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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-98-75

PRICES, DELAY , AND THE DYNAMICS OF TRADE *

Diego Moreno and John Wooders

Abstract

We characterize trading patterns and their dynamics in a market in which trade is bilateral, finding a trading partner is costly, prices are determined by bargaining, and preferences are private information. We also determine how the trading pattern depends on the market composition. Our analysis reveals that market equilibria may be inefficient and may exhibit delay. As the market becomes frictionless the welfare loss due to inefficiency vanishes; delay persists, however, and in this respect frictionless markets are not competitive.

Keywords: Trade dynamics, matching, bargaining, delay, asymmetric information, decentralized trade.

Moreno (dmoreno@eco.uc3m.es), Departamento de Economía, Universidad Carlos III de Madrid.
Wooders (jwooders@bpa.arizona.edu), Department of Economics, University of Arizona.

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1 Introduction

In many markets, when (or whether) an agent trades, and at what price, depends on his own characteristics (his value, or the cost or quality of his good), as well as on the characteristics of the other traders. In the market for new assistant professors of economics, for example, high-quality job candidates tend to leave the market (i.e., to accept job offers) earlier than low-quality candidates. In the clothing market, high-value buyers purchase the new fall fashions as soon as the clothes enter stores, whereas low-value buyers purchase later in the season once the clothes go on sale. The distribution of the characteristics of active traders also varies over time: In the market for new assistant professors, for example, the proportion of active candidates that are of high quality is larger when the market opens than when it closes. In these markets the "trading pattern" at each date (i.e., which types of buyers and sellers trade), and the "market composition" at each date (i.e., the characteristics of active traders), are determined endogenously.

In this paper we introduce a simple model of a market, and we characterize the trading patterns that arise in equilibrium as well as their dynamics (that is, when more than one trading pattern arises, we identify the possible transitions from one trading pattern to the next). We also determine how the trading pattern depends on the market composition. With these results in hand, we establish that market equilibria may be inefficient (and may even exhibit trading delay), and we obtain results on the competitiveness of nearly frictionless markets.

We study a market for an indivisible good that operates over a finite number of periods. There are two types of buyers, whose values are either "high" or "low," initially present in the market in given proportions. All sellers can supply a unit of the good at equal cost. After the market opens there is no further entry. Each period, active traders are randomly matched and bargain bilaterally. In the bargaining game one of the traders is randomly selected to make a take-it-or-leave-it price proposal. Bargaining is under incomplete information, as a seller does not know whether his partner has a high or a low value.

A variety of trading patterns are possible in this market. A "pure" trading pattern (where traders of the same type bargain the same way) specifies (i) whether sellers' price offers are accepted by high-value buyers, (ii) whether sellers' price offers are accepted by low-value buyers, (iii) whether low-value buyers' price offers are accepted by sellers, and (iv) whether high-value buyers' price offers are accepted by sellers. Hence in a given date there are 16 different "pure" trading patterns. In addition, there are "mixed" trading patterns in which traders of the same type bargain differently.

There are two cases of interest: the "high cost" case where sellers have a cost above the value of low-value buyers (but below the value of high-value buyers), and the "low cost" case where the sellers' cost is below the values of both types of buyers. For both the high cost and the low cost case, we establish that a market equilibrium exists, and that as the discount factor approaches one and the time horizon becomes infinite, transaction prices converge to the competitive price.

In the high cost case the market equilibrium is unique and symmetric (except for rejected offers): high-value buyers and sellers always trade, but low-value buyers never trade.

In the low cost case, the case of primary interest, a richer set of trading patterns can arise. We establish that a market equilibrium exhibits at most three (pure) trading patterns over the life of the market. An important variable in determining which trading pattern arises at each date is the proportion of high-value buyers in the market. In periods where high-value buyers are abundant (i.e., when their proportion exceeds a critical threshold we identify), the trading pattern is either *separating* (high-value buyers trade, but low-value buyers do not trade) or *partially-separating* (high-value buyers trade, and low-value buyers trade only when they propose). In periods where high-value buyers are scarce, the trading pattern is *pooling* (both types of buyers trade). We establish that the proportion of high-value buyers in the market is (weakly) decreasing over time, and hence the pooling trading pattern is absorbing. Moreover, for discount factors near or equal to one, the transitions from one trading pattern to the next are in a particular order: from separating to partially-separating to pooling. When the market transits from one pure trading pattern to the next, however, there may be a single intervening period in which the trading pattern is mixed. (Thus, unlike in the high cost case, in this case market equilibria need not be symmetric.)

Our analysis reveals two properties of market equilibria in the low cost case that are in sharp contrast with Walrasian equilibria: market equilibria may be inefficient and may exhibit delay. Since efficiency in the low cost case requires that both types of buyers always trade when matched, the equilibrium is inefficient whenever the separating or partially separating trading pattern arise. Market equilibria exhibit delay when the trading pattern is either separating or partially separating at the market open, and it is pooling by the market close; e.g., low-value buyers do not trade at the market open, but do trade at later periods. A sufficient condition for market equilibria to be inefficient is that high-value buyers are abundant when the market opens. If in addition the time horizon is sufficiently long, then there is also delay. Hence both inefficiency and delay occur for a non-negligible subset of the parameter space. As the market becomes frictionless the welfare loss due to inefficiency vanishes; delay persists, however, and in this respect frictionless markets are not competitive.

Related Literature

Our results on trading patterns and their dynamics are novel and have no counterpart in the literature. Our findings that market equilibria may be inefficient and may exhibit delay, and that transaction prices are competitive as frictions vanish relate to results already in the literature. We discuss these connections.

Equilibrium in a market is competitive: There is now a large literature studying whether decentralized markets are competitive as frictions vanish (see, for example, Rubinstein and Wolinsky (1985), and Gale (1987)). With the important exception of Binmore and Herrero (1988), who study markets with a single type of buyer and a single type of seller, the literature has focused on stationary equilibria. In the present paper we study dynamic markets with heterogenous traders.

Equilibrium in markets where agents are asymmetrically informed: Wolinsky (1990) studies convergence to rational expectations equilibrium as frictions vanish when some traders are uninformed about the quality of the traded good. In Wolinsky's model traders may bargain either "tough" or "soft." If both traders in a match bargain tough, the outcome is no trade. Otherwise, they trade at one of three *exogenously* given prices. The pair of bargaining positions determines at which of the three prices they trade. Adopting Wolinsky's model of bargaining, in the low cost case Serrano and Yosha (1996) show that when frictions are small every match ends with trade (in our terminology, the trading pattern is pooling), and therefore that market equilibria are efficient. Our results on trading patterns and their dynamics reveal that both the separating and partially-separating trading patterns arise in equilibrium even when frictions are small (in fact, even when frictions vanish). Therefore market equilibria may be inefficient and exhibit delay even when frictions are small. Our results differs from Serrano and Yosha's because we place no *a priori* restrictions on the prices a trader can offer.[1]

Market Efficiency and Equilibrium Delay: Samuelson (1992) models the decision of traders to terminate bargaining. In a model of decentralized trade and Nash bargaining, Sattinger (1995) shows that equilibrium is not efficient. Jackson and Palfrey (1999) show that there is a robust distribution of buyer and seller values for which equilibrium is inefficient for every bargaining game in a general class. Although inefficiency also arises in our model, we show that as the market becomes frictionless the welfare loss vanishes and each trader obtains his competitive equilibrium utility. It is an open question whether similar results on the efficiency of frictionless markets hold in Jackson and Palfrey's general framework.

Our work also relates to a large literature studying price dispersion and sales. Varian (1980), for example, shows that sales provide a means for sellers to price discriminate between informed and uninformed consumers. In Varian's model informed buyers pay the lowest offered price, while uninformed buyers purchase from each firm

[1] Wolinsky's two bargaining position model also imposes a monotonicity restriction on bargaining behavior as a trader's strategy is simply the number of periods in which he bargains tough (after which he *forever* bargains soft). Example 2 shows that the monotonicity of bargaining strategies is not a feature of equilibrium in our model, e.g., a seller may raise his price offer from one period to the next.

with equal likelihood. In our model price discrimination arises as a consequence of the differential willingness of high-value and low-value buyers to endure delay. When equilibrium delay arises, low-value buyers postpone trading until sellers lower their price offers from the high-value buyer reservation price to the low-value buyer reservation price, i.e., until sellers offer a "sale" price.

The paper is organized as follows: we describe our model in Section 2. In Section 3 we establish that a market equilibrium exists. We study the properties of market equilibria in Section 4. In Section 5 we conclude. Appendix A contains the proof of existence of equilibrium. The remaining proofs are in Appendix B.

2 The Model

A market for a single indivisible commodity opens for $T+1$ periods, which we denote by the positive integers from 0 to T. Each seller is endowed with a single unit of the indivisible good. Each buyer is endowed with one unit of money. Buyers and sellers preferences are characterized by, respectively, their values and costs: All sellers (S) have the same cost, $c \geq 0$, whereas there are two types of buyers, "high-value" (H) and "low-value" (L), whose values are, respectively, u^H and u^L, where $1 \geq u^H > u^L \geq 0$. At period $t = 0$ there is a continuum of traders; no new traders enter the market subsequently. Buyers and sellers are initially present in equal measures; high-value and low-value buyers are present in the population of buyers in proportions b_0^H and $b_0^L = 1 - b_0^H$, respectively. We assume throughout that $u^H > c$ and $b_0^H \in (0,1)$. If a buyer whose value is u^τ trades with a seller at the price p in time t they obtain a utility of $\delta^t(u^\tau - p)$ and $\delta^t(p - c)$, respectively. Here $\delta \in (0,1]$, the discount factor, expresses the traders' impatience. A buyer or a seller who never trades obtains a utility of zero.

Each period every buyer (seller) remaining in the market meets a randomly selected seller (buyer) with probability α, where $0 < \alpha < 1$. A matched seller does not observe the buyer's value. When a buyer and a seller meet, one of them is selected randomly (with probability $\frac{1}{2}$) to propose a price at which to trade. If the proposed price is accepted by the other party, then the agents trade at that price and both leave the market. Otherwise, the agents remain in the market at the next date and wait for a new match. An agent who is not matched in the current period also remains in the market at the next date. A trader observes only the outcome of his own matches.

A *strategy* for a trader of type $\tau \in \{H, L, S\}$ is a vector of real numbers indicating the trader's price offers and reservation prices at each date $(p_0^\tau, ..., p_T^\tau; r_0^\tau, ..., r_T^\tau) = (p^\tau, r^\tau) \in \mathbb{R}^{2(T+1)}$. The vector of prices specifies the price that an agent would propose at each date if matched and selected to propose a price; the vector of reservation prices specifies the maximum (minimum) price that a buyer (seller) would accept at each date if responding to a price offer. A *strategy distribution* is a vector $(p, r, \lambda) = [(p^{H_i}, r^{H_i}, \lambda^{H_i})_{i=1}^{n^H}, (p^{L_i}, r^{L_i}, \lambda^{L_i})_{i=1}^{n^L}, (p^{S_i}, r^{S_i}, \lambda^{S_i})_{i=1}^{n^S}]$, where $\sum_{k=1}^{n^\tau} \lambda^{\tau_k} = 1$ for each $\tau \in \{H, L, S\}$, $\lambda^{\tau_k} > 0$ is the proportion of type τ players using strategy $(p^{\tau_k}, r^{\tau_k}) \in \mathbb{R}^{2(T+1)}$, and n^τ is the (countable) number of distinct strategies used by (a positive measure of) type τ traders.

We do not restrict attention to symmetric strategy distributions (i.e., different agents of the same type τ may follow different strategies). Indeed, allowing asymmetric strategy distributions is necessary to guarantee existence of a market equilibrium (see Example 3). We consider only strategies in which a trader does not condition his actions in the current match on the history of his prior matches. This restriction is inconsequential, since for any equilibrium in which the players' strategies depend on histories, there is another equilibrium in history-independent strategies which is equivalent (i.e., transaction prices, trading patterns, and market compositions are the same). For simplicity, we restrict attention to strategy distributions where only countably many distinct strategies are used. As we shall see, however, for discount factors near or equal to one, in equilibrium at most two different strategies are played by each type of trader.

2.1 Laws of Motion

Given a strategy distribution (p, r, λ), for $\tau \in \{H, L, S\}$ and $k \leq n^\tau$ let $\lambda_t^{\tau_k}$ denote the proportion of agents following the k-th type τ strategy out of the total measure of agents of type τ who remain in the market at time t. (Throughout, we use i, j, and k, respectively, to index the strategies of buyers, sellers, and generic traders.) This proportion can be computed for $t \in \{0, \ldots, T\}$, given $\lambda_0^{\tau_k} = \lambda^{\tau_k}$, as

$$\lambda_{t+1}^{\tau_k} = \frac{\lambda_t^{\tau_k} \mu_t^{\tau_k}}{\sum_{l=1}^{n^\tau} \lambda_t^{\tau_l} \mu_t^{\tau_l}},$$

where $\mu_t^{\tau_k}$ denotes the probability that a trader who is in the market at t and who follows the strategy $(p_t^{\tau_k}, r_t^{\tau_k})$ remains in the market at the next period. This probability is computed as follows: For $x, y \in \mathbb{R}$ denote by $I(x, y)$ the *indicator function*, whose value is 1 if $x \geq y$, and 0 otherwise. Writing $B = \{H, L\}$ for the set of buyer types, then for $\tau \in B$ we have

$$\mu_t^{\tau_i} = 1 - \frac{\alpha}{2} \sum_{j=1}^{nS} \lambda_t^{S_j} I(p_t^{\tau_i}, r_t^{S_j}) - \frac{\alpha}{2} \sum_{j=1}^{nS} \lambda_t^{S_j} I(r_t^{\tau_i}, p_t^{S_j}).$$

For sellers, this probability is given by

$$\mu_t^{S_j} = 1 - \frac{\alpha}{2} \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(r_t^{\tau_i}, p_t^{S_j}) - \frac{\alpha}{2} \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(p_t^{\tau_i}, r_t^{S_j}),$$

where b_t^τ, the proportion of the buyers of type τ out of the total measure of buyers remaining in the market at time t, can be computed for $t > 0$, given b_0^τ, as

$$b_t^\tau = \frac{b_{t-1}^\tau \sum_{i=1}^{n^\tau} \lambda_{t-1}^{\tau_i} \mu_{t-1}^{\tau_i}}{b_{t-1}^H \sum_{i=1}^{n^H} \lambda_{t-1}^{H_i} \mu_{t-1}^{H_i} + b_{t-1}^L \sum_{i=1}^{n^L} \lambda_{t-1}^{L_i} \mu_{t-1}^{L_i}}.$$

Since there is a continuum of traders, the market evolves deterministically, even though a trader's own market experience is stochastic.

2.2 Value Functions

Given a strategy distribution (p, r, λ), the expected utility at time t of an agent of type $\tau \in \{H, L, S\}$ who is using strategy τ_k is computed recursively, given $V_{T+1}^{\tau_k} = 0$, as

$$V_t^{\tau_k} = \frac{\alpha}{2}(P_t^{\tau_k} + R_t^{\tau_k}) + (1 - \alpha)\delta V_{t+1}^{\tau_k}.$$

In this expression, $P_t^{\tau_k}$ $(R_t^{\tau_k})$ is the expected utility to a trader of type τ following the k-th type τ strategy who is matched at t and selected to propose (respond to) a price offer. These expected utilities can be calculated for $\tau \in B$ as

$$P_t^{\tau_i} = (u^\tau - p_t^{\tau_i}) \sum_{j=1}^{n^S} \lambda_t^{S_j} I(p_t^{\tau_i}, r_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(p_t^{\tau_i}, r_t^{S_j}))\delta V_{t+1}^{\tau_i},$$

and

$$R_t^{\tau_i} = \sum_{j=1}^{n^S} (u^\tau - p_t^{S_j})\lambda_t^{S_j} I(r_t^{\tau_i}, p_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^{\tau_i}, p_t^{S_j}))\delta V_{t+1}^{\tau_i}.$$

For sellers we have

$$P_t^{S_j} = (p_t^{S_j} - c) \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(r_t^{\tau_i}, p_t^{S_j}) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(r_t^{\tau_i}, p_t^{S_j}))\delta V_{t+1}^{S_j},$$

and

$$R_t^{S_j} = \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i}(p_t^{\tau_i} - c) I(p_t^{\tau_i}, r_t^{S_j}) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(p_t^{\tau_i}, r_t^{S_j}))\delta V_{t+1}^{S_j}.$$

Note that $\lambda_t^{S_j}$ is the probability that a buyer matched at t is matched with a seller following the j-th seller strategy. Similarly, $b_t^\tau \lambda_t^{\tau_i}$ is the probability that a seller matched at t is matched with a buyer of type τ following the i-th type τ buyer strategy.

2.3 Equilibrium

A strategy distribution (p, r, λ) is a *market equilibrium* if for each $t \in \{0, \dots, T\}$, each $\tau \in B$ and $i \in \{1, \dots, n^\tau\}$, and each $j \in \{1, \dots, n^S\}$

$$(E.1) \qquad \begin{aligned} u^\tau - r_t^{\tau_i} &= \delta V_{t+1}^{\tau_i}, \\ r_t^{S_j} - c &= \delta V_{t+1}^{S_j}, \end{aligned}$$

and

$$(E.2) \qquad \begin{aligned} p_t^{\tau_i} &\in \arg\max_x (u^\tau - x) \sum_{j=1}^{n^S} \lambda_t^{S_j} I(x, r_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(x, r_t^{S_j}))\delta V_{t+1}^{\tau_i}, \\ p_t^{S_j} &\in \arg\max_x (x - c) \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(r_t^{\tau_i}, x) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(r_t^{\tau_i}, x))\delta V_{t+1}^{S_j}. \end{aligned}$$

Condition $E.1$ requires that at each date a trader's reservation price makes him indifferent between accepting or rejecting an offer of his reservation price. Condition

$E.2$ ensures that price offers are optimal. Given the recursive nature of our setting, in a market equilibrium traders' strategies are globally optimal, i.e., no trader can do better by changing his reservation prices or price offers simultaneously at more than one date.

In a market equilibrium, at each date traders form their expectations of the proportion of buyers of each type remaining in the market (b_t^τ), and the proportion of traders following each of the strategies being played ($\lambda_t^{\tau k}$), on the basis of the strategy distribution being played. Moreover, each trader maximizes his expected utility at each of his information sets. Thus, the notion of market equilibrium is in the spirit of sequential (or Bayes perfect) equilibrium, when agents do not take their own observation of a deviation from equilibrium play as evidence that play of a positive measure of agents has deviated from equilibrium. (See Osborne and Rubinstein (1990), pages 154-162, for a discussion of this issue for related models.)

3 Existence of Market Equilibria

In this section we establish that market equilibria exist under general conditions. It might seem that one could calculate a market equilibrium via backward induction. Computing a traders' reservation price and optimal price offer at a date t, however, requires knowing the market composition (i.e., the proportion of traders of each type present in the market) at t, as well as his expected utility if he remains in the market at $t+1$. Since the market composition at date t is determined by the trading patterns (and the traders' strategies) *prior* to t, a market equilibrium cannot be computed by backward induction.

For some parameter configurations it is easy to guess an equilibrium sequence of trading patterns (e.g., in the high cost case, or in the low cost case if the initial proportion of high-value buyers is small). In general, however, this is a difficult task: although the number of "pure" trading patterns that may arise in equilibrium is small (as we shall see in the next section), there is a continuum of mixed trading patterns, differing in the proportions of traders following different strategies. These mixed trading patterns cannot be neglected since for some parameter values the unique market equilibrium has "mixed" trading patterns (see Example 3). Thus, guessing equilibrium trading patterns in order to establish existence of equilibrium does not seem viable.

Notwithstanding this difficulty, we establish in Theorem 1 that a market equilibrium always exists.

Theorem 1. *A market equilibrium exists.*

Proof: See Appendix A.

Theorem 1 is established using a fixed point argument: We construct a mapping which for arbitrary sequences describing the trading patterns, market compositions, and reservation prices at each date, provides

(i) the trading patterns arising when traders make optimal price offers for the given sequence of market compositions and reservation prices, and

(ii) the sequence of market compositions and reservation prices that results from the sequence of trading patterns obtained in (i).

As this description suggests, the "equilibrium mapping" is a composition of two mappings. The first mapping turns out to be an upper hemicontinuous non-empty compact convex valued correspondence. The second mapping is a continuous function. In general, the result of this composition need not yield a convex valued correspondence, a property required to use Kakutani's Fixed Point Theorem. Nevertheless, we are able to establish existence of a fixed point using Cellina's Theorem. From a fixed point of this mapping we construct a strategy distribution which we show is a market equilibrium.

4 Properties of Market Equilibria

We study the properties of market equilibria for the two cases of interest: the high cost case (i.e., $u^H > c > u^L$), and the low cost case (i.e., $u^H > u^L > c$). We study these two cases in turn.

4.1 Properties of market equilibria in the high cost case

Supply and demand schedules in this case are illustrated below in Figure 1. Beginning with this case allows us to discuss the workings of our model in a simple environment and facilitates understanding the subtleties that arise in the more interesting case where there are gains to trade between sellers and both types of buyers.

Figure 1 goes here.

Market equilibria in this case have a simple structure: at every date high-value (low-value) buyers offer a price equal to (below) the seller reservation price, and sellers offer a price equal to the high-value-buyer reservation price. Thus, only high-value buyers, and the sellers they are matched with, trade. We provide an informal discussion of these results.

Let (p, r, λ) be a market equilibrium. As an agent who does not trade while the market is open obtains a utility of zero (i.e., $V^{\tau}_{T+1} = 0$), by $E1$ reservation prices at the last date are $r^H_T = u^H$, $r^L_T = u^L$, and $r^S_T = c$. Hence $r^H_T > r^S_T > r^L_T$. It is easy to see that high-value (low-value) buyers offer at date T a price equal to (below) the seller-reservation price: A high-value (low-value) buyer obtains a utility of $u^H - r^S_T = u^H - c > 0$ ($u^L - r^S_T = u^L - c < 0$) offering r^S_T, the lowest price accepted by sellers, and obtains $\delta V^H_{T+1} = 0$ ($\delta V^L_{T+1} = 0$) with a lower price offer. Thus, $p^H_T = r^S_T$ ($p^L_T < r^S_T$). Sellers offer at date T the high-value-buyer reservation price (i.e., the highest price accepted by high-value buyers): a seller who offers r^H_T obtains an expected utility of $b^H_T(r^H_T - c) = b^H_T(u^H - c) > 0$, whereas he obtains $r^L_T - c = u^L - c < 0$ offering r^L_T.[2] Thus, $p^S_T = r^H_T$. Hence the pattern of trade at date T is *separating*: all matched high-value buyers trade, low-value buyers do not trade, and sellers only trade when matched to a high-value buyer. Therefore traders' expected utilities at T are $V^H_T = \frac{1}{2}(u^H - c)$, $V^L_T = 0$, and $V^S_T = \frac{1}{2}b^H_T(u^H - c)$.

[2]Note that b^H_T is strictly positive since a measure $(1-\alpha)^T b^H_0 > 0$ of high-value buyers has never been matched before T, and therefore at least this measure of high-value buyers remains in the market at T.

Now, using $E1$ again we calculate traders' reservation prices at $T-1$ to obtain $r^H_{T-1} = u^H - \delta\frac{1}{2}(u^H - c)$, $r^S_{T-1} = c + \delta\frac{1}{2}b^H_T(u^H - c)$, and $r^L_{T-1} = u^L$. Thus $r^H_{T-1} > r^S_{T-1} > r^L_{T-1}$, regardless of the value of b^H_T, and the same pattern of trade arises at date $T-1$. In fact, it can be shown by induction that reservation prices satisfy this inequality at every date t, *independently* of b^H_t, and therefore that the pattern of trade is separating at every date.

Given the initial proportion of high-value buyers in the market and knowing the pattern of trade at each date, we can compute the entire evolution of the market composition (i.e., the sequence $\{b^H_t\}^T_{t=0}$). Knowing the trading pattern and the market composition at each date, the sequence of reservation prices is then computed recursively. Transaction prices are the seller-reservation price when high-value buyers propose, and the high-value-buyer reservation price when sellers propose. The market equilibrium is therefore unique and symmetric, except for low-value-buyer price offers (which are not determined).

When traders are sufficiently patient (i.e., δ is close to one) and the time horizon is sufficiently long, transaction prices at a given date are close to the competitive price (the sellers' cost in this case). Intuitively this is because when the time horizon is long, high-value buyers eventually become so scarce that the seller-reservation price approaches their cost. Since the probability of a future match is close to one (because the time horizon is long and the matching probabilities are constant), if high-value buyers do not discount future utilities very much, then their cost of waiting is small, and therefore their reservation price also approaches the sellers' cost.

These findings are summarized in Proposition 1.

Proposition 1. *Assume* $u^H > c > u^L$.

(P.1) Let (p, r, λ) *be a market equilibrium and* $\bar{t} \in \{0, \ldots, T\}$.

Reservation Prices:

(P1.1.1) $r^{\tau_i}_{\bar{t}} = r^{\tau}_{\bar{t}}$ *for every* $\tau \in \{H, L, S\}$ *and* $i \leq n^{\tau}$.

(P1.1.2) $r^H_{\bar{t}} > r^S_{\bar{t}} > r^L_{\bar{t}}$.

Price Offers:

(P1.1.3) $p^{H_i}_{\bar{t}} = r^S_{\bar{t}}$ *for* $i \leq n^H$, $p^{L_i}_{\bar{t}} < r^S_{\bar{t}}$ *for* $i \leq n^L$, *and* $p^{S_j}_{\bar{t}} = r^H_{\bar{t}}$ *for* $j \leq n^S$.

Market Composition:

(P1.1.4) $b^H_{t+1} = \frac{(1-\alpha)b^H_t}{(1-\alpha)b^H_t+1-b^H_t} < b^H_t$.

(P.2) For $\delta \in (0,1]$, and $T < \infty$, let $r(\delta, T)$ be the sequence of equilibrium reservation prices.

Transaction Prices as Frictions Vanish:

(P1.2.1) $\lim_{\delta\to 1}\lim_{T\to\infty} r^S_t(\delta,T) = \lim_{T\to\infty}\lim_{\delta\to 1} r^S_t(\delta,T) = c$.

(P1.2.2) $\lim_{\delta\to 1}\lim_{T\to\infty} r^H_t(\delta,T) = \lim_{T\to\infty}\lim_{\delta\to 1} r^H_t(\delta,T) = c$.

Example 1 below illustrates the results of Proposition 1.

Example 1. Figure 2 shows equilibrium transaction prices for a market that opens for 10 periods and whose parameter values are the ones specified.

Figure 2 goes here.

Seller price offers are not monotonic, as price offers at first decrease as time passes, but later increase. There are two effects at work: The first effect is that as time passes high-value buyers become scarce, which lowers the reservation price of both sellers and high-value buyers. The second effect is that as time passes the end of the horizon approaches, which raises the reservation price of high-value buyers. At the market open the first effect dominates and the reservation price of high-value buyers is falling, while near the market close the second effect dominates and high-value buyer reservation price is rising.

The mean transaction price (weighted by the volume of trade) is .4308 which is near reservation prices in the first few periods since most trade occurs within the first few periods. In the competitive equilibrium of this market, the price is .2 and the entire surplus of .7520 goes to high-value buyers. In contrast, in the market equilibrium sellers capture 29% of the total (discounted) surplus of .6834, in spite of the fact that frictions are relatively small (the probability that an agent is never matched is $\alpha^{T+1} = \frac{1}{1024}$). Even when $\delta = 1$, sellers capture 15% of the total surplus of .7513. Finally, the equilibrium is efficient since all matches between sellers and high-value buyers end with trade.

4.2 Properties of market equilibria in the low cost case

Figure 3 below illustrates the supply and demand schedules for this case. We identify the trading patterns which may arise in equilibrium and identify the transitions among them. We also relate the trading pattern to the market composition, and describe how the market composition evolves over time. We show that if the initial proportion of high-value buyers in the market is above a critical threshold we identify, then in a market equilibrium trade is inefficient. If in addition the time horizon is long then a market equilibrium also exhibits delay. Finally, we show that as market frictions vanish (i.e., as the discount factor approaches one and the time horizon grows long), equilibrium transaction prices converge to a competitive equilibrium price. We establish that in the limit equilibrium delay persists, although its cost vanishes.

Figure 3 goes here.

In order to illustrate the difficulties that arise in the analysis of the present case, assume, for the purpose of discussion, that in a market equilibrium (i) traders of the same type have the same reservation price, (ii) sellers offer either the high-value-buyer reservation price r_t^H, or the low-value-buyer reservation price r_t^L, and (iii) $r_t^H > r_t^L$.[3] When a seller offers r_t^H at date t, he trades only with high-value buyers and obtains an expected utility of

$$b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c).$$

(Recall that $\delta V_{t+1}^S = r_t^S - c$ by $E1$.) A seller who offers r_t^L at date t trades with both types of buyers, and obtains $r_t^L - c$. Therefore it is optimal for a seller to offer the high-value-buyer reservation price if

$$b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c) \geq r_t^L - c,$$

i.e.,

$$b_t^H(r_t^H - r_t^S) \geq r_t^L - r_t^S.$$

[3] Each of these facts is proven in Appendix B.

In other words, sellers offer the high-value-buyer reservation price if the probability that the current partner is a high-value buyer times the gains to trade with high-value buyers is greater than the gains to trade with low-value buyers. (In both cases, the gains are calculated relative to the reservation prices, rather than the actual values or costs.) Writing π_t for the ratio $(r_t^L - r_t^S)/(r_t^H - r_t^S)$, which measures the relative gains to trade of sellers with low-value buyers versus high-value buyers, the inequality above can be written as

$$b_t^H \geq \pi_t.$$

Hence, in contrast to the low-cost case where the pattern of trade is separating regardless of the market composition, in the present case the pattern of trade at date t depends on the market composition. Further, the market composition at date t is determined in turn by the trading patterns prior to t. Thus, the entire sequence of trading patterns and market compositions must be determined simultaneously.

In spite of this difficulty, we identify the basic properties of market equilibria in propositions 2 through 4. Proposition 2 establishes some basic facts about equilibrium price offers and reservation prices.

Proposition 2. *Assume that $u^H > u^L > c$. Let (p, r, λ) be a market equilibrium and let $\bar{t} \in \{0, \dots, T\}$.*

Reservation Prices:

(P2.1.1) $r_{\bar{t}}^{\tau_i} = r_{\bar{t}}^{\tau}$ *for every* $\tau \in \{H, L, S\}$ *and* $i \leq n^{\tau}$.

(P2.1.2) $r_{\bar{t}}^H > \max\{r_{\bar{t}}^L, r_{\bar{t}}^S\}$.

High-Value-Buyer Price Offers:

(P2.2) $p_{\bar{t}}^{H_i} = r_{\bar{t}}^S$, *for every* $i \leq n^H$.

Low-Value-Buyer Price Offers:

(P2.3.1) $p_{\bar{t}}^{L_i} \leq r_{\bar{t}}^S$ *for every* $i \leq n^L$.

(P2.3.2) There is $\varepsilon(\alpha, T) > 0$ *such that for* $\delta > 1 - \varepsilon(\alpha, T)$:

(i) If $p_{\bar{t}}^{L_i} < r_{\bar{t}}^S$ *for some* $i \leq n^L$, *then* $p_t^{L_i} < r_t^S$ *for every* $t < \bar{t}$ *and* $i \leq n^L$.

(ii) If $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ *for some* $i \leq n^L$, *then* $p_t^{L_i} = r_t^S$ *for every* $t > \bar{t}$ *and* $i \leq n^L$.

Seller Price Offers:

(P2.4.1) $p_{\bar{t}}^{S_j} \in \{r_{\bar{t}}^L, r_{\bar{t}}^H\}$ *for every* $j \leq n^S$.

(P2.4.2) If $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ *for some* $j \leq n^S$, *then* $p_t^{S_j} = r_t^L$ *for every* $t > \bar{t}$ *and* $j \leq n^S$.

(P2.4.3) If $p_{\bar{t}}^{S_j} = r_{\bar{t}}^H$ *for some* $j \leq n^S$, *then* $p_t^{S_j} = r_t^H$ *for every* $t < \bar{t}$ *and* $j \leq n^S$.

Seller and Low-Value-Buyer Price Offers:

(P2.5) If $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ *for some* $j \leq n^S$, *then* $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ *for every* $i \leq n^L$.

In a market equilibrium all traders of the same type have identical reservation prices ($P2.1.1$). The high-value-buyer reservation price is above both the low-value-buyer and seller reservation prices ($P2.1.2$). High-value buyers offer sellers their reservation price ($P2.2$). If the discount factor is sufficiently large, low-value buyers may initially offer sellers a price below their reservation price, but once a positive proportion of low-value buyers offers the seller reservation price, then all low-value buyers offer this price at every subsequent date ($P2.3.2$). Similarly, seller's may initially offer the high-value-buyer reservation price ($P2.4.3$), but once a positive proportion of sellers offers the low-value-buyer reservation price, all sellers offer this price at every subsequent date ($P2.4.2$). Finally, if at date t a positive proportion of sellers offer the low-value buyer reservation price, then at date t all low-value buyers offer sellers their reservation price ($P2.5$).

TRADING PATTERNS

We begin by discussing which trading patterns may arise in equilibrium. A "pure" trading pattern, in which agents of the same type make the same price offers, specifies whether sellers' price offers are accepted by high-value buyers, whether sellers' price offers are accepted by low-value buyers, whether low-value buyers' price offers are accepted by sellers, and whether high-value buyers' price offers are accepted by sellers. There are 16 possible pure trading patterns.

Proposition 2 implies that at most three of these pure trading patterns may arise in equilibrium: Sellers' price offers are accepted by high-value buyers ($P2.1.2$ and $P2.4.1$). High-value buyers' price offers are accepted by sellers ($P2.2$). Hence $P2.1.2$, $P2.2$, and $P2.4.1$ rule out all but four of the feasible pure trading patterns. In addition, $P2.5$ rules out the trading pattern in which sellers' price offers are accepted by both types of buyers, but low-value buyers' price offers are not accepted by sellers. Thus, only three pure trading patterns may arise in equilibrium: a *separating* (S) trading pattern, where only matches between high-value buyers and sellers end with trade; a *partially-separating* (PS) trading pattern, where matches between high-value buyers and sellers end with trade and matches between low-value buyers and sellers end with trade only if the buyer proposes; and a *pooling* (P) trading pattern, where all matches end with trade. The relation between price offers and reservation prices in each of these trading patterns are described in Table I.

Trading Patterns	**Price Offers**		
	Sellers	High-Value	Low-Value
Separating	$p_t^S = r_t^H$	$p_t^H = r_t^S$	$p_t^L < r_t^S$
Partially-Separating	$p_t^S = r_t^H$	$p_t^H = r_t^S$	$p_t^L = r_t^S$
Pooling	$p_t^S = r_t^L$	$p_t^H = r_t^S$	$p_t^L = r_t^S$

TABLE I: Equilibrium Pure Trading Patterns when $u^H > u^L > c$.

In addition to the three "pure" trading patterns, an equilibrium *may* also have "mixed" ones (i.e., ones in which traders of the same type make different price offers). In particular, an equilibrium may have "*S-PS*" trading patterns and "*PS-P*" trading patterns. The *S-PS* trading pattern is the same as *S*, except that low-value buyers "mix," i.e., a positive proportion offer the seller reservation price, and a positive proportion offer a price below the seller reservation price. The *PS-P* trading pattern is the same as *PS*, except that sellers "mix," i.e., a positive proportion offer the high-value-buyer reservation price and a positive proportion offer the low-value-buyer reservation price.

Proposition 2 ensures that the *PS-P* trading pattern arises in at most one period (*P*2.4.3). Moreover, when the discount factor is sufficiently high, the *S-PS* trading pattern arises also in at most one period, since by *P*2.3.2 once a positive proportion of low-value buyers offer the seller reservation price, at subsequent periods all low-value buyers offer this price.

Dynamics of Trading Patterns

Proposition 2 also yields conclusions concerning the order in which trading patterns arise in equilibrium. $P2.4.2$ establishes that if at date t a positive proportion of sellers offer the low-value-buyer reservation price, then at every subsequent date all sellers offer this price. Hence the S, S-PS, and PS trading patterns (when they arise) precede the PS-P and P trading patterns. $P2.3.2$ establishes that when the discount factor is sufficiently close to one the S trading pattern precedes all the other trading patterns. Furthermore, $P2.4.2$ and $P2.3.2$ imply, respectively, that the S-PS mixed trading pattern precedes PS, and the PS-P mixed trading pattern precedes P.

It can be shown that if trading patterns S and P are both visited, then pattern PS must also be visited. Also, PS is always visited unless the market opens at P. The mixed trading patterns may be skipped, although the subset of the parameter space where all market equilibria exhibit mixed trading patterns is not negligible (see Example 3). The dynamics of trading patterns are illustrated in Figure 4.

Figure 4 goes here.

Market Composition

The market composition at date t is described by b_t^H, the proportion of high-value buyers in the market. Proposition 3 below relates this proportion to the trading pattern and the dynamics of the market composition. Denote by π^* the ratio $(u^L - c)/(u^H - c)$.

Proposition 3. *Assume that $u^H > u^L > c$. Let (p, r, λ) be a market equilibrium and let $\bar{t} \in \{0, \ldots, T\}$.*

The Critical Threshold (π^*):

(P3.1.1) If $b_{\bar{t}}^H < \pi^*$, then $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ for every $j \leq n^S$ and $b_{\bar{t}+1}^H = b_{\bar{t}}^H$.

(P3.1.2) If $b_{\bar{t}}^H = \pi^*$, then $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ for every $i \leq n^L$, and either

(i) $b_{\bar{t}+1}^H < b_{\bar{t}}^H$; or

(ii) $b_{\bar{t}+1}^H = b_{\bar{t}}^H$, and $p_t^{S_j} = r_t^L$ for every $j \leq n^S$ and $t \geq \bar{t}$.

(P3.1.3) If $b_t^H > \pi^*$, then $p_t^{S_j} = r_t^H$ for every $j \leq n^S$, and $b_{t+1}^H < b_t^H$.

The Critical Threshold is Eventually Reached :

(P3.2) There is $\bar{T} = \bar{T}(b_0^H, \alpha, \pi^*)$ such that if $T > \bar{T}$, then $b_t^H \leq \pi^*$ for $t \geq \bar{T}$.

Proposition 3 therefore establishes that trading patterns and the dynamics of market composition are governed by the relation of the proportion of high-value buyers in the market to the critical threshold π^*. If the proportion of high-value buyers in the market is below π^*, then the trading pattern is P ($P3.1.1$). If the proportion of high-value buyers in the market equals π^*, then either the trading pattern is PS or PS-P and the proportion of high-value buyers in the market is less than π^* at the next period, or the trading pattern is P (and remains at P at every subsequent period) and the proportion of high-value buyers is π^* at the next period ($P3.1.2$). $P3.1.1$ and $P3.1.2$ imply that if the proportion of high-value buyers in the market equals π^* at some date, at the next date the trading pattern must be P. If the proportion of high-value buyers in the market is greater than π^*, then the trading pattern is either S, S-PS, or PS ($P3.1.3$). Finally, $P3.2$ ensures that if the time horizon is sufficiently long, then eventually the proportion of high-value buyers in the market is less than or equal to π^*. Hence, by $P3.1.1$ and $P3.1.2$, if the time horizon is sufficiently long, then the trading pattern is eventually P.

Dynamics of Market Composition

In both the S and the PS trading pattern, as well as in the mixed trading patterns S-PS and PS-P, the proportion of high-value buyers in the market is falling: in S, each period a fraction α of high-value buyers exit the market, while no low-value buyer exits; in PS a fraction α of high-value buyers and a fraction $\frac{\alpha}{2}$ of low-value buyers exit the market each period. In the trading pattern P the same fraction α of each type of buyer exits the market at each date, and hence the proportion of high-value buyers in the market remains constant. Thus, the proportion of high-value buyers in the market decreases (quickly in S, and more slowly in PS), but once P is reached (i.e., once this proportion falls below π^*), it becomes stationary.

A numerical example in which all three trading patterns arise in equilibrium is given in Example 2.

Example 2. Figure 5 shows an equilibrium in which all three pure trading patterns arise for a market that opens for 10 periods and whose parameter values are the ones specified.

Figure 5 goes here.

The top graph in Figure 5 shows transaction prices. The trading pattern is S for periods 0 to 2, is PS for periods 3 to 7, and is P for periods 8 and 9. In period 8 the good goes on "sale" as sellers switch from offering the high-value-buyer reservation price to the low-value-buyer reservation price. Low-value buyers trade with delay: they do not trade at period 0 through 2; they trade only if they propose in periods 3 through 7; and they trade whether they propose or respond in periods 8 or 9. The bottom graph shows the evolution of the market composition and the ratio π_t.

The set of competitive prices for the market in Example 2 is the interval $[.2, .4]$. We focus on the competitive price of .3, since in a market equilibrium all transactions are at this price as frictions vanish (see Proposition 4). Table II shows the division of the surplus in three different settings: at the competitive equilibrium price of .3; in the market equilibrium displayed in Figure 5; and under efficient trading, i.e., when each match ends with trade (here the distribution of the surplus is computed when each match ends with trade at the price of .3).

Interestingly, in this market equilibrium sellers capture more than twice the surplus than they capture in the competitive equilibrium. The market equilibrium is not efficient since low-value buyers do not trade when matched in periods 0 through 2 and trade only if they propose in periods 3 through 7. The efficiency loss resulting from delay ($.0045 = .6943 - .6898$) is small since only 6% of the buyers are low-value at the market open.

	High-value	Low-value	Seller	Total
Competitive Equil.	.6580 (86%)	.0060 (.7%)	.1000 (13%)	.7640 (100%)
Efficient Trading	.5980 (86%)	.0055 (.7%)	.0909 (13%)	.6943 (100%)
Market Equil.	.4738 (68%)	.0033 (.4%)	.2128 (30%)	.6898 (100%)

TABLE II: The Division of Surplus

PRICES AS FRICTION VANISH

Proposition 4 below establishes that, as frictions vanish, transaction prices converge to the competitive equilibrium price that splits the gains between low-value buyers and sellers equally, i.e., the price $(u^L + c)/2$. For each $\delta \in (0,1]$ and integer T, denote by $r(\delta, T)$ the set of all sequences of equilibrium reservation prices, and by $V(\delta, T)$ the set of all sequences of equilibrium expected utilities. These sets are non-empty by Theorem 1.

Proposition 4: *Assume that $u^H > u^L > c$. Then for every $\bar{t}$ and $\tau \in \{H, L, S\}$*

$$\lim_{\delta \to 1} \lim_{T \to \infty} r_{\bar{t}}^{\tau}(\delta, T) = \lim_{T \to \infty} \lim_{\delta \to 1} r_{\bar{t}}^{\tau}(\delta, T) = \frac{u^L + c}{2};$$

i.e., transaction prices are competitive as frictions vanish. Furthermore,

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_{\bar{t}}^{\tau}(\delta, T) = \lim_{T \to \infty} \lim_{\delta \to 1} V_{\bar{t}}^{\tau}(\delta, T) = u^{\tau} - \frac{u^L + c}{2},$$

for each $\tau \in \{H, L\}$, and

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_{\bar{t}}^{S}(\delta, T) = \lim_{T \to \infty} \lim_{\delta \to 1} V_{\bar{t}}^{S}(\delta, T) = \frac{u^L + c}{2} - c;$$

i.e., as frictions vanish each agent obtains his competitive equilibrium utility.

DELAY

Although transaction prices converge to a competitive price as market frictions vanish, delay persists in the limit and, in this sense, the market outcome is not competitive. Consider a market in which $b_0^H > \pi^*$ and let (p, r, λ) be a market equilibrium. By Proposition 3, so long as the proportion of buyers in the market is above π^* then the trading pattern is either S or PS. Define the sequence $\{\underline{b}_t^H\}$ as $\underline{b}_0^H = b_0^H$ and, for $t \geq 0$

$$\underline{b}_{t+1}^H = \frac{(1-\alpha)\underline{b}_t^H}{(1-\alpha)\underline{b}_t^H + 1 - \underline{b}_t^H}.$$

The sequence $\{\underline{b}_t^H\}$ describes the evolution of the market composition as though the trading pattern is always S. Since the proportion of high-value buyers in the market falls more quickly in S than in the other trading patterns, then $b_t^H \geq \underline{b}_t^H$ for all $t \geq 0$. Therefore, if $\bar{t}$ is smallest integer such that $\underline{b}_{\bar{t}}^H \leq \pi^*$, then we have $b_t^H > \pi^*$ for $t < \bar{t}$; hence the trading pattern is either S or PS for periods 0 through $\bar{t}-1$. If the time horizon T is sufficiently long that P is eventually reached, then low-value buyers trade with delay: low-value buyers do not trade when responding prior to $\bar{t}$, but do trade if responding when P is reached. (If the market opens at the S trading pattern, then low-value buyers do not trade at all if matched initially, but always trade if matched once P is reached.) Since $\bar{t}$ is independent of the time horizon T and the discount factor, equilibrium delay persists for at least $\bar{t}$ periods even as frictions vanish.

Equilibrium delay can be made to persist arbitrary long, since $\bar{t}$ can be made arbitrarily large by choosing b_0^H near 1. Nonetheless, by Proposition 4 each trader receives his competitive equilibrium utility as market frictions vanish, and therefore delay becomes costless.

SYMMETRY

We conclude by discussing the asymmetries that may arise in equilibrium. By Proposition 2 all high-value buyers follow the same strategy in equilibrium. Also, the equilibrium strategies of sellers must be the same at every date, except at the PS-P mixed trading pattern (if it arises) where both the low-value and the high-value buyer reservation prices are offered by a positive proportion of sellers. Note, however, that reaching this mixed trading pattern requires that the proportion of high-value buyers exactly equal π^* at some date (see $P3.1.2$).

As for low-value buyers, they all offer the same price except in the S or S-PS trading pattern. In S, low-value buyers offer prices below the seller reservation price (which sellers reject). Hence low-value buyer price offers are not determined, and therefore there are asymmetric market equilibria in which low-value buyers make different (rejected) price offers. Nonetheless, if a market equilibrium exhibits only asymmetries of this kind, there is also a symmetric market equilibrium which generates the same trading pattern, market composition and transaction prices at each date. There is a more significant asymmetry when the S-PS mixed trading pattern arises. In this case, a positive proportion of low-value buyers offer the seller reservation price, and a positive proportion offer prices below the seller reservation price. As Example 3 shows, there are markets whose unique equilibrium exhibits the S-PS trading pattern. Hence existence of a symmetric market equilibrium cannot be guaranteed.

Example 3: Consider a market with parameter values as given in Example 2, except that the initial proportion of high-value buyers is now $b_0^H = .92$. Also let the market open only for two periods (i.e., $T = 1$). Note that whatever the trading pattern is at date 0, the proportion of high-value buyers at date 1 satisfies

$$b_1^H \geq \frac{(1-\alpha)b_0^H}{(1-\alpha)b_0^H + 1 - b_0^H} = .85185 > \pi^* = \frac{.4 - .2}{1 - .2} = .25.$$

Hence at date 1 sellers offer $p_1^S = r_1^H = u^H = 1$. Also as $r_1^L > r_1^S$ (because $r_1^L = u^L > c = r_1^S$), low-value buyers offer $p_1^L = r_1^S = c = .2$. Therefore, in a market equilibrium the trading pattern at date 1 is PS, and the traders' expected utilities are $V_1^H = \frac{\alpha}{2}(u^H - c) = \frac{1}{4}(1 - .2) = \frac{1}{5}$, $V_1^L = \frac{\alpha}{2}(u^L - c) = \frac{1}{4}(.4 - .2) = \frac{1}{20}$, and $V_1^S = \frac{\alpha}{2}b_1^H(u^H - c) = \frac{1}{4}b_1^H(1 - .2) = \frac{1}{5}b_1^H$, where b_1^H remains to be determined.

We must now determine the traders' strategies at date 0. By Proposition 2, three trading patterns are possible: S, S-PS, or PS. Suppose that the pattern of trade at date 0 is S. Then $b_1^H = .85185$, and $r_0^S = c + \delta V_1^S = .2 + .9(\frac{1}{5})(.85185) = .35333$. Since $r_0^L = u^L - \delta V_1^L = .4 - .9(\frac{1}{20}) = .355$, we have $r_0^S < r_0^L$. But then low-value buyers must offer the seller reservation price at date 0 (see Lemma 2.2), and therefore the pattern of trade is not S.

Suppose that the pattern of trade at date 0 is PS. Then

$$b_1^H = \frac{(1-\alpha)b_0^H}{(1-\alpha)b_0^H + (1 - \frac{\alpha}{2})(1 - b_0^H)} = .88462,$$

and $r_0^S = .2 + .9(\frac{1}{5})(.88462) = .35923 > .355 = r_0^L$. But then low-value buyers must offer a price below the seller reservation price at date 0 (see Lemma 2.3), and therefore the pattern of trade is not PS.

Since a market equilibrium exists, then the trading pattern at date 0 must be S-PS. Indeed, suppose that a proportion $\lambda^{L_1} = .29029$ of low-value buyers offer $p_0^{L_1} = r_0^S$, whereas a proportion $\lambda^{L_2} = 1 - \lambda^{L_1}$ of low-value buyers offer $p_0^{L_2} < r_0^S$. Then

$$b_1^H = \frac{(1-\alpha)b_0^H}{(1-\alpha)b_0^H + (1-\frac{\alpha}{2}\lambda^{L_1})(1-b_0^H)} = .86111,$$

and $r_0^S = .2 + .9(\frac{1}{5})(.86111) = .355 = r_0^L$; both $p_0^{L_1}$ and $p_0^{L_2}$ are optimal offers (i.e., low-value buyers are indifferent between trading or not trading at the sellers' reservation price). Hence, the strategy distribution described is a market equilibrium.

5 Concluding Remarks

Previous work studying the properties of decentralized markets in which traders are asymmetrically informed (e.g., Samuelson (1992), Serrano and Yosha (1996)) imposes ex-ante restrictions on transaction prices (forcing each transaction to be at one of at most three possible prices). Such restrictions seem unnatural in models whose aim is to develop a theory of price formation. Ex-ante price restrictions may artificially restrict the possibilities for trade: even if a buyer and a seller, when bargaining, have gains to trade relative to continuing to search, there may be not be a feasible price below the buyer's and above the seller's reservation prices. Price restrictions may also qualitatively affect the results (e.g., Serrano and Yosha (1996) find that when frictions are small equilibrium is efficient, while we find that the equilibrium may be inefficient). Ex-ante price restrictions also seem inconsistent with decentralized trading since an external authority must be relied upon to enforce them.

In our framework, transaction prices, the pattern of trade, and the distribution of the characteristics of the active traders are determined endogenously. Our results contribute to understanding how in markets these variables are interrelated and how they evolve dynamically over time. Our findings illustrate that markets may exhibit interesting dynamics, and these dynamics persist as frictions vanish even though transaction prices become competitive. The model we introduce is useful for investigating how the institutional setting (i.e., the bargaining rules) and the nature of uncertainty (i.e., whether it is one-sided or two-sided) influence market dynamics and the properties of market equilibria. These are important issues which we leave for future research.

6 Appendix A: Existence of Market Equilibria

We establish existence of a market equilibrium by means of a fixed point argument. Market outcomes are completely described by a triple (z, β, ρ) specifying, respectively, the trading pattern, market composition, and reservation prices at each date. As we establish in Appendix B, in equilibrium high-value buyers offer the seller reservation price ($L6.1$); low value buyers offer the seller reservation price or less ($L2$); and sellers offer either the high-value or low-value buyer reservation price ($L3.1$). Thus, we can simplify the representation of trading patterns by focusing on the proportion z_t^L of low-value buyers who offer the seller reservation price and the proportion z_t^S of sellers who offer the low-value buyer reservation price. (Then $1 - z_t^L$ is the proportion of low-value buyers offering a price below r_t^S and $1 - z_t^S$ is the proportion of sellers offering the price r_t^H.) The sequence of equilibrium trading patterns is represented as $z = (z_0, \ldots, z_T)$, where $z_t = (z_t^L, z_t^S) \in [0,1]^2$. The market composition at each date is described by $\beta = (\beta_0, \ldots, \beta_T)$, where $\beta_t \in [0,1]$ is the proportion of buyers in the market at date t who have a high value. Reservation prices at each date are given by $\rho = (\rho_0, \ldots, \rho_T)$, where $\rho_t = (\rho_t^H, \rho_t^L, \rho_t^S) \in [0,1]^3$.

The strategy of the proof of existence is as follows: we construct a mapping φ which for each *arbitrary* triple (z, β, ρ) provides the trading patterns that result when traders' price offers are optimal, and the market composition and reservation prices resulting from these new trading patterns. As we shall see the mapping φ is upper hemicontinuous and non-empty valued, but it may not be convex valued. Hence we cannot apply Kakutani's Fixed Point Theorem. Cellina (1969) has shown, however, that if for each (z,β,ρ), $\varphi(z,\beta,\rho)$ is the image of a convex set under a continuous function, then φ has a fixed point. Specifically, Cellina establishes the following theorem:[4]

Theorem *(Cellina 1969, Theorem 2). Let K be a non-empty compact convex subset of a Banach space. Let φ and γ be two upper hemicontinuous correspondences from K into K such that for each $x \in K$, $\varphi(x)$ is closed and $\gamma(x)$ is convex. Let f be a continuous function from the graph of γ into K such that for each $x \in K$, $\varphi(x) = \{f(x,y) \mid y \in \gamma(x)\}$. Then φ has a fixed point in K.*

With this result in hand we prove Theorem 1.

Proof of Theorem 1: Let $T < \infty$, $(u^H, u^L, c) \in [0,1]^3$ with $u^H > \max\{u^L, c\}$, $b_0^H \in (0,1)$, $\delta \in (0,1]$, and $\alpha \in (0,1)$. Write $\underline{\beta} = (1-\alpha)^T b_0^H$, and denote by K the set of triples (z,β,ρ) such that $z \in [0,1]^{2(T+1)}$, $\beta \in [\underline{\beta}, b_0^H]^{T+1}$, and $\rho \in [0,1]^{3(T+1)}$ satisfies $\rho_T = (\rho_T^H, \rho_T^L, \rho_T^S) = (u^H, u^L, c)$, and for $t < T$, $\rho_t^H - \rho_t^L \geq 0$, and $\rho_t^H - \rho_t^S \geq (1 - \alpha\delta\frac{1-\delta^{T-t}(1-\alpha)^{T-t}}{1-\delta(1-\alpha)})(u^H - c)$. Note that $K \subset [0,1]^{6(T+1)}$ is a non-empty compact convex set.

The mapping $\varphi : K \twoheadrightarrow K$ is constructed as follows: Let $\gamma : K \twoheadrightarrow K$ be given for $(z,\beta,\rho) \in K$ by $\gamma(z,\beta,\rho) = (\gamma^z(z,\beta,\rho), \beta, \rho)$, where

$$\gamma_t^z(z,\beta,\rho) = \begin{cases} \{(1,1)\} & \text{if } \beta_t(\rho_t^H - \rho_t^S) < \rho_t^L - \rho_t^S \\ \{1\} \times [0,1] & \text{if } \beta_t(\rho_t^H - \rho_t^S) = \rho_t^L - \rho_t^S \\ \{(1,0)\} & \text{if } \beta_t(\rho_t^H - \rho_t^S) > \rho_t^L - \rho_t^S > 0 \\ [0,1] \times \{0\} & \text{if } \rho_t^L - \rho_t^S = 0 \\ \{(0,0)\} & \text{if } \rho_t^L - \rho_t^S < 0, \end{cases}$$

for each $t \in \{0,\ldots,T\}$. Note that $\beta_t(\rho_t^H - \rho_t^S) > 0$ for $t \in \{0,\ldots,T\}$, whenever $(z,\beta,\rho) \in K$. Hence γ is well defined. Also note that γ is an upper hemicontinuous non-empty compact convex valued correspondence.

Now let D be the graph of γ (i.e., the set $\{(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) \mid (\bar{z},\bar{\beta},\bar{\rho}) \in \gamma(z,\beta,\rho)\}$), and let $f : D \to K$ be given for $(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) \in D$ by $f(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) = (\bar{z}, g(\bar{z}), h(\bar{z},\bar{\beta},\bar{\rho}))$,

[4]See also Border (1985), Theorem 15.1, page 72.

where g is defined as $g_0(\bar{z}) = b_0^H$, and for $t > 0$

$$g_t(\bar{z}) = \frac{(1-\alpha)g_{t-1}(\bar{z})}{(1-\alpha)g_{t-1}(\bar{z}) + (1-\frac{\alpha}{2}(\bar{z}_{t-1}^L + \bar{z}_{t-1}^S))(1-g_{t-1}(\bar{z}))},$$

and h is defined as $h_T(\bar{z},\bar{\beta},\bar{\rho}) = (u^H, u^L, c)$ and for $t < T$

$$h_t(\bar{z},\bar{\beta},\bar{\rho}) = \begin{bmatrix} (1-\delta)u^H + \delta(\frac{\alpha}{2}\bar{\rho}_{t+1}^S + \frac{\alpha}{2}\bar{z}_{t+1}^S\bar{\rho}_{t+1}^L + (1-\frac{\alpha}{2}(1+\bar{z}_{t+1}^S))\bar{\rho}_{t+1}^H) \\ (1-\delta)u^L + \delta(\frac{\alpha}{2}\bar{z}_{t+1}^L\bar{\rho}_{t+1}^S + (1-\frac{\alpha}{2}\bar{z}_{t+1}^L)\bar{\rho}_{t+1}^L) \\ (1-\delta)c + \delta(\frac{\alpha}{2}(\bar{z}_{t+1}^S(\bar{\rho}_{t+1}^L - \bar{\rho}_{t+1}^S) + (1-\bar{z}_{t+1}^S)\bar{\beta}_{t+1}(\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^S)) + \bar{\rho}_{t+1}^S) \end{bmatrix}.$$

The function g gives the market composition that results when the sequence of trading patterns is given by $\bar{z}$ and the initial proportion of high-value buyers is b_0^H. The function h gives the reservation prices that result when the sequence of trading patterns, market compositions, and reservation prices is given by $\bar{z}$, $\bar{\beta}$, and $\bar{\rho}$, respectively. We show that f is well defined, i.e., that for each $(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) \in D$, $f(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) \in K$.

Let $(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) \in D$. We show that $f(z,\beta,\rho;\bar{z},\bar{\beta},\bar{\rho}) = (\hat{z},\hat{\beta},\hat{\rho}) \in K$. Clearly $\hat{z} = \bar{z} \in \gamma^z(z,\beta,\rho) \subset [0,1]^{2(T+1)}$. We prove by induction that

$$(1-\alpha)^t b_0^H \leq \hat{\beta}_t \leq b_0^H,$$

for $t \in \{0,\dots,T\}$, therefore establishing that $\hat{\beta} \in [\underline{\beta}, b_0^H]^{T+1}$. (Note that for $t \leq T$, $(1-\alpha)^t b_0^H \geq (1-\alpha)^T b_0^H = \underline{\beta}$.) Since $\hat{\beta}_0 = b_0^H$, assume that the claim holds at $\bar{t} \geq 0$. We show that it holds at $\bar{t}+1$. By the definition of g we have

$$\hat{\beta}_{\bar{t}+1} = \frac{(1-\alpha)\hat{\beta}_{\bar{t}}}{(1-\alpha)\hat{\beta}_{\bar{t}} + (1-\frac{\alpha}{2}(\bar{z}_{\bar{t}}^L + \bar{z}_{\bar{t}}^S))(1-\hat{\beta}_{\bar{t}})}.$$

Since $\hat{\beta}_{\bar{t}+1}$ is increasing in both $\bar{z}_{\bar{t}}^L$ and $\bar{z}_{\bar{t}}^S$, and $\bar{z}_{\bar{t}}^L \leq 1$ and $\bar{z}_{\bar{t}}^S \leq 1$, we have

$$\hat{\beta}_{\bar{t}+1} \leq \frac{(1-\alpha)\hat{\beta}_{\bar{t}}}{(1-\alpha)\hat{\beta}_{\bar{t}} + (1-\alpha)(1-\hat{\beta}_{\bar{t}})} = \hat{\beta}_{\bar{t}} \leq b_0^H.$$

Also since $\bar{z}_{\bar{t}}^L \geq 0$ and $\bar{z}_{\bar{t}}^S \geq 0$, and $0 < \alpha\hat{\beta}_{\bar{t}} < 1$, we have

$$\hat{\beta}_{\bar{t}+1} \geq \frac{(1-\alpha)\hat{\beta}_{\bar{t}}}{(1-\alpha)\hat{\beta}_{\bar{t}} + (1-\hat{\beta}_{\bar{t}})} = \frac{(1-\alpha)\hat{\beta}_{\bar{t}}}{1-\alpha\hat{\beta}_{\bar{t}}} \geq (1-\alpha)\hat{\beta}_{\bar{t}} \geq (1-\alpha)^{\bar{t}+1} b_0^H.$$

Finally, we show that $\hat{\rho} \in [0,1]^{3(T+1)}$, and satisfies $\hat{\rho}_T = (u^H, u^L, c)$, and for $t < T$, $\rho_t^H - \rho_t^L \geq 0$, and $\rho_t^H - \rho_t^S \geq (1 - \alpha\delta\frac{1-\delta^{T-t}(1-\alpha)^{T-t}}{1-\delta(1-\alpha)})(u^H - c)$. By the definition of h,

$\hat{\rho}_T = (u^H, u^L, c)$. We show that $0 \leq \hat{\rho}_t^\tau \leq 1$ for $t \in \{0, \ldots, T\}$ and $\tau \in \{L, H, S\}$, and therefore that $\hat{\rho} \in [0,1]^{3(T+1)}$. Since $(\bar{z}, \bar{\beta}, \bar{\rho}) \in K$, then $\bar{\rho}_{t+1}^S \leq 1$, $\bar{\rho}_{t+1}^L \leq 1$, and $\bar{\rho}_{t+1}^H \leq 1$, and since in the expressions for $\hat{\rho}_t^H$ and $\hat{\rho}_t^L$ the coefficients on $\bar{\rho}_{t+1}^S$, $\bar{\rho}_{t+1}^L$, and $\bar{\rho}_{t+1}^H$ sum to one, we have

$$\hat{\rho}_t^\tau \leq (1-\delta)u^\tau + \delta \leq 1,$$

for $\tau \in \{H, L\}$. Rewriting the expression for $\hat{\rho}_t^S$ as

$$\hat{\rho}_t^S = (1-\delta)c + \delta(\frac{\alpha}{2}[\bar{z}_{t+1}^S \bar{\rho}_{t+1}^L + (1-\bar{z}_{t+1}^S)((1-\bar{\beta}_{t+1}^H)\bar{\rho}_{t+1}^S + \bar{\beta}_{t+1}^H \bar{\rho}_{t+1}^H)] + (1-\frac{\alpha}{2})\bar{\rho}_{t+1}^S),$$

the same argument yields

$$\hat{\rho}_t^S \leq (1-\delta)c + \delta \leq 1.$$

Also $(\bar{z}, \bar{\beta}, \bar{\rho}) \in K$ implies that $\bar{\rho}_{t+1}^S \geq 0$, $\bar{\rho}_{t+1}^L \geq 0$, and $\bar{\rho}_{t+1}^H \geq 0$, and since the coefficients of these terms in the expressions for $\hat{\rho}_t^H$, $\hat{\rho}_t^L$, and $\hat{\rho}_t^S$ are nonnegative, we have $\hat{\rho}_t^\tau \geq 0$ for $\tau \in \{H, L, S\}$.

Now for $t < T$, we have

$$\hat{\rho}_t^H - \hat{\rho}_t^L = (1-\delta)(u^H - u^L) + \delta[\frac{\alpha}{2}(1-\bar{z}_{t+1}^L)(\bar{\rho}_{t+1}^S - \bar{\rho}_{t+1}^L) + (1-\frac{\alpha}{2}(1+\bar{z}_{t+1}^S))(\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^L)].$$

Since $(1-\delta)(u^H - u^L) \geq 0$ and $(1-\bar{z}_{t+1}^L)(\bar{\rho}_{t+1}^S - \bar{\rho}_{t+1}^L) \geq 0$ (because $\bar{z}_{t+1}^L < 1$ implies, by the definition of γ, that $\bar{\rho}_{t+1}^L \leq \bar{\rho}_{t+1}^S$), and $\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^L \geq 0$ (because $(\bar{z}, \bar{\beta}, \bar{\rho}) \in K$), we have $\hat{\rho}_t^H - \hat{\rho}_t^L \geq 0$.

Also for $t < T$, we have

$$\begin{aligned}
\hat{\rho}_t^H - \hat{\rho}_t^S &= (1-\delta)(u^H - c) + \delta[1 - \frac{\alpha}{2}(1+\bar{z}_{t+1}^S) - \frac{\alpha}{2}(1-\bar{z}_{t+1}^S)\bar{\beta}_{t+1})](\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^S) \\
&\geq (1-\delta)(u^H - c) + \delta(1-\alpha)(\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^S).
\end{aligned}$$

Since $(\bar{z}, \bar{\beta}, \bar{\rho}) \in K$, we have

$$\bar{\rho}_{t+1}^H - \bar{\rho}_{t+1}^S \geq (1 - \alpha\delta\frac{1-\delta^{T-t-1}(1-\alpha)^{T-t-1}}{1-\delta(1-\alpha)})(u^H - c),$$

and therefore

$$\begin{aligned}
\hat{\rho}_t^H - \hat{\rho}_t^S &\geq ((1-\delta) + \delta(1-\alpha)(1 - \alpha\delta\frac{1-\delta^{T-t-1}(1-\alpha)^{T-t-1}}{1-\delta(1-\alpha)}))(u^H - c) \\
&= (1 - \alpha\delta\frac{1-\delta^{T-t}(1-\alpha)^{T-t}}{1-\delta(1-\alpha)})(u^H - c).
\end{aligned}$$

Hence f is well defined, and since both g and h are continuous functions, f is a continuous function. Now let φ be given for $(z, \beta, \rho) \in K$ by

$$\varphi(z, \beta, \rho) = \{f(z, \beta, \rho; \bar{z}, \bar{\beta}, \bar{\rho}) \mid (\bar{z}, \bar{\beta}, \bar{\rho}) \in \gamma(z, \beta, \rho)\}.$$

Clearly φ is an upper hemicontinuous closed valued correspondence. Cellina's Theorem therefore implies that φ has a fixed point.

Let (z, β, ρ) be a fixed point of φ. We construct a market equilibrium (p, r, λ) as follows: We use binary strings $m = (m_0, \ldots, m_T) \in \{0,1\}^{T+1}$ to index low-value buyers and sellers strategies. For $\tau \in \{S, L\}$ and $m = (m_0, \ldots, m_T) \in \{0,1\}^{T+1}$ define

$$\lambda^{\tau m} = \prod_{t=0}^{T} (z_t^\tau)^{m_t} (1 - z_t^\tau)^{1-m_t},$$

and let $M^\tau = \{m \in \{0,1\}^{T+1} \mid \lambda^{\tau m} > 0\}$. Note that $\sum_{m \in M^\tau} \lambda^{\tau m} = 1$.

HIGH-VALUE BUYERS: All high-value buyers follow the same strategy, given by $r_t^H = \rho_t^H$ and $p_t^H = \rho_t^S$ for $t \in \{0, \ldots, T\}$. Hence $\lambda^H = 1$.

LOW-VALUE BUYERS: Let $x = (x_0, \ldots, x_T)$ be an arbitrary vector of real numbers such that $x_t < \rho_t^S$ for $t \in \{0, \ldots, T\}$. For $m \in M^L$ define the low-value buyer strategy (p^{Lm}, r^{Lm}) as $r_t^{Lm} = \rho_t^L$ and

$$p_t^{Lm} = \begin{cases} \rho_t^S & \text{if } m_t = 1 \\ x_t & \text{otherwise,} \end{cases}$$

for $t \in \{0, \ldots, T\}$.

SELLERS: For $m \in M^S$ define the seller strategy (p^{Sm}, r^{Sm}) as $r_t^{Sm} = \rho_t^S$, and

$$p_t^{Sm} = \begin{cases} \rho_t^L & \text{if } m_t = 1 \\ \rho_t^H & \text{otherwise,} \end{cases}$$

for $t \in \{0, \ldots, T\}$.

For $t \in \{0, \ldots, T\}$ and $\tau \in \{L, S\}$, define $M_t^\tau = \{m \in M^\tau \mid m_t = 1\}$, i.e., M_t^L (M_t^S) contains the indexes corresponding to low-value buyer (seller) strategies which offer sellers (low-value buyers) their reservation price at date t. Straightforward calculations (which we omit) show that under the laws of motion given in Section 2.1, the strategy distribution (p, r, λ) defined above satisfies

$$\sum_{m \in M_t^\tau} \lambda_t^{\tau m} = z_t^\tau,$$

for each $t \in \{0, \ldots, T\}$ and $\tau \in \{L, S\}$. In other words, at each date t the proportion of low-value buyers (sellers) in the market who offer sellers (low-value buyers) their reservation price is z_t^L (z_t^S).

We prove that (p, r, λ) is a market equilibrium. Given (p, r, λ), let b_t^H be computed according to the laws of motion developed in Section 2.1. We show by induction that $b_t^H = \beta_t$ for $t \in \{0, \ldots, T\}$. Clearly $b_0^H = g_0(z) = \beta_0$. Assume that $b_{\bar{t}}^H = g_{\bar{t}}(z) = \beta_{\bar{t}}$ for $\bar{t} \geq 0$; we show that $b_{\bar{t}+1}^H = \beta_{\bar{t}+1}$. By definition we have

$$b_{\bar{t}+1}^H = \frac{b_{\bar{t}}^H \mu_{\bar{t}}^H}{b_{\bar{t}}^H \mu_{\bar{t}}^H + b_{\bar{t}}^L \sum_{m \in M^L} \lambda_{\bar{t}}^{Lm} \mu_{\bar{t}}^{Lm}}.$$

In this expression, $\mu_{\bar{t}}^H$ is given by

$$\mu_{\bar{t}}^H = 1 - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} I(p_{\bar{t}}^H, r_{\bar{t}}^{S_{m'}}) - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} I(r_{\bar{t}}^H, p_{\bar{t}}^{S_{m'}}).$$

For each $m' \in M^S$ we have $p_{\bar{t}}^H = \rho_{\bar{t}}^S = r_{\bar{t}}^{S_{m'}}$, and $p_{\bar{t}}^{S_{m'}} \leq r_{\bar{t}}^H$ (because $p_{\bar{t}}^{S_{m'}} \in \{\rho_{\bar{t}}^L, \rho_{\bar{t}}^H\}$, and $(z, \beta, \rho) \in K$ implies $\rho_{\bar{t}}^L \leq \rho_{\bar{t}}^H = r_{\bar{t}}^H$). Thus, $I(p_{\bar{t}}^H, r_{\bar{t}}^{S_{m'}}) = I(r_{\bar{t}}^H, p_{\bar{t}}^{S_{m'}}) = 1$ for each $m' \in M^S$, and hence

$$\mu_{\bar{t}}^H = 1 - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} = 1 - \alpha.$$

Also for $\mu_{\bar{t}}^{Lm}$ we have

$$\mu_{\bar{t}}^{Lm} = 1 - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} I(p_{\bar{t}}^{Lm}, r_{\bar{t}}^{S_{m'}}) - \frac{\alpha}{2} \sum_{m' \in M^S} \lambda_{\bar{t}}^{S_{m'}} I(r_{\bar{t}}^{Lm}, p_{\bar{t}}^{S_{m'}}).$$

Note that for each $m' \in M^S$, $m_{\bar{t}} = 1$ implies $I(p_{\bar{t}}^{Lm}, r_{\bar{t}}^{S_{m'}}) = 1$, and $m_{\bar{t}} = 0$ implies $I(p_{\bar{t}}^{Lm}, r_{\bar{t}}^{S_{m'}}) = 0$; also $I(r_{\bar{t}}^{Lm}, p_{\bar{t}}^{S_{m'}}) = 1$ whenever $m' \in M_{\bar{t}}^S$, and $I(r_{\bar{t}}^{Lm}, p_{\bar{t}}^{S_{m'}}) = 0$ whenever $m' \notin M_{\bar{t}}^S$. Therefore we have

$$\mu_{\bar{t}}^{Lm} = 1 - \frac{\alpha}{2} m_{\bar{t}} - \frac{\alpha}{2} \sum_{m' \in M_{\bar{t}}^S} \lambda_{\bar{t}}^{S_{m'}}.$$

Substituting in the expression for $b^H_{\bar{t}+1}$, and noticing that $b^H_{\bar{t}} = \beta_{\bar{t}}$ by the induction hypothesis, we get

$$b^H_{\bar{t}+1} = \frac{\beta_{\bar{t}}(1-\alpha)}{\beta_{\bar{t}}(1-\alpha) + (1-\beta_{\bar{t}})\sum_{m\in M^L} \lambda^{Lm}_{\bar{t}}(1 - \frac{\alpha}{2}m_{\bar{t}} - \frac{\alpha}{2}\sum_{m'\in M^S_{\bar{t}}} \lambda^{Sm'}_{\bar{t}})}.$$

Since $m_{\bar{t}} = 1$ if $m \in M^L_{\bar{t}}$ and $m_{\bar{t}} = 0$ if $m \notin M^L_{\bar{t}}$, we have

$$\sum_{m\in M^L} \lambda^{Lm}_{\bar{t}} m_{\bar{t}} = \sum_{m\in M^L_{\bar{t}}} \lambda^{Lm}_{\bar{t}}.$$

Also as noted earlier $\sum_{m\in M^\tau_{\bar{t}}} \lambda^{\tau m}_{\bar{t}} = z^\tau_{\bar{t}}$ for each $\tau \in \{L, S\}$. Hence, the expression for $b^H_{\bar{t}+1}$ simplifies to

$$b^H_{\bar{t}+1} = \frac{\beta_{\bar{t}}(1-\alpha)}{\beta_{\bar{t}}(1-\alpha) + (1-\beta_{\bar{t}})(1 - \frac{\alpha}{2}z^L_{\bar{t}} - \frac{\alpha}{2}z^S_{\bar{t}})}.$$

Therefore $b^H_{\bar{t}+1} = g_{\bar{t}+1}(z) = \beta_{\bar{t}+1}$.

Now we establish by induction that (p, r, λ) satisfies Condition $E1$ for $t \in \{0, \ldots, T\}$. Since $V^\tau_{T+1} = 0$ for $\tau \in \{H, L, S\}$, the definition of h yields

$$u^H - r^H_T = u^H - \rho^H_T = 0 = \delta V^H_{T+1},$$

and

$$u^L - r^{Lm}_T = u^L - \rho^L_T = 0 = \delta V^L_{T+1},$$

for $m \in M^L$. For $m \in M^S$

$$r^{Sm}_T - c = \rho^S_T - c = 0 = \delta V^S_{T+1}.$$

Thus, $E1$ holds at T.

Suppose that $E1$ is satisfied at $\bar{t} + 1 \le T$; we show that it is satisfied at $\bar{t}$, i.e., $u^H - r^H_{\bar{t}} = \delta V^H_{\bar{t}+1}$. For high-value buyers we have

$$P^H_{\bar{t}+1} = (u^H - p^H_{\bar{t}+1}) \sum_{m\in M^S} \lambda^{Sm}_{\bar{t}+1} I(p^H_{\bar{t}+1}, r^{Sm}_{\bar{t}+1}) + (1 - \sum_{m\in M^S} \lambda^{Sm}_{\bar{t}+1} I(p^H_{\bar{t}+1}, r^{Sm}_{\bar{t}+1}))\delta V^H_{\bar{t}+2}.$$

Since $p^H_{\bar{t}+1} = \rho^S_{\bar{t}+1} = r^{Sm}_{\bar{t}+1}$ for each $m \in M^S$, $I(p^H_{\bar{t}+1}, r^{Sm}_{\bar{t}+1}) = 1$ for each $m \in M^S$. Therefore

$$P^H_{\bar{t}+1} = u^H - \rho^S_{\bar{t}+1}.$$

Also

$$R^H_{\bar{t}+1} = \sum_{m\in M^S} (u^H - p^{S_m}_{\bar{t}+1})\lambda^{S_m}_{\bar{t}+1} I(r^H_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}) + (1 - \sum_{m\in M^S} \lambda^{S_m}_{\bar{t}+1} I(r^H_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}))\delta V^H_{\bar{t}+2}.$$

Since $p^{S_m}_{\bar{t}+1} = \rho^L_{\bar{t}+1} \le \rho^H_{\bar{t}+1} = r^H_{\bar{t}+1}$ when $m_{\bar{t}+1} = 1$, and $p^{S_m}_{\bar{t}+1} = \rho^H_{\bar{t}+1} = r^H_{\bar{t}+1}$ when $m_{\bar{t}+1} = 0$, we have $I(r^H_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}) = 1$ for each $m \in M^S$. Thus

$$\begin{aligned} R^H_{\bar{t}+1} &= \sum_{m\in M^S_{\bar{t}+1}} (u^H - \rho^L_{\bar{t}+1})\lambda^{S_m}_{\bar{t}+1} + \sum_{m\in M^S\backslash M^S_{\bar{t}+1}} (u^H - \rho^H_{\bar{t}+1})\lambda^{S_m}_{\bar{t}+1} \\ &= u^H - (z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - z^S_{\bar{t}+1})\rho^H_{\bar{t}+1}). \end{aligned}$$

>From the definition of $V^H_{\bar{t}+1}$ in Section 2.2, and as $\delta V^H_{\bar{t}+2} = u^H - \rho^H_{\bar{t}+1}$ by the induction hypothesis, we have

$$\begin{aligned} \delta V^H_{\bar{t}+1} &= \delta[\frac{\alpha}{2}(P^H_{\bar{t}+1} + R^H_{\bar{t}+1}) + (1-\alpha)\delta V^H_{\bar{t}+2}] \\ &= \delta[\frac{\alpha}{2}(u^H - \rho^S_{\bar{t}+1} + u^H - (z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - z^S_{\bar{t}+1})\rho^H_{\bar{t}+1})) + (1-\alpha)(u^H - \rho^H_{\bar{t}+1})] \\ &= \delta u^H - \delta[\frac{\alpha}{2}\rho^S_{\bar{t}+1} + \frac{\alpha}{2} z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - \frac{\alpha}{2}(1 + z^S_{\bar{t}+1}))\rho^H_{\bar{t}+1}] \\ &= u^H - \rho^H_{\bar{t}}, \end{aligned}$$

where the last equality holds by the definition of h. Therefore $E1$ holds at $\bar{t}$ for $\tau = H$.

We now establish that E1 holds at $\bar{t}$ for each low-value buyer strategy $m \in M^L$, i.e., $u^L - r^{L_m}_{\bar{t}} = \delta V^{L_m}_{\bar{t}+1}$. For $m \in M^L$, we have

$$P^{L_m}_{\bar{t}+1} = (u^L - p^{L_m}_{\bar{t}+1}) \sum_{m'\in M^S} \lambda^{S_{m'}}_{\bar{t}+1} I(p^{L_m}_{\bar{t}+1}, r^{S_{m'}}_{\bar{t}+1}) + (1 - \sum_{m'\in M^S} \lambda^{S_{m'}}_{\bar{t}+1} I(p^{L_m}_{\bar{t}+1}, r^{S_{m'}}_{\bar{t}+1}))\delta V^{L_m}_{\bar{t}+2}.$$

Since $p^{L_m}_{\bar{t}+1} = \rho^S_{\bar{t}+1} = r^{S_{m'}}_{\bar{t}+1}$ for each $m' \in M^S$ if $m_{\bar{t}+1} = 1$, and $p^{L_m}_{\bar{t}+1} = x_{\bar{t}+1} < \rho^S_{\bar{t}+1} = r^{S_{m'}}_{\bar{t}+1}$ if $m_{\bar{t}+1} = 0$, and since $\delta V^{L_m}_{\bar{t}+2} = u^L - \rho^L_{\bar{t}+1}$ by the induction hypothesis, we have

$$P^{L_m}_{\bar{t}+1} = \begin{cases} u^L - \rho^S_{\bar{t}+1} & \text{if } m_{\bar{t}+1} = 1 \\ u^L - \rho^L_{\bar{t}+1} & \text{if } m_{\bar{t}+1} = 0. \end{cases}$$

We show that $P^{L_m}_{\bar{t}+1}$ can be written as

$$P^{L_m}_{\bar{t}+1} = u^L - [z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} + (1 - z^L_{\bar{t}+1})\rho^L_{\bar{t}+1}].$$

If $z^L_{\bar{t}+1} = 0$, then $m_{\bar{t}+1} = 0$, and therefore

$$P^{L_m}_{\bar{t}+1} = u^L - \rho^L_{\bar{t}+1} = u^L - [z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} + (1 - z^L_{\bar{t}+1})\rho^L_{\bar{t}+1}].$$

If $z^L_{\bar{t}+1} = 1$, then $m_{\bar{t}+1} = 1$ and therefore

$$P^{L_m}_{\bar{t}+1} = u^L - \rho^S_{\bar{t}+1} = u^L - [z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} + (1 - z^L_{\bar{t}+1})\rho^L_{\bar{t}+1}].$$

If $z^L_{\bar{t}+1} \in (0,1)$, then $\rho^S_{\bar{t}+1} = \rho^L_{\bar{t}+1}$ by the definition of γ, and hence

$$P^{L_m}_{\bar{t}+1} = u^L - [z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} + (1 - z^L_{\bar{t}+1})\rho^L_{\bar{t}+1}]$$

whether $m_{\bar{t}+1} = 1$ or $m_{\bar{t}+1} = 0$.

For $m \in M^L$, we have

$$\begin{aligned}
R^{L_m}_{\bar{t}+1} &= \sum_{m' \in M^S} (u^L - p^{S_{m'}}_{\bar{t}+1})\lambda^{S_{m'}}_{\bar{t}+1} I(r^{L_m}_{\bar{t}+1}, p^{S_{m'}}_{\bar{t}+1}) + (1 - \sum_{m' \in M^S} \lambda^{S_{m'}}_{\bar{t}+1} I(r^{L_m}_{\bar{t}+1}, p^{S_{m'}}_{\bar{t}+1}))\delta V^{L_m}_{\bar{t}+2} \\
&= \sum_{m' \in M^S_{\bar{t}}} \lambda^{S_{m'}}_{\bar{t}+1}(u^L - \rho^L_{\bar{t}+1}) + (1 - \sum_{m' \in M^S_{\bar{t}}} \lambda^{S_{m'}}_{\bar{t}+1})\delta V^{L_m}_{\bar{t}+2} \\
&= u^L - \rho^L_{\bar{t}+1},
\end{aligned}$$

where the last equality holds since, by the induction hypothesis, $\delta V^{L_m}_{\bar{t}+2} = u^L - \rho^L_{\bar{t}+1}$.

Summing up, we have

$$\begin{aligned}
\delta V^{L_m}_{\bar{t}+1} &= \delta[\frac{\alpha}{2}(P^{L_m}_{\bar{t}+1} + R^{L_m}_{\bar{t}+1}) + (1-\alpha)\delta V^{L_m}_{\bar{t}+2}] \\
&= \delta[\frac{\alpha}{2}(u^L - z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} - (1 - z^L_{\bar{t}+1})\rho^L_{\bar{t}+1} + u^L - \rho^L_{\bar{t}+1}) + (1-\alpha)(u^L - \rho^L_{\bar{t}+1})] \\
&= \delta u^L - \delta[\frac{\alpha}{2} z^L_{\bar{t}+1}\rho^S_{\bar{t}+1} + (1 - \frac{\alpha}{2} z^L_{\bar{t}+1})\rho^L_{\bar{t}+1}] \\
&= u^L - \rho^L_{\bar{t}},
\end{aligned}$$

where the last equality follows from the definition of h. Since $m \in M^L$ was arbitrary, therefore $E1$ holds at $\bar{t}$ for all $m \in M^L$.

We show that E1 holds at $\bar{t}$ for each seller strategy $m \in M^S$, i.e., $c - r^{S_m}_{\bar{t}} = \delta V^{S_m}_{\bar{t}+1}$. For $m \in M^S$, we have

$$\begin{aligned}
P^{S_m}_{\bar{t}+1} &= (p^{S_m}_{\bar{t}+1} - c)[b^H_{\bar{t}+1} I(r^H_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}) + (1 - b^H_{\bar{t}+1}) \sum_{m' \in M^L} \lambda^{L_{m'}}_{\bar{t}+1} I(r^{L_{m'}}_{\bar{t}+1}, p^{S_m}_{\bar{t}+1})] \\
&\quad + (1 - b^H_{\bar{t}+1} I(r^H_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}) - (1 - b^H_{\bar{t}+1}) \sum_{m' \in M^L} \lambda^{L_{m'}}_{\bar{t}+1} I(r^{L_{m'}}_{\bar{t}+1}, p^{S_m}_{\bar{t}+1}))\delta V^{S_m}_{\bar{t}+2}.
\end{aligned}$$

Since $p^{S_m}_{\bar{t}+1} = \rho^L_{\bar{t}+1} \leq \rho^H_{\bar{t}+1} = r^H_{\bar{t}+1}$ if $m_{\bar{t}+1} = 1$, and $p^{S_m}_{\bar{t}+1} = \rho^H_{\bar{t}+1} = r^H_{\bar{t}+1}$ if $m_{\bar{t}+1} = 0$, and since $\delta V^{S_m}_{\bar{t}+2} = \rho^S_{\bar{t}+1} - c$ by the induction hypothesis, we have

$$P^{S_m}_{\bar{t}+1} = \begin{cases} \rho^L_{\bar{t}+1} - c & \text{if } m_{\bar{t}+1} = 1 \\ \beta_{\bar{t}+1}(\rho^H_{\bar{t}+1} - c) + (1 - \beta_{\bar{t}+1})(\rho^S_{\bar{t}+1} - c) & \text{if } m_{\bar{t}+1} = 0, \end{cases}$$

where we have replaced $b^H_{\bar{t}+1}$ with $\beta_{\bar{t}+1}$. We show that $P^{S_m}_{\bar{t}+1}$ can be written as

$$P^{S_m}_{\bar{t}+1} = z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - z^S_{\bar{t}+1})(\beta_{\bar{t}+1}\rho^H_{\bar{t}+1} + (1 - \beta_{\bar{t}+1})\rho^S_{\bar{t}+1}) - c.$$

This clearly holds if either $z^S_{\bar{t}+1} = 0$ (and hence $m_{\bar{t}+1} = 0$) or $z^S_{\bar{t}+1} = 1$ (and hence $m_{\bar{t}+1} = 1$). If $z^S_{\bar{t}+1} \in (0, 1)$ then $\beta_{\bar{t}+1}(\rho^H_{\bar{t}+1} - \rho^S_{\bar{t}+1}) = \rho^L_{\bar{t}+1} - \rho^S_{\bar{t}+1}$ by the definition of γ, which is the same as $\rho^L_{\bar{t}+1} = \beta_{\bar{t}+1}\rho^H_{\bar{t}+1} + (1 - \beta_{\bar{t}+1})\rho^S_{\bar{t}+1}$. Hence

$$\begin{aligned} P^{S_m}_{\bar{t}+1} &= \rho^L_{\bar{t}+1} - c = \beta_{\bar{t}+1}(\rho^H_{\bar{t}+1} - c) + (1 - \beta_{\bar{t}+1})(\rho^S_{\bar{t}+1} - c) \\ &= z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - z^S_{\bar{t}+1})(\beta_{\bar{t}+1}\rho^H_{\bar{t}+1} + (1 - \beta_{\bar{t}+1})\rho^S_{\bar{t}+1}) - c. \end{aligned}$$

For $m \in M^S$ we have

$$\begin{aligned} R^{S_m}_{\bar{t}+1} &= b^H_{\bar{t}+1}(p^H_{\bar{t}+1} - c)I(p^H_{\bar{t}+1}, r^{S_m}_{\bar{t}+1}) + (1 - b^H_{\bar{t}+1}) \sum_{m' \in M^L} \lambda^{L_{m'}}_{\bar{t}+1}(p^{L_{m'}}_{\bar{t}+1} - c) \\ &\quad + (1 - b^H_{\bar{t}+1}I(p^H_{\bar{t}+1}, r^{S_m}_{\bar{t}+1}) - (1 - b^H_{\bar{t}+1}) \sum_{m' \in M^L} \lambda^{L_{m'}}_{\bar{t}+1} I(p^{L_{m'}}_{\bar{t}+1}, r^{S_m}_{\bar{t}+1}))\delta V^{S_m}_{\bar{t}+2}. \end{aligned}$$

Using that $p^H_{\bar{t}+1} = \rho^S_{\bar{t}+1} = r^{S_m}_{\bar{t}+1}$, $p^{L_{m'}}_{\bar{t}+1} = \rho^S_{\bar{t}+1} = r^{S_m}_{\bar{t}+1}$ if $m'_{\bar{t}+1} = 1$, $p^{L_{m'}}_{\bar{t}+1} = x_{\bar{t}+1} < \rho^S_{\bar{t}+1} = r^{S_m}_{\bar{t}+1}$ if $m'_{\bar{t}+1} = 0$, and $\delta V^{S_m}_{\bar{t}+2} = \rho^S_{\bar{t}+1} - c$ by the induction hypothesis we have

$$\begin{aligned} R^{S_m}_{\bar{t}+1} &= \beta_{\bar{t}+1}(\rho^S_{\bar{t}+1} - c) + (1 - \beta_{\bar{t}+1}) \sum_{m' \in M^L_{\bar{t}+1}} \lambda^{L_{m'}}_{\bar{t}+1}(\rho^S_{\bar{t}+1} - c) \\ &\quad + (1 - \beta_{\bar{t}+1} - (1 - \beta_{\bar{t}+1}) \sum_{m' \in M^L_{\bar{t}+1}} \lambda^{L_{m'}}_{\bar{t}+1})(\rho^S_{\bar{t}+1} - c). \end{aligned}$$

Hence $R^{S_m}_{\bar{t}+1} = \rho^S_{\bar{t}+1} - c$.

Substituting these formulas into the expression for $V^{S_m}_{\bar{t}+1}$ and using that $\delta V^{S_m}_{\bar{t}+2} = \rho^S_{\bar{t}+1} - c$ by the induction hypothesis, we have

$$\begin{aligned} \delta V^{S_m}_{\bar{t}+1} &= \delta\frac{\alpha}{2}(P^{S_m}_{\bar{t}+1} + R^{S_m}_{\bar{t}+1}) + (1 - \alpha)\delta^2 V^{S_m}_{\bar{t}+2} \\ &= \delta[\frac{\alpha}{2}(z^S_{\bar{t}+1}\rho^L_{\bar{t}+1} + (1 - z^S_{\bar{t}+1})(\beta_{\bar{t}+1}\rho^H_{\bar{t}+1} + (1 - \beta_{\bar{t}+1})\rho^S_{\bar{t}+1}) + \rho^S_{\bar{t}+1} - 2c) + (1 - \alpha)(\rho^S_{\bar{t}+1} - c \\ &= \delta[\frac{\alpha}{2}(z^S_{\bar{t}+1}(\rho^L_{\bar{t}+1} - \rho^S_{\bar{t}+1}) + (1 - z^S_{\bar{t}+1})\beta_{\bar{t}+1}(\rho^H_{\bar{t}+1} - \rho^S_{\bar{t}+1})) + \rho^S_{\bar{t}+1}] - \delta c \\ &= \rho^S_{\bar{t}} - c, \end{aligned}$$

where the last equality follows from the definition of h. Therefore $E1$ holds at $\bar{t}$ for all $m \in M^S$.

Finally we show that (p, r, λ) satisfies $E2$ for $t \in \{0, \dots, T\}$. For buyers, $E2$ requires that p_t^τ maximize

$$P_t^\tau(x) = (u^\tau - x) \sum_{m \in M^S} \lambda_t^{S_m} I(x, r_t^{S_m}) + (1 - \sum_{m \in M^S} \lambda_t^{S_m} I(x, r_t^{S_m}))\delta V_{t+1}^\tau.$$

(Note that for low value buyers $P_t^\tau(x)$ does not depend upon the particular low-value buyer strategy being followed, since $V_{t+1}^{L_m} = u^L - \rho_t^L$ by $E1$, and so we write $P_t^L(x)$ instead of $P_t^{L_m}(x)$.) Since $r_t^{S_m} = \rho_t^S$ for $m \in M^S$, $P_t^\tau(x)$ is given by

$$P_t^\tau(x) = (u^\tau - x)I(x, \rho_t^S) + (1 - I(x, \rho_t^S))(u^\tau - \rho_t^\tau).$$

If $x > \rho_t^S$, then $P_t^\tau(\rho_t^S) > P_t^\tau(x)$, as

$$P_t^\tau(\rho_t^S) = u^\tau - \rho_t^S > u^\tau - x = P_t^\tau(x).$$

If $\rho_t^\tau > \rho_t^S$, then $P_t^\tau(\rho_t^S) > P_t^\tau(x)$ for $x < \rho_t^S$, as

$$P_t^\tau(\rho_t^S) = u^\tau - \rho_t^S > u^\tau - \rho_t^\tau = P_t^\tau(x).$$

If $\rho_t^\tau = \rho_t^S$, then $P_t^\tau(\rho_t^S) = P_t^\tau(x) > P_t^\tau(x')$ for $x < \rho_t^S < x'$, as

$$P_t^\tau(x) = u^\tau - \rho_t^\tau = u^\tau - \rho_t^S > u^\tau - x' = P_t^\tau(x').$$

Finally, if $\rho_t^\tau < \rho_t^S$, then $P_t^\tau(x) > P_t^\tau(x')$ for $x < \rho_t^S \le x'$, as

$$P_t^\tau(x) = u^\tau - \rho_t^\tau > u^\tau - \rho_t^S \ge u^\tau - x' = P_t^\tau(x').$$

For high-value buyers we have $\rho_t^H \ge \rho_t^S$ and therefore $p_t^H = \rho_t^S$ maximizes $P_t^H(x)$. For low-value buyers, let $m \in M^L$ be arbitrary. If $z_t^L = 1$, then $\rho_t^L \ge \rho_t^S$ by the definition of γ and so $p_t^{L_m} = \rho_t^S$ maximizes $P_t^L(x)$. If $0 < z_t^L < 1$, then $\rho_t^L = \rho_t^S$ and therefore $p_t^{L_m} = \rho_t^S$ (if $m_t = 1$) and $p_t^{L_m} = x_t < \rho_t^S$ are both maximizers of $P_t^L(x)$. If $z_t^L = 0$, then $\rho_t^L \le \rho_t^S$ and therefore $p_t^{L_m} = x_t < \rho_t^S$ maximizes $P_t^L(x)$.

Finally, we show that sellers strategies satisfy $E2$. Let $m \in M^S$. We must show that for each $t \in \{0, \dots, T\}$, $p_t^{S_m}$ maximizes

$$\begin{aligned} P_t^{S_m}(x) &= (x - c)(b_t^H I(r_t^H, x) + (1 - b_t^H) \sum_{m' \in M^L} \lambda_t^{L_{m'}} I(r_t^{L_{m'}}, x)) \\ &\quad + (1 - b_t^H I(r_t^H, x) + (1 - b_t^H) \sum_{m' \in M^L} \lambda_t^{L_{m'}} I(r_t^{L_{m'}}, x))\delta V_{t+1}^{S_m}. \end{aligned}$$

As shown earlier, $b_t^H = \beta_t$. Since $r_t^H = \rho_t^H$, $r_t^{L_{m'}} = \rho_t^L$ for each $m' \in M^L$, and since $\delta V_{t+1}^{S_m} = \rho_t^S - c$ by $E1$, this expression reduces to

$$P_t^{S_m}(x) = \rho_t^S - c + (x - \rho_t^S)[\beta_t I(\rho_t^H, x) + (1-\beta_t) I(\rho_t^L, x)].$$

Note that for $x < \rho_t^L$, since $\rho_t^L \leq \rho_t^H$ we have

$$P_t^{S_m}(x) = x - c < \rho_t^L - c = P_t^{S_m}(\rho_t^L).$$

Also note that since $\rho_t^S \leq \rho_t^H$, then for $x > \rho_t^H$, we have

$$P_t^{S_m}(x) = \rho_t^S - c \leq \rho_t^S - c + (\rho_t^H - \rho_t^S)\beta_t = P_t^{S_m}(\rho_t^H).$$

Finally, for $\rho_t^L < x < \rho_t^H$, we have

$$P_t^{S_m}(x) = \rho_t^S - c + (x - \rho_t^S)\beta_t < \rho_t^S - c + (\rho_t^H - \rho_t^S)\beta_t = P_t^{S_m}(\rho_t^H).$$

Thus for arbitrary x, $P_t^{S_m}(x) \leq \max\{P_t^{S_m}(\rho_t^L), P_t^{S_m}(\rho_t^H)\}$.

If $z_t^S = 1$, then $0 < (\rho_t^H - \rho_t^S)\beta_t \leq \rho_t^L - \rho_t^S$ and therefore

$$P_t^{S_m}(\rho_t^H) = \rho_t^S - c + (\rho_t^H - \rho_t^S)\beta_t \leq \rho_t^S - c + \rho_t^L - \rho_t^S = P_t^{S_m}(\rho_t^L).$$

Hence $p_t^{S_m} = \rho_t^L$ maximizes $P_t^{S_m}(x)$. If $0 < z_t^S < 1$ then $0 < (\rho_t^H - \rho_t^S)\beta_t = \rho_t^L - \rho_t^S$ and therefore

$$P_t^{S_m}(\rho_t^H) = \rho_t^S - c + (\rho_t^H - \rho_t^S)\beta_t = \rho_t^L - c = P_t^{S_m}(\rho_t^L).$$

Hence both $p_t^{S_m} = \rho_t^L$ (if $m_t = 1$) and $p_t^{S_m} = \rho_t^H$ (if $m_t = 0$) maximize $P_t^{S_m}(x)$. If $z_t^S = 0$, then $(\rho_t^H - \rho_t^S)\beta_t \geq \rho_t^L - \rho_t^S$ and therefore

$$P_t^{S_m}(\rho_t^H) = \rho_t^S - c + (\rho_t^H - \rho_t^S)\beta_t \geq \rho_t^S - c + \rho_t^L - \rho_t^S = P_t^{S_m}(\rho_t^L).$$

Hence $p_t^{S_m} = \rho_t^H$ maximizes $P_t^{S_m}(x)$. □

7 Appendix B: Proofs of propositions 1 to 4

Before proving propositions 1 to 4, we establish a number of lemmas. Throughout assume that (p, r, λ) is a market equilibrium.

Lemma 1 establishes that in a market equilibrium all type τ traders have identical reservation prices and expected utilities.

Lemma 1. *For each* $\tau \in \{H, L, S\}$, *each* $k, k' \in \{1, \dots, n^\tau\}$ *and each* $t = 0, \dots, T:$

$(L1.1)$ $r_t^{\tau_k} = r_t^{\tau_{k'}}$,

$(L1.2)$ $R_t^{\tau_k} = R_t^{\tau_{k'}}$,

$(L1.3)$ $P_t^{\tau_k} = P_t^{\tau_{k'}}$, and

$(L1.4)$ $V_t^{\tau_k} = V_t^{\tau_{k'}}$.

Proof: We show that if $V_{\bar{t}+1}^{\tau_k} = V_{\bar{t}+1}^{\tau_{k'}}$ for $\tau \in \{H, L, S\}$, and $k, k' \in \{1, \dots, n^\tau\}$, then $L1.1 - L1.4$ hold at $\bar{t}$. This establishes the Lemma as $V_{T+1}^{\tau_k} = V_{T+1}^{\tau_{k'}} = 0$ for $\tau \in \{H, L, S\}$, and $k, k' \in \{1, \dots, n^\tau\}$.

Assume that $V_{\bar{t}+1}^{\tau_k} = V_{\bar{t}+1}^{\tau_{k'}}$ for $\tau \in \{H, L, S\}$, and $k, k' \in \{1, \dots, n^\tau\}$; then for $\tau \in B$, $E.1$ implies

$$r_{\bar{t}}^{\tau_i} = u^\tau - \delta V_{\bar{t}+1}^{\tau_i} = u^\tau - \delta V_{\bar{t}+1}^{\tau_{i'}} = r_{\bar{t}}^{\tau_{i'}}.$$

For $\tau = S$, $E.1.$ implies

$$r_{\bar{t}}^{S_j} = c + \delta V_{\bar{t}+1}^{S_j} = c + \delta V_{\bar{t}+1}^{S_{j'}} = r_{\bar{t}}^{S_{j'}}.$$

Hence $L1.1$ holds at $\bar{t}$.

Since $r_{\bar{t}}^{\tau_k} = r_{\bar{t}}^{\tau_{k'}}$ and $V_{\bar{t}+1}^{\tau_k} = V_{\bar{t}+1}^{\tau_{k'}}$, then $R_{\bar{t}}^{\tau_k} = R_{\bar{t}}^{\tau_{k'}}$, and therefore $L1.2$ holds at $\bar{t}$.

For $\tau \in B$, $V_{\bar{t}+1}^{\tau_i} = V_{\bar{t}+1}^{\tau_{i'}}$ and $r_{\bar{t}}^{S_j} = r_{\bar{t}}^{S_{j'}}$ implies $P_{\bar{t}}^{\tau_i} = P_{\bar{t}}^{\tau_{i'}}$, since otherwise either $p_{\bar{t}}^{\tau_i}$ or $p_{\bar{t}}^{\tau_{i'}}$ does not satisfy $E.2$. An analogous argument shows $P_{\bar{t}}^{S_j} = P_{\bar{t}}^{S_{j'}}$; hence $L1.3$ holds at $\bar{t}$. Finally, trader τ_k's expected utility at $\bar{t}$ is

$$V_{\bar{t}}^{\tau_k} = \frac{\alpha}{2}(P_{\bar{t}}^{\tau_k} + R_{\bar{t}}^{\tau_k}) + (1 - \alpha)\, \delta V_{\bar{t}+1}^{\tau_k}.$$

Since $L1.2$ and $L1.3$ hold at $\bar{t}$, and since $V_{\bar{t}+1}^{\tau_k} = V_{\bar{t}+1}^{\tau_{k'}}$, $L1.4$ holds at $\bar{t}$. □

Hereafter we write r_t^τ, R_t^τ, P_t^τ, and V_t^τ for the equilibrium reservation prices and expected utilities of a trader of type $\tau \in \{H, L, S\}$ at time $t \leq T$. Also we denote by $P_t^\tau(x)$ the expected utility of a buyer of type τ who is matched and proposes a price of x at date t and follows his equilibrium strategy thereafter, i.e.,

$$P_t^\tau(x) = (u^\tau - x) I(x, r_t^S) + (1 - I(x, r_t^S)) \delta V_{t+1}^\tau.$$

Analogously, we denote by $R_t^\tau(x)$ the expected utility of a buyer of type τ who is matched, employs a reservation price of x at date t, and follows his equilibrium strategy thereafter, i.e.,

$$R_t^\tau(x) = \sum_{j=1}^{n^S}(u^\tau - p_t^{S_j})\lambda_t^{S_j} I(x, p_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(x, p_t^{S_j}))\delta V_{t+1}^\tau.$$

Neither $P_t^\tau(x)$ nor $R_t^\tau(x)$ depend on which equilibrium strategy a type τ buyer might be playing since by Lemma 1 buyers of the same type have identical continuation payoff V_{t+1}^τ.

For sellers, we denote by $P_t^S(x)$ the expected utility of a matched seller who proposes a price of x at date t and follows his equilibrium strategy thereafter, i.e.,

$$P_t^S(x) = (x - c)\sum_{\tau \in B} b_t^\tau I(r_t^\tau, x) + (1 - \sum_{\tau \in B} b_t^\tau I(r_t^\tau, x))\delta V_{t+1}^S.$$

Analogously, we denote by $R_t^S(x)$ denote the expected utility of a matched seller who employs a reservation price of x at date t and follows his equilibrium strategy thereafter, i.e.,

$$R_t^S(x) = \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau}(p^{\tau_i} - c)\lambda_t^{\tau_i} I(p_t^{\tau_j}, x) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(p_t^{\tau_j}, x))\delta V_{t+1}^S.$$

Condition $E.2$ can be written as $p_t^{\tau_k} \in \arg\max_x P_t^\tau(x)$ for $\tau \in \{H, L, S\}$ and $k \leq n^\tau$.[5] Further, if r_t^τ satisfies $E.1$, then for any x we have $R_t^\tau(r_t^\tau) \geq R_t^\tau(x)$; i.e., $r_t^\tau \in \arg\max_x R_t^\tau(x)$. This follows from $r_t^\tau = u^\tau - \delta V_{t+1}^\tau$, as $R_t^\tau(r_t^\tau) - R_t^\tau(x)$ can be written as

$$\sum_{i=1}^{n^S} \lambda_t^{S_j}(r_t^\tau - p^{S_j})\{I(r_t^\tau, p_t^{S_j}) - I(x, p_t^{S_j})\},$$

which is always non-negative since $I(r_t^\tau, p_t^{S_j}) - I(x, p_t^{S_j}) > 0$ implies $r_t^\tau - p_t^{S_j} > 0$, and $I(r_t^\tau, p_t^{S_j}) - I(x, p_t^{S_j}) < 0$ implies $r_t^\tau - p_t^{S_j} < 0$.

Lemma 2 characterizes buyers' optimal price offers.

[5] Note that the sequences $\{\lambda_t^{\tau_k}\}_{t=0}^T$ for $\tau \in \{H, L, S\}$ and $k \in \{1, \ldots, n_\tau\}$, and $\{b_t^\tau\}_{t=0}^T$ for $\tau \in \{H, L\}$ are unaffected by a single trader offering a price different from his equilibrium offer.

Lemma 2. *For each $\tau \in B$, each $i \leq n^\tau$ and each $t = 0, \ldots, T$:*

$(L2.1)$ $p_t^{\tau_i} \leq r_t^S$, and

$(L2.2)$ $r_t^\tau > r_t^S$ implies $p_t^{\tau_i} = r_t^S$, and

$(L2.3)$ $r_t^\tau < r_t^S$ implies $p_t^{\tau_i} < r_t^S$.

Proof: Let $\tau \in B$, $i \leq n^\tau$ and $t \in \{0, \ldots, T\}$. We prove $L2.1$. Suppose that $p_t^{\tau_i} \geq r_t^S$. Condition $E.2$ implies that $P_t^\tau(p_t^{\tau_i}) \geq P_t^\tau(x)$ for $x \geq 0$. In particular, $P_t^\tau(p_t^{\tau_i}) \geq P_t^\tau(r_t^S)$, i.e.,

$$u^\tau - p_t^{\tau_i} \geq u^\tau - r_t^S;$$

hence $p_t^{\tau_i} \leq r_t^S$. Thus $p_t^{\tau_i} \geq r_t^S$ implies $p_t^{\tau_i} = r_t^S$; i.e., $p_t^{\tau_i} \leq r_t^S$. Therefore $L2.1$ holds

We prove $L2.2$. Suppose $p_t^{\tau_i} \neq r_t^S$; then $L2.1$ implies $p_t^{\tau_i} < r_t^S$. By $E.2$ we have

$$u^\tau - r_t^\tau = \delta V_{t+1}^\tau = P_t^{\tau_i}(p_t^{\tau_i}) \geq P_t^{\tau_i}(r_t^S) = u^\tau - r_t^S,$$

which yields $r_t^\tau \leq r_t^S$.

Finally, we prove $L2.3$. Suppose $p_t^{\tau_i} \geq r_t^S$; then $L2.1$ implies $p_t^{\tau_i} = r_t^S$. Let x be such that $r_t^S > x \geq 0$. By $E.2$ we have

$$u^\tau - r_t^S = P_t^\tau(p_t^{\tau_i}) \geq P_t^\tau(x) = u^\tau - r_t^\tau,$$

which implies $r_t^\tau \geq r_t^S$. $\square$

For each t such that $r_t^H - r_t^S > 0$, we write π_t for the ratio $(r_t^L - r_t^S)/(r_t^H - r_t^S)$. Lemma 3 characterizes sellers' optimal offers.

Lemma 3. *For each $j \leq n^S$ and each $t = 0, \ldots, T$, if $r_t^H > \max\{r_t^L, r_t^S\}$ then*

$(L3.1)$ $p_t^{S_j} \in \{r_t^L, r_t^H\}$,

$(L3.2)$ $b_t^H < \pi_t$ implies $p_t^{S_j} = r_t^L$,

$(L3.3)$ $b_t^H = \pi_t$ implies $P_t^S(r_t^H) = P_t^S(r_t^L)$, and

$(L3.4)$ $b_t^H > \pi_t$ implies $p_t^{S_j} = r_t^H$.

Proof: Let $j \leq n^S$ and $t \leq T$, and assume that $r_t^H > \max\{r_t^L, r_t^S\}$. We establish $L3.1$.

If $p_t^{S_j} \leq r_t^L$, then $I(r_t^H, p_t^{S_j}) = I(r_t^L, p_t^{S_j}) = 1$. By $E.2$ we have

$$p_t^{S_j} - c = P_t^S(p_t^{S_j}) \geq P_t^S(r_t^L) = r_t^L - c,$$

and therefore $p_t^{S_j} = r_t^L$.

If $r_t^L < p_t^{S_j} \leq r_t^H$, then

$$b_t^H(p_t^{S_j} - c) + b_t^L \delta V_{t+1}^S = P_t^S(p_t^{S_j}) \geq P_t^S(r_t^H) = b_t^H(r_t^H - c) + b_t^L \delta V_{t+1}^S.$$

As $b_t^H > 0$ (because $b_0^H \in (0,1)$ and $\alpha < 1$), it follows that $p_t^{S_j} \geq r_t^H$. Hence $p_t^{S_j} = r_t^H$.

We show that $p_t^{S_j} \leq r_t^H$, which establishes $L3.1$. Suppose $p_t^{S_j} > r_t^H$; then $E.2$ implies

$$r_t^S - c = P_t^S(p_t^{S_j}) \geq P_t^S(r_t^H) = b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c);$$

i.e., $r_t^S \geq r_t^H$. This contradicts $r_t^H > \max\{r_t^L, r_t^S\}$, and proves $p_t^{S_j} \leq r_t^H$.

Now we prove $L3.2 - L3.4$. If $r_t^L > r_t^S$, then $\pi_t > 0$. Since $r_t^H > r_t^L$, the definitions of $P_t^S(x)$ and π_t yield

$$P_t^S(r_t^H) = b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c) = \left(\frac{b_t^H}{\pi_t} - 1\right)(r_t^L - r_t^S) + P_t^S(r_t^L).$$

If $b_t^H < \pi_t$, then $P_t^S(r_t^H) < P_t^S(r_t^L)$ and therefore $p_t^{S_j} = r_t^L$. If $b_t^H = \pi_t$, then $P_t^S(r_t^H) = P_t^S(r_t^L)$. Finally, if $b_t^H > \pi_t$, then $P_t^S(r_t^H) > P_t^S(r_t^L)$ and therefore $p_t^{S_j} = r_t^H$.

If $r_t^L \leq r_t^S$, then $\pi_t \leq 0$ and therefore $b_t^H > \pi_t$. We must show that $p_t^{S_j} = r_t^H$. Since $b_t^H > 0$ and $r_t^H > r_t^S \geq r_t^L$, we have $b_t^H r_t^H + (1 - b_t^H) r_t^S > r_t^L$, and therefore

$$P_t^S(r_t^H) = b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c) > r_t^L - c = P_t^S(r_t^L);$$

hence $p_t^{S_j} = r_t^H$. □

Lemmas 4 and 5 establish some inequalities between reservation prices and between expected utilities for the different types of traders.

Lemma 4. *For each* $t = 0, \ldots, T$:

($L4.1$) $r_t^H > r_t^L$, *and*

($L4.2$) $V_t^H - V_t^L < u^H - u^L$.

Proof: We establish Lemma 4 by induction. As $V_{T+1}^H = V_{T+1}^L = 0$, $E.1$ implies $r_T^\tau = u^\tau$ for $\tau \in B$; hence $r_T^H = u^H > u^L = r_T^L$, and therefore $L4.1$ holds for $t = T$. Also as $r_T^S = c$, we have $r_T^H > \max\{r_T^L, r_T^S\}$, and $L3.1$ implies $p_T^S \in \{r_T^H, r_T^L\}$; hence $V_T^H \leq \frac{\alpha}{2}(u^H - c) + \frac{\alpha}{2}(u^H - u^L)$, and $V_T^L = \frac{\alpha}{2}(u^L - c)$. Thus, as $\alpha < 1$ we get

$$V_T^H - V_T^L \leq \alpha(u^H - u^L) < u^H - u^L,$$

and therefore $L4.2$ holds for $t = T$.

Assume that $L4.1$ and $L4.2$ hold for $t = k + 1 \leq T$. We show that they hold for $t = k$. By $E.1$ we have $r_k^\tau = u^\tau - \delta V_{k+1}^\tau$, and therefore

$$r_k^H = u^H - \delta V_{k+1}^H = (u^H - u^L + \delta V_{k+1}^L - \delta V_{k+1}^H) + u^L - \delta V_{k+1}^L > u^L - \delta V_{k+1}^L = r_k^L,$$

where the strict inequality follow from the induction hypothesis. Thus $L4.1$ holds for $t = k$.

We show that $V_k^H - V_k^L < u^H - u^L$. Let $i \leq n^H$. Since $P_k^L \geq P_k^L(p_k^{H_i})$ and $R_k^L \geq R_k^L(r_k^{H_i})$ by $E1$ and $E2$, respectively, we have

$$V_k^L = \frac{\alpha}{2}(P_k^L + R_k^L) + (1-\alpha)\delta V_{k+1}^L \geq \frac{\alpha}{2}(P_k^L(p_k^{H_i}) + R_k^L(r_k^{H_i})) + (1-\alpha)\delta V_{k+1}^L.$$

Also the induction hypothesis yields

$$\begin{aligned} P_k^H - P_k^L(p_k^{H_i}) &= I(p_k^{H_i}, r_k^S)(u^H - u^L) + (1 - I(p_k^{H_i}, r_k^S))\delta(V_{k+1}^H - V_{k+1}^L) \\ &\leq u^H - u^L, \end{aligned}$$

and

$$\begin{aligned} R_k^H - R_k^L(r_k^{H_i}) &= (u^H - u^L)\sum_{j=1}^{n^S} \lambda_k^{S_j} I(r_k^H, p_k^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_k^{S_j} I(r_k^H, p_k^{S_j}))\delta(V_{k+1}^H - V_{k+1}^L) \\ &\leq u^H - u^L. \end{aligned}$$

Therefore

$$\begin{aligned} V_k^H - V_k^L &\leq \frac{\alpha}{2}(P_k^H - P_k^L(p_k^{H_i}) + R_k^H - R_k^L(r_k^{H_i})) + (1-\alpha)\delta(V_{k+1}^H - V_{k+1}^L) \\ &< u^H - u^L. \end{aligned}$$

Hence $L4.2$ holds for $t = k$. □

Lemma 5. *For each* $t = 0, \ldots, T$:

(L5.1) $r_t^H > r_t^S$, and

(L5.2) $V_t^H + V_t^S \leq \alpha(u^H - c)\frac{1-\delta^{T-t+1}(1-\alpha)^{T-t+1}}{1-\delta(1-\alpha)}$.

Proof: First we show that $r_t^H > r_t^S$ implies $P_t^H + R_t^S \leq u^H - c$. Suppose that $r_t^H > r_t^S$; then $p_t^{H_i} = r_t^S$ for each $i \leq n^H$ by $L2.2$. Thus

$$P_t^H = u^H - r_t^S,$$

and

$$(p_t^{H_i} - c)I(p_t^{H_i}, r_t^S) + (1 - I(p_t^{H_i}, r_t^S))(r_t^S - c) = r_t^S - c.$$

Since $p_t^{L_i} \leq r_t^S$ for $i \leq n^L$ by $L2.1$, we have

$$(p_t^{L_i} - c)I(p_t^{L_i}, r_t^S) + (1 - I(p_t^{L_i}, r_t^S))(r_t^S - c) = r_t^S - c.$$

Hence

$$R_t^S = \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i}(p_t^{\tau_i} - c)I(p_t^{\tau_i}, r_t^S) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(p_t^{\tau_i}, r_t^S))(r_t^S - c) = r_t^S - c,$$

and therefore $P_t^H + R_t^S = u^H - r_t^S + r_t^S - c = u^H - c$.

Next we establish that $r_t^H > r_t^S$ implies $R_t^H + P_t^S \leq u^H - c$. Suppose $r_t^H > r_t^S$; since $r_t^H > r_t^L$ by $L4.1$, then $r_t^H > \max\{r_t^L, r_t^S\}$ and therefore $p_t^{S_j} \in \{r_t^H, r_t^S\}$ by $L3.1$. If $p_t^{S_j} = r_t^L$, then $L3.4$ implies $b_t^H \leq \pi_t$, and since $b_t^H > 0$, we have $r_t^L - r_t^S > 0$ and therefore $p_t^{S_j} > r_t^S$. If $p_t^{S_j} = r_t^H$, then as $r_t^H > r_t^S$ we also have $p_t^{S_j} > r_t^S$. Thus $r_t^H > r_t^S$ implies $I(r_t^H, p_t^{S_j}) \geq I(r_t^L, p_t^{S_j})$ for $j \leq n^S$, and therefore we have

$$I(r_t^H, p_t^{S_j})(p_t^{S_j} - c) + (1 - I(r_t^H, p_t^{S_j}))(r_t^S - c) \geq I(r_t^L, p_t^{S_j})(p_t^{S_j} - c) + (1 - I(r_t^L, p_t^{S_j}))(r_t^S - c);$$

Hence (recall $\sum_{j=1}^{n^S} \lambda_t^{S_j} = 1$)

$$\begin{aligned} P_t^S &= \sum_{\tau \in B} b_t^\tau [I(r_t^\tau, p_t^{S_j})(p_t^{S_j} - c) + (1 - I(r_t^\tau, p_t^{S_j}))(r_t^S - c)] \\ &\leq I(r_t^H, p_t^{S_j})(p_t^{S_j} - c) + (1 - I(r_t^H, p_t^{S_j}))(r_t^S - c). \end{aligned}$$

Thus

$$P_t^S \leq \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^H, p_t^{S_j})(p_t^{S_j} - c) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^H, p_t^{S_j}))(r_t^S - c).$$

Also

$$R_t^H = \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^H, p_t^{S_j})(u^H - p_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^H, p_t^{S_j}))(u^H - r_t^H).$$

Summing up we obtain $R_t^H + P_t^S \leq u^H - c$.

We establish Lemma 5 by induction. Since $V_{T+1}^{T_k} = 0$, $E.1$ implies $r_T^H = u^H > c = r_T^S$, and therefore $L5.1$ holds for $t = T$. Moreover, since $r_T^H > r_T^S$, we have $P_T^H + R_T^S \leq u^H - c$, and $R_T^H + P_T^S \leq u^H - c$. Hence, since $\alpha < 1$, for $t = T$ we have

$$V_T^S + V_T^H = \frac{\alpha}{2}(P_T^S + R_T^S + P_T^H + R_T^H) \leq \alpha(u^H - c) = \alpha(u^H - c)\frac{1 - \delta^{T-t+1}(1-\alpha)^{T-t+1}}{1 - \delta(1-\alpha)}.$$

Therefore $L5.2$ holds for $t = T$.

Assume that $r_{k+1}^H > r_{k+1}^S$, and $V_{k+1}^H + V_{k+1}^S \leq \alpha(u^H - c)\frac{1-\delta^{T-k}(1-\alpha)^{T-k}}{1-\delta(1-\alpha)}$ for $k+1 \leq T$. We show that $r_k^H > r_k^S$ and $V_k^H + V_k^S < \alpha(u^H - c)\frac{1-\delta^{T-k+1}(1-\alpha)^{T-k+1}}{1-\delta(1-\alpha)}$. Since $\delta > 0$ and $\alpha < 1$ implies $\alpha\frac{1-\delta^{T-k+1}(1-\alpha)^{T-k+1}}{1-\delta(1-\alpha)} < 1$, the induction hypothesis yields $V_{k+1}^H + V_{k+1}^S < u^H - c$, and therefore $E.1$ yields

$$r_k^H - r_k^S = (u^H - \delta V_{k+1}^H) - (\delta V_{k+1}^S + c) > 0.$$

Hence $r_k^H > r_k^S$, and therefore $P_k^H + R_k^S \leq u^H - c$, and $R_k^H + P_k^S \leq u^H - c$. Thus

$$\begin{aligned} V_k^S + V_k^H &= \frac{\alpha}{2}(P_k^S + R_k^S + P_k^H + R_k^H) + (1-\alpha)\delta(V_{k+1}^S + V_{k+1}^H) \\ &\leq \alpha(u^H - c)(1 + (1-\alpha)\delta\frac{1 - \delta^{T-k}(1-\alpha)^{T-k}}{1 - \delta(1-\alpha)}) \\ &= \alpha(u^H - c)\frac{1 - \delta^{T-k+1}(1-\alpha)^{T-k+1}}{1 - \delta(1-\alpha)}, \end{aligned}$$

which establishes the lemma. □

Lemma 6 establishes a number of basic results that are frequently used in subsequent arguments.

Lemma 6. *For each $t = 0, \ldots, T$:*

(L6.1) $p_t^{H_i} = r_t^S$, for each $i \leq n^H$;

(L6.2) $\mu_t^{H_i} = 1 - \alpha$, for each $i \leq n^H$;

(L6.3) $P_t^H = u^H - r_t^S$;

($L6.4$) $R_t^L = u^L - r_t^L$;

($L6.5$) $R_t^S = r_t^S - c$.

Proof: $L6.1$ is a direct implication of $L2.2$ and $L5.1$. In order to prove $L6.2$, note that $L6.1$ implies $I(p_t^{H_i}, r_t^S) = 1$ for each $j \leq n^H$, and since $r_t^H > \max\{r_t^L, r_t^S\}$ by $L4.1$ and $L5.1$, we have $r_t^L \leq p_t^{S_j} \leq r_t^H$, and therefore $I(r_t^H, p_t^{S_j}) = 1$ for each $j \leq n^S$ by $L3.1$. Hence

$$\mu_t^{H_i} = 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{t-1}^{S_j} I(p_t^{H_i}, r_t^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{t-1}^{S_j} I(r_t^H, p_t^{S_j}) = 1 - \alpha.$$

Also note that again $r_t^L \leq p_t^{S_j}$ for each $j \leq n^S$ implies $I(r_t^L, p_t^{S_j})(u^L - p_t^{S_j}) = I(r_t^L, p_t^{S_j})(u^L - r_t^L)$, and therefore using $E.1$ we have

$$R_t^L = \sum_{j=1}^{n^S}(u^L - p_t^{S_j})\lambda_t^{S_j} I(r_t^L, p_t^{S_j}) + (1 - \sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^L, p_t^{S_j}))(u^L - r_t^L) = u^L - r_t^L,$$

which establishes $L6.4$.

Finally, since $p_t^{\tau_i} \leq r_t^S$ by $L2.1$, we have $I(p_t^{\tau_i}, r_t^S)(p_t^{\tau_i} - c) = I(p_t^{\tau_i}, r_t^S)(r_t^S - c)$ for each $\tau \in B$ and $i \leq n^\tau$. Hence $E.1$ implies

$$R_t^S = \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau}(p^{\tau_i} - c)\lambda_t^{\tau_i} I(p_t^{\tau_j}, r_t^S) + (1 - \sum_{\tau \in B} b_t^\tau \sum_{i=1}^{n^\tau} \lambda_t^{\tau_i} I(p_t^{\tau_j}, r_t^S))(r_t^S - c) = r_t^S - c,$$

which establishes $L6.5$. □

Since in a market equilibrium all high-value buyers follow the same strategy by $L1.1$ and $L6.1$, henceforth we refer to the high-value buyer strategy as (p^H, r^H). Lemma 7 establishes that the proportion of high-value buyers in the market does not increase over time.

Lemma 7. *The sequence* $\{b_t^H\}_{t=0}^{T+1}$ *is non-increasing.*

Proof: For $t \in \{0, \dots, T\}$, b_{t+1}^H is given by

$$\begin{aligned} b_{t+1}^H &= \frac{b_t^H \sum_{i=1}^{n^H} \lambda_t^{H_i} \mu_t^{H_i}}{b_t^H \sum_{i=1}^{n^H} \lambda_t^{H_i} \mu_t^{H_i} + b_t^L \sum_{i=1}^{n^L} \lambda_t^{L_i} \mu_t^{L_i}} \\ &= \frac{(1-\alpha) b_t^H}{(1-\alpha) b_t^H + b_t^L \sum_{i=1}^{n^L} \lambda_t^{L_i} \mu_t^{L_i}}, \end{aligned}$$

where we have used the fact that $\mu_t^{H_i} = 1 - \alpha$ by $L6.2$. Since

$$\begin{aligned}\mu_t^{L_i} &= 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_t^{S_j} I(p_t^{L_i}, r_t^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^L, p_t^{S_j}) \\ &= 1 - \frac{\alpha}{2} I(p_t^{L_i}, r_t^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^L, p_t^{S_j}) \leq 1 - \alpha,\end{aligned}$$

for each $i \leq n^L$, we have

$$b_{t+1}^H \leq \frac{(1-\alpha)b_t^H}{(1-\alpha)b_t^H + (1-\alpha)b_t^L} = b_t^H. \ \square$$

In lemmas 8 to 12 we for the low cost case which have implications for the dynamics of trading patterns. Lemma 8 relates π_t and π^* when sellers offer the low-value-buyer reservation price.

Lemma 8. *Assume that $u^H > u^L > c$.*

($L8.1$) If $p_t^{S_j} = r_t^L$ for each $j \leq n^S$ and $\pi_t = \pi^*$, then $\pi_{t-1} = \pi^*$.

($L8.2$) If $p_t^{S_j} = r_t^L$ for each $j \leq n^S$ and $\pi_t < \pi^*$, then $\pi_{t-1} < \pi^*$.

Proof: Suppose that $p_t^{S_j} = r_t^L$ for each $j \leq n^S$. By $E.1$, the reservation price of a buyer of type $\tau \in B$ at $t-1$ is

$$r_{t-1}^\tau = u^\tau - \delta[\frac{\alpha}{2}(P_t^\tau + R_t^\tau) + (1-\alpha)(u^\tau - r_t^\tau)].$$

Also the reservation price of a seller at $t-1$ is

$$r_{t-1}^S = c + \delta[\frac{\alpha}{2}(P_t^S + R_t^S) + (1-\alpha)(r_t^S - c)].$$

Since $p_t^{S_j} = r_t^L$ for each $j \leq n^S$ we have $P_t^S = r_t^L - c$ and $R_t^H = u^H - r_t^L$. Furthermore, since $P_t^S \geq P_t^S(r_t^S) = r_t^S - c$ by $E.2$, we have $r_t^L \geq r_t^S$; hence $P_t^L = u^L - r_t^S$. (If $r_t^L > r_t^S$, then $L2.1$ implies $p_t^{L_i} = r_t^S$ for every $i \leq n^L$, and therefore $P_t^L = u^L - r_t^S$; if $r_t^L = r_t^S$, then $p_t^{L_i} \leq r_t^S$ for every $i \leq n^L$ by $L2.2$, and we have also $P_t^L = u^L - r_t^L = u^L - r_t^S$.)

Substituting P_t^τ and R_t^τ from above and noticing $L6.3 - L6.5$ yields for $\tau \in B$,

$$r_{t-1}^\tau - r_{t-1}^S = (1-\delta)(u^\tau - c) + \delta(1-\alpha)(r_t^\tau - r_t^S).$$

Since $u^L - c = \pi^*(u^H - c)$ and $r_t^L - r_t^S = \pi_t(r_t^H - r_t^S)$ we have

$$\begin{aligned}\pi_{t-1} &= \frac{(1-\delta)(u^L - c) + \delta(1-\alpha)(r_t^L - r_t^S)}{(1-\delta)(u^H - c) + \delta(1-\alpha)(r_t^H - r_t^S)} \\ &= \pi^* \frac{(1-\delta)(u^H - c) + \delta(1-\alpha)\frac{\pi_t}{\pi^*}(r_t^H - r_t^S)}{(1-\delta)(u^H - c) + \delta(1-\alpha)(r_t^H - r_t^S)}.\end{aligned}$$

Thus, if $\pi_t = \pi^*$ then $\pi_{t-1} = \pi^*$; hence $L8.1$ holds. If $\pi_t < \pi^*$, then since $r_t^H > r_t^S$ by $L5.1$, we have $\pi_{t-1} < \pi^*$; hence $L8.2$ holds. □

Lemma 9 establishes a dynamic relation between the proportion of high-value buyers in the market and ratio π_t.

Lemma 9. *Assume that $u^H > u^L > c$.*

($L9.1$) If $b_t^H \geq \pi^* \geq \pi_t$, then $\pi^* \geq \pi_{t-1}$.

($L9.2$) If $b_t^H > \pi^* \geq \pi_t$, then $\pi^* > \pi_{t-1}$.

($L9.3$) If $b_t^H \geq \pi^* \geq \pi_t$ and $p_t^{S_j} = r_t^H$ for some $j \leq n^S$, then $\pi^* > \pi_{t-1}$.

Proof: Suppose that $b_t^H \geq \pi^* \geq \pi_t$. $L3.3$, $L3.4$ and $L4.1$ imply

$$P_t^S = P_t^S(r_t^H) = b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c).$$

Since $R_t^S = r_t^S - c$ by $L6.5$, using $E.1$ we have

$$\begin{aligned}r_{t-1}^S &= c + \delta[\frac{\alpha}{2}(b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c) + r_t^S - c) + (1-\alpha)(r_t^S - c)] \\ &= (1-\delta)\,c + \delta[\frac{\alpha}{2}b_t^H(r_t^H - r_t^S) + r_t^S].\end{aligned}$$

Let $a \in [0,1]$ denote the proportion of sellers (out of the total measure of sellers in the market at date t) who offer r_t^H at date t. Then by $L3.1$ a proportion $1-a$ of sellers offer r_t^L. Thus, $R_t^H = u^H - ar_t^H - (1-a)r_t^L$, and since $P_t^H = u^H - r_t^S$ by $L6.4$, $E.1$ yields

$$\begin{aligned}r_{t-1}^H &= u^H - \delta[\frac{\alpha}{2}(u^H - ar_t^H - (1-a)r_t^L) + \frac{\alpha}{2}(u^H - r_t^S) + (1-\alpha)(u^H - r_t^H)] \\ &= (1-\delta)u^H + \delta\{\frac{\alpha}{2}[r_t^S + ar_t^H + (1-a)r_t^L] + (1-\alpha)r_t^H\}.\end{aligned}$$

Thus

$$\begin{aligned}r_{t-1}^H - r_{t-1}^S &= (1-\delta)(u^H - c) + \delta(1 - \frac{\alpha}{2})(r_t^H - r_t^S) \\ &\quad -\delta\frac{\alpha}{2}[b_t^H(r_t^H - r_t^S) + (1-a)(r_t^H - r_t^L)].\end{aligned}$$

Note that if $a < 1$, then $b_t^H \leq \pi_t$ by $L3.4$, and since $b_t^H \geq \pi^* \geq \pi_t$ we have $b_t^H = \pi^* = \pi_t$. Hence $b_t^H(r_t^H - r_t^S) = r_t^L - r_t^S$, and therefore

$$b_t^H(r_t^H - r_t^S) + (1-a)(r_t^H - r_t^L) = r_t^H - r_t^S - a(r_t^H - r_t^L).$$

Since $R_t^L = u^L - r_t^L$ by $L6.4$, and $I(p_t^{L_i}, r_t^S)(u^L - p_t^{L_i}) = I(p_t^{L_i}, r_t^S)(u^L - r_t^S)$ for $i \leq n^L$ by Lemma 2, $E.1$ yields

$$\begin{aligned} r_{t-1}^L &= u^L - \delta\{\frac{\alpha}{2}[I(p_t^{L_i}, r_t^S)(u^L - r_t^S) + (1 - I(p_t^{L_i}, r_t^S))(u^L - r_t^L) + (u^L - r_t^L)] \\ &\quad + (1-\alpha)\,(u^L - r_t^L)\} \\ &= (1-\delta)\,u^L - \delta[\frac{\alpha}{2}I(p_t^{L_i}, r_t^S)(r_t^L - r_t^S) - r_t^L]. \end{aligned}$$

Hence

$$r_{t-1}^L - r_{t-1}^S = (1-\delta)\left(u^L - c\right) + \delta(1 - \frac{\alpha}{2}I(p_t^{L_i}, r_t^S))(r_t^L - r_t^S) - \delta\frac{\alpha}{2}b_t^H(r_t^H - r_t^S).$$

Suppose that $p_t^{L_i} = r_t^S$ for every $i \leq n^L$; then

$$r_{t-1}^L - r_{t-1}^S = (1-\delta)\left(u^L - c\right) + \delta(1 - \frac{\alpha}{2})(r_t^L - r_t^S) - \delta\frac{\alpha}{2}b_t^H(r_t^H - r_t^S).$$

Noticing that $u^L - c = \pi^*(u^L - c)$ and $r_t^L - r_t^S = \pi_t(r_t^H - r_t^S)$, we have

$$\pi_{t-1} = \pi^* \frac{(1-\delta)\left(u^H - c\right) + \delta\left(1 - \frac{\alpha}{2}\right)\frac{\pi_t}{\pi^*}(r_t^H - r_t^S) - \delta\frac{\alpha}{2}\frac{b_t^H}{\pi^*}(r_t^H - r_t^S)}{(1-\delta)\left(u^H - c\right) + \delta\left(1 - \frac{\alpha}{2}\right)\left(r_t^H - r_t^S\right) - \delta\frac{\alpha}{2}[b_t^H(r_t^H - r_t^S) + (1-a)(r_t^H - r_t^L)]}.$$

If $a = 1$, then since $\pi_t \leq \pi^* < 1$, we have $\pi_{t-1} < \pi^*$. Note that $b_t^H > \pi^* \geq \pi_t$ implies $a = 1$ by $L3.4$, and therefore the conclusion of $L9.2$ holds. If $a < 1$, then as shown above $b_t^H(r_t^H - r_t^S) + (1-a)(r_t^H - r_t^L) = r_t^H - r_t^S - a(r_t^H - r_t^L)$, and therefore

$$\pi_{t-1} = \pi^* \frac{(1-\delta)\left(u^H - c\right) + \delta\left(1-\alpha\right)\frac{\pi_t}{\pi^*}(r_t^H - r_t^S)}{(1-\delta)\left(u^H - c\right) + \delta\left(1-\alpha\right)\left(r_t^H - r_t^S\right) + \delta\frac{\alpha}{2}a(r_t^H - r_t^L)}.$$

Since $\pi^* \geq \pi_t$ and $r_t^H - r_t^L > 0$ by $L4.1$, we have $\pi^* \geq \pi_{t-1}$. Hence the conclusion of $L9.1$ holds. Moreover, if there is $j \leq n^S$ such that $p_t^{S_j} = r_t^H$, then $a > 0$, and since $\delta > 0$, we have $\pi^* > \pi_{t-1}$; therefore the conclusion of $L9.3$ holds. Hence the conclusions of $L9.1$, $L9.2$ and $L9.3$ hold when $p_t^{L_i} = r_t^S$ for every $i \leq n^L$.

for $\tau \in \{L, S\}$, and $V_{k+1}^S \geq V_{k+1}^L$ by the induction hypothesis.

For $j \leq n^S$ we have

$$\begin{aligned} P_k^S &= (p_k^{S_j} - c)\sum_{\tau \in B} b_k^\tau I(r_k^\tau, p_k^{S_j}) + (1 - \sum_{\tau \in B} b_k^\tau I(r_k^\tau, p_k^{S_j}))\delta V_{k+1}^S \\ &= (r_k^S - c) + \sum_{\tau \in B} b_k^\tau I(r_k^\tau, p_k^{S_j})(p_k^{S_j} - r_k^S). \end{aligned}$$

Hence

$$P_k^S = (r_k^S - c) + \sum_{j=1}^{n^S} \lambda_k^{S_j} \sum_{\tau \in B} b_k^\tau I(r_k^\tau, p_k^{S_j})(p_k^{S_j} - r_k^S).$$

Thus, since $R_k^S = r_k^S - c$ by $L6.5$, we have

$$P_k^S + R_k^S = 2(r_k^S - c) + \sum_{j=1}^{n^S} \lambda_k^{S_j} \sum_{\tau \in B} b_k^\tau I(r_k^L, p_k^{S_j})(p_k^{S_j} - r_k^S).$$

For $i \leq n^L$ we have

$$\begin{aligned} P_k^L &= I(p_k^{L_i}, r_k^S)(u^L - p_k^{L_i}) + (1 - I(p_k^{L_i}, r_k^S))(u^L - r_k^L) \\ &= (u^L - r_k^L) + I(p_k^{L_i}, r_k^S)(r_k^L - p_{k-1}^{L_i}), \end{aligned}$$

where the last equality follows again from Lemma 2. Hence

$$P_k^L = (u^L - r_k^L) + \sum_{i=1}^{n^L} \lambda_k^{L_i} I(p_k^{L_i}, r_k^S)(r_k^L - r_k^S).$$

Thus, since $R_k^L = u^L - r_k^L$ by $L6.4$, we have

$$P_k^L + R_k^L = 2(u^L - r_k^L) + \sum_{i=1}^{n^L} \lambda_k^{L_i} I(p_k^{L_i}, r_k^S)(r_k^L - r_k^S).$$

Suppose that $p_k^{S_j} = r_k^H$ for each $j \leq n^S$. Then we have must show

$$2(r_k^S - c) + b_k^H(r_k^H - r_k^S) \geq 2(u^L - r_k^L) + (r_k^L - r_k^S)\sum_{i=1}^{n^L} \lambda_k^{L_i} I(p_k^{L_i}, r_k^S),$$

which can be written using $E.1$ as

$$2\delta(V_{k+1}^S - V_{k+1}^L) \geq (r_k^L - r_k^S)\sum_{i=1}^{n^L} \lambda_k^{L_i} I(p_k^{L_i}, r_k^S) - b_k^H(r_k^H - r_k^S)$$

Since $V_{k+1}^S \geq V_{k+1}^L$ by the induction hypothesis and $r_k^H > r_k^S$ by $L5.1$, this inequality holds if $r_k^L \leq r_k^S$. If $r_k^L > r_k^S$, then since $0 \leq \sum_{i=1}^{n^L} \lambda_k^{L_i} I(p_k^{L_i}, r_k^S) \leq 1$, it suffices to show

$$(r_k^L - r_k^S) - b_k^H(r_k^H - r_k^S) \leq 0;$$

i.e.,

$$\pi_k - b_{k-1}^H \leq 0,$$

which holds as $p_k^{S_j} = r_k^H$ implies $b_k^H \geq \pi_k$ by $L3.2$ and $L3.3$.

Suppose that $p_k^{S_j} = r_k^L$, for some $j \leq n^S$. Then

$$P_k^S + R_k^S = (r_k^L - r_k^S) + 2(r_k^S - c).$$

Also $\pi_k \geq b_k^H > 0$ by $L3.2$, and since $r_k^H > r_k^S$ by $L4.1$, we have $r_k^L > r_k^S$. Thus $p_k^{L_i} = r_k^S$ for each $i \leq n^L$ by $L2.1$, and therefore

$$P_k^L + R_k^L = 2(u^L - r_k^L) + (r_k^L - r_k^S).$$

Hence we must show

$$-2c + (r_k^L + r_k^S) \geq 2u^L - (r_k^L + r_k^S);$$

i.e.,

$$(r_k^S - c) - (u^L - r_k^L) \geq 0;$$

i.e.,

$$\delta(V_{k+1}^S - V_{k+1}^L) \geq 0,$$

which holds by the induction hypothesis. □

Lemma 12 establishes that if at some date there are no gains to trade between low-value buyers and sellers, then there are no gains to trade at prior dates.

Lemma 12: *Assume that $u^H > u^L > c$. There is $\varepsilon(\alpha, T) > 0$ such that if $\delta \in [1 - \varepsilon(\alpha, T), 1]$, then $r_t^L - r_t^S \leq 0$ implies $r_{t-1}^L - r_{t-1}^S < 0$.*

Proof: Assume $r_t^L - r_t^S \leq 0$; Lemma 2 implies $I(p_t^{L_i}, r_t^S)(u^L - p_t^{L_i}) = I(p_t^{L_i}, r_t^S)(u^L - r_t^L)$, therefore $E.1$ implies

$$P_t^L = I(p_t^{L_i}, r_t^S)(u^L - p_t^{L_i}) + (1 - I(p_t^{L_i}, r_t^S))\delta V_{t+1}^L = u^L - r_t^L.$$

Since $R_t^L = u^L - r_t^L$ by $L6.4$, we have

$$V_t^L = \frac{\alpha}{2}(P_t^L + R_t^L) + (1-\alpha)\delta V_{t+1}^L = u^L - r_t^L.$$

Thus, using $E.1$ we have

$$r_{t-1}^L = u^L - \delta V_t^L = (1-\delta)u^L + \delta r_t^L.$$

For sellers, since $r_t^H - r_t^S > 0$ by $L5.1$ and $r_t^L - r_t^S \leq 0$, we have $\pi_t \leq 0 < b_t^H$. Thus, $p_t^{S_j} = r_t^H > r_t^L$ for every $j \leq n^S$ by $L3.4$, and $E.1$ implies

$$P_t^S = b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c).$$

Since $R_t^S = r_t^S - c$ by $L6.5$, we have

$$\begin{aligned} V_t^S &= \frac{\alpha}{2}(P_t^S + R_t^S) + (1-\alpha)\delta V_{t+1}^S \\ &= \frac{\alpha}{2}(b_t^H(r_t^H - c) + (1 - b_t^H)(r_t^S - c) + r_t^S - c) + (1-\alpha)(r_t^S - c). \\ &= (r_t^S - c) + \frac{\alpha}{2} b_t^H (r_t^H - r_t^S), \end{aligned}$$

Thus, again by $E.1$

$$r_{t-1}^S = c + \delta V_t^S = (1-\delta)c + \delta r_t^S + \delta \frac{\alpha}{2} b_t^H (r_t^H - r_t^S).$$

Therefore,

$$r_{t-1}^L - r_{t-1}^S = (1-\delta)(u^L - c) + \delta(r_t^L - r_t^S) - \delta\frac{\alpha}{2} b_t^H (r_t^H - r_t^S).$$

Since $r_t^L - r_t^S \leq 0$, in order to prove $r_{t-1}^L - r_{t-1}^S < 0$, it suffices to show

$$\frac{\alpha\delta}{2} b_t^H (r_t^H - r_t^S) > (1-\delta)(u^L - c).$$

Using $E.1$, this inequality can be written as

$$\frac{\alpha\delta}{2} b_t^H [(u^H - c) - \delta(V_{t+1}^H + V_{t+1}^S)] > (1-\delta)(u^L - c).$$

Since $b_t^H > \pi_t$, $L10.1$ implies $b_t^H \geq \pi^*$. (For if $b_t^H < \pi^*$, then $\pi^* = \pi_t < b_t^H$, a contradiction.) Thus by $L5.2$ we have

$$\begin{aligned}\frac{\alpha\delta}{2}b_t^H[(u^H-c)-\delta(V_{t+1}^H+V_{t+1}^S)] &\geq \frac{\alpha\delta}{2}\pi^*(u^H-c)[1-\delta\alpha\frac{1-\delta^{T-t+1}(1-\alpha)^{T-t+1}}{1-\delta(1-\alpha)}] \\ &\geq \frac{\alpha\delta}{2}(u^L-c)[1-\delta\alpha\frac{1-\delta^{T+1}(1-\alpha)^{T+1}}{1-\delta(1-\alpha)}].\end{aligned}$$

Thus, since $u^L - c > 0$, the inequality $r_{t-1}^L - r_{t-1}^S < 0$ holds whenever

$$\psi(\alpha, T, \delta) = \frac{\alpha\delta}{2}(1-\delta\alpha\frac{1-\delta^{T+1}(1-\alpha)^{T+1}}{1-\delta(1-\alpha)}) - 1 + \delta > 0.$$

Note that given $\alpha \in (0,1)$ and T, $\psi(\alpha, T, \delta)$ is continuous on $\delta \in [0,1]$. Also $\psi(\alpha, T, 1) > 0$. Hence there is $\varepsilon(\alpha, T) > 0$ such that for $\delta \in [1-\varepsilon(\alpha,T), 1]$ we have $\psi(\alpha, T, \delta) > 0$, and therefore $r_{t-1}^L - r_{t-1}^S < 0$. $\square$

We are now ready to prove propositions 1 to 4.

Proof of Proposition 1: $P1.1.1$ holds by $L1.1$. We establish $P1.1.2$ by induction. At date T we have $r_T^H = u^H > r_T^S = c > r_T^L = u^L$. Assume that $r_{k+1}^H > r_{k+1}^S \geq c > r_{k+1}^L = u^L$ for $k+1 \leq T$; we show that $r_k^H > r_k^S \geq c > r_k^L = u^L$. As $r_{k+1}^H > r_{k+1}^S > r_{k+1}^L$, $L2.1$ and $L2.3$ imply $p_{k+1}^H = r_{k+1}^S > p_{k+1}^L$. Also $\pi_{k+1} < 0 < b_{k+1}^H$ implies $p_{k+1}^{S_j} = r_{k+1}^H$ for each $j \leq n^S$ by $L3.4$. Thus, using $E1$ we calculate the buyer reservation prices at date k as

$$r_k^H = (1-\delta)u^H + \delta[r_{k+1}^H - \frac{\alpha}{2}(r_{k+1}^H - r_{k+1}^S)],$$

and

$$r_k^L = (1-\delta)u^L + \delta r_{k+1}^L = u^L.$$

The seller reservation price at date k is given by

$$r_k^S = (1-\delta)c + \delta[r_{k+1}^S + \frac{\alpha}{2}b_{k+1}^H(r_{k+1}^H - r_{k+1}^S)] > c > u^L = r_k^L.$$

Since $u^H > c$ and $r_{k+1}^H > r_{k+1}^S$, we have

$$r_k^H - r_k^S = (1-\delta)(u^H-c) + \delta[1-\frac{\alpha}{2}(1+b_{k+1}^H)](r_{k+1}^H - r_{k+1}^S) > 0.$$

Hence $r_k^H > r_k^S \geq c > r_k^L = u^L$.

Now, $P1.1.3$ is implied by $P1.1.2$, $L2.1$, $L2.3$, and $L3.4$. In order to prove $P1.1.4$, note that $P1.1.3$ yields for $i \leq n^L$

$$\mu_t^{L_i} = 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_t^{S_j} I(p_t^{L_i}, r_t^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_t^{S_j} I(r_t^L, p_t^{S_j}) = 1;$$

and since $\mu_t^{H_i} = 1 - \alpha$ for $i \leq n^H$ by $L6.2$, we have

$$\begin{aligned} b_t^H &= \frac{b_{t-1}^H \sum_{i=1}^{n^H} \lambda_{t-1}^{H_i}\mu_{t-1}^{H_i}}{b_{t-1}^H \sum_{i=1}^{n^H} \lambda_{t-1}^{H_i}\mu_{t-1}^{H_i} + b_{t-1}^L \sum_{i=1}^{n^L} \lambda_{t-1}^{L_i}\mu_{t-1}^{L_i}} \\ &= \frac{(1-\alpha)b_{t-1}^H}{(1-\alpha)b_{t-1}^H + 1 - b_{t-1}^H} < b_{t-1}^H. \end{aligned}$$

We prove $P1.2.1$ and $P1.2.2$. All transactions are at either the high-value-buyer or the seller reservation price. These prices are determined, for $t < T$, by the system of difference equations

$$\begin{bmatrix} r_t^H \\ r_t^S \end{bmatrix} = (1-\delta)\begin{bmatrix} u^H \\ c \end{bmatrix} + \delta \begin{bmatrix} 1-\frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2}b_{t+1}^H & 1-\frac{\alpha}{2}b_{t+1}^H \end{bmatrix} \begin{bmatrix} r_{t+1}^H \\ r_{t+1}^S \end{bmatrix},$$

where $r_T^H = u^H$ and $r_T^S = c$. Thus, since $1 - \frac{\alpha}{2}(1 + b_k^H) \leq 1 - \frac{\alpha}{2}$ for each k, we have

$$\begin{aligned} r_t^H - r_t^S &= (1-\delta)\left(u^H - c\right) + \left[1 - \frac{\alpha}{2}(1 + b_{t+1}^H)\right]\left(r_{t+1}^H - r_{t+1}^S\right) \\ &\leq (u^H - c)[(1-\delta)\frac{1 - \delta^{T-t}(1-\frac{\alpha}{2})^{T-t}}{1 - \delta(1-\frac{\alpha}{2})} + \delta^{T-t}(1-\frac{\alpha}{2})^{T-t}]. \end{aligned}$$

Also from above we have for each t

$$b_t^H = \frac{(1-\alpha)b_{t-1}^H}{(1-\alpha)b_{t-1}^H + 1 - b_{t-1}^H} = \frac{(1-\alpha)^t b_0^H}{1 - [1-(1-\alpha)^t]b_0^H}.$$

Then, since $\sum_{k=0}^{T-t}\delta^k < \frac{1}{1-\delta}$, $\frac{1}{1-[1-(1-\alpha)^k]b_0^H} < \frac{1}{1-b_0^H}$, and since $(1-\alpha)^k < (1-\frac{\alpha}{2})^k$ for $k > 0$, we have

$$\begin{aligned} r_t^S &= (1-\delta)c + \delta[r_{t+1}^S + \frac{\alpha}{2}b_{t+1}^H(r_{t+1}^H - r_{t+1}^S)] \\ &= c + \frac{\alpha}{2}\sum_{k=t+1}^{T} \delta^{k-t} b_k^H (r_k^H - r_k^S) \\ &< c + \frac{\alpha}{2}(u^H - c)\frac{b_0^H}{(1-b_0^H)[1-\delta(1-\frac{\alpha}{2})]}\eta(\delta, T), \end{aligned}$$

where

$$\begin{aligned}\eta(\delta, T) &= \sum_{k=t+1}^{T} \delta^{k-t}(1-\frac{\alpha}{2})^{k}[1-\delta+\frac{\alpha}{2}\delta^{T-k+1}(1-\frac{\alpha}{2})^{T-k}] \\ &= (1-\delta)\delta(1-\frac{\alpha}{2})^{t+1}\frac{1-\delta^{T-t}(1-\frac{\alpha}{2})^{T-t}}{1-\delta(1-\frac{\alpha}{2})}+\frac{\alpha}{2}(1-\frac{\alpha}{2})^{T}(T-t)\,\delta^{T-t+1}.\end{aligned}$$

Since $\lim_{T\to\infty}(T-t)(1-\frac{\alpha}{2})^{T}=0$, we have

$$\lim_{T\to\infty}\lim_{\delta\to 1}\eta(\delta, T)=\lim_{\delta\to 1}\lim_{T\to\infty}\eta(\delta, T)=0.$$

Hence, since $r_t^S \geq c$, we have $\lim_{T\to\infty}\lim_{\delta\to 1} r_t^S = \lim_{\delta\to 1}\lim_{T\to\infty} r_t^S = c$. Therefore, $P1.2.1$ holds.

Also, from above, we have

$$r_t^H = \left(u^H - c\right)(1-\frac{\alpha}{2})^{T-t} + r_t^S.$$

Hence, as $r_t^H > r_t^S \geq c$, $\lim_{T\to\infty}\lim_{\delta\to 1} r_t^H = \lim_{\delta\to 1}\lim_{T\to\infty} r_t^H = c$. Therefore $P1.2.2$ holds. $\square$

Proof of Proposition 2: Some of the properties listed in Proposition 2 are direct implications of the previous lemmas: $P2.1.1$ has been established in Lemma 1 ($L1.1$); $P2.1.2$ is implied by $L4.1$ and $L5.1$; $P2.2$ is $L6.1$; $P2.3.1$ is implied by Lemma 2; $P2.4.1$ is implied by $P2.1.2$ and $L3.1$.

We prove $P2.3.2$. By Lemma 12 there is $\varepsilon(\alpha, T) > 0$ such that if $\delta > 1-\varepsilon(\alpha, T)$, then $r_t^L \leq r_t^S$ implies $r_{t-1}^L < r_{t-1}^S$. Suppose that $\delta \in [1-\varepsilon(\alpha, T), 1]$. If $p_{\bar{t}}^{L_i} < r_{\bar{t}}^S$ for some $i \leq n^L$, then $L2.1$ implies $r_{\bar{t}}^L \leq r_{\bar{t}}^S$; hence then $r_{\bar{t}-1}^L < r_{\bar{t}-1}^S$, and by induction $r_t^L < r_t^S$ for $t < \bar{t}$; therefore $L2.3$ implies $p_t^{L_i} < r_t^S$ for every $t < \bar{t}$ and $i \leq n^L$. Now let $\bar{t}$ and $i \leq n^L$ be such that $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$, suppose by way of contradiction that $p_{\bar{t}}^{L_{i'}} = r_{\bar{t}}^S$ for some $i' \leq n^L$, and $p_{\hat{t}}^{L_{i''}} < r_{\hat{t}}^S$ for some $i'' \leq n^L$ and $\hat{t} > \bar{t}$; then the previous argument implies $p_t^{L_i} < r_t^S$ for every $t < \hat{t}$ and $i \leq n^L$; in particular, $p_{\bar{t}}^{L_{i'}} < r_{\bar{t}}^S$, which is a contradiction.

We now prove $P2.4.2$. Assume that $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ for some $j \leq n^S$. Then we have $b_{\bar{t}}^H \leq \pi^*$, for if $b_{\bar{t}}^H > \pi^*$ then $b_{\bar{t}}^H > \pi_{\bar{t}}$ by $L10.3$, and therefore we would have $p_{\bar{t}}^{S_j} = r_{\bar{t}}^H$ for each $j \leq n^S$ by $L3.4$ and $P2.1.2$, which is a contradiction. Suppose that $b_{\bar{t}}^H < \pi^*$;

then $L10.1$ implies $b_t^H < \pi_t$ for each $t \geq \bar{t}$, and therefore $p_t^{S_j} = r_t^L$ for every $t \geq \bar{t}$ and $j \leq n^S$ by $L3.2$. Suppose that $b_{\bar{t}}^H = \pi^*$; then either $b_{\bar{t}+1}^H < \pi^*$ or $b_t^H = \pi^*$ for $t \geq \bar{t}$ by $L10.2$. If $b_{\bar{t}+1}^H < \pi^*$, then $L10.1$ again implies $b_t^H < \pi_t$ for each $t \geq \bar{t}+1$ and therefore $p_t^{S_j} = r_t^L$ for every $t \geq \bar{t}+1$ and $j \leq n^S$, by $L3.2$. If $b_t^H = \pi^*$ for $t \geq \bar{t}$, since high-value buyers always trade when they are matched (by $P2.2$, $P2.1.2$ and $P2.4.1$), then low-value buyers must also trade when matched; hence $p_t^{S_j} = r_t^L$ for every $t \geq \bar{t}+1$ and $j \leq n^S$.

We establish $P2.4.3$. Suppose by way of contradiction that $p_{\bar{t}}^{S_{j'}} = r_{\bar{t}}^H$ for some $j' \leq n^S$, and $p_{\hat{t}}^{S_{j''}} = r_{\hat{t}}^L$ for some $j'' \leq n^S$ and $\hat{t} < \bar{t}$. Then $P2.4.2$ implies $p_t^{S_j} = r_t^L$ for every $t > \hat{t}$ and $j \leq n^S$. In particular, $p_{\bar{t}}^{S_{j'}} = r_{\bar{t}}^L$, which is a contradiction.

Finally, we prove $P2.5$. If $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ for some $j \leq n^S$ then $L3.4$ implies $b_{\bar{t}}^H \leq \pi_{\bar{t}}$, and since $b_{\bar{t}}^H > 0$ and $r_{\bar{t}}^H - r_{\bar{t}}^S > 0$ (by $L5.1$), we have $r_{\bar{t}}^L - r_{\bar{t}}^S > 0$. Hence $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ for every $i \leq n^L$ by $L2.1$. □

Proof of Proposition 3: We prove $P3.1.1$. Assume $b_{\bar{t}}^H < \pi^*$; then $b_{\bar{t}}^H < \pi_{\bar{t}}$ by $L10.1$, and $p_{\bar{t}}^{S_j} = r_{\bar{t}}^L$ by $L3.2$; hence $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ by $P2.5$ for every $i \leq n^L$. Thus $I(p_{\bar{t}}^{L_i}, r_{\bar{t}}^S) = I(r_{\bar{t}}^L, p_{\bar{t}}^{S_j}) = 1$, and therefore

$$\mu_{\bar{t}}^{L_i} = 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{\bar{t}}^{S_j} I(p_{\bar{t}}^{L_i}, r_{\bar{t}}^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{\bar{t}}^{S_j} I(r_{\bar{t}}^L, p_{\bar{t}}^{S_j}) = 1 - \alpha.$$

Since $\mu_{\bar{t}}^{H_i} = 1 - \alpha$ by $L5.2$, we have

$$b_{\bar{t}+1}^H = \frac{(1-\alpha)b_{\bar{t}}^H}{(1-\alpha)b_{\bar{t}}^H + (1-\alpha)b_{\bar{t}}^L} = b_{\bar{t}}^H.$$

Hence $P3.1.1$ holds.

We establish $P3.1.2$. Assume $b_{\bar{t}}^H = \pi^*$. Then by $L10.2$ either $b_{\bar{t}+1}^H < \pi^*$ or $b_{\bar{t}}^H = \ldots = b_T^H = \pi^*$. If $b_{\bar{t}+1}^H < \pi^* = b_{\bar{t}}^H$, then $b_{\bar{t}+1}^H < \pi_{\bar{t}+1}$ (by $L10.1$), and $p_{\bar{t}+1}^{S_j} = r_{\bar{t}+1}^L$ for every $j \leq n^S$ by $L3.2$. Hence $\pi_{\bar{t}} = \pi^* > 0$ by $L8.1$, and therefore $r_{\bar{t}}^L > r_{\bar{t}}^S$ (because $r_{\bar{t}}^H > r_{\bar{t}}^S$ by $P2.1.2$), and $L2.1$ implies $p_{\bar{t}}^{L_i} = r_{\bar{t}}^S$ for every $i \leq n^L$.

If $b_{\bar{t}}^H = \ldots = b_T^H = \pi^*$, since matched high-value buyers trade (and therefore exit the market) by $P2.2$, $P2.1.2$ and $P2.4.1$, then matched low-value buyers must also trade; hence for $t \geq \bar{t}$, $p_t^{L_i} = r_t^S$ for every $i \leq n^L$ and $p_t^{S_j} = r_t^L$ for every $j \leq n^S$. Thus, $P3.1.2$ holds.

We show that $P3.1.3$ holds. Suppose that $b_{\bar{t}}^H > \pi^*$; then $b_{\bar{t}}^H > \pi_{\bar{t}}$ by Lemma $L10.3$, and therefore $p_{\bar{t}}^{S_j} = r_{\bar{t}}^H$ for each $j \le n^S$ by $L3.4$; since $r_{\bar{t}}^H > r_{\bar{t}}^L$ by $L4.1$, then $I(r_{\bar{t}}^L, p_{\bar{t}}^{S_j}) = 0$ for each $j \le n^S$, and therefore for each $i \le n^L$ we have

$$\begin{aligned}
\mu_{\bar{t}}^{L_i} &= 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{\bar{t}}^{S_j} I(p_{\bar{t}}^L, r_{\bar{t}}^{S_j}) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{\bar{t}}^{S_j} I(r_{\bar{t}}^L, p_{\bar{t}}^{S_j}) \\
&= 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_{\bar{t}}^{S_j} I(p_{\bar{t}}^L, r_{\bar{t}}^{S_j}) \ge 1 - \frac{\alpha}{2}.
\end{aligned}$$

Thus, since $\mu_{\bar{t}}^{H_i} = 1 - \alpha$ by $L6.2$, we get

$$\begin{aligned}
b_{\bar{t}+1}^H &= \frac{(1-\alpha)b_{\bar{t}}^H}{(1-\alpha)b_{\bar{t}}^H + b_{\bar{t}}^L \sum_{i=1}^{n^L} \lambda_{\bar{t}}^{L_i} \mu_{\bar{t}}^{L_i}} \\
&\le \frac{(1-\alpha)b_{\bar{t}}^H}{(1-\alpha)b_{\bar{t}}^H + (1-\frac{\alpha}{2})(1-b_{\bar{t}}^H)} < b_{\bar{t}}^H.
\end{aligned}$$

Therefore $P3.1.3$ holds.

Finally, we establish $P3.2$. Define the sequence $\{\underline{b}_t\}$ by $\underline{b}_0 = b_0^H$, and for $t > 0$

$$\underline{b}_{t+1} = \frac{(1-\alpha)\underline{b}_t}{(1-\alpha)\underline{b}_t + (1-\frac{\alpha}{2})(1-\underline{b}_t)}.$$

We show that $b_t^H > \pi^*$ implies $b_{t+1}^H \le \underline{b}_{t+1}$. Assume $b_t^H > \pi^*$; we show by induction that $b_k^H \le \underline{b}_k$ for $k \le t+1$. By construction $b_0^H \le \underline{b}_0$. Assume that $b_k^H \le \underline{b}_k$ for $k \le t$; we show that $b_{k+1}^H \le \underline{b}_{k+1}$. Since $\{b_t^H\}$ is non-increasing by Lemma 7 and $k \le t$, then $b_k^H \ge b_t^H > \pi^*$. Therefore $P3.1.3$ implies $p_k^{S_j} = r_k^H$, and hence $I(r_k^L, p_k^{S_j}) = 0$, for every $j \le n^S$. Therefore

$$\mu_k^{L_i} = 1 - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_k^{S_j} I(p_k^{L_i}, r_k^S) - \frac{\alpha}{2}\sum_{j=1}^{n^S} \lambda_k^{S_j} I(r_k^L, p_k^{S_j}) \ge 1 - \frac{\alpha}{2},$$

for every $i \le n^L$. Thus, since $\mu_k^{H_i} = 1 - \alpha$ by $L6.2$, we have

$$\begin{aligned}
b_{k+1}^H &= \frac{(1-\alpha)b_k^H}{(1-\alpha)b_k^H + b_k^L \sum_{i=1}^{n^L} \lambda^{L_i} \mu_k^{L_i}} \\
&\le \frac{(1-\alpha)\underline{b}_k^H}{(1-\alpha)\underline{b}_k^H + (1-\frac{\alpha}{2})(1-\underline{b}_k^H)} = \underline{b}_1.
\end{aligned}$$

We now prove $P3.2$. If $b_0^H \le \pi^*$, the $P3.2$ holds for $\bar{T} = 0$. If $b_0^H > \pi^*$, let $\bar{T} = \bar{T}(b_0^H, \alpha, \pi^*)$ be the first integer such that $\underline{b}_{\bar{T}} < \pi^*$. Such integer exists, since

$$\underline{b}_t = \frac{(1-\alpha)^t b_0^H}{(1-\alpha)^t b_0^H + (1-\frac{\alpha}{2})^t (1-b_0^H)},$$

and therefore $\{\underline{b}_t\}$ converges to zero. Suppose $b_{\bar{T}}^H > \pi^*$; then $b_{\bar{T}-1}^H > \pi^*$ (by Lemma 7), and therefore $b_{\bar{T}}^H \leq \underline{b}_{\bar{T}} < \pi^*$, which is a contradiction. Thus $b_{\bar{T}}^H \leq \pi^*$, and since $\{b_t^H\}$ is a non-increasing sequence (Lemma 7), $b_t^H \leq \pi^*$ for $t \geq \bar{T}$. Hence $P3.2$ holds. $\square$

Proof of Proposition 4: By $P3.2$ there is $\bar{T} = \bar{T}(b_0^H, \alpha, \pi^*)$ such that $b_t^H \leq \pi^*$ for $t \geq \bar{T}$. Thus, $P3.1.1$ and $P3.1.2$ imply $p_t^{S_j} = r_t^L < r_t^H$ for every $j \leq n^S$ and $t \geq \bar{T}+1$, and therefore $p_t^{L_i} = r_t^S$ for every $i \leq n^L$ and $t \geq \bar{T}+1$ by $P2.5$. Also $p_t^{H_i} = r_t^S$ for every $i \leq n^H$ and t by $P2.2$. Thus, for each $\delta \in [0,1]$ and T, let $r \in r(\delta, T)$ be a sequence of equilibrium reservation prices and let V be the corresponding sequence of expected utilities. Since $r_t^\tau = u^\tau - \delta V_{t+1}^\tau$ for $\tau \in \{H, L\}$, and $r_t^S = c + \delta V_{t+1}^S$ by $E1$, traders expected utilities for $t \geq \bar{T}+1$ are given by the system of difference equations

$$\begin{bmatrix} V_t^H \\ V_t^L \\ V_t^S \end{bmatrix} = \frac{\alpha}{2} \begin{bmatrix} 2u^H - u^L - c \\ u^L - c \\ u^L - c \end{bmatrix} + \delta \begin{bmatrix} 1-\alpha & \frac{\alpha}{2} & -\frac{\alpha}{2} \\ 0 & 1-\frac{\alpha}{2} & -\frac{\alpha}{2} \\ 0 & -\frac{\alpha}{2} & 1-\frac{\alpha}{2} \end{bmatrix} \begin{bmatrix} V_{t+1}^H \\ V_{t+1}^L \\ V_{t+1}^S \end{bmatrix}$$

Thus, for every date after $\bar{T} = \bar{T}(b_0^H, \alpha, \pi^*)$ traders expected utilities are uniquely determined. Noting that V_t^S and V_t^L are determined independently of V_t^H, and using $V_{T+1}^L = V_{T+1}^S = 0$, we can solve for V_t^S and V_t^L, to obtain

$$V_t^S = V_t^L = \frac{\alpha}{2}(u^L - c)\frac{1 - [\delta(1-\alpha)]^{T-t+1}}{1 - \delta(1-\alpha)}.$$

Thus, for $t \geq \bar{T}+1$ we have

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^S = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^S = \frac{u^L - c}{2},$$

and

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^L = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^L = \frac{u^L - c}{2}.$$

For high-value buyers, since $V_{T+1}^H = 0$, the above system yields for $t \geq \bar{T}+1$

$$V_t^H = \frac{\alpha}{2}(2u^H - u^L - c)\frac{1 - [\delta(1-\alpha)]^{T-t+1}}{1 - \delta(1-\alpha)}.$$

Hence

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^H = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^H = u^H - \frac{u^L + c}{2}.$$

Now let $t < \bar{T} + 1$. A trader of type τ who is in the market at date t obtains an expected utility of V_t^τ by following his equilibrium strategy; thus the expected utility to a trader who remains in the market at t must satisfy $V_t^\tau \geq \delta^{\bar{T}-t} V_{\bar{T}+1}^\tau$, for otherwise he benefits from a deviation where he makes unacceptable offers and rejects any offers until date $\bar{T}+1$, following his equilibrium strategy thereafter. Also $V_t^S + V_t^H \leq u^H - c$ by $L5.2$. Thus

$$\delta^{\bar{T}-t} V_{\bar{T}+1}^S \leq V_t^S \leq u^H - c - V_t^H \leq u^H - c - \delta^{\bar{T}-t} V_{\bar{T}+1}^H,$$

and therefore

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^S = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^S = \frac{u^L - c}{2}.$$

Also

$$\delta^{\bar{T}-t} V_{\bar{T}+1}^H \leq V_t^H \leq u^H - c - V_t^S \leq u^H - c - \delta^{\bar{T}-t} V_{\bar{T}+1}^S;$$

hence

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^H = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^H = u^H - \frac{u^L + c}{2}.$$

For low-value buyers we have $V_t^S \geq V_t^L \geq \delta^{\bar{T}-t} V_{\bar{T}+1}^L$, where the first inequality follows from Lemma 11 and the second was established above. Thus,

$$\lim_{\delta \to 1} \lim_{T \to \infty} V_t^L = \lim_{T \to \infty} \lim_{\delta \to 1} V_t^L = \frac{u^L - c}{2},$$

Furthermore, since $r_t^S = c + \delta V_{t+1}^S$ and $r_t^\tau = u^\tau - \delta V_{t+1}^\tau$ for $\tau \in \{H, L\}$ by $E1$, the above limits imply

$$\lim_{\delta \to 1} \lim_{T \to \infty} r_t^\tau = \lim_{T \to \infty} \lim_{\delta \to 1} r_t^\tau = \frac{u^L + c}{2},$$

for $\tau \in \{H, L, S\}$ and t. □

References

[1] Binmore, K. and M. Herrero (1988): "Matching and Bargaining in Dynamic Markets," *Review of Economic Studies* **55**, 17-31.

[2] Border, K. (1985): *Fixed point theorems with applications to economics and game theory*, Cambridge Universiy Press.

[3] Cellina, A. (1969): "Approximation of set valued functions and fixed point theorems," *Annali di Matematica Pura ed Aournal of Economic Theory* **43**, 20-54.

[4] Gale, D. (1987): "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory* **43**, 20-54.

[5] Jackson, M. and T. Palfrey. (1998): "Efficiency and Voluntary Implementation in Markets with Repeated Parewise Bargaining," *Econometrica* **66**, 1353-1388.

[6] Rubinstein, A. and A. Wolinsky. (1985): "Equilibrium in a Market with Sequential Bargaining," *Econometrica* **53**, 1133-1150.

[7] Samuelson, L. (1992): "Disagreement in Markets with Matching and Bargaining," *Review of Economic Studies* **59**, 177-186.

[8] Sattinger, M. (1995): "Search and the Efficient Assignment of Workers to Jobs," *International Economic Review* **36**, 283-302.

[9] Serrano, R. and O. Yosha (1996): "Decentralized Information and the Walrasian Outcome: A Pairwise Meetings Market with Private Values," mimeo.

[10] Varian, H. (1980): "A Model of Sales," *The American Economic Review* **70**, 651-659.

[11] Wolinsky, A. (1990): "Information Revelation in a Market with Pairwise Meetings," *Econometrica* **58**, 1-23.

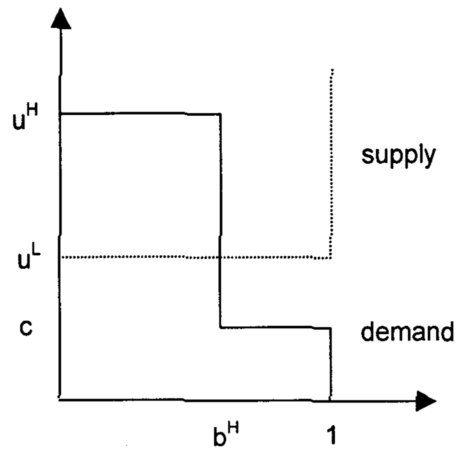


Figure 1

Transaction Prices

$(b_0^H = .94,\ u^H = 1, u^L = .1, c = .2; \delta = .9, \alpha = .5)$

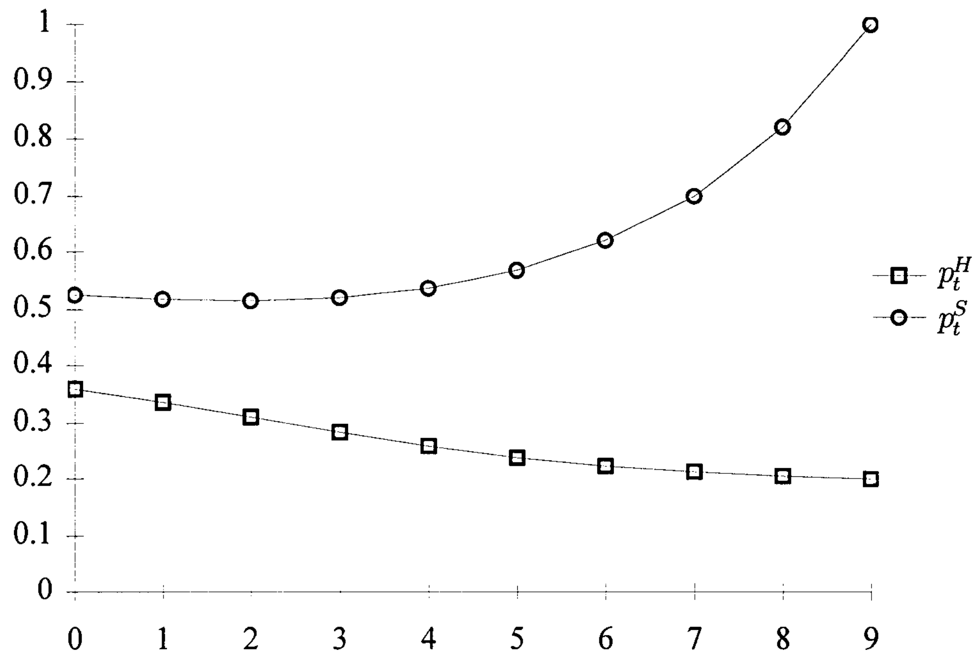


Figure 2

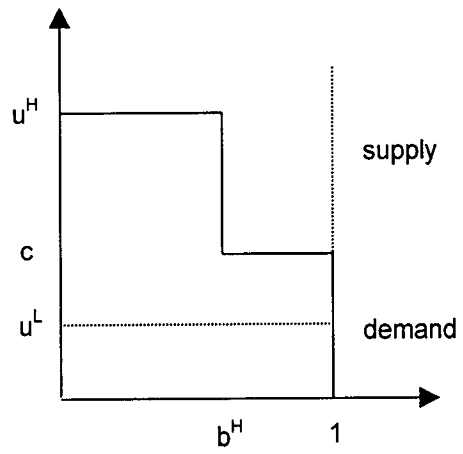


Figure 3

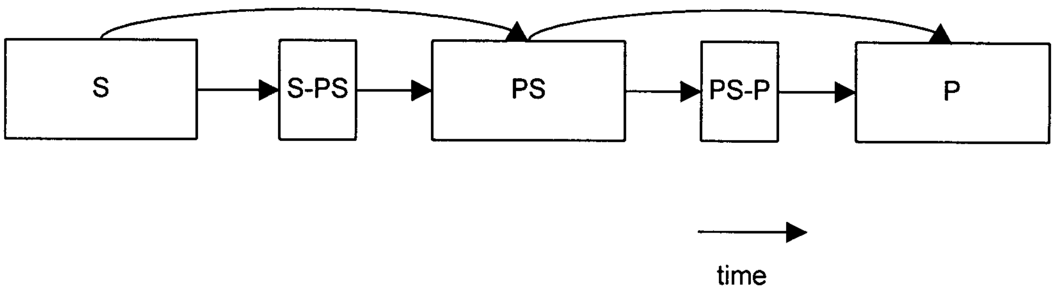
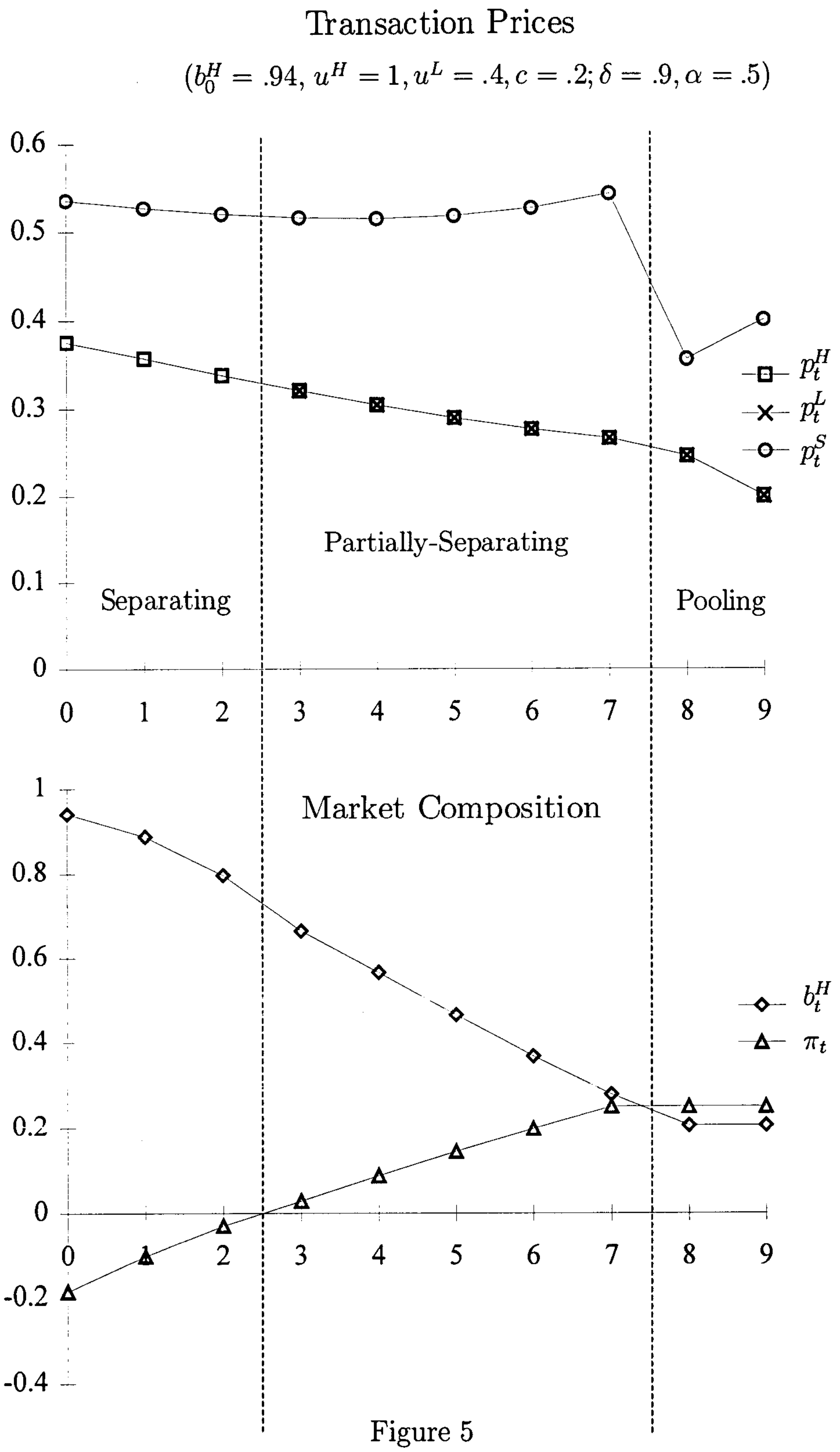


Figure 4



Transaction Prices
$(b_0^H = .94,\ u^H = 1, u^L = .4, c = .2; \delta = .9, \alpha = .5)$
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9
p_t^H
p_t^L
p_t^S
Separating
Partially-Separating
Pooling
Market Composition
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
b_t^H
π_t

Figure 5

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