Statistical distribution of the estimator of Weibull modulus

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The Weibull statistic [1] has been widely used to study the inherent scatter existing in the strength properties of many advanced materials [2–7], as well as in the fracture toughness of steels in the ductile-brittle transition region [8, 9]

The two-parameter Weibull distribution function is given by:

$$\boldsymbol{F} = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^{m}\right] \tag{1}$$

where $\boldsymbol{F}$ is the probability of rupture under uniaxial tensile stress σ, $\boldsymbol{m}$ the shape parameter or Weibull modulus and σ_0 the scale parameter. From the results of a limited number of tests and by applying standard statistical techniques (maximum likelihood, generalized regression, moments method, etc.), estimations of the parameters $\boldsymbol{m}$ and σ_0 can be obtained. Obviously these estimation values are subject to uncertainties, so, for design purposes, it is necessary to calculate the appropriated confidence intervals of the estimators.

The confidence interval of the Weibull modulus estimation, $\hat{\boldsymbol{m}}$, can be obtained from the percentage points, $\boldsymbol{l}_\alpha$, of the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$, defined as:

$$\Pr\left[\frac{\hat{\boldsymbol{m}}}{\boldsymbol{m}} \leq \boldsymbol{l}_\alpha\right] = \alpha \tag{2}$$

Thus, the limits of the interval for a confidence level γ are $\boldsymbol{l}_{\frac{1-\gamma}{2}}$ and $\boldsymbol{l}_{\frac{\gamma}{2}}$.

The percentage points, $\boldsymbol{l}_\alpha$, were numerically calculated by Thoman ***et al.*** [10] without any assumption about the statistical distribution of the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$, and they were published in the form of tables.

To obtain the statistical distribution of the pivotal variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$, a simulation procedure based on the Monte Carlo method may be used. In this procedure, a set of $\boldsymbol{n}$ values (sample size) are generated as:

$$\sigma_{\boldsymbol{i}} = \sigma_0 \cdot \ln\left(\frac{1}{\boldsymbol{R}}\right)^{\frac{1}{m}} \tag{3}$$

where $\boldsymbol{R}$ is a random variable with uniform distribution in the [0, 1] interval. From each sample so obtained, estimations of the Weibull modulus are computed using the maximum likelihood method, and from these estimations, the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$ may also be built. Repeated application of this procedure provides a statistical distribution of this latter variable. Thoman ***et al.*** [10] showed that, if the method of maximum likelihood is used to estimate $\boldsymbol{m}$, the distribution of the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$ is independent of the true values of the parameters $\boldsymbol{m}$ and σ_0. Therefore, in order to make the simulation, any values of these parameters can be chosen ($\boldsymbol{m} = 1$ and $\sigma_0 = 1$, for example).

To describe the statistical behavior of $\hat{\boldsymbol{m}}$ by means of a conventional probability distribution function, Gong [11] assumed that this variable follows a Log-normal distribution, with mean value, $\boldsymbol{M}$, and standard deviation, $\boldsymbol{S}$, both of them depending on sample size, $\boldsymbol{n}$. Barbero ***et al.*** [12] proposed a three-parameter Weibull distribution for $\hat{\boldsymbol{m}}/\boldsymbol{m}$. To obtain a better approximation, the authors now propose that the variable $\ln\left(\frac{\hat{\boldsymbol{m}}}{\boldsymbol{m}}\right)$, named $\boldsymbol{X}$ throughout this work, follows a three parameter Weibull distribution.

The aim of this letter is to compare the results deriving from the above distributions with those obtained numerically. The authors calculated the percentage points, repeating 20 000 times the numerical procedure stated above, for each sample size, increasing progressively this latter from 5 to 120. From the numerical results, the mean values, $\boldsymbol{M}$ and $\boldsymbol{M}_{\mathrm{x}}$ and the standard deviations, $\boldsymbol{S}$ and $\boldsymbol{S}_{\mathrm{x}}$, of the variables $\hat{\boldsymbol{m}}/\boldsymbol{m}$ and $\boldsymbol{X}$, respectively, were calculated and fitted to the sample size, $\boldsymbol{n}$, with the following four-parameter functions:

$$\boldsymbol{M} = 0.9807 + 1.7001 \cdot \left(\frac{1}{\ln(1.0408 \cdot \boldsymbol{n})}\right)^{2.5873} \tag{4a}$$

$$\boldsymbol{S} = -0.1357 + 0.5297 \cdot \left(\frac{1}{\ln(0.3087 \cdot \boldsymbol{n})}\right)^{0.7303} \tag{4b}$$

$$\boldsymbol{M}_{\mathrm{x}} = -0.01455 + 5.87953 \cdot \left(\frac{1}{\ln(2.7293 \cdot \boldsymbol{n})}\right)^{3.18323} \tag{5a}$$

$$\boldsymbol{S}_{\mathrm{x}} = -0.03669 + 5.5248 \cdot \left(\frac{1}{\ln(4.79698 \cdot \boldsymbol{n})}\right)^{2.12069} \tag{5b}$$

Figs 1 and 2 show the comparison between the numerical and fitted values for the mean and standard deviation of the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$ (Fig. 1) and $\ln\left(\frac{\hat{\boldsymbol{m}}}{\boldsymbol{m}}\right)$ (Fig. 2). Excellent agreement is observed in the range of sample sizes analysed ($\boldsymbol{n} = 5$–120).

To describe the statistical behavior of the variable $\hat{\boldsymbol{m}}/\boldsymbol{m}$, recently Barbero ***et al.*** [12] proposed a

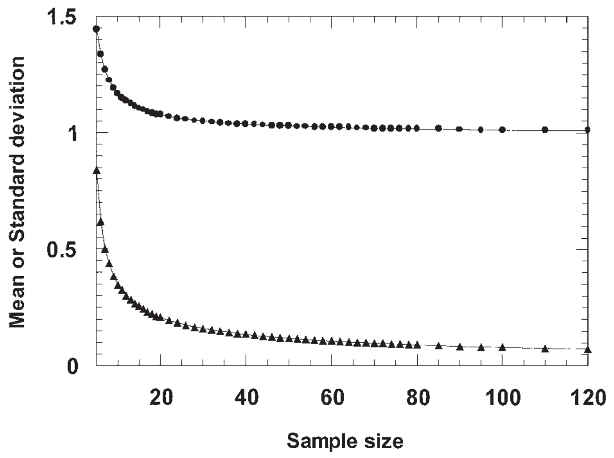

Figure 1 Mean value (circles) and standard deviation (triangles) of variable $\hat{m}/m$ as a function of sample size. The solid lines in the figure are the fitted lines according to Equations 4a and b.

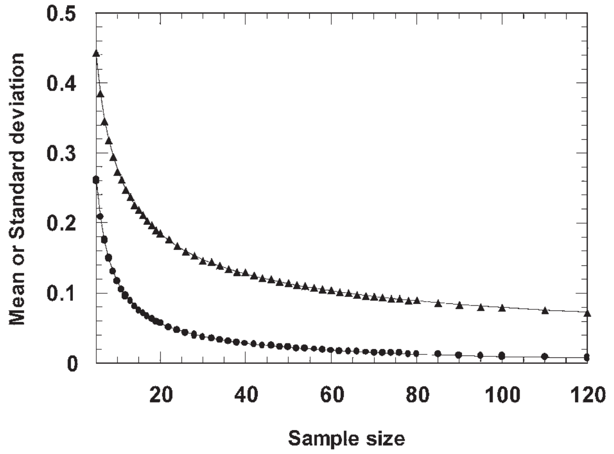

Figure 2 Mean (circles) and standard deviation (triangles) of variable $\ln \hat{m}/m$ as a function of sample size. The solid lines in the figure are the fitted lines according to Equations 5a and b.

three-parameter Weibull distribution:

$$F(\hat{m}/m) = 1 - \exp\left[-\cdot\left(\frac{\hat{m}/m - P_1}{P_2}\right)^{P_3}\right] \qquad (6)$$

where P_1, P_2, P_3 are, respectively, the position, scale and form parameters that were fitted as a functions of the sample size, n, by:

$$P_i = a_{i1} + a_{i2} \cdot (\ln n)^{a_{i3}} \quad (i = 1,\ 2,\ 3) \qquad (7)$$

the parameters a_{i1}, a_{i2} and a_{i3} being those shown in Table I.

Now the authors propose a new way to describe the statistical behavior of the variable $\hat{m}/m$, assuming that the variable $X\,(X = \ln(\frac{\hat{m}}{m}))$ follows a three parameter

TABLE I Parameter of Equation 7

Parameter	a_{i1}	a_{i2}	a_{i3}
P_1	0.65303	0.00467	2.33393
P_2	2.47938	−1.65201	0.20487
P_3	−1.13169	1.52229	0.59986

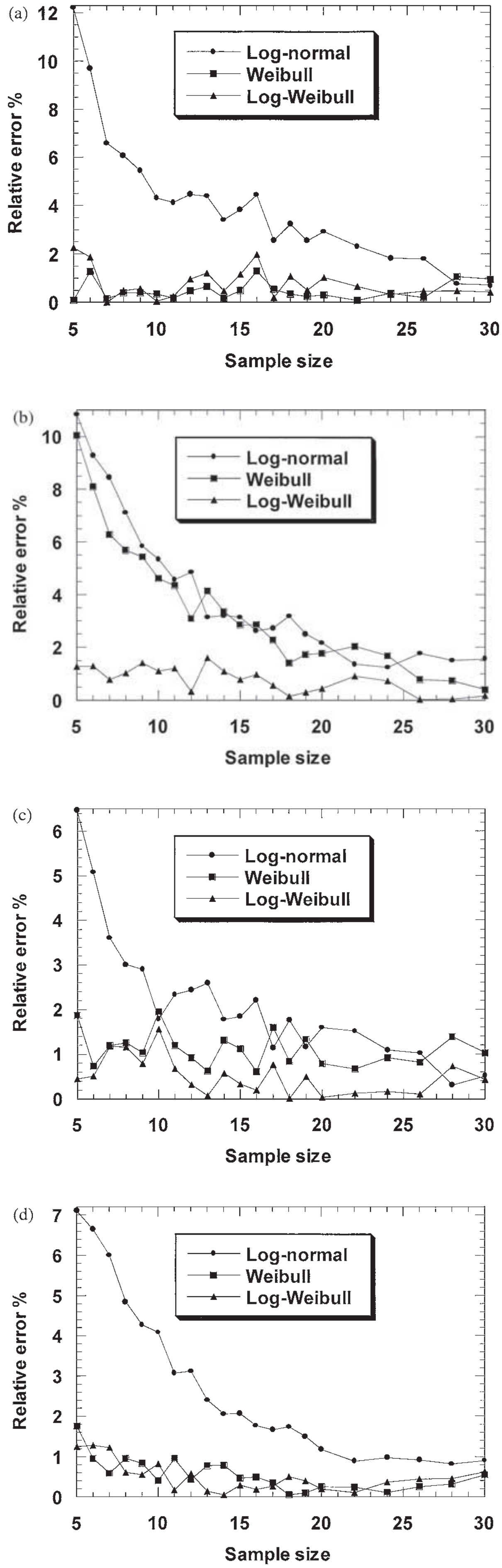

Figure 3 Relative error between the percentage points numerically obtained and those fitted of variable $\hat{m}/m$. a) $\alpha = 0.975$. b) $\alpha = 0.025$. c) $\alpha = 0.950$. d) $\alpha = 0.050$.

TABLE II Parameter of Equation 9

Parameter	b_{i1}	b_{i2}	b_{i3}
Q_1	−1.50972	0.70844	0.39005
Q_2	−1.25257	2.74062	−0.38729
Q_3	−1.12766	2.84699	0.29276

Weibull distribution:

$$F(X) = 1 - \exp\left[- \cdot \left(\frac{X - Q_1}{Q_2} \right)^{Q_3} \right] \qquad (8)$$

where Q_1, Q_2, Q_3 may be fitted as a function of the sample size, n, by:

$$Q_1 = b_{i1} + b_{i2} \cdot (\ln n)^{b_{i3}} \quad (i = 1, 2, 3) \qquad (9)$$

The parameters b_{i1}, b_{i2}, y b_{i3} are given in Table II.

To compare the cited approximations, the limits of two confidence intervals were analyzed. The selected confidence levels were $\gamma = 0.9$ and $\gamma = 0.95$, that are widely used for design purposes.

The corresponding limits are the percentage points, l_α, of the variable $\hat{m}/m$, for $\alpha = 0.05$ and $\alpha = 00.95$ (case $\gamma = 0.9$) and $\alpha = 0.025$ and $\alpha = 0.975$ (case $\gamma = 0.95$). The relative errors between the percentage points numerically obtained, $(l_\alpha)_{\mathrm{num}}$, and those fitted, $(l_\alpha)_{\mathrm{fit}}$, by the above mentioned methods (Log-normal [11], Weibull [12], and that proposed in this paper, (hereinafter named Log-Weibull) were calculated using

$$\mathrm{error}(\%) = \mathrm{abs}\left(1 - \frac{(l_\alpha)_{\mathrm{nur}}}{(l_\alpha)_{\mathrm{fit}}} \right) \times 100 \qquad (10)$$

The results obtained are shown in Fig. 3. Log-normal distribution leads to higher errors than Weibull and Log-Weibull distributions, particularly for small sample sizes. For n values larger than 30, all approximations are similar, with maximum errors less than 2%. In general, Log-Weibull distribution gives errors less than 2%, independently of the sample size.

Acknowledgment
The authors are indebted to the Fundación Ramón Areces (Área de Materiales, IX Concurso Nacional) for its financial support of this research.

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