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{f}\big(\omega + \frac{2\pi}{T_s}n\big) = T_s\sum_{m=-\infty}^{\infty} f(mT_s)\frac{e^{-imT_s\omega}}{\sqrt{2\pi}}\,.$$

Thus we have obtained the Poisson summation formula applied to $\widehat{f}$ with period $2\pi/T_s$. From this formula we deduce that the spectrum of the sequence $\{f(mT_s)\}_{m\in\mathbb{Z}}$, i.e., the sampled signal, is precisely (up to a scale factor) the $\frac{2\pi}{T_s}$-periodized version of the spectrum $\widehat{f}$ of f.

As a consequence, in the *oversampling case*, where $0 < T_s \leq 1$, we can recover the spectrum of f from the spectrum of the sampled signal, and hence, recover the signal f. In terms of the WSK sampling theorem, the explanation is easy: if a signal is bandlimited to the interval $[-\pi, \pi]$, is also bandlimited to any interval $[-\pi\sigma, \pi\sigma]$ with $\sigma \geq 1$. This situation is depicted in figure 3.

In the so-called *undersampling case*, where $T_s > 1$, we cannot obtain the spectrum of f from the spectrum of the sampled signal because the copies of $\widehat{f}$ overlap in $\widehat{f_p}$. Hence,

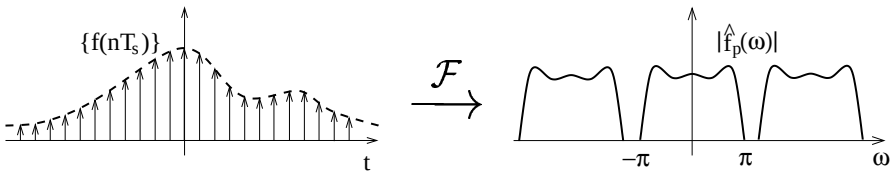

Figure 3: Oversampling.

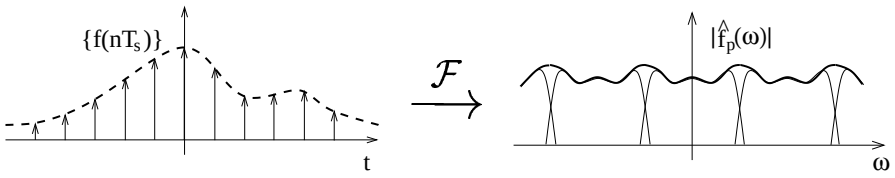

Figure 4: Undersampling.

it is impossible to recover the signal from its samples. The alluded overlap produces the *aliasing phenomenon*, i.e.. some frequencies go under the name of another ones. As pointed out by Hamming in [60, p. 14], this is a familiar phenomenon to the watchers of TV and western movies. As the stage coach starts up, the wheels start going faster and faster, but then they gradually slow down, stop, go backwards, slow down, stop, go forward, etc. This effect is due solely to the sampling the picture makes of the real scene. The undersampling situation is depicted in figure 4.

This undersampling/oversampling discussion clarifies the crucial role of the critical Nyquist period which is given by $T_s = 1/\sigma$ whenever $\operatorname{supp} \widehat{f} \subset [-\pi\sigma, \pi\sigma]$.

Some comments about the Poisson summation formula are in order:

(a) Poisson summation formula is a fundamental way to link a function f with its Fourier transform $\widehat{f}$ or vice versa. Namely

$$\sum_{n=-\infty}^{\infty} f\big(t + \frac{2\pi}{T} n\big) = \frac{T}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widehat{f}(mT)\, e^{imTt}\,. \tag{31}$$

Whenever $f \in L^1(\mathbb{R})$, the left-hand in (31) denotes a $\frac{2\pi}{T}$-periodic function belonging to $L^1[0, 2\pi/T]$. Expanding this function in Fourier series with respect to the orthonormal basis $\big\{e^{imTt}/\sqrt{\frac{2\pi}{T}}\big\}_{m\in\mathbb{Z}}$ we obtain the right-hand of (31) (see [54, 64] for the details). Under smooth hypotheses on f, it can be proved that (31) also holds pointwise (see, for instance, references [54, 99]).

(b) The following formalism, very familiar in the engineering literature, can be used to deduce the WSK sampling formula. Namely, it is a common use to write the

sampled signal $\{f(n)\}_{n\in\mathbb{Z}}$ (we are assuming by simplicity that $T_s = 1$) as

$$f_s(t) := \sum_{n=-\infty}^{\infty} f(n)\delta(t-n) = (f * \Delta)(t)\,,$$

where $\Delta := \sum_{n=-\infty}^{\infty} \delta(t-n)$ denotes the so-called *Dirac's comb* or *train of deltas* at the integers. We want to recover the signal f from its sampled signal f_s by using an appropriate filtering device, i.e.,

$$f(t) = (f_s * g)(t) = \sum_{n=-\infty}^{\infty} f(n)g(t-n)\,,$$

for an appropriate impulse response g. By taking Fourier transform we obtain

$$\widehat{f}(\omega) = \widehat{f_s}(\omega)\widehat{g}(\omega) = \widehat{g}(\omega) \sum_{n=-\infty}^{\infty} \widehat{f}(\omega + 2\pi n)\,,$$

where we have used the Fourier transform for f_s given by Poisson summation formula. Whenever $\operatorname{supp} \widehat{f} \subset [-\pi, \pi]$, the appropriate $\widehat{g}$ is $\chi_{[-\pi,\pi]}$ and consequently $g(t) = \operatorname{sinc}(t)$ i.e., an ideal low-pass filter. All the steps in the above reasoning can be made rigorous at the light of the theory of distributions [54].

2. Robust reconstruction

The actual computation of the cardinal series (17) presents some numerical difficulties since the cardinal sine function behaves like $1/t$ as $|t| \to \infty$. An easy example is the given by the numerical calculation of $f(1/2)$, for a function f in PW_π, from a noisy sequence of samples $\{f(n) + \delta_n\}$. The error in this case $\left|\sum_n \frac{(-1)^n \delta_n}{\pi(n-\frac{1}{2})}\right|$, even when $|\delta_n| \leq \delta$, could be infinity!

One way to overcome this difficulty is again the *oversampling technique*, i.e., sampling the signal at a frequency higher than that given by its bandwidth. In this way we obtain sampling functions converging to zero at infinity faster than the cardinal sine functions. Indeed, consider the bandlimited function

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma}^{\pi\sigma} F(\omega) e^{i\omega t} d\omega \quad \text{with } F \in L^2[-\pi\sigma, \pi\sigma] \quad \text{and } \sigma < 1\,.$$

Extending F to be zero in $[-\pi, \pi] \setminus [-\pi\sigma, \pi\sigma]$, we have

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) \frac{e^{-in\omega}}{\sqrt{2\pi}} \quad \text{in } L^2[-\pi, \pi]\,.$$

Let $\theta(\omega)$ be a smooth function taking the value 1 on $[-\pi\sigma, \pi\sigma]$, and the value 0 outside $[-\pi, \pi]$. As a consequence,

$$F(\omega) = \theta(\omega)F(\omega) = \sum_{n=-\infty}^{\infty} f(n)\theta(\omega)\frac{e^{-in\omega}}{\sqrt{2\pi}} \quad \text{in } L^2[-\pi, \pi]\,,$$

and the sampling expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n)$$

holds, where S is the inverse Fourier transform $\mathcal{F}^{-1}$ of $\theta/\sqrt{2\pi}$ and, consequently, $S(t-n) = \mathcal{F}^{-1}[\theta(\omega)e^{-in\omega}/\sqrt{2\pi}](t)$. Furthermore, using the properties of the Fourier transform we see that the smoother θ is, the faster the decay of S is. However, the new sampling functions $\{S(t-n)\}_{n=-\infty}^{\infty}$ are no longer orthogonal.

Next, we consider a particular example. Assume that $\sigma = 1-\epsilon$ with $0 < \epsilon < 1$, and consider for θ the trapezoidal function

$$\theta(\omega) = \begin{cases} 1 & \text{if } |\omega| \le \pi(1-\epsilon), \\ \frac{1}{\epsilon}\big(1 - \frac{|\omega|}{\pi}\big) & \text{if } \pi(1-\epsilon) \le |\omega| \le \pi, \\ 0 & \text{if } |\omega| \ge \pi. \end{cases}$$

One can easily obtain that $S(t) = \dfrac{\sin \epsilon\pi t}{\epsilon\pi t}\,\dfrac{\sin \pi t}{\pi t}$. Note that, in this case, S behaves like $1/t^2$ as $|t| \to \infty$. The corresponding sampling expansion takes the form

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\frac{\sin \epsilon\pi(t-n)}{\epsilon\pi(t-n)}\,\frac{\sin \pi(t-n)}{\pi(t-n)}\,.$$

In this example, if each sample $f(n)$ is subject to an error δ_n such that $|\delta_n| \le \delta$, then the total error in the above calculated $f(t)$ is bounded by a constant depending only on δ and ϵ [99, p. 211]. Thus we have obtained a robust reconstruction for f by using the oversampling technique.

C. Sampling by using other type of samples

A sampling series in PW_π may also contain samples from a transformed version of the signal as, for instance, its derivative or its Hilbert transform. This is the so-called *multi-channel sampling* setting: the signal is processed through various channels before being sampled. This idea is in Shannon's famous paper [113], where he suggests taking samples of the signal and its derivative. General methods for multi-channel sampling go back to Papoulis' work [97], as pointed out in the expository paper [22]. As this author

says: "for certain applications, data about a given bandlimited signal can be available from several sources". As we will see in the below examples, in the multi-channel case the sampling points can occur at a density below the Nyquist one, but maintaining the overall "number" of samples.

1. Using samples from the derivative

Now we prove that it is possible to recover a signal $f \in PW_\pi$ by using its samples $\{f(2n)\}_{n\in\mathbb{Z}}$, taken at a half of the Nyquist rate, along with the samples $\{f'(2n)\}_{n\in\mathbb{Z}}$ taken from its first derivative. Namely

• *Any function $f \in PW_\pi$ can be recovered from the sequences of samples $\{f(2n)\}_{n\in\mathbb{Z}}$ and $\{f'(2n)\}_{n\in\mathbb{Z}}$ by means of the formula*

$$f(t) = \sum_{n=-\infty}^{\infty} \big\{ f(2n) + (t-2n) f'(2n) \big\} \left[\frac{\sin \frac{\pi}{2}(t-2n)}{\frac{\pi}{2}(t-2n)} \right]^2 \tag{32}$$

To this end, we consider $F \in L^2[-\pi,\pi]$ the Fourier transform of f. The following Fourier expansions in $L^2[-\pi,\pi]$ hold

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) \frac{e^{-in\omega}}{\sqrt{2\pi}} \quad \text{and} \quad F(\omega-\pi) = \sum_{n=-\infty}^{\infty} (-1)^n f(n) \frac{e^{-in\omega}}{\sqrt{2\pi}} \,.$$

As a consequence, the function $S(\omega) = \dfrac{1}{2}[F(\omega) + F(\omega-\pi))]$ admits the Fourier expansion

$$S(\omega) = \sum_{n=-\infty}^{\infty} f(2n) \frac{e^{-i2n\omega}}{\sqrt{2\pi}} \ \text{ in } L^2[0,\pi] \,.$$

In a similar way, since

$$f'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} i\omega F(\omega) e^{it\omega} d\omega \,,$$

we obtain the Fourier expansion

$$i\omega F(\omega) = \sum_{n=-\infty}^{\infty} f'(n) \frac{e^{-in\omega}}{\sqrt{2\pi}} \ \text{ in } \ L^2[-\pi,\pi] \,.$$

Therefore, the function $R(\omega) = \dfrac{i}{2}[\omega F(\omega) + (\omega-\pi)F(\omega-\pi)]$ has the Fourier expansion

$$R(\omega) = \sum_{n=-\infty}^{\infty} f'(2n) \frac{e^{-i2n\omega}}{\sqrt{2\pi}} \ \text{ in } L^2[0,\pi] \,.$$

Gathering together both expansions, for $\omega \in [0,\pi]$, we have

$$\begin{pmatrix} S(\omega) \\ R(\omega) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i\omega & i(\omega-\pi) \end{pmatrix} \begin{pmatrix} F(\omega) \\ F(\omega-\pi) \end{pmatrix}$$

or inverting the matrix

$$\begin{pmatrix} F(\omega) \\ F(\omega-\pi) \end{pmatrix} = \frac{2i}{\pi} \begin{pmatrix} i(\omega-\pi) & -1 \\ -i\omega & 1 \end{pmatrix} \begin{pmatrix} S(\omega) \\ R(\omega) \end{pmatrix} \tag{33}$$

Therefore, introducing this splitting of F into (29) and after some calculations we find,

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(\omega) e^{it\omega} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{0} \sum_{-\infty}^{\infty} [\frac{2}{\pi}(\omega+\pi) f(2n) + \frac{2i}{\pi} f'(2n)] \frac{e^{-i2n\omega}}{\sqrt{2\pi}} e^{it\omega} d\omega \\ &+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi} \sum_{-\infty}^{\infty} [\frac{2}{\pi}(\pi-\omega) f(2n) - \frac{2i}{\pi} f'(2n)] \frac{e^{-i2n\omega}}{\sqrt{2\pi}} e^{it\omega} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \left\{ \int_{-\pi}^{\pi} \sqrt{\frac{2}{\pi}} (1 - \frac{|\omega|}{\pi}) f(2n) e^{i(t-2n)\omega} d\omega \right. \\ &\left. + \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{(-i \operatorname{sgn}\omega)}{\sqrt{2\pi}} f'(2n) e^{i(t-2n)\omega} d\omega \right\} . \end{aligned}$$

The desired result comes by using the Fourier duals (see for instance [64, p. 203])

$$\operatorname{sinc}\left(\frac{t}{2}\right) \sin\left(\frac{\pi t}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{(-i \operatorname{sgn}\omega)}{\sqrt{2\pi}} e^{it\omega} d\omega \tag{34}$$

and

$$\operatorname{sinc}^2\left(\frac{t}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sqrt{\frac{2}{\pi}} \Big(1 - \frac{|\omega|}{\pi}\Big) e^{it\omega} d\omega \, .$$

Some additional comments may throw light on this derivative sampling result:

i. This case corresponds to a two-channel sampling: the signal f is filtered with two filters with transfer functions $H_1(\omega) = 1$ and $H_2(\omega) = i\omega$ where $\omega \in [-\pi,\pi]$, respectively, before sampling with a sampling period $T_s = 2$.

ii. The multi-channel approach used in [64, Ch. 12] and [106] is implicitly in the proof of this easier example. Indeed, the matricial relation (33) can be understood as a linear, bijective and bounded operator from the external direct sum $L^2[0,\pi] \oplus L^2[0,\pi]$ onto $L^2[-\pi,\pi]$. Recall that in this external direct sum the norm is given by

$$\|(F,G)\|^2 = \|F\|^2_{L^2[0,\pi]} + \|G\|^2_{L^2[0,\pi]} \, .$$

The orthonormal basis in $L^2[0,\pi] \oplus L^2[0,\pi]$ given by

$$\{(e^{-i2n\omega}/\sqrt{\pi}, 0), (0, e^{-i2n\omega}/\sqrt{\pi})\}_{n\in\mathbb{Z}}$$

is transformed, via (33), into a Riesz basis of $L^2[-\pi,\pi]$, a concept that will appear later in Section II.E.1.

iii. Reconstruction formulas from samples of $f \in PW_\pi$ and its first $p-1$ derivatives taken at the sampling period $T_s = p$ can be obtained in a similar way. See [65, p. 58] for the equivalent sampling formula to (32) in this more general setting.

2. Using samples from the Hilbert transform

In this section we show that a function $f \in PW_\pi$ can be recovered either from the samples of its Hilbert transform $\{\widetilde{f}(n)\}_{n\in\mathbb{Z}}$, or from both sequences of samples $\{f(2n)\}_{n\in\mathbb{Z}}$ and $\{\widetilde{f}(2n)\}_{n\in\mathbb{Z}}$, where $\widetilde{f}$ stands for the Hilbert transform of f.

First of all we introduce the Hilbert transform for functions in $L^2(\mathbb{R})$, by giving the following motivation: in the case of a real signal $f \in L^2(\mathbb{R})$, its Fourier transform $\widehat{f}$ can be written as $\widehat{f}(\omega) = A(\omega) + i\,B(\omega)$ where A and B are even and odd functions respectively. Therefore, $\widehat{f}$, and consequently f, are determined by the values of $\widehat{f}$ on $[0,\infty)$, i.e., by $\widehat{f}\cdot u$ where u denotes the Heaviside function

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0\,, \\ 0 & \text{if } t < 0\,. \end{cases}$$

The appropriate tool which allows us to take into account this feature about f is, as we will see later, its associated analytic signal defined by means of its Hilbert transform.

a. The Hilbert transform in $L^2(\mathbb{R})$

The *Hilbert transform* in $L^2(\mathbb{R})$ can be defined as

$$\begin{aligned} H : L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}) \\ f &\longrightarrow H(f) = \widetilde{f} := \mathcal{F}^{-1}[(-i\,\mathrm{sgn})\widehat{f}]\,, \end{aligned}$$

where i stands for the imaginary unity and sgn denotes the *signum function*, i.e., $\mathrm{sgn}(t)$ equals 1 if $t > 0$ and -1 if $t < 0$. It is straightforward to obtain the main properties of the Hilbert transform in $L^2(\mathbb{R})$ by using those of the Fourier transform in $L^2(\mathbb{R})$. Namely,

1) H is well defined, i.e., $H(f) \in L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R})$, and is linear by using the properties of the Fourier transform in $L^2(\mathbb{R})$.

2) H is an isometry in $L^2(\mathbb{R})$ since $\|H(f)\|_2 = \|\widehat{H(f)}\|_2 = \|\widehat{f}\|_2 = \|f\|_2$

3) H is bijective and moreover, $H^{-1} = -H$. Indeed,

$$\mathcal{F}[H(Hf)] = (-i\,\mathrm{sgn})\widehat{Hf} = (-i\,\mathrm{sgn})(-i\,\mathrm{sgn})\widehat{f} = -\widehat{f}\,,$$

and hence, $H^2 = -Id$.

4) If f is a real-valued signal in $L^2(\mathbb{R})$, the same occurs with its Hilbert transform $\widetilde{f}$. Indeed, we can write

$$\widetilde{f}(t) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} (-i\,\mathrm{sgn}\,\omega)\widehat{f}(\omega)\, e^{i\omega t}\, d\omega\,,$$

in the mean sense. Hence,

$$\overline{\widetilde{f}(t)} = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} i\,\mathrm{sgn}\,\omega\, \widehat{f}(-\omega)\, e^{-i\omega t}\, d\omega\,.$$

The change of variable $\xi = -\omega$ allows us to conclude that $\overline{\widetilde{f}(t)} = \widetilde{f}(t)$ for almost all t in $\mathbb{R}$.

Another equivalent definition for the Hilbert transform is the given by

$$\begin{aligned}\widetilde{f}(t) &:= \frac{1}{\pi}[p.v.(\frac{1}{x}) * f](t) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{t-x}\,dx \\ &= \lim_{\delta\to 0^+} \frac{1}{\pi}\int_{|u|>\delta} \frac{f(t-u)}{u} du\,,\end{aligned}$$

where $p.v.$ denotes the Cauchy principal value of the integral. It allows us to enlarge the definition of the Hilbert transform to other functional spaces as the $L^p(\mathbb{R})$ with $1 < p < \infty$.

The Paley-Wiener space PW_π is closed under the Hilbert transform, i.e., for $f \in PW_\pi$ its Hilbert transform $\widetilde{f}$ is also in PW_π. Indeed, given $f \in PW_\pi$ we can write its Hilbert transform as

$$\widetilde{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (-i\,\mathrm{sgn}\,\omega)\,\widehat{f}(\omega) e^{it\omega} d\omega\,. \tag{35}$$

Since $|i\,\mathrm{sgn}\,\omega| = 1$, it easily follows that the sequence $\left\{ i\,\mathrm{sgn}\,\omega\, \frac{e^{-in\omega}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$. For a fixed $t \in \mathbb{R}$, we expand the Fourier kernel with respect to this basis obtaining

$$\begin{aligned}\frac{e^{-it\omega}}{\sqrt{2\pi}} &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\langle e^{-it\omega}, i\,\mathrm{sgn}\,\omega\, e^{-in\omega} \right\rangle_{L^2[-\pi,\pi]} i\,\mathrm{sgn}\,\omega\, \frac{e^{-in\omega}}{\sqrt{2\pi}} \\ &= -\sum_{n=-\infty}^{\infty} \mathrm{sinc}\,\frac{1}{2}(t-n)\, \sin\frac{\pi}{2}(t-n) i\,\mathrm{sgn}\,\omega\, \frac{e^{-in\omega}}{\sqrt{2\pi}} \quad \text{in} \quad L^2[-\pi,\pi]\,,\end{aligned} \tag{36}$$

where we have used formula (34) to derive the coefficients of the expansion. This expansion allows us to prove the following sampling result in PW_π

• *Any function $f \in PW_\pi$ can be recovered from the sequence of samples $\{\widetilde{f}(n)\}_{n\in\mathbb{Z}}$ of its Hilbert transform by means of the formula*

$$f(t) = -\sum_{n=-\infty}^{\infty} \widetilde{f}(n)\ \mathrm{sinc}\,\frac{1}{2}(t-n)\sin\frac{\pi}{2}(t-n)\,. \tag{37}$$

To this end, consider

$$f(t) = \langle \widehat{f}, \frac{e^{-it\omega}}{\sqrt{2\pi}}\rangle_{L^2[-\pi,\pi]}\,.$$

Introducing the expansion obtained in (36) and taking into account the continuity of the inner product with respect to the $L^2[-\pi,\pi]$ convergence, we can take out the series. Thus, we obtain

$$\begin{aligned} f(t) &= -\sum_{n=-\infty}^{\infty} \mathrm{sinc}\,\frac{1}{2}(t-n)\sin\frac{\pi}{2}(t-n)\langle \widehat{f}, i\,\mathrm{sgn}\,\omega\,\frac{e^{-in\omega}}{\sqrt{2\pi}}\rangle_{L^2[-\pi,\pi]} \\ &= -\sum_{n=-\infty}^{\infty} \mathrm{sinc}\,\frac{1}{2}(t-n)\sin\frac{\pi}{2}(t-n)\,\widetilde{f}(n)\,. \end{aligned}$$

This sampling formula can be interpreted as a single-channel sampling result: the signal f is filtered with a filter whose transfer function is $\widetilde{H}(\omega) = -i\,\mathrm{sgn}\,\omega$, $\omega \in [-\pi,\pi]$, before sampling with a sampling period $T_s = 1$.

Having in mind that $H^2(f) = -f$, if we apply (37) to $\widetilde{f} \in PW_\pi$ we obtain the dual sampling formula

$$\widetilde{f}(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,\frac{1}{2}(t-n)\sin\frac{\pi}{2}(t-n)\,.$$

Next, we introduce the concept of analytic signal associated with a real-valued signal f in $L^2(\mathbb{R})$.

b. Analytic signal

Given a real signal $f \in L^2(\mathbb{R})$, its associated *analytic signal* is the signal in $L^2(\mathbb{R})$ defined as

$$f_a := f + i\widetilde{f}\,.$$

The Fourier transform of the analytic signal f_a satisfies

$$\widehat{f_a} = \widehat{f} + i(-i\,\mathrm{sgn})\widehat{f} = 2\widehat{f}\cdot u\,,$$

and consequently, $\mathrm{supp}\,\widehat{f_a} \subseteq [0,+\infty)$.

In signal processing, the analytic signal is used, for example, to define the *instantaneous frequency* of a real-valued signal f. We can write its analytic signal as $f_a(t) = A(t)\, e^{i\varphi(t)}$ in order to define its instantaneous frequency as $\omega := \varphi'(t)$. Thus, for a fixed time u, the Wigner-Ville time-frequency distribution of f_a given by

$$P_V f_a(u,\xi) := \int_{-\infty}^{\infty} f_a\big(u+\frac{\tau}{2}\big)\, \overline{f_a\big(u-\frac{\tau}{2}\big)}\, e^{-i\tau\xi}\, d\tau\,,$$

is typically concentrated in a neighborhood of the instantaneous frequency $\xi = \varphi'(u)$ since

$$\varphi'(u) = \frac{\int_{-\infty}^{\infty} \xi\, P_V f_a(u,\xi)\, d\xi}{\int_{-\infty}^{\infty} P_V f_a(u,\xi)\, d\xi}$$

(see [82, p. 108]).

From now on, we confine ourselves to use analytic signals for sampling purposes. For instance, for sampling efficiently a *bandpass signal.*

A signal $f \in L^2(\mathbb{R})$ is a bandpass signal if the support of its Fourier transform satisfies $\operatorname{supp} \widehat{f} \subseteq [-\omega_0 - \sigma\pi, -\omega_0] \cup [\omega_0, \omega_0 + \sigma\pi]$, *where* $\omega_0 > 0$.

Whithout loss of generality we take $\sigma = 1$. Naturally, one can apply the WSK formula with the sampling period $T_s = \frac{\pi}{\omega_0+\pi} < 1$ in order to recover f from its samples $\{f(nT_s)\}$. Next, we show that we can recover a bandpass signal f by sampling the signal itself and its Hilbert transform with a sampling period $T_s = 2$. To this end, we use the following reasoning involving the analytic signal f_a. Namely, the analytic signal f_a of the bandpass signal f satisfies

$$f_a(t) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0}^{\omega_0+\pi} 2\, \widehat{f}(\omega) e^{i\omega t}\, d\omega\,.$$

As a consequence of the sampling formulas (19) and (20) we have

$$f_a(t) = \sum_{n=-\infty}^{\infty} f_a(2n) e^{i\omega_1(t-2n)} \frac{\sin\pi(\frac{t}{2}-n)}{\pi(\frac{t}{2}-n)}\,, \tag{38}$$

where $\omega_1 = \omega_0 + \pi/2$. Having in mind that $f = \Re f_a$ we obtain that

• *Any real bandpass signal f such that* $\operatorname{supp} \widehat{f} \subset [-\omega_0 - \pi, -\omega_0] \cup [\omega, \omega_0 + \pi]$ *can be expanded as*

$$f(t) = \sum_{n=-\infty}^{\infty} \Big\{ f(2n)\cos\omega_1(t-2n) - \widetilde{f}(2n)\sin\omega_1(t-2n) \Big\}\, \frac{\sin\frac{\pi}{2}(t-2n)}{\frac{\pi}{2}(t-2n)}\,.$$

In particular, taking $\omega_0 = 0$ in (38) we obtain the following sampling result

• *Any real function $f \in PW_\pi$ can be recovered by using its samples $\{f(2n)\}_{n\in\mathbb{Z}}$ and those $\{\widetilde{f}(2n)\}_{n\in\mathbb{Z}}$ of its Hilbert transform by means of the formula*

$$f(t) = \sum_{n=-\infty}^{\infty} \left\{ f(2n)\cos\frac{\pi}{2}(t-2n) - \widetilde{f}(2n)\sin\frac{\pi}{2}(t-2n) \right\} \frac{\sin\frac{\pi}{2}(t-2n)}{\frac{\pi}{2}(t-2n)}\,.$$

The final comment is concerning the term *analytic signal* used in this section. An analytic signal f_a is not, obviously, an analytic function in the context of complex analysis. However, if we define the function

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty 2\widehat{f}(\omega) e^{iz\omega} d\omega\,, \quad z = t + iy \in \mathbb{C},$$

we obtain an analytic function on the upper half-plane $\mathbb{C}^+ = \{z = t + iy \in \mathbb{C},\ y > 0\}$. Besides, its boundary function $\lim_{y\to 0} F(t+iy)$ coincides almost everywhere with $f_a \in L^2(\mathbb{R})$, the analytic signal associated with $f \in L^2(\mathbb{R})$. The mathematical details, involving the Hardy space $H^2(\mathbb{C}^+)$, can be found in [40].

For more information about the Hilbert transform and its uses in sampling purposes, we refer the interested reader to [21, 98, 135, 136]. A unified approach to sampling theorems for derivatives and Hilbert transforms can be found in [119].

D. Zeros of bandlimited functions

The problem of signal recovering can also be considered from a different point of view. As we know the signals in the space PW_π are entire functions of exponential type at most π whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. Although entire functions are not completely determined by the location of their zeros, as can be seen from the Hadamard factorization theorem [132, p. 74], bandlimited functions are, as can be deduced from a Titchmarsh's theorem to which we will refer later on: any bandlimited function is uniquely determined by its zeros up to an exponential factor depending on its spectral interval $[a, b]$. If the spectral interval is of the form $[-a, a]$, this exponential factor reduces to a constant. Recall that it is the case for real-valued bandlimited signals as it was pointed out in Section II.B.1.

The referred Titchmarsh's theorem [122] providing the needed mathematical foundation reads as follows

• *Let $F \in L^1[a, b]$ and define the entire function f to be*

$$f(z) = \int_a^b F(w)\, e^{zw} dw\,.$$

Then f has infinitely many zeros, $\{z_n\}_{n\in\mathbb{N}}$, with nondecreasing absolute values, such that

$$f(z) = f(0)\, e^{\frac{a+b}{2}z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)\,,$$

where the infinite product is conditionally convergent.

In the above result, it is assumed that a and b are the effective lower and upper limits of the integral, in the sense that there are no numbers $\alpha > a$ and $\beta < b$ such that $F(\omega) = 0$ (a.e.) in $[a, \alpha]$ or $[\beta, b]$.

If f is bandlimited to $[-a, a]$, then

$$f(z) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) ,$$

provided $f(0) \neq 0$, or

$$f(z) = Az^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) ,$$

if $z = 0$ is a zero of f of order m.

Notice that the zeros in Titchmarsh theorem may be complex. This poses a difficulty from a practical point of view, as complex zeros are harder to detect than real zeros. Whenever they are real, this theorem provides a useful tool for signal recovering, usually referred to as *real-zero interpolation* [18, 86]. One way to deal only with real zeros is by using the so-called *sine wave crossings* technique [86] involved in the following result from Duffin and Schaeffer [38], which reads as follows

• *Let f be an entire function of exponential type at most γ such that $|f(x)| \leq 1$ on the real axis. Then for every real α the function $\cos(\gamma z + \alpha) - f(z)$ has only real zeros, or vanishes identically. Moreover all the zeros are simple, excepts perhaps at points on the real axis where $f(x) = \pm 1$.*

For a deeper study of the oscillatory properties of Paley-Wiener functions see references [64, 96, 124].

E. Irregular sampling

The WSK expansion in PW_π (17) can also be written as

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{G(z)}{G'(n)(z-n)} , \tag{39}$$

where $G(z) = \sin \pi z / \pi$. The latter expression exhibits the Lagrange type interpolatory character of the WSK result. Its expresses the possibility of recovering a certain kind of signal from a sequence of *regularly spaced* samples.

From a practical point of view it is interesting to have a similar result, but for a sequence of samples taken with a *nonuniform* distribution along the real line (a straightforward application of this result would be the recovering of signals from samples affected by time-jitter error, i.e., taken at points $t_n = n + \delta_n$, with δ_n some measurement uncertainty).

Intuitively speaking, nonuniform sampling is the natural way for the discrete representation of a signal. For example, let us assume there is a signal with high instantaneous frequency regions and low instantaneous frequency in other regions. It is more efficient to sample the low frequency regions at a lower rate than the high frequency regions.

An appropriate question to get such a result would be how close should the sample points be to the regular sample points so that a similar equation to (39) still holds. A first answer to this question was given by Paley and Wiener [96], who proved that if the sequence of sample points, $\{t_n\}_{n\in\mathbb{Z}}$, satisfies

$$D := \sup_{n\in\mathbb{Z}} |t_n - n| < \tau\,, \tag{40}$$

where $\tau = 1/\pi^2$, and the sequence is symmetric, i.e., $t_{-n} = t_n$ $(n \geq 1)$, then any $f \in PW_\pi$ can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)}\,,$$

where now

$$G(z) = (z - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{t_n^2}\right)\,.$$

Later, Levinson [79] extended condition (40) to $\tau = 1/4$ and nonsymmetric sequences. This result is related with the "maximum" perturbation of the Hilbert basis $\{e^{-in\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ of the square-integrable function space $L^2[-\pi,\pi]$, in such a way that the perturbed sequence $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis, a concept which will be introduced later, of the same space. Kadec proved that Levinson's result, $\tau = 1/4$, is optimal, in the sense that if $D = 1/4$ counterexamples can be found (see [132, pp. 42-44] for the details).

1. Introducing Riesz bases

In order to apply Riesz bases for irregular sampling purposes in Paley-Wiener spaces, we briefly remind the more important features of these bases, giving elementary proofs when available.

A Riesz basis $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathbb{H}$ is a basis obtained from an orthonormal one $\{e_n\}_{n=1}^{\infty}$ by means of a bounded invertible operator $T : \mathbb{H} \longrightarrow \mathbb{H}$, i.e., $T(e_n) = x_n$ for each $n \in \mathbb{N}$.

Next we draw up a list of the most important properties concerning Riesz bases:

i) For each $x \in \mathbb{H}$ there exists a unique sequence of scalars $\{c_n\}_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} c_n x_n$$

in the $\mathbb{H}$ norm sense.

Indeed, for each $x \in \mathbb{H}$ there is a unique $y \in \mathbb{H}$ such that $x = T(y)$. Expanding y in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ we obtain

$$x = T(y) = T\Big(\sum_{n=1}^{\infty}\langle y, e_n\rangle e_n\Big) = \sum_{n=1}^{\infty}\langle y, e_n\rangle T(e_n) = \sum_{n=1}^{\infty} c_n x_n\,,$$

with $c_n = \langle y, e_n\rangle$. As a consequence, the sequence $\{x_n\}_{n=1}^{\infty}$ forms a complete set in $\mathbb{H}$.

ii) For each $n \in \mathbb{N}$, the coefficient functional defined as

$$\begin{aligned} f_n : \mathbb{H} &\longrightarrow \mathbb{C} \\ x &\longrightarrow f_n(x) = c_n = \langle y, e_n\rangle\,, \end{aligned}$$

is linear and bounded in $\mathbb{H}$.

It easily follows from Cauchy-Schwarz inequality

$$|\langle y, e_n\rangle| \leq \|y\|\,\|e_n\| = \|T^{-1}(x)\| \leq \|T^{-1}\|\,\|x\|\,.$$

iii) For every $x \in \mathbb{H}$, its sequence of coefficients $\{c_n\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$.

iv) As a consequence of the Riesz representation theorem, for each $n \in \mathbb{N}$ there exists a unique $y_n \in \mathbb{H}$ such that $f_n(x) = \langle x, y_n\rangle$ for every $x \in \mathbb{H}$. Thus, for very $x \in \mathbb{H}$ we have the unique representation $x = \sum_{n=1}^{\infty}\langle x, y_n\rangle x_n$.

v) The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are biorthonormal, i.e., $\langle x_m, y_n\rangle = \delta_{n,m}$. It follows from the uniqueness of the coefficients since $x_m = \sum_{n=1}^{\infty}\langle x_m, y_n\rangle x_n$.

vi) The sequence $\{y_n\}_{n=1}^{\infty}$ also forms a Riesz basis for $\mathbb{H}$ and the expansions

$$x = \sum_{n=1}^{\infty}\langle x, y_n\rangle x_n = \sum_{n=1}^{\infty}\langle x, x_n\rangle y_n\,,$$

hold for every $x \in \mathbb{H}$. A proof of this result and more specific account of Riesz bases can be found in [132, pp. 19-36].

2. The Riesz bases setting

In what follows $\{t_n\}_{n\in\mathbb{Z}}$ will denote a sequence of real numbers such that

$$D := \sup_{n\in\mathbb{Z}} |t_n - n| < \frac{1}{4}\,.$$

As a consequence of Kadec's $\frac{1}{4}$-theorem [132, p. 42], $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$. A necessary and sufficient condition about the sequence $\{t_n\}_{n\in\mathbb{Z}}$ in order to be $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ a Riesz basis for $L^2[-\pi,\pi]$ was given by Pavlov in [100].

Consider

$$G(z) = (z - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{t_n}\right) \left(1 - \frac{z}{t_{-n}}\right), \tag{41}$$

an entire, well-defined function, whose set of zeros is $\{t_n\}_{n\in\mathbb{Z}}$, as it will be made clear along the proof of the following theorem, the so-called Paley-Wiener-Levinson sampling theorem, hereafter PWL sampling theorem.

• *Any $f \in PW_\pi$ can be recovered from its sample values $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange type interpolation series*

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)},$$

which is absolutely and, uniformly convergent in horizontal strips of $\mathbb{C}$ (in particular in $\mathbb{R}$).

For the proof, let $\{h_n(\omega)\}_{n\in\mathbb{Z}}$ be the unique biorthonormal basis of $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$, i.e., for every $m, n \in \mathbb{Z}$,

$$\langle h_n, \frac{e^{-it_m\omega}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} = \delta_{nm}\,.$$

Thus, every $\widehat{f} \in L^2[-\pi,\pi]$ can be expressed as

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, h_n \rangle_{L^2[-\pi,\pi]} \frac{e^{-it_n\omega}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, \frac{e^{-it_n\omega}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} h_n(\omega)\,.$$

By using the Fourier duality in PW_π we get

$$f(z) = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, h_n \rangle_{L^2[-\pi,\pi]} \mathcal{F}^{-1}\left(\frac{e^{-it_n\omega}}{\sqrt{2\pi}}\right)(z) = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, \frac{e^{-it_n\omega}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} \mathcal{F}^{-1}(h_n)(z)\,.$$

By setting $g_n = \mathcal{F}^{-1}(h_n)$ and taking into account that $\langle \widehat{f}, h_n \rangle_{L^2[-\pi,\pi]} = \langle f, g_n \rangle_{PW_\pi}$ and that $\langle \widehat{f}, \frac{e^{-it_n\omega}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} = f(t_n)$, we can rewrite

$$f(z) = \sum_{n=-\infty}^{\infty} \langle f, g_n \rangle_{PW_\pi} \operatorname{sinc}(z - t_n) = \sum_{n=-\infty}^{\infty} f(t_n) g_n(z)\,.$$

Now,

$$g_n(z) = \mathcal{F}^{-1}(h_n)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} h_n(\omega)\, e^{iz\omega}\, d\omega$$

is an entire function, bandlimited to $[-\pi,\pi]$ whose only zeros are $\{t_m\}_{m\neq n}$. Suppose by contrary that $s \notin \{t_m\}_{m\neq n}$ is a zero of g_n. Using the classical Paley-Wiener theorem, the function

$$g(z) = \frac{z-t_n}{z-s}\, g_n(z)$$

belongs to PW_π and vanish at every t_n. Taking into account the completeness of a Riesz basis this implies that $g \equiv 0$, a contradiction.

Therefore, as a consequence of Titchmarsh's theorem, we have

$$g_n(z) = A_n \frac{G(z)}{z-t_n}\,.$$

Notice that by setting $n = 0$, for instance, the above formula shows that $G(z)$ is an entire function, as stated at the beginning of this result. Since $g_n(t_n) = 1$ then $A_n = 1/G'(t_n)$ and hence,

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z-t_n)}\,.$$

The series is convergent in the norm of PW_π and, consequently, uniformly in horizontal strips of $\mathbb{C}$. The absolute convergence of the series comes from the fact that a Riesz basis is also an unconditional basis.

Note that, as a byproduct in the proof of the PWL Theorem, we deduce that the sequences $\left\{\frac{G(z)}{G'(t_n)(z-t_n)}\right\}_{n\in\mathbb{Z}}$ and $\left\{\frac{\sin \pi(z-t_n)}{\pi(z-t_n)}\right\}_{n\in\mathbb{Z}}$ are biorthonormal Riesz bases in PW_π.

A general theory for nonorthogonal sampling formulas by using Riesz bases instead of orthonormal bases can be developed. The main steps involved in the theory are pointed out in next section.

3. A unified approach to nonorthogonal sampling formulas

The Riesz bases setting is the appropriate framework to get nonorthogonal sampling formulas while retaining the Riesz basis property in a unified way. The procedure closely parallels the one given for orthogonal formulas in Section II. Due to this parallelism, we only highlight a sequence of the most important results.

Throughout this section, $\{\phi_n(x)\}_{n=1}^{\infty}$ and $\{\phi_n^*(x)\}_{n=1}^{\infty}$ will be denote a pair of biorthonormal Riesz bases for a fixed $L^2(I)$ space. Note that the sequences of their conjugate functions $\{\overline{\phi_n}(x)\}_{n=1}^{\infty}$ and $\{\overline{\phi_n^*}(x)\}_{n=1}^{\infty}$ are also a pair of biorthonormal Riesz bases for $L^2(I)$.

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of functions $S_n : \Omega \subset \mathbb{R} \longrightarrow \mathbb{C}$ and $\{t_n\}_{n=1}^{\infty}$ a sequence in Ω verifying conditions [C1.] and [C2.] in Section II.A.

Define the kernel $K(x,t)$ as

$$K(x,t) = \sum_{n=1}^{\infty} S_n(t)\overline{\phi_n^*}(x)\,, \qquad (x,t) \in I \times \Omega\,. \tag{42}$$

Note that, as a function of x, $K(\cdot,t)$ belongs to $L^2(I)$ since $\{\overline{\phi_n}\}_{n=1}^{\infty}$ is a Riesz basis for $L^2(I)$. By using this kernel $K(x,t)$, define on $L^2(I)$ the linear integral transform

$$f(t) := \int_I F(x)\,K(x,t)dx\,, \quad \text{for } F \in L^2(I)\,. \tag{43}$$

This transform is again a bijective isometry between $L^2(I)$ and its range $\mathcal{H}$, provided we endow this space with the norm $\|f\|_{\mathcal{H}} := \|F\|_{L^2(I)}$. The following properties for $\mathcal{H}$ hold:

a) $\{S_n(t)\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, Indeed, it is the image by (43) of the Riesz basis $\{\phi_n(x)\}_{n=1}^{\infty}$ of $L^2(I)$. Besides, its biorthonormal basis $\{S_n^*(t)\}_{n=1}^{\infty}$ is the corresponding image by (43) of $\{\phi_n^*(x)\}_{n=1}^{\infty}$.

b) $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a RKHS space whose reproducing kernel $k(t,s)$ is given by

$$k(t,s) = \langle K(\cdot,t), K(\cdot,s)\rangle_{L^2(I)} = \sum_{n=1}^{\infty} S_n(t)\overline{S_n^*(s)}\,.$$

c) Expanding any $f \in \mathcal{H}$ in the Riesz basis $\{S_n(t)\}_{n=1}^{\infty}$, i.e.,

$$f = \sum_{n=1}^{\infty}\langle f, S_n^*\rangle S_n = \sum_{n=1}^{\infty}\langle F, \phi_n^*\rangle\, S_n = \sum_{n=1}^{\infty}\frac{f(t_n)}{a_n}S_n\,,$$

we obtain the (nonorthogonal) sampling expansion

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{S_n(t))}{a_n}\,. \tag{44}$$

The convergence of the series (44) is absolute, and uniform on subsets of Ω where $\|K(\cdot,t)\|_{L^2(I)} = \sqrt{k(t,t)}$ is bounded.

We illustrate the proposed method with a couple of examples:

(1) Consider a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real numbers satisfying Kadec's condition. It can be proved [96] that, for any fixed $t \in \mathbb{R}$, we can expand the Fourier kernel in $L^2[-\pi,\pi]$ as

$$\frac{e^{itx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty}\frac{G(t)}{(t-t_n)G'(t_n)}\frac{e^{it_n x}}{\sqrt{2\pi}}\,, \tag{45}$$

where G stands for (41), the infinite product of the sequence $\{t_n\}_{n\in\mathbb{Z}}$. Regarding expansion (45), see also the proof of the PWL sampling theorem in previous section.

Taking $S_n(t) = \dfrac{G(t)}{(t-t_n)G'(t_n)}$ and the sampling points $\{t_n\}_{n\in\mathbb{Z}}$, (44) is nothing more than the statement of the PWL sampling theorem in PW_π.

(2) Now, let $\{t_n\}_{n=1}^{\infty}$ be the sequence of positive roots of the equation $\sin 2\pi t = 1/t$. It is proven in [88] that $\{\cos[(x+\pi)t_n]\}_{n=1}^{\infty}$ forms a Riesz basis for $L^2[-\pi,\pi]$. Its biorthonormal basis is given by

$$\left\{\frac{1}{\alpha_n}\cos[(x-\pi)t_n]\right\}_{n=1}^{\infty},$$

where the normalization constants are $\alpha_n = \frac{\sin 2\pi t_n}{2t_n} + \pi\cos 2\pi t_n > 0$. For each fixed $t\in\mathbb{R}$ one gets the expansion

$$\cos[(x+\pi)t] = \sum_{n=1}^{\infty}\frac{t\sin 2\pi t - 1}{\alpha_n(t^2-t_n^2)}\cos[(x+\pi)t_n]\,, \quad \text{in } L^2[-\pi,\pi]\,.$$

Therefore, taking $S_n(t) = \dfrac{t\sin 2\pi t - 1}{\alpha_n(t^2-t_n^2)}$ and the sampling points $\{t_n\}_{n=1}^{\infty}$ we obtain the following nonorthogonal sampling result

• *Any function f of the form*

$$f(t) = \int_{-\pi}^{\pi} F(x)\cos[(x+\pi)t]dx\,, \quad F\in L^2[-\pi,\pi]\,,$$

can be expanded as the sampling formula

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{t\sin 2\pi t - 1}{\alpha_n(t^2-t_n^2)}\,.$$

The corresponding RKHS $\mathcal{H}$ has the reproducing kernel

$$k(t,s) = \frac{\sin 2\pi(t+s)}{2(t+s)} + \frac{\sin 2\pi(t-s)}{2(t-s)}\,.$$

We close the section with a pertinent comment:

The constructive method proposed here is limited to those integral transforms whose kernel $K(x,t)$ can be written as (42). However, it can be proved (see [53]) that under plausible hypotheses the integral kernel adopts the required form (42). Namely, consider an integral transform

$$f(t) = \int_I F(x)\,K(x,t)dx\,, \quad F\in L^2(I)\,,$$

where $t \in \Omega$ and the integral kernel belongs to $L^2(I)$ for every fixed $t \in \Omega$. Assume that, for the functions f in the range space of this integral transform, a sampling formula like (44) holds pointwise in Ω, with $\{f(t_n)/a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, and the sampling functions $\{S_n(t)\}_{n=1}^{\infty}$ satisfying the two conditions:

i. $\sum_{n=1}^{\infty} |S_n(t)|^2 < \infty$ for each $t \in \Omega$, and

ii. $\sum_{n=1}^{\infty} \alpha_n S_n(t) = 0$ for every $t \in \Omega$ with $\{\alpha_n\} \in \ell^2$, implies $\alpha_n = 0$ for all n.

Then the kernel of the integral transform can be expressed as

$$K(x,t) = \sum_{n=1}^{\infty} S_n(t)\overline{\phi_n^*}(x)\,,$$

where $\{\phi_n^*\}_{n=1}^{\infty}$ is, in general, a Riesz basis for $L^2(I)$. This result includes the particular case where $\{\phi_n^*\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(I)$.

We return to the case of irregular sampling in PW_π. Kadec condition about the sampling points $\{t_n\}_{n\in\mathbb{Z}}$ can be relaxed by using exponential frames in $L^2[-\pi,\pi]$. In next section, we introduce frames in a separable Hilbert space, giving also an account of their most important properties.

4. Introducing frames

First, we state the definition of a *frame* in a Hilbert space:

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathbb{H}$ is said to be a frame if there exist constants $0 < A \leq B$, called the frame bounds, such that for any $x \in \mathbb{H}$ the frame inequality

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \tag{46}$$

holds.

If $A = B$, then the frame is called a *tight frame*. If the removal of one element x_m renders the sequence $\{x_n\}_{n\neq m}$ no longer a frame, then it is called an *exact frame*. The left-hand in the frame inequalities (46) shows that a frame is a complete sequence in $\mathbb{H}$. An orthonormal basis is a tight frame with $A = B = 1$.

For our sampling purposes, we gather the most important properties in frame theory in the following list:

i) For an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{H}$, the following are equivalent

   (a) $\{x_n\}_{n=1}^{\infty}$ is a frame with frame bounds A and B, and

   (b) the *frame operator* defined as $S(x) := \sum_{n=1}^{\infty}\langle x, x_n\rangle x_n$ is a bounded positive operator in $\mathbb{H}$ with $AI \leq S \leq BI$, where I denotes the identity operator in $\mathbb{H}$.

Recall that T is a positive operator in $\mathbb{H}$ ($T \geq 0$) if $\langle T(x), x\rangle \geq 0$ for all $x \in \mathbb{H}$. $T \leq S$ means that $S - T \geq 0$. See [8, p. 467] for a proof.

ii) S^{-1} exists and is positive in $\mathbb{H}$, and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

iii) $\{S^{-1}(x_n)\}_{n=1}^{\infty}$ is also a frame, called the *dual frame*, with frame bounds B^{-1} and A^{-1} [8, p. 468].

iv) Any $x \in \mathbb{H}$ can be written in terms of the dual frame as

$$x = \sum_{n=1}^{\infty} \langle x, S^{-1}(x_n)\rangle x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle S^{-1}(x_n)\,. \tag{47}$$

Indeed, as S^{-1} is the operator frame for the dual frame we have

$$x = S(S^{-1}(x)) = S\Big(\sum_{n=1}^{\infty} \langle x, S^{-1}(x_n)\rangle S^{-1}(x_n)\Big) = \sum_{n=1}^{\infty} \langle x, S^{-1}(x_n)\rangle x_n\,.$$

We get the other representation for $x \in \mathbb{H}$ by considering $x = S^{-1}(S(x))$.

v) If $\{x_n\}_{n=1}^{\infty}$ is a tight frame in $\mathbb{H}$, then the operator frame is $S = AI$, and for every $x \in \mathbb{H}$ the representation

$$x = \frac{1}{A}\sum_{n=1}^{\infty} \langle x, x_n\rangle x_n \tag{48}$$

holds.

vi) Suppose that there exists a sequence of scalars $\{b_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} b_n x_n$. Then,

$$\sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |a_n - b_n|^2\,,$$

where $a_n = \langle x, S^{-1}(x_n)\rangle$. That is, the coefficients obtained via the dual frame representation (47) have a minimum norm property in $\ell^2(\mathbb{N})$ [8, p. 468].

vii) A sequence $\{x_n\}_{n=1}^{\infty}$ of a separable Hilbert space $\mathbb{H}$ is a Riesz basis if and only if it is an exact frame (see [8, p. 471] or [132, p. 188] for a proof).

viii) If $\{x_n\}_{n=1}^{\infty}$ is an exact frame, then $\{x_n\}_{n=1}^{\infty}$ and $\{S^{-1}(x_n)\}_{n=1}^{\infty}$ are biorthonormal sequences, i.e., $\langle x_n, S^{-1}(x_m)\rangle = \delta_{nm}$ [8, p. 471].

In the frame setting we retain the representation property, i.e., every $x \in \mathbb{H}$ may be written as $x = \sum_{n=1}^{\infty}\langle x, S^{-1}(x_n)\rangle x_n$, but sacrificing uniqueness of the representation unlike orthonormal or Riesz bases.

It is worth to point out that, in finite dimensional spaces, proper frames correspond to spanning sets, no necessary linearly independent. If $\{x_n\}_{n=1}^M$ is an spanning set for $\mathbb{C}^N$ where $M > N$, there exist constants $A, B > 0$ such that for all $x \in \mathbb{C}^N$

$$A\|x\|^2 \leq \sum_{n=1}^{M} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 .$$

Frame theory goes back to 1952 when a seminal paper by Duffin and Schaeffer was published in the context of Paley-Wiener spaces [39]. It was revived, in the last decade, in connection with wavelet theory and has proved to be a fundamental tool in irregular sampling. The reader interested in deeper knowledge in frame theory might address to references [32, 36, 132].

5. The frame setting

Assume that, for a real sequence $\{t_n\}_{n\in\mathbb{Z}}$, the family $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a frame in $L^2[-\pi,\pi]$. Then, there exist two constants $0 < A \leq B$ such that

$$A\|\varphi\|^2_{L^2[-\pi,\pi]} \leq \sum_{n=-\infty}^{\infty} \left|\langle \varphi, \frac{e^{-it_n\omega}}{\sqrt{2\pi}}\rangle_{L^2[-\pi,\pi]}\right|^2 \leq B\|\varphi\|^2_{L^2[-\pi,\pi]} ,$$

for each $\varphi \in L^2[-\pi,\pi]$. Taking $f = \mathcal{F}^{-1}(\varphi)$ in PW_π we obtain

$$A\|\varphi\|^2_{L^2[-\pi,\pi]} \leq \sum_{n=-\infty}^{\infty} |f(t_n)|^2 \leq B\|\varphi\|^2_{L^2[-\pi,\pi]} ,$$

or

$$A\|f\|^2_{PW_\pi} \leq \sum_{n=-\infty}^{\infty} |f(t_n)|^2 \leq B\|f\|^2_{PW_\pi} ,$$

by using the Fourier duality. Since $f(t_n) = \langle f, \mathrm{sinc}(\cdot - t_n)\rangle_{PW_\pi}$, we deduce that $\left\{\dfrac{\sin\pi(t-t_n)}{\pi(t-t_n)}\right\}_{n\in\mathbb{Z}}$ is a frame in PW_π. Let $\{h_n\}_{n\in\mathbb{Z}}$ be its dual frame. Then, as a consequence of the representation property (47), for every $f \in PW_\pi$ we have the sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)h_n(t) .$$

The problem with this sampling formula is that we do not know the dual frame $\{h_n\}_{n\in\mathbb{Z}}$. We would like to have a method to recover $f \in PW_\pi$ from the available information, that is, the sequence of samples $\{f(t_n)\}_{n\in\mathbb{Z}}$, or equivalently the frame operator $S(f) = \sum_{n=-\infty}^{\infty} f(t_n)\,\mathrm{sinc}(\cdot - t_n)$. As we will see in next section an iterative algorithm, essentially the Richardson method, will allow us to recover f from the operator frame evaluated at f, $S(f)$.

An explanation of the oversampling technique seen in Section III.B.2 can be given in the light of frame theory. Namely,

• *The sequence* $\{\sigma \operatorname{sinc} \sigma(t-n)\}_{n\in\mathbb{Z}}$ *is a tight frame with bound* $A = 1$ *for every Paley-Wiener space* $PW_{\pi\sigma}$ *with* $\sigma < 1$.

To this end, let f be a function in $PW_{\pi\sigma}$, and let F be its Fourier transform supported in $[-\pi\sigma, \pi\sigma]$. Extending F to be zero in $[-\pi, \pi] \setminus [-\pi\sigma, \pi\sigma]$, we have

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) \frac{e^{-in\omega}}{\sqrt{2\pi}} \quad \text{in } L^2[-\pi, \pi] .$$

Applying Parseval's equality in $L^2[-\pi, \pi]$ and Fourier's duality in $PW_{\pi\sigma}$ we get

$$\|f\|^2_{PW_{\pi\sigma}} = \|F\|^2_{L^2[-\pi,\pi]} = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \sum_{n=-\infty}^{\infty} |\langle f, \sigma \operatorname{sinc} \sigma(\cdot - n)\rangle|^2 ,$$

which proves our assertion. Note that $\sigma \operatorname{sinc} \sigma(t-s)$ is the reproducing kernel in $PW_{\pi\sigma}$ (21). As a corollary,

• *Any signal* f *in* $PW_{\pi\sigma}$ *with* $\sigma < 1$, *can be expanded by using the tight frame representation* (48) *as*

$$f(t) = \sigma \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \sigma(t-n) .$$

Finally, we close the section by giving sufficient conditions on the real sampling points $\{t_n\}_{n\in\mathbb{Z}}$ to guarantee that the sequence $\{e^{-it_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a frame in $L^2[-\pi, \pi]$, or equivalently, $\{\operatorname{sinc}(t-t_n)\}_{n\in\mathbb{Z}}$ is a frame in PW_π. The first result in this direction is due to Duffin and Schaeffer [39] and reads as follows:

Suppose that there exist constants $0 < \epsilon < 1$, $\alpha, L > 0$ *so that the sampling sequence* $\{t_n\}_{n\in\mathbb{Z}}$ *satisfies* $|t_n - t_m| \geq \alpha$ *for* $n \neq m$ *and*

$$\sup_{n\in\mathbb{Z}} |t_n - \epsilon n| \leq L .$$

Then, $\{\operatorname{sinc}(t-t_n)\}_{n\in\mathbb{Z}}$ *is a frame in* PW_π.

Condition $|t_n - t_m| \geq \alpha$ for $n \neq m$ (the sampling set $\{t_n\}_{n\in\mathbb{Z}}$ is said to be *separated* or *uniformly discrete*) implies by itself the existence of a constant $B > 0$ such that $\sum_{n=-\infty}^{\infty} |f(t_n)|^2 \leq B\|f\|_{PW_\pi}$ for every f in PW_π [99, p. 219]. Both conditions together imply the existence of a constant $A > 0$ such that $A\|f\|_{PW_\pi} \leq \sum_{n=-\infty}^{\infty} |f(t_n)|^2$ for every f in PW_π.

The second result, which proof can be found in [99, pp. 219-231], is the following:

Suppose that a uniformly discrete set $\{t_n\}_{n\in\mathbb{Z}}$ *satisfies that there exists a constant* k *such that*

$$\|f\|_\infty \leq k \sup_n |f(t_n)| , \quad \textit{for all } f \in PW_\pi ,$$

then the sequence $\{\mathrm{sinc}(t-t_n)\}_{n\in\mathbb{Z}}$ *is a frame in* PW_π.

(The renowned mathematician A. Beurling called this new condition *balayage*).

F. Iterative algorithms

The iterative method allowing to recover $f \in PW_\pi$ from the frame operator $S(f)$ is, from a functional analysis point of view, the inversion of a linear operator by means of a Neumann series. Recall that if T is a continuous linear transformation of a Banach space E into itself such that $\|T\| < 1$, then $(I-T)^{-1}$ exists and is continuous. Moreover, it can be given by the series

$$(I-T)^{-1} = I + T + T^2 + T^3 + \cdots = \sum_{n=0}^{\infty} T^n\,,$$

which converges in the operator norm topology (see, for instance, [91, p. 431]). Using the above result we prove a version of the so-called extrapolated Richardson method, i.e., an iterative method to find the solution f of a linear system $\mathcal{A}f = h$.

• *Let* $\mathcal{A}$ *be a bounded operator on a Banach space* E *such that* $\|f - \mathcal{A}(f)\| \le \gamma\,\|f\|$ *for all* $f \in E$ *with* $\gamma < 1$. *Then* $\mathcal{A}$ *is invertible on* E *and any* f *can be recovered from* $\mathcal{A}(f)$ *by the following iteration algorithm: set* $f_0 = \mathcal{A}(f)$ *and* $f_{n+1} = f_n + \mathcal{A}(f - f_n)$ *for* $n \ge 0$, *then* $f = \lim_{n\to\infty} f_n$. *After* n *iterations, the error estimate is given by* $\|f - f_n\| \le \gamma^{n+1}\,\|f\|$.

Indeed, since $\|I - \mathcal{A}\| \le \gamma < 1$ then, $I - (I - \mathcal{A}) = \mathcal{A}$ is invertible and $\mathcal{A}^{-1} = \sum_{k=0}^{\infty}(I-\mathcal{A})^k$. Therefore,

$$g_{n+1} := \sum_{k=0}^{n+1}(I-\mathcal{A})^k\mathcal{A}(f) \to f\,, \quad \text{as } n \to \infty\,.$$

On the other hand we can write

$$\begin{aligned} g_{n+1} &= \mathcal{A}(f) + \sum_{k=1}^{n+1}(I-\mathcal{A})^k\mathcal{A}(f) = \mathcal{A}(f) + (I-\mathcal{A})\sum_{k=0}^{n}(I-\mathcal{A})^k\mathcal{A}(f) \\ &= g_n + \mathcal{A}(f - g_n) \end{aligned}$$

for $n \ge 0$ and $g_0 = \mathcal{A}(f)$. Hence, we have obtained the convergence of the proposed iterative algorithm to f. Moreover, regarding its convergence rate we have

$$\|f - g_n\| = \Big\| \sum_{k=n+1}^{\infty} (I-\mathcal{A})^k\mathcal{A}(f)\Big\| = \|(I-\mathcal{A})^{n+1}\mathcal{A}^{-1}\mathcal{A}(f)\| \le \gamma^{n+1}\,\|f\|\,,$$

obtaining the desired result.

Now we put to use this general iterative algorithm to recover bandlimited signals from a frame in PW_π. Assume that $\{\mathrm{sinc}(t-t_n)\}_{n\in\mathbb{Z}}$ is a frame in PW_π with frame bounds A and B. Let S be the frame operator given by

$$S(f) = \sum_{n=-\infty}^{\infty} f(t_n)\,\mathrm{sinc}(\cdot - t_n)\,,$$

and consider the new operator $\mathcal{S} := \dfrac{2}{A+B}\,S$. We prove that we can use this operator in the above iterative algorithm. To this end, since $AI \leq S \leq BI$, we have

$$\frac{2A}{A+B}\|f\|^2 \leq \frac{2}{A+B}\langle S(f), f\rangle \leq \frac{2B}{A+B}\|f\|^2\,.$$

Therefore,

$$\|f\|^2 - \frac{2A}{A+B}\|f\|^2 \geq \|f\|^2 - \frac{2}{A+B}\langle S(f), f\rangle\,.$$

As a consequence,

$$\langle (I-\mathcal{S})(f), f\rangle \leq \frac{B-A}{A+B}\|f\|^2\,.$$

In a similar way we prove that

$$-\frac{B-A}{A+B}\|f\|^2 \leq \langle (I-\mathcal{S})(f), f\rangle\,.$$

Since $I-\mathcal{S}$ is a bounded self-adjoint operator we deduce [91, p. 371]

$$\|I-\mathcal{S}\| = \sup_{\|f\|=1} |\langle (I-\mathcal{S})(f), f\rangle| \leq \frac{B-A}{A+B} = \gamma < 1\,.$$

For more details about frames and irregular sampling see the references [10, 11, 16, 45].

Some comments about iterative algorithms for sampling purposes are in order:

a) One can see the crucial role played by the frame bounds in the convergence of the above algorithm. Thus, it is of practical importance to obtain sharply estimates for A and B. If only a crude upper bound B and the existence of a lower band $A > 0$ are known, the frame algorithm can still be used by using a relaxation parameter $\lambda > 0$ (see [45, 57] for more details).

b) If we are able to construct an approximation of the identity operator in PW_π by using a sequence of samples $\{f(t_n)\}_{n\in\mathbb{Z}}$, we could apply the iterative algorithm to recover f. For instance, let $\{t_n\}_{n\in\mathbb{Z}}$ be a strictly increasing real sequence with $\lim_{n\to\pm\infty} t_n = \pm\infty$. Consider $\delta = \sup_{n\in\mathbb{Z}}(t_{n+1}-t_n)$, the maximal gap between samples, and $\{z_n\}_{n\in\mathbb{Z}}$ the sequence of midpoints, i.e., $z_n = (t_n + t_{n+1})/2$. In the

case when $\delta < 1$, one can obtain an approximation of the identity operator in PW_π by setting

$$\mathcal{A}(f) := P_{PW_\pi}\Big(\sum_{n=-\infty}^{\infty} f(t_n)\chi_{[z_{n-1},z_n)}\Big)\,,$$

i.e., we interpolate f by a step function first, followed by the orthogonal projection onto PW_π. Indeed, it can be proved that $\|f - \mathcal{A}(f)\| \leq \delta\|f\|$ for every $f \in PW_\pi$ (see [45, 99] for the proof).

In [45] one can find another approximations of the identity operator in PW_π. Let $\{t_n\}_{n\in\mathbb{Z}}$ be a sequence as above with maximal gap between samples δ. If we define $w_n = (t_{n+1} - t_{n-1})/2$ it is proved that the sequence

$$\{\sqrt{w_n}\,\mathrm{sinc}(t - t_n)\}_{n\in\mathbb{Z}}\,,$$

forms a frame for PW_π with frame bounds $(1-\delta)^2$ and $(1+\delta)^2$. Consequently, we can recover any function $f \in PW_\pi$ from

$$\mathcal{A}(f) := \frac{1}{1+\delta^2} \sum_{n=-\infty}^{\infty} w_n f(t_n)\,\mathrm{sinc}(\cdot - t_n)\,,$$

by means of the mentioned iterative algorithm with a rate of convergence $\gamma = \frac{2\delta}{1+\delta^2}$. The amplitude factor $\sqrt{w_n}$ compensates for the non-uniformity of the density of samples (see [45] for the proof).

c) The standard frame algorithm can be used in combination with acceleration methods like Chebyshev acceleration or conjugate gradient acceleration, allowing a reduction in the number of iterations [57].

d) The iterative techniques also work in higher dimensional settings [43].

The interested reader can also consult the related references [42, 46, 56, 57, 58, 59].

IV. Sampling stationary stochastic processes

A stochastic process $\{X(t) : t \in \mathbb{R}\}$ defined on a probability space $(\Omega, \mathcal{A}, p)$ is said to be a stationary (wide sense) stochastic process continuous in mean square if it verifies the following assumptions:

(a) $X(t) \in L^2(\mathcal{A}; \mathbb{C})$, i.e., $\|X(t)\|^2 = E[|X(t)|^2] < \infty$, and $E[X(t)] = 0$, for each $t \in \mathbb{R}$ where E denotes the expectation of a random variable.

(b) $\{X(t)\}$ is stationary (wide sense), i.e., $R_X(t+u, t) = R_X(u)$ for all $t \in \mathbb{R}$, where R_X stands for the autocorrelation function given by $R_X(t, t') = E[X(t)\overline{X(t')}]$.

(c) The mapping defined as

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & L^2(\mathcal{A};\mathbb{C}) \\ t & \longrightarrow & X(t) \end{array}$$

is continuous when $L^2(\mathcal{A};\mathbb{C})$ is endowed with its usual norm $\|U\|^2 = E[|U|^2]$.

It is known that such a process admits an integral representation where the function to be integrated is scalar and the measure takes values in the $L^2(\mathcal{A};\mathbb{C})$ space [51, 108, 117]. Moreover, whenever the process is bandlimited, it can be expanded as a Shannon sampling series [51, 80, 108, 117].

The main aim in this section is to capture the main features from the latter definition, i.e., stationarity and continuity, in order to obtain this class of results in an abstract Hilbert space setting. Most of the ideas included here are taken from the reference [52]. We begin by giving the definition of a *generalized stationary process.*

A generalized stationary process is a family $\{x_t\}_{t\in\mathbb{R}} \subset \mathbb{H}$ *satisfying the two following conditions:*

i) The function $r(u) = \langle x_{u+t}, x_t\rangle_{\mathbb{H}}$ *is well-defined for all* $u \in \mathbb{R}$ *(Stationarity) ;*

ii) The function r *is continuous at* 0 *(Continuity).*

The function r is the so-called *autocorrelation function* of the process. Observe that whenever condition i) holds then, condition ii) implies that $\{x_t\}_{t\in\mathbb{R}}$ is a continuous process (in the $\mathbb{H}$-norm). Indeed,

$$\|x_t - x_s\|^2_{\mathbb{H}} = \langle x_t - x_s, x_t - x_s\rangle = 2r(0) - 2\Re r(t-s)\,.$$

Consequently, the continuity of $\{x_t\}_{t\in\mathbb{R}}$ in the $\mathbb{H}$ norm is equivalent to the continuity of r at 0. In particular, a generalized stationary process is weak continuous and consequently r is continuous in $\mathbb{R}$.

On the other hand, condition i) implies that r is a function of *positive type* since for all choices of $N \in \mathbb{N}$, $t_1, \ldots, t_N \in \mathbb{R}$, and $c_1, \ldots, c_N \in \mathbb{C}$ we have

$$\sum_{m,n=1}^{N} r(t_m - t_n) c_m \overline{c_n} = \sum_{m,n=1}^{N} \langle x_{t_m}, x_{t_n}\rangle c_m \overline{c_n} = \Big\langle \sum_{m=1}^{N} c_m x_{t_m}, \sum_{m=1}^{N} c_m x_{t_m} \Big\rangle \geq 0\,.$$

Since r is a continuous function of positive type, by using Bochner's theorem [107, p. 385] we obtain that

- *There exists a positive finite measure* μ *on* $\mathcal{B}_{\mathbb{R}}$*, the Borel sets in* $\mathbb{R}$*, such that*

$$r(u) = \int_{-\infty}^{\infty} e^{iu\omega}\, d\mu(\omega)\,.$$

μ is the so-called *spectral measure* associated with the process $\{x_t\}_{t\in\mathbb{R}}$.

Let $\mathbb{H}_X$ denote the Hilbert space spanned by the process $\{x_t\}_{t\in\mathbb{R}}$ in $\mathbb{H}$, and consider the space L^2_μ of all complex valued measurable functions f such that $\int_{-\infty}^{\infty} |f(\omega)|^2\, d\mu(\omega) < \infty$. Then

• *There exists an isometric isomorphism $\widetilde{\Phi}$ between the spaces L^2_μ and $\mathbb{H}_X$ with corresponding elements $e^{it\omega}$ and x_t*

To this end, define $\widetilde{\Phi} : L^2_\mu \to \mathbb{H}_X$ by

$$\widetilde{\Phi}(g) = \sum_{k=1}^{n} a_k x_{t_k} \quad \text{whenever} \quad g(\omega) = \sum_{k=1}^{n} a_k e^{it_k\omega}\,.$$

Clearly, for $g(\omega) = \sum_{k=1}^{n} a_k e^{it_k\omega}$ and $g'(\omega) = \sum_{j=1}^{m} b_j e^{it_j\omega}$ we get

$$\begin{aligned}
\langle \widetilde{\Phi}(g), \widetilde{\Phi}(g')\rangle_{\mathbb{H}} &= \langle \sum_{k=1}^{n} a_k x_{t_k}, \sum_{j=1}^{m} b_j x_{t_j}\rangle_{\mathbb{H}} = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \overline{b_j} r(t_k - t_j) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \overline{b_j} \int_{-\infty}^{\infty} e^{i(t_k - t_j)\omega}\, d\mu(\omega) \\
&= \int_{-\infty}^{\infty} \Big(\sum_{k=1}^{n} a_k e^{it_k\omega} \Big) \Big(\sum_{j=1}^{m} \overline{b_j} e^{-it_j\omega} \Big)\, d\mu(\omega) \\
&= \int_{-\infty}^{\infty} g(\omega)\overline{g'(\omega)}\, d\mu(\omega) = \langle g, g'\rangle_{L^2_\mu}\,.
\end{aligned}$$

A standard limit process allow us to extend $\widetilde{\Phi}$ to an isometric linear map on the closed linear manifold generated by $\{e^{it\omega} : t \in \mathbb{R}\}$, i.e., on all of L^2_μ. Clearly it maps L^2_μ onto $\mathbb{H}_X$.

Now we derive the Shannon sampling theorem for bandlimited generalized stationary processes. A generalized stationary process is said to be *bandlimited to* $[-\pi, \pi]$ if $\operatorname{supp}\mu \subseteq [-\pi, \pi]$, i.e., $r(u) = \int_{-\pi}^{\pi} e^{iu\omega}\, d\mu(\omega)$.

• *Let $\{x_t\}_{t\in\mathbb{R}}$ be a bandlimited generalized stationary process to $[-\pi, \pi]$ and suppose that $\mu(\{-\pi, \pi\}) = 0$. Then, the following sampling formula holds*

$$x_t = \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} x_n\,, \tag{49}$$

where the series converges in $\mathbb{H}$ for each $t \in \mathbb{R}$.

Indeed, for each $t \in \mathbb{R}$, we have in $L^2[-\pi, \pi]$

$$e^{it\omega} = \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} e^{in\omega}\,. \tag{50}$$

The Dirichlet-Jordan test [143, p. 57] ensures that convergence is also uniform on intervals $[-\pi+\delta, \pi-\delta]$, with $\delta > 0$. Consequently, the series in (50) converges everywhere in $(-\pi, \pi)$, and μ-almost everywhere in $[-\pi, \pi]$. Besides, since the bounded function $e^{it\omega}$ has Fourier coefficients $O(1/n)$ as $|n| \to \infty$, the partial sums in (50) are uniformly bounded in $[-\pi, \pi]$ [143, p. 90]. From the bounded convergence theorem for μ we get

$$\int_{-\pi}^{\pi} \left| e^{it\omega} - \sum_{n=-N}^{N} \frac{\sin \pi(t-n)}{\pi(t-n)} e^{in\omega} \right|^2 d\mu(\omega) \to 0$$

when N goes to ∞. We have convergence in the L^2_μ-sense, and by using the isometry $\tilde{\Phi}$ we obtain the desired expansion.

In particular, when the measure μ is absolutely continuous with respect to the Lebesgue measure on $[-\pi, \pi]$, i.e., $d\mu = s(\omega)d\omega$ with $s \in L^1[-\pi, \pi]$ the so-called *spectral density* of the process, it implies that $\mu(\{-\pi, \pi\}) = 0$ and the following corollary holds

• *If the measure μ is absolutely continuous with respect to Lebesgue measure on $[-\pi, \pi]$, then the sampling formula* (49) *holds.*

Finally, it is worth to point out that formula (49) works for generalized stationary processes whose μ measure is not absolutely continuous with respect to the Lebesgue measure. An easy example is given by $\{x_t = e^{iat}h\}_{t\in\mathbb{R}}$ where $a \in (-\pi, \pi)$ and $h \in \mathbb{H}$ with $\|h\| = 1$. In this case, $r(u) = e^{iau}$ and $\mu = \delta_a$, the Dirac delta at the point a, which is not absolutely continuous with respect to the Lebesgue measure on $[-\pi, \pi]$.

Closing the section we provide some hints to go into more detail:

a) The first comment is concerning the integral representation of a generalized stationary process $\{x_t\}_{t\in\mathbb{R}}$ by means of an orthogonal countably additive measure Φ on $\mathcal{B}_\mathbb{R}$ and taking values in $\mathbb{H}$ such that

$$x_t = \int_{\infty}^{\infty} e^{it\omega}\, d\Phi(\omega)\,, \quad t \in \mathbb{R}. \tag{51}$$

Recall that a countably additive measure $\Phi : \mathcal{B}_\mathbb{R} \longrightarrow \mathbb{H}$ satisfies

$$\Phi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Phi(A_n)$$

in the norm of $\mathbb{H}$, for every disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{B}_\mathbb{R}$. The isometry $\tilde{\Phi}$ defines the measure Φ. Indeed, let B be a Borel set in $\mathbb{R}$. Setting $\Phi(B) = \tilde{\Phi}(\chi_B)$, where χ_B is the characteristic function of B, we obtain a countably aditive measure. This measure takes orthogonal values for any disjoint Borel subsets since the following equality holds $\langle \Phi(B), \Phi(B') \rangle_\mathbb{H} = \langle \chi_B, \chi_{B'} \rangle_{L^2_\mu}$.

b) In general, we can consider a process $\{x_t\}_{t\in\mathbb{R}}$ represented by (51) where the countably additive measure Φ is not necessarily orthogonal. These processes are the so-called *harmonizable processes*. In the case of bandlimited harmonizable processes the sampling formula (49) still remains valid whenever supp $\Phi \subseteq [-\pi, \pi]$ and $\Phi(\{-\pi\}) = \Phi(\{\pi\}) = 0 \in \mathbb{H}$. Indeed, the convergence in (50) is Φ-almost everywhere and bounded. The bounded convergence theorem for Φ applied to the expansion (50) allows us to interchange the series with the integral obtaining the sampling expansion for the process. Indeed

$$\begin{aligned} x_t &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} e^{in\omega}\, d\Phi(\omega) \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} x_n\,. \end{aligned}$$

Technical details about the integral of an scalar function with respect to a vectorial measure Φ have been obviously omitted. We refer the interested reader to reference [9] for the details and proofs of convergence results.

We finish off with a note on harmonizable processes. Stationarity is an unacceptable restriction in many problems such as signal detection. Searching a relaxation of stationarity while still retaining the methods of harmonic analysis lead to Loève [81] to introduce the concept of *harmonizability*. An historical evolution of this concept and its mathematical treatment can be found in [105]. In [30], the importance of harmonizable stochastic processes in system analysis is stressed by showing that the output of a wide class of systems is a harmonizable process. See also [35, 101] for topics related with harmonizable processes and sampling.

V. At the end of the walk

The author is indebted to all those who, with their books, papers, and surveys, have contributed to the revitalization of this beautiful and relevant topic in applied mathematics. Let me mention, as a *sampling* of references the surveys [13, 23, 63, 70], the papers [10, 11, 26, 28, 45, 90, 123], and the books [14, 64, 67, 83, 84, 135]. Besides, handling books on related subjects, such as wavelets or harmonic analysis is a highly recommended exercise in order to place sampling theory in more general contexts. In this respect see, for example, the books [12, 37, 82, 87, 104, 120].

To conclude these notes we venture to include a personal list of sampling topics or group of topics not mentioned in previous sections. By no means should it be understood as an updated state-of-the-art in Sampling Theory: it only intends to orientate those curious readers into more advanced sampling problems from different points of view.

• Many bandlimited signals one meets in practical applications do not have finite energy (they do not belong to any $PW_{\pi\sigma}$) and the techniques in Section III do not apply. Naturally, in this case it is necessary to specify the exact meaning of the word bandlimited. Some generalizations of the concept of bandlimited signal have appeared in the literature. In particular, if we allow the Fourier transform to be taken in the sense of Schwartz distributions, then the class of bandlimited signals can be enlarged tremendously. Any complex exponential signal $e_\omega(t) = e^{it\omega}$ can be regarded as a bandlimited signal, since its Fourier transform is essentially the Dirac delta function $\delta(x-\omega)$, which is a generalized function with compact support at $\{\omega\}$. Sampling theorems for signals that are bandlimited in the distributional sense can be found, for instance, in references [31, 50, 68, 125]. Other generalizations of the concept of bandlimited signal can be found in [29, 78, 111, 133, 135].

• Another interesting issue is to enlarge the set of classical bandlimited functions by considering new spaces where the WSK sampling theorem still applies. This leads to the study of *Bernstein spaces* B_σ^p where $\sigma > 0$ and $1 \leq p \leq \infty$, defined as the set of all entire functions of exponential type at most σ and whose restriction to $\mathbb{R}$ belongs to $L^p(\mathbb{R})$. It also leads to the general *Paley-Wiener classes* PW_σ^p, defined as the set of functions f with an integral representation

$$f(z) = \int_{-\sigma}^{\sigma} F(x)\, e^{izx}\, dx\,, \quad \text{with } F \in L^p[-\sigma, \sigma]\,.$$

In the particular case $p = 2$, both classes coincide, i.e., $PW_\sigma^2 = B_\sigma^2$. More specific accounts of these spaces and their properties can be found in [17, 64, 132, 135].

• Also, it is surprising the strong relationship between the WKS sampling theorem and other fundamental results in mathematics, such as Poisson's summation formula or Cauchy's integral formula. In recent years, many authors have drawn new relationships by showing the equivalence of the WSK sampling theorem, or any of its generalizations, with other important mathematical results like the Euler-MacLaurin formula, the Abel-Plana summation formula, Plancherel's theorem, the maximun modulus principle or the Phragmén-Lindelöt principle, among others. We refer the interested reader to sources [25, 26, 27, 64, 66, 103].

• In practice, sampling expansions incur in several types of errors. *Truncation error* results when only a finite number of samples are used; *aliasing error* occurs when the bandlimiting condition is violated, or when an inappropriate bandwidth is used for our signal; *amplitude error* arises when we only know approximations of the samples of our signal; *time-jitter error* is caused by sampling at instants which differ from the theoretical ones given by the corresponding sampling at hand; finally, *information loss error* arises when some sampled data or fractions thereof are missing. Concerning this topic see [26, 44, 64, 70, 71, 83, 135] and references therein.

• Bandlimited functions cannot be timelimited, i.e., they are defined for all $t \in \mathbb{R}$. Indeed, any $f \in PW_\pi$ is an entire function and, as a consequence of the isolated zeros

principle, it cannot be zero on any interval of the real line unless f is itself the zero function. Also, due to the same mathematical reason, a bandlimited function can be extrapolated. As pointed out by Prof. Higgins in his book [64, Ch. 17], bandlimited signals are the "mathematical model" of a "real signal". In other words, a real signal is considered to be known only in so far as we can make measurements or observations of it. Although a Paley-Wiener function is not exactly timelimited, it can be considered nearly timelimited in the sense that most of its energy is concentrated on a bounded time interval. This leads to the study of the energy concentration of a signal, and consequently, to the prolate spheroidal functions, and the uncertainty principle in signal analysis. Further discussions and details about this topic can be found in references [64, 75, 114, 115, 116, 123].

• Another interesting question is that concerning the density of sampling points required to have a stable sampling in $PW_B = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp} \widehat{f} \subseteq B\}$. A sequence of sampling points $\{t_n\}$ is a *set of stable sampling* for PW_B if there exists a constant K, independent of $f \in PW_B$, such that

$$\|f\|_{L^2} \leq K \|f(t_n)\|_{\ell^2} ,$$

for every $f \in PW_B$. Hence, errors in the output of a sampling and reconstruction process are bounded by errors in the input. Although bandlimited functions are entire functions and, as a consequence, are completely determined by their values in a sequence of sampling points with an accumulation point (in particular, in any segment of the real line), sampling in practice is meaningless in the absence of the stable sampling condition. Note that whenever we are dealing with frames in PW_B (which includes, in particular, orthonormal and Riesz bases) the involved sampling set is stable. This is not the case when we are dealing with a *set of uniqueness* in PW_B, i.e., $f(t_n) = 0$ for every n implies that f is the zero function. Notice that the set of uniqueness condition is equivalent to the sequence of complex exponentials $\{e^{it_n x}\}$ being a complete set in $L^2(B)$. Although samples taken at a set of uniqueness determine elements of PW_B uniquely, it does not lead to any process by which we can reconstruct a function by its samples. For example, any finite set of M vectors $\{x_n\}_{n=1}^M$ is always a frame in the space generated by their linear combinations. When M increases, the frame bounds may go respectively to 0 and $+\infty$, and this illustrates the fact that in infinite dimensional spaces, a family of vectors may be complete and still not yield a stable signal representation.

For a set of stable sampling for PW_B, its density $D(t_n)$, defined (when the limit exists) by

$$D(t_n) := \lim_{r \to \infty} \frac{\#\{t_n : t_n \in [-r, r]\}}{2r} ,$$

with $\#$ denoting the cardinality of a set, satisfies $D(t_n) \geq m(B)/2\pi$, where $m(B)$ stands for the Lebesgue measure of the set B. The critical density $m(B)/2\pi$ is called the *Nyquist-Landau sampling rate*, below which stable reconstruction is not possible. In

the case when $B = [-\pi\sigma, \pi\sigma]$, the Nyquist-Landau density coincides with the Nyquist one σ. In the multi-channel setting, the Nyquist-Landau density is smaller than the Nyquist one.

Furthermore, if $\{t_n\}$ is a *set of interpolation* for PW_B then $D(t_n) \leq m(B)/2\pi$. Recall that $\{t_n\}$ is a set of interpolation for PW_B if the moment problem $f(t_n) = a_n$ for every n has a solution whenever $\{a_n\} \in \ell^2$. This is the case for Riesz bases [132, p. 169] and, as a consequence, the density $D(t_n)$ coincides with the Nyquist-Landau one in the Riesz bases setting. For more details see [13, 64, 76, 77, 99, 112, 132]

• An extension of Shannon's model has been proposed recently: it is the sampling in shift-invariant or spline-like spaces. This is achieved by simply replacing the sinc function by another generating function φ. Accordingly, the basic approximation space $V(\varphi)$ is specified as

$$V(\varphi) := \left\{ s(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t-k) : \{c_k\} \in \ell^2 \right\}.$$

As pointed out in [123], this allows in practice for simpler and more realistic interpolation models, which can be used in conjunction with a much wider class of anti-aliasing prefilters that are not necessarily ideal low-pass. Measured signals in applications have frequency components that decay for higher frequencies, but these signals are not bandlimited in the strict sense. As a consequence, sampling in shift-invariant spaces that are not bandlimited is a suitable and realistic model for many applications. See [123] and references therein for this new topic. For irregular sampling in shift-invariant spaces see [2, 3, 4].

To close this work, just a final comment: the coverage of sampling theory in these notes is by no means intended to be exhaustive, and the author apologizes in advance for any important omission.

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