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Capital requirements: Are they the best solution?

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Abstract

General risk functions are becoming very important in finance and insurance. Many risk functions are interpreted as initial capital requirements that a manager must add and invest in a risk-free security in order to protect his clients wealth. Nevertheless, until now it has not been proved that an alternative investment will be outperformed by the riskless asset.

This paper deals with a complete arbitrage free market and a general expectation bounded risk measure and analyzes whether the investment in the riskless asset of the capital requirements is optimal. It is shown that it is not optimal in many important cases. For instance, if the risk measure is the *CVaR* and we consider the assumptions of the *CAPM* or the Black and Scholes model. Furthermore, the Black and Scholes model the explicit expression of the optimal strategy is provided, and it is composed of several put options. If the confidence level of the *CVaR* is close to 100% then the optimal strategy becomes a classical portfolio insurance strategy. This may be a surprising and important finding for both researchers and practitioners. In particular, managers can discover how to reduce the level of initial capital requirements by trading options.

Keywords: Risk Measure, Capital Requirement, Optimal Strategy, Portfolio Insurance.
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1 Introduction

General risk functions are becoming very important in finance and insurance. Since Artzner *et al.* (1999) introduced the axioms and properties of the "Coherent Measures of Risk" many authors have extended the discussion. The recent development of new markets (insurance or weather linked derivatives, commodity derivatives, energy/electricity markets, etc.) and products (inflation-linked bonds, equity indexes annuities or unit-links, hedge funds, etc.), the necessity of managing new types of risk (credit risk, operational risk, etc.) and the (often legal) obligation of providing initial capital requirements have made it necessary to overcome the variance as the most used risk measure and to introduce more general risk functions.[1] Hence, it is not surprising that the recent literature presents many interesting contributions focusing on new methods for measuring risk levels. Among others, Föllmer and Schied (2002) have defined the Convex Risk Measures, Goovaerts *et al.* (2004) have introduced the Consistent Risk Measures, Rockafellar *et al.* (2006) have defined the General Deviations and the Expectation Bounded Risk Measures, and Brown and Sim (2009) have introduced the Satisfying Measures.

Many classical actuarial and financial problems have been revisited by using new risk functions. So, with regard to portfolio choice and asset allocation problems, amongst many others authors, Alexander *et al.* (2006) compare the minimization of the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) for a portfolio of derivatives, Calafiore (2007) studies "robust" efficient portfolios in discrete probability spaces, and Schied (2007) deals with optimal investment with convex risk measures.

Pricing and hedging issues in incomplete markets have also been studied (Föllmer and Schied, 2002, Nakano, 2004, Staum, 2004, etc.), as well as Equity Linked Annuities hedging issues (Barbarin and Devolder, 2005), Optimal Reinsurance Problems (Balbás *et al.*, 2009), and other practical topics.

However, it seems that hedging problems have not been studied with general risk functions, unless they are related to pricing or asset allocation issues. Coherent, expectation bounded or convex risk measures are usually interpreted as capital requirements. They provide

[1] It has been proved that the variance is not compatible with the Second Order Stochastic Dominance if asymmetries and/or heavy tails are involved (Ogryczak and. Ruszczynski, 1999).

regulators and supervisors with the capital reserve that a manager must add in order to protect the clients wealth. This reserve must be invested in a riskless asset, though, until now, nobody has proved that the investment of the reserve in a riskless asset will outperform every alternative hedging strategy. In a recent paper Artzner *et al.* (2009) consider the possibility of investing the capital requirement in an alternative “eligible asset”, though they do not present concrete examples. They also analyze risk measurement problems with a collection of “several eligible assets”. The authors highlight that this approach may be interesting if several currencies are involved in the portfolio, in which case, a risk-free asset per currency may be called eligible.

Balbás *et al.*, 2009 have shown that for linear pricing principles the optimal reinsurance may be a stop-loss contract, though risk levels can be given by expectation bounded risk measures. Actuaries know that a stop-loss reinsurance may be understood as a “European option” whose underlying asset is the global amount paid by the insurer to his customers (claims). On the other hand, a classical viewpoint uses European puts so as to provide investors with “Portfolio Insurance”. Moreover, the empirical evidence seems to point out that classical “portfolio insurance strategies” also perform well in practice if risk levels are measured by VaR and $CVaR$ (Annaert *et al.*, 2009).

The present paper considers a general measure of risk and analyzes whether the investment of the capital requirement in the risk-free security outperforms the remaining feasible hedging investments. According to the ideas above, it could make sense to study the effectiveness of investing this money in adequate derivatives.

The article’s outline is as follows. Section 2 will present the notations and the general framework we are going to deal with. We will consider a complete arbitrage-free market and an expectation bounded risk measure ρ. The manager must add an initial capital reserve so as to make it vanish the risk level indicated by ρ. Instead of assuming that the reserve must be devoted to buy the risk-free security, we will provide a general optimization problem that minimizes the global risk under a budget constraint, since the added capital cannot be larger than the capital requirement indicated by ρ.

General optimality conditions will be presented in Section 3. In particular, Theorem 1 shows that the optimal added strategy y^* can be characterized by a system of equations that involves the sub-gradient of ρ and the Stochastic Discount Factor of the market.

Corollary 2 and its remarks show that y^* is often far of being a risk-free security. For instance y^* is a risky security if the sub-gradient of ρ is composed of essentially bounded random variables and the stochastic discount factor is not essentially bounded. Examples of risk functions are the $CVaR$, the Dual Power Transform (DPT), the absolute deviation, or the absolute down-side semi-deviation. Examples of pricing models are the Capital Asset Pricing Model ($CAPM$), the Black and Scholes model and many others. Since y^* eliminates the risk without being a risk-free security we have decided to call it "Shadow Riskless Asset".[2]

We focus on the $CVaR$ and the Black and Scholes model in Section 4. On the one hand, the $CVaR$ is becoming a risk function very appreciated by both researchers and practitioners owing to interesting properties. So, $CVaR$ respects the Second Order Stochastic Dominance (Ogryczak and. Ruszczynski, 2002) and is coherent and expectation bounded (Rockafellar *et al.*, 2006). On the other hand, the Black and Scholes model is also very popular and used by managers, so it may be worth to provide them with practical methods to reduce the capital requirements.

The most important result in Section 4 is Theorem 4, along with its remarks, which show that the shadow riskless asset is a combination of European and digital puts, that will be close to a single put option (a portfolio insurance strategy) if the level of confidence of the $CVaR$ is close to 100%. In some sense this may be a surprising finding that yields managers with a very important and practical conclusion. They can reduce the capital requirements by purchasing and selling options in a suitable manner.

Section 5 points out the most important conclusions of the paper.

2 Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure μ. Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the Banach

[2] We have taken the expression shadow riskless asset from Ingersoll (1987), where the author constructs a hedging strategy in a pricing model without interest rate. The expression was later used in Balbás and Ibáñez (1998), where the authors dealt with interest rate risk linked problems.

space of $\mathbb{R}-$valued random variables y on Ω such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}\left(\right)$ representing the mathematical expectation ($\mathbb{E}\left(|y|^q\right) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem (Horvàth, 1966), we have that L^q is the dual space of L^p.

Consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $\left(\mathcal{F}_t\right)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We will assume that the market is complete, *i.e.*, every final pay-off $y \in L^p$ may be reached by an adapted to the filtration $\left(\mathcal{F}_t\right)_{t \in \mathcal{T}}$ price process of a self-financing portfolio $\left(S_t\right)_{t \in \mathcal{T}}$,[3] in the sense that $S_T = y$, *a.s.* Consequently, there is a continuous pricing rule

$$\Pi : L^p \longrightarrow \mathbb{R}$$

providing us with the price $\Pi\left(y\right)$ of every $y \in L^p$.

The completeness of the model implies the existence of a riskless asset. Thus, denote by $r_f \geq 0$ the risk-free rate and the equality

$$\Pi\left(k\right) = k e^{-r_f T} \tag{1}$$

must hold for every $k \in \mathbb{R}$.

According to the Riesz Representation Theorem there exists a unique $z_\pi \in L^q$ such that

$$\Pi\left(y\right) = e^{-r_f T} \mathbb{E}\left(y z_\pi\right)$$

for every $y \in L^p$. Moreover, to prevent the existence of arbitrage, the strict inequality

$$z_\pi > 0$$

a.s. must hold (Duffie, 1988). If we assume that $p = 2$ then z_π is usually called "Stochastic Discount Factor" (SDF), and it is closely related to the Market Portfolio of the $CAPM$ (Duffie, 1988). In this paper we will not impose $p = 2$ but z_π will be still called SDF.

Expression (1) implies that

$$k e^{-r_f T} = \Pi\left(k\right) = e^{-r_f T} k \mathbb{E}\left(z_\pi\right),$$

[3] Almost all the classical pricing models in finance are complete. Interesting examples are, among others, the Black and Scholes and the Heston models for pricing equity linked derivatives.

which leads to

$$\mathbb{E}(z_\pi) = 1. \tag{2}$$

Let

$$\rho : L^p \longrightarrow \mathbb{R}$$

be the general risk function that a trader uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; -\mathbb{E}(yz) \leq \rho(y), \ \forall y \in L^p\}. \tag{3}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma(L^q, L^p)$ −compact,[4] and

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{4}$$

holds for every $y \in L^p$. Furthermore, we will also impose

$$\Delta_\rho \subset \{z \in L^q; \mathbb{E}(z) = 1\}. \tag{5}$$

Summarizing, we have:

Assumption 1. The set Δ_ρ given by (3) is convex and $\sigma(L^q, L^p)$ −compact, (4) holds for every $y \in L^p$ and (5) holds. □

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, and bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for convex functions, it is easy to prove that the fulfillment of Assumption 1 holds if ρ is continuous and satisfies:

a)

$$\rho(y + k) = \rho(y) - k \tag{6}$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

b)

$$\rho(\alpha y) = \alpha \rho(y)$$

[4] See Horvàth (1966) for further details about $\sigma(L^q, L^p)$ −compact sets.

for every $y \in L^p$ and $\alpha > 0$.

c)

$$\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$$

for every $y_1, y_2 \in L^p$.

d)

$$\rho(y) \geq -\mathbb{E}(y)$$

for every $y \in L^p$.[5]

It is easy to see that if ρ is continuous and satisfies Properties a), b), c) and d) above then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L_+^q = \{z \in L^q; \mu(z \geq 0) = 1\}. \tag{7}$$

Particular interesting examples are the Conditional Value at Risk ($CVaR$) of Rockafellar *et al.* (2006), the Dual Power Transform (DPT) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Furthermore, following the original idea of Rockafellar *et al.* (2006) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho(y) = \sigma(y) - \mathbb{E}(y) \tag{8}$$

is continuous and satisfies a), b), c) and d) if $\sigma : L^p \longrightarrow \mathbb{R}$ is a continuous deviation, that is, if σ satisfies b), c),

e)

$$\sigma(y + k) = \sigma(y)$$

for every $y \in L^p$ and $k \in \mathbb{R}$, and

f)

$$\sigma(y) \geq 0$$

[5] Actually, the properties above are almost similar to those used by Rockafellar *et al.* (2006) in order to introduce their Expectation Bounded Risk Measures. These authors also impose a), b), c) and d), work with $p = 2$, allow for $\rho(y) = \infty$, and impose $\rho(y) > -\mathbb{E}(y)$ if y is not constant.

for every $y \in L^p$.

Particular examples are the classical $p-$deviation given by

$$\rho(y) = \left[\mathbb{E}\left(|\mathbb{E}(y) - y|^p\right)\right]^{1/p}, \tag{9}$$

or the downside $p-$semi-deviation given by

$$\rho(y) = \left[\mathbb{E}\left(|Max\ \{\mathbb{E}(y) - y, 0\}|^p\right)\right]^{1/p}, \tag{10}$$

among many others.

3 Characterizing the shadow riskless asset

Suppose that the random variable $y_0 \in L^p$ represents a trader's final wealth. Its final risk will be given by $\rho(y_0)$, which justifies that this quantity may be an adequate final value (at T) of the capital requirement.[6] Indeed, (6) leads to

$$\rho(y_0 + \rho(y_0)) = 0 \tag{11}$$

and the risk will vanish if the amount $\rho(y_0)\, e^{-r_f T}$ is invested in the riskless security. Our purpose is to study whether the investment above in the riskless asset is the best solution so as to make the risk vanish. Until now it has not been proved that an alternative investment will be outperformed by the riskless asset.

Consequently, consider the pay-off $y \in L^p$ added by the trader to his initial portfolio $y_0 \in L^p$. Suppose that $C > 0$ gives (the value at T of) the highest amount of money that will be invested to reduce the risk level.[7] Then the trader will choose y so as to solve

$$\left\{\begin{array}{l} Min\ \rho(y + y_0 - \mathbb{E}(y z_\pi)) \\ \mathbb{E}(y z_\pi) \leq C \\ y \geq 0 \end{array}\right. . \tag{12}$$

[6] *i.e.*, $\rho(y)\, e^{-r_f T}$ should be the initial cash reserve (or capital requirement) invested in the risk-free asset.

[7] If $\rho(y_0) > 0$ then (11) shows that $C = \rho(y_0)$ could be a suitable choice for C.

Problem (12) considers the global risk level $\rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right)$ that the trader is facing, so it has to incorporate the value $\mathbb{E}\left(yz_\pi\right)$ of the added portfolio that will have to be paid and will reduce the trader wealth. One might also skip the price of y and solve

$$\begin{cases} Min\ \rho\left(y+y_0\right) \\ \mathbb{E}\left(yz_\pi\right)=C \\ y\geq 0 \end{cases} \tag{13}$$

though (6) points out that the solution y^* of (12) also solves (13) as long as $\mathbb{E}\left(y^* z_\pi\right)=C$. Some results below will show that (13) is frequently solved by the solution of (12).

As we said and justified in the introduction of the paper, the solution y^* of (12) will be called "shadow riskless asset".

Problem (12) does not consider any utility function. It only focuses on the capital needed by the trader in order to reduce risk levels. Nevertheless, there are many relationships between utility functions and risk functions, as pointed out by Ogryczak and Ruszczynski (1999) and (2002), among others.

Problem (12) is not differentiable because ρ is not differentiable either. Recent literature has developed several optimization methods that may solve this caveat (see, among others, Ruszczynski and Shapiro, 2006). In this paper we will follow a procedure quite parallel to that used in Balbás *et al.* (2009), where the authors deal with a mathematical programming problem leading to optimal reinsurance contracts. Some duality linked properties and Theorem 1 below will not be proved due to their analogy with similar results of the mentioned paper.

In particular, bearing in mind Assumption 1, (12) is equivalent to the infinite-dimensional linear optimization problem

$$\begin{cases} Min\ \theta \\ \theta+\mathbb{E}\left(\left(y+y_0\right)z\right)-\mathbb{E}\left(yz_\pi\right)\geq 0, \quad \forall z\in\Delta_\rho \\ \mathbb{E}\left(yz_\pi\right)\leq C \\ \theta\in\mathbb{R},\ y\geq 0 \end{cases} \tag{14}$$

$\theta\in\mathbb{R}$ and $y\in L^p$ being the decision variables, in the sense that y solves (12) if and only if there exists $\theta \in \mathbb{R}$ such that (θ, y) solves (14), in which case

$$\theta = \rho\left(y + y_0 - \mathbb{E}\left(y z_\pi\right)\right)$$

holds. Furthermore, following Balbás *et al.* (2009), one can show that Problem

$$\begin{cases} Max \ \ -C\lambda - \mathbb{E}\left(y_0 z\right) \\ z \leq \left(1+\lambda\right) z_\pi \\ \lambda \in \mathbb{R},\ \lambda \geq 0,\ \ z \in \Delta_\rho \end{cases} \tag{15}$$

is the dual of (14), $\lambda \in \mathbb{R}$ and $z \in \Delta_\rho$ being the decision variables. Finally, the following primal-dual relationships hold

Theorem 1 *Suppose that $y^* \in L^p$ and $(\lambda^*, z^*) \in \mathbb{R} \times L^q$. Then, they solve (12) and (15) if and only if the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} \lambda^*\left(C - \mathbb{E}\left(y^* z_\pi\right)\right) = 0 \\ C - \mathbb{E}\left(y^* z_\pi\right) \geq 0 \\ \mathbb{E}\left(\left(y^* + y_0\right) z\right) \geq \mathbb{E}\left(\left(y^* + y_0\right) z^*\right), \qquad\qquad \forall z \in \Delta_\rho \\ \mathbb{E}\left(\left(\left(1+\lambda^*\right) z_\pi - z^*\right) y^*\right) = 0 \\ \left(1+\lambda^*\right) z_\pi - z^* \geq 0 \\ y^* \in L^p,\ \ y^* \geq 0,\ \ \lambda \in \mathbb{R},\ \lambda \geq 0,\ \ z^* \in \Delta_\rho \end{cases} \tag{16}$$

are fulfilled. Moreover, the dual solution is attainable if (12) is bounded. □

As already said, until now nobody has proved that the investment in the riskless asset is the best way to make the risk level $\rho(y_0)$ vanish. Next let us use the latter theorem so as to prove that the riskless asset and the shadow riskless asset will be often different.

Corollary 2 *Suppose that y^* solves (12) and (λ^*, z^*) solves (15). Then,*

a) If $\lambda^* > 0$ then y^* saturates the budget constraint ($C - \mathbb{E}(y^* z_\pi) = 0$). In particular, y^* also solves Problem (13).

b) If $\lambda^* = 0$ then $z^* = z_\pi$.

c) If $\mu(y^* > 0) = 1$ then $\lambda^* = 0$ and $z^* = z_\pi$.

Proof. $a)$ is obvious, so let us prove $b)$ and $c)$. If $\lambda^* = 0$ then the dual constraint leads to $z^* \leq (1 + \lambda^*) z_\pi = z_\pi$, and therefore $z^* = z_\pi$ because both random variables have the same expectation (see (2) and (5)). Besides, if $\mu(y^* > 0) = 1$ then the fourth equation in (16) implies that $z^* = (1 + \lambda^*) z_\pi$. Taking expectations and bearing in mind (2) and (5) we have that $1 = 1 + \lambda^*$. □

Remark 1 *The previous corollary points out that the shadow riskless asset y^* will frequently be a risky asset. Indeed, if it were the riskless asset $y^* = C$ then Statement $c)$ would lead to $z_\pi \in \Delta_\rho$ which does not hold for many important risk measures and pricing models. For instance, since $L^p \subset L^1$, suppose that ρ may be extended to the whole space L^1. Important expectation bounded risk measures satisfy this condition. Among others, the DPT of Wang (2000), given by*

$$DPT_a(y) = \int_0^1 VaR_{1-t}(y)\, g_a'(t)\, dt$$

for every $y \in L^1$, $a > 1$ being an arbitrary constant and

$$g_a : (0,1) \longrightarrow (0,1)$$

given by

$$g_a(t) = 1 - (1-t)^a,$$

the $CVaR$ and the measure (8) if σ is the $1-$deviation (or absolute deviation) or the $1-$down-side semi-deviation (or down-side absolute semi-deviation) (see (9) and (10)).[8] *In such a case (3) points out that $\Delta_\rho \subset L^\infty$, and therefore the elements in Δ_ρ are essentially bounded. But there are many pricing models whose Stochastic Discount Factor is not essentially bounded. For instance, the $CAPM$, where $p = 2$ and the SDF is "almost similar" to the Market Portfolio (Chamberlain and Rothschild, 1983, or Duffie, 1988), or the Black and Scholes model, where $p = 2$ once again, and z_π is unbounded as will be seen in Section* 4. □

Remark 2 *Statement $b)$ shows that $\lambda^* > 0$ will hold as long as z_π is unbounded and $\Delta_\rho \subset L^\infty$. Then Statement $a)$ implies that the solution y^* of (12) will also solve (13), and it is not a riskless asset according to the previous remark.* □

[8] Let us remark that these three risk measures respect the second order stochastic dominance (Ogryczak and Ruszczynski, 1999 and 2002).

4 Dealing with the CVaR and the Black and Scholes model

Henceforth we will assume that $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ being the level of confidence. According to Rockafellar *et al.* (2006) we have that

$$\Delta_{CVaR_{\mu_0}} = \left\{ z \in L^{\infty}; \mathbb{E}(z) = 1,\ 0 \leq z \leq \frac{1}{1-\mu_0} \right\}. \tag{17}$$

Hence, bearing in mind (7), $CVaR_{\mu_0}$ is a coherent and expectation bounded measure of risk. Moreover, Ogryczak and Ruszczynski (2002) have shown that $CVaR_{\mu_0}$ is consistent with the second order stochastic dominance. All these properties provoke that $CVaR_{\mu_0}$ is becoming a very popular risk measure for both academics and practitioners.[9]

First of all let us adapt (16) to the particular case we are dealing with. The third condition in (16) shows that the dual solution z^* must solve the mathematical programming problem

$$\begin{cases} Min\ \mathbb{E}\left(\left(y^* + y_0\right) z\right) \\ \mathbb{E}(z) = 1 \\ z \leq \dfrac{1}{1-\mu_0} \\ z \geq 0 \\ z \in L^{\infty} \end{cases} \tag{18}$$

We will need the following result:

Lemma 3 *If $y^* \in L^p$ and z^* is (18)-feasible then z^* solves (18) if and only if there exist $\alpha \in \mathbb{R}$, $\alpha_1, \alpha_2 \in L^p$ and a measurable partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ such that*

$$\begin{cases} y^* + y_0 = \alpha - \alpha_1 + \alpha_2 & \\ \alpha_i \geq 0 & i = 1,2 \\ \alpha_1 = \alpha_2 = 0 & on\ \Omega_0 \\ z^* = \dfrac{1}{1-\mu_0}\ and\ \alpha_2 = 0 & on\ \Omega_1 \\ z^* = 0\ and\ \alpha_1 = 0 & on\ \Omega_2 \end{cases} \tag{19}$$

[9] VaR_{μ_0} is also important and used in practice, but its major caveat is that it is not sub-additive, and therefore it is not clear that it will favor diversification. This problem has motivated several authors so as to study particular conditions that lead to a sub-additive VaR_{μ_0}. See, amongst others, García *et al.* (2007).

holds.

Proof. Problem (18) is obviously linear, in the sense that both the objective function and the constraints are linear. Thus, its Karush-Kuhn-Tucker conditions are sufficient optimality conditions. Besides, the Slater Qualification holds, since there are random variables z that are feasible and

$$\mu(0 < z < \frac{1}{1-\mu_0}) = 1$$

(for instance, take the zero-variance random variable $z = 1$). Thus the Karush-Kuhn-Tucker conditions of (18) are also necessary optimality conditions (Luenberger, 1969).

Furthermore, the dual space of L^∞ is composed of those finitely additive measures that have bounded variation and are $\mu-$continuous (Horvàth, 1966). Thus, according to Luenberger (1969), the Karush-Kuhn-Tucker conditions of (18) hold if and only if there exist $\alpha \in \mathbb{R}$, and two measures in the dual of L^∞ such that (19) is satisfied. In particular, $\alpha_1 = \alpha - (y^* + y_0)$ in Ω_1 and vanishes outside Ω_1, which proves that $\alpha_1 \in L^p$. Similarly, $\alpha_2 \in L^p$. $\square$

As an obvious consequence we can modify the third condition in (16) using (19), and we will have new necessary and sufficient optimality conditions for (12) and (15).

Let us now focus on the Black and Scholes model. Consequently, suppose that y_0 is the final value (at T) of a Geometrical Brownian Motion (GBM). Then it is known that y_0 has a log-normal distribution. Without loss of generality we can simplify the structure of the probability space $(\Omega, \mathcal{F}, \mu)$. Indeed, assume that $\Omega = (0,1)$ and μ is the Lebesgue measure on the Borel $\sigma-$algebra of this set. Then we can take

$$y_0(\omega) = Exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right) \tag{20}$$

for $\omega \in (0,1)$ $a.s.$, r and σ denoting the drift and the volatility of the GBM, respectively (Wang, 2000). Obviously, $\Phi : \mathbb{R} \longmapsto (0,1)$ is the cumulative distribution function of the standard normal distribution and is given by the well-known expression

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du,$$

for every $x \in \mathbb{R}$.

Actually, if (20) represents the trader final wealth then $\mu(y_0 > 0) = 1$, and the risk level $\rho(y_0)$ is strictly negative. Hence, no capital requirements should be added. Nevertheless, one can consider the fund manager whose asset final value is given by (20) but whose liability equals a positive amount M with maturity at T. Then the manager final pay-off is given by $y_0 - M$, that can be negative, and $\rho(y_0 - M) > 0$ may hold. Thus (12) and (13) should be modified and $y_0 - M$ should play the role of y_0. However, due to (6), the solution of both problems remains the same if y_0 replaces $y_0 - M$, so we do not miss anything if we take y_0 as in (20) and deal with (12) and (13).

Taking into account (20) it may be immediately verified that y_0 is a continuous and strictly increasing function (with respect to the ω variable) such that

$$Lim_{\omega \to 0} y_0(\omega) = 0, \tag{21}$$

and

$$Lim_{\omega \to 1} y_0(\omega) = \infty.$$

It is also easy to see (Wang, 2000) that z_π is the first derivative of the one to one increasing and convex function

$$(0,1) \ni \omega \longmapsto g(\omega) = \Phi\left(a + \Phi^{-1}(\omega)\right) \in (0,1), \tag{22}$$

where

$$a = \frac{r - r_f}{\sigma}\sqrt{T} \tag{23}$$

is positive because we assume, as usual, that $r > r_f$. Computing the derivative in (22) we have that

$$z_\pi(\omega) = Exp\left(-\frac{a^2}{2} - a\Phi^{-1}(\omega)\right) \tag{24}$$

$\omega \in (0,1)$, which easily allows us to verify that z_π is continuous and strictly decreasing,

$$Lim_{\omega \to 0} z_\pi(\omega) = \infty, \tag{25}$$

and

$$Lim_{\omega \to 1} z_\pi(\omega) = 0. \tag{26}$$

Corollary 2 and its remarks have shown the existence of an alternative investment y^* outperforming the riskless asset. Next let us compute y^*.

Theorem 4 *Under the assumptions and notations above, if y^* solves (12) then it also solves (13), and there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$0 < \beta < \alpha,$$

and

$$y^* = \begin{cases} 0 & if\ y_0 > \alpha \\ \alpha - y_0 & if\ \beta < y_0 \leq \alpha \\ 0 & if\ y_0 \leq \beta \end{cases} \tag{27}$$

Proof. Consider the dual solution (λ^*, z^*). (17) implies that $\Delta_{CVaR_{\mu_0}} \subset L^\infty$, while (25) shows that z_π is not bounded. Then, Corollary 2 implies that $\lambda^* > 0$. Furthermore, Statement a) in the same corollary shows that y^* also solves Problem (13).

Since $(1+\lambda^*)\, z_\pi$ is continuous and strictly decreasing (25) and (26) show the existence of $\gamma_1 \in (0,1)$ such that $(1+\lambda^*)\, z_\pi(\gamma_1) = \dfrac{1}{1-\mu_0}$, $(1+\lambda^*)\, z_\pi(\omega) > \dfrac{1}{1-\mu_0}$ for $\omega \in (0,\gamma_1)$ and $(1+\lambda^*)\, z_\pi(\omega) < \dfrac{1}{1-\mu_0}$ for $\omega \in (\gamma_1, 1)$. In particular, $z^*(\omega) < (1+\lambda^*)\, z_\pi(\omega)$ in $(0,\gamma_1)$, which, along with the fourth and fifth equations in (16) imply that $y^*(\omega) = 0$ in $(0,\gamma_1)$. On the other hand, being y_0 continuous and strictly increasing, take $\beta = y_0(\gamma_1)$ and we have that $y_0 \leq \beta$ if and only if $(0,\gamma_1] \ni \omega$, *i.e.*, the third part of (27) has been proved.[10]

Consider the partition $(0,1) = \Omega_0 \cup \Omega_1 \cup \Omega_2$ of (19). Notice that the fourth equation in (19) and the fifth one in (16) lead to $\Omega_1 \subset (0,\gamma_1]$. Notice also that $y_0 = \alpha - \alpha_1$ in Ω_1, whereas $y_0 = \alpha + \alpha_2$ in $(0,\gamma_1] \setminus \Omega_1$, since α_1 vanishes outside Ω_1 and y^* vanishes in $(0,\gamma_1]$. Being $\alpha_1, \alpha_2 \geq 0$ we conclude that y_0 increases from Ω_1 to $(0,\gamma_1] \setminus \Omega_1$. Since y_0 is strictly increasing there will exist $\tilde{\gamma}_1 \leq \gamma_1$ such that $\Omega_1 = (0, \tilde{\gamma}_1]$.

Let us see that $(\tilde{\gamma}_1, \gamma_1] \subset \Omega_2$. Indeed, otherwise in a non-null subset of $(\tilde{\gamma}_1, \gamma_1]$ we would have $y_0 = \alpha + \alpha_2 = \alpha$ (α_2 vanishes outside Ω_2), but this is a contradiction because y_0 is strictly increasing and cannot achieve any concrete value with strictly positive probability.

Assume for a few moments that Ω_0 is void. Then $\Omega_2 = (\tilde{\gamma}_1, 1)$ and $z^* = 0$ in $(\tilde{\gamma}_1, 1)$ (last condition in (19)). Since $(1+\lambda^*)\, z_\pi > 0$ (see (24)), the fourth equation in (16) implies $y^* = 0$ in $(0,1)$. Then $C > 0$ and $\lambda^* > 0$ provoke that the first equality in (16) does not hold, and we are facing a contradiction.

[10] y^* may be modified in $\{\gamma_1\}$ because it is a $\mu-$null set.

Consequently Ω_0 is not a null set. Let us see that $\tilde{\gamma}_1 = \gamma_1$. Indeed, we know that $\Omega_0 \subset (\gamma_1, 1)$. Fix λ^*. According to (15), z^* must solve

$$Min \; \{\mathbb{E}\,(y_0 z)\,;\; z \leq (1+\lambda^*)\, z_\pi, z \in \Delta_\rho\}\,. \tag{28}$$

If $\tilde{\gamma}_1 < \gamma_1$ then take $v = Inf\ (\Omega_0)$, $u = Sup\ (\Omega_0)$ and

$$\tilde{z} = \begin{cases} z^*, & \omega \in \Omega_1 = (0, \tilde{\gamma}_1] \\ z^*\,(\omega + v - \tilde{\gamma}_1)\,, & \tilde{\gamma}_1 < \omega < \tilde{\gamma}_1 + u - v \\ 0, & otherwise \end{cases}$$

$\tilde{z}$ trivially satisfies the constraints of (28) because so does z^*, z^* vanishes on Ω_2 and z_π is strictly decreasing. On the other hand, $\mathbb{E}\,(y_0 \tilde{z}) < \mathbb{E}\,(y_0 z^*)$ trivially holds because y_0 is strictly increasing, so z^* does not solve (28). Hence, $\tilde{\gamma}_1 = \gamma_1$.

Applying an argument similar to that of the paragraph above it is easy to show the existence of $\gamma_2 > \gamma_1$ such that $\Omega_0 = (\gamma_1, \gamma_2)$. Moreover, $y^* = \alpha - y_0$ in (γ_1, γ_2) implies that $y_0\,(\omega) \leq \alpha$ for $\omega \in (\gamma_1, \gamma_2)$, because $y^* \geq 0$. Since y_0 is continuous and strictly increasing one has that

$$\alpha \geq y_0\,(\gamma_2) > y_0\,(\gamma_1) = \beta > 0.$$

Finally, if $y_0\,(\omega) > \alpha$ then $\omega > \gamma_2$, so $\omega \in \Omega_2$, $z^* = 0$ (last equation in (19)), the fifth equation in (16) holds in terms of strict inequality, and the fourth equation in (16) shows that y^* vanishes. □

Remark 3 *Notice that the solution y^* above may be given by*

$$y^* = y^*_\alpha - y^*_\beta - (\alpha - \beta)\, y^*_{D\beta},$$

y^*_α denoting the European put option with maturity at T and strike α, y^*_β denoting the similar put with strike β, and $y^*_{D\beta}$ denoting the digital put option with maturity at T and strike β, whose pay-off is

$$y^*_{D\beta} = \begin{cases} 0 & if\ y_0 > \beta \\ 1 & if\ y_0 \leq \beta \end{cases}$$

Then the shadow riskless asset is a combination of three put options.

Remark 4 *In order to apply our finding in practice we have to provide the values of β and α. Suppose for a few moments that we know the value of the dual solution λ^*. Then* *theorem's proof and (20) point out that β may be computed in practice by*

$$\beta = Exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\gamma_1)\right),$$

where, according to the theorem's proof and (24),

$$\gamma_1 = z_\pi^{-1}\left(\frac{1}{(1-\mu_0)(1+\lambda^*)}\right) = \Phi\left(\frac{2L(1-\mu_0) + 2L(1+\lambda^*) - a^2}{2a}\right),$$

and a is given by (23).

Since the theorem's proof is constructive it also yields and algorithm leading to the computation of λ^*. Indeed, take in a first iteration $\gamma_1 = 1 - \mu_0$ and

$$1 + \lambda^* = \frac{1}{(1-\mu_0)z_\pi(\gamma_1)}. \tag{29}$$

In the theorem's proof this choice means that we are taking

$$z^* = \begin{cases} \frac{1}{(1-\mu_0)} & \omega \leq \gamma_1 \\ 0 & otherwise \end{cases}$$

We know that this choice does not provide the dual solution because it implies that Ω_0 is void (see the theorem's proof). Anyway, we can compute the (minus) objective of (15) in the proposed solution,

$$C\lambda^* + \mathbb{E}(z^* y_0). \tag{30}$$

Then, choose a "small enough step" $\varepsilon > 0$ and consider $\gamma_1 = 1 - \mu_0 - \varepsilon$. Take λ^* as in (29) and

$$z^* = \begin{cases} \frac{1}{(1-\mu_0)} & \omega \leq \gamma_1 \\ (1+\lambda^*) z_\pi & \gamma_1 < \omega \leq \gamma_2 \\ 0 & otherwise \end{cases},$$

where γ_2 must be selected so as to reach

$$\mathbb{E}(z^*) = \frac{\gamma_1}{1-\mu_0} + (1+\lambda^*)\int_{\gamma_1}^{\gamma_2} y_0(\omega) z_\pi(\omega) d\omega = 1.$$

Notice that the integral may be calculated by numerical methods. Then compute the (minus) objective of (15) as indicated in (30). If the value of (30) has decreased with respect to the previous one then we already reached the desired value λ^*. Otherwise take $\gamma_1 = 1 - \mu_0 - 2\varepsilon$ and repeat a new iteration of the algorithm.

Once β has been computed one can calculate α because the price of y^* must equal $Ce^{-r_f T}$, i.e., the following equation

$$\Pi\left(y_\alpha^*\right) = Ce^{-r_f T} + \Pi\left(y_\beta^*\right) + (\alpha - \beta)\,\Pi\left(y_{D\beta}^*\right)$$

must hold. □

Remark 5 *The risk measure $CVaR_{\mu_0}$ may be also given by (Rockafellar et al., 2006)*

$$CVaR_{\mu_0}(y) = \frac{1}{1-\mu_0}\int_0^{1-\mu_0} VaR_{1-t}(y)\,dt,$$

for every $y \in L^p$. Accordingly, since $VaR(y)$ only focuses on "the worst" values of y (on the left tail of y), so does $CVaR_{\mu_0}(y)$. Thus, it is not so surprising that y^* vanishes if y_0 achieves high values, since they are not affecting the global risk level.

A little bit more shocking is that y^* also vanishes if y_0 achieves its lowest values. Notice that the theorem's proof leads to $\beta = y_0(\gamma_1)$, and, according to the previous remark,

$$0 < \gamma_1 < 1 - \mu_0.$$

Therefore,

$$Lim_{\mu_0 \longmapsto 1}\gamma_1 = 0,$$

which, along with (21) and $\beta = y_0(\gamma_1)$, imply that

$$Lim_{\mu_0 \longmapsto 1}\beta = 0.$$

Thus, for a high level of confidence the lowest values of y_0 become very important, and y^* almost becomes the European put option y_α^*. The limit value of α as μ_0 tends to 1 may be computed from $\Pi\left(y_\alpha^*\right) = Ce^{-r_f T}$. □

Remark 6 *There are several classical strategies providing "portfolio insurance". Maybe the most popular one is the purchase of an appropriate European put option. Theorem 4 highlights that for high levels of confidence the use of portfolio insurance strategies may be adequate to protect the investor's risk. It is consistent with some empirical findings of recent literature. For instance, the test implemented by Annaert et al. (2009) seems to reveal that some put option-linked portfolio insurance strategies are not outperformed by other hedging methods. The authors use stochastic dominance criteria and VaR and $CVaR$ in their empirical test.* □

5 Conclusions

The paper has considered a complete arbitrage free market and a general expectation bounded risk measure, and has analyzed whether it is optimal to invest the capital requirements in the riskless asset. Once the optimal strategy, or shadow riskless asset, has been characterized, it has been shown that it is not the riskless security in many important cases. For instance, if the risk measure is the $CVaR$ or the absolute deviation or down-side semi-deviation and we consider the assumptions of the $CAPM$ or the Black and Scholes model. Furthermore, for the $CVaR$ and the Black and Scholes model the explicit expression of the shadow riskless asset has been provided, and it is composed of a long European put plus a short European put plus a short digital put. If the confidence level of the $CVaR$ is close to 100% then the shadow riskless asset becomes an European put option. This may be a surprising and important finding for both researchers and practitioners. In particular, managers can discover how to significantly reduce the level of initial capital requirements by trading options.

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