Conditionally heteroscedastic unobserved component models and their reduced form ☆

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A B S T R A C T

The reduced form of the local level model with conditionally heteroscedastic GARCH(1,1) noises is analyzed. We show that the IMA GARCH model is a good alternative but its conditional heteroscedasticity is weaker than this of the unobserved disturbances.

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1. Introduction

When economic and financial time series have stochastic trends, it is common taking differences and fitting ARMA models to the corresponding stationary transformation. Alternatively, the series may be represented by unobserved component models. In the presence of Gaussian disturbances, both models are equivalent in the sense that they have the same autocorrelation function (acf); see, for example, (Harvey, 1989). In this paper, we analyze the relationship between them when the underlying stochastic trends are conditionally heteroscedas tic; see, for example, (Diebold, 2004) for conditionally heteroscedastic ARIMA models and (Stock and Watson, 2007) for unobserved component models with conditionally heteroscedastic noises.

We focus on the following local level model (LLM) in which the series of interest, y_t, is composed by a transitory component, ε_t, and a stochastic level, μ_t,

$$y_t = \mu_t + \varepsilon_t, \tag{1a}$$

$$\mu_t = \mu_{t-1} + \eta_t, \tag{1b}$$

where ε_t and η_t are mutually independent and serially uncorrelated processes, with zero means and variances σ_ε^2 and σ_η^2, respectively. Taking first differences in model (1), we obtain the following stationary representation

$$\Delta y_t = \eta_t + \Delta\varepsilon_t. \tag{2}$$

Alternatively, model (2) can be represented by the following IMA (1,1) model

$$\Delta y_t = a_t + \theta a_{t-1}, \tag{3}$$

where, if Δy_t is invertible, $\theta = [(q^2+4q)^{1/2} - 2 - q]/2$, with $q = \sigma_\eta^2/\sigma_\varepsilon^2$ being the signal to noise ratio, and the reduced form disturbance, a_t, is uncorrelated with zero mean and positive variance, $\sigma_a^2 = -\frac{\sigma_\varepsilon^2}{\theta}$. Our objective is to analyze the properties of a_t when ε_t and η_t are GARCH(1,1).[1]

2. Properties of the local level model

Conditionally heteroscedastic series are characterized by having excess kurtosis and positive autocorrelations of squares. Therefore, in this section, we derive these two moments for Δy_t. Consider model 1 and assume that the noises have symmetric distributions around zero, and finite fourth order moments. Then, the excess kurtosis of Δy_t is given by

$$\overline{\kappa}_{\Delta y} = \frac{q^2\,\overline{\kappa}_\eta + 2\overline{\kappa}_\varepsilon + 6(\overline{\kappa}_\varepsilon + 2)\rho_1^{\varepsilon^2}}{(q+2)^2}, \tag{4}$$

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[1] The results for stochastic volatility noises are similar.

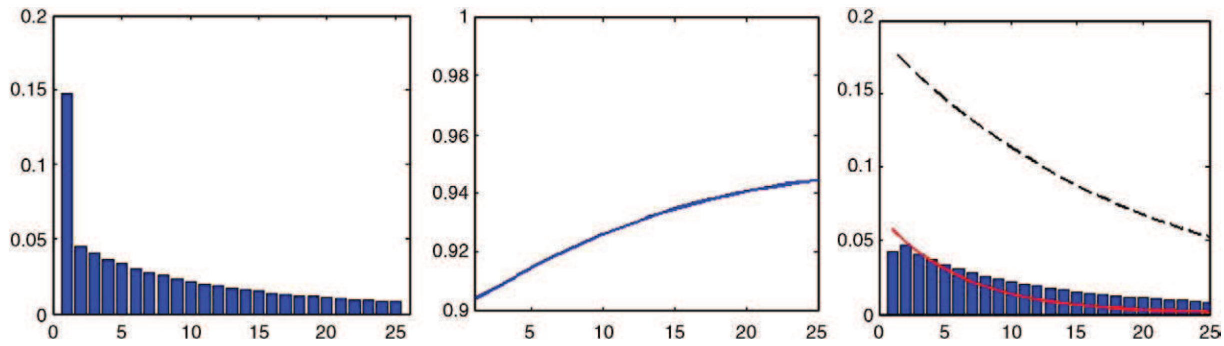

Fig. 1. Autocorrelations of $(\Delta y_t)^2$ (left column), their rate of decay defined as the ratio $\rho_\tau^{(\Delta y)2}/\rho_{\tau-1}^{(\Delta y)2}$ (central column) and autocorrelations of ε_t^2 in solid lines, η_t^2 in dashed lines and a_t^2 in bars (right column). The parameters of the model are fixed to $\gamma_1=0.1$, $\gamma_2=0.85$, $\alpha_1=0.05$, $\alpha_2=0.8$ and $q=1$.

where $\overline{\kappa}_\varepsilon$ and $\overline{\kappa}_\eta$ are the excess kurtoses of ε_t and η_t, respectively, and $\rho_1^{\varepsilon^2}$ is the first order autocorrelation of ε_t^2. Note that q plays an important role in determining the relative influence of the excess kurtosis of each noise on $\overline{\kappa}_{\Delta y}$. The acf of $(\Delta y_t)^2$ is given by

$$\rho_\tau^{(\Delta y)^2} = \frac{q^2(\overline{\kappa}_\eta+2)\rho_\tau^{\eta^2} + (\overline{\kappa}_\varepsilon+2)(\rho_{\tau-1}^{\varepsilon^2} + 2\rho_\tau^{\varepsilon^2} + \rho_{\tau+1}^{\varepsilon^2})}{(\overline{\kappa}_{\Delta y}+2)(q+2)^2}, \tau\geq 1. \quad (5)$$

Note that for Gaussian noises, $\rho_1^{(\Delta y)^2}=(q+2)^{-2}$, which is the squared first order autocorrelation of Δy_t given by $\rho_1^{\Delta y}=-(q+2)^{-1}$; see (Maravall, 1983). However, when ε_t and η_t are not Gaussian, $\rho_1^{(\Delta y)^2}$ differs from $(\rho_1^{\Delta y})^2$. Finally, given that the acf of squares of both disturbances converge to zero, the acf of $(\Delta y_t)^2$ also converges to zero.

Consider now that ε_t and η_t are GARCH(1,1) noises[2] given by $\varepsilon_t=\varepsilon_t^\dagger h_t^{1/2}$ and $\eta_t=\eta_t^\dagger q_t^{1/2}$, with $\varepsilon_t^\dagger$ and $\eta_t^\dagger$ mutually independent Gaussian processes and

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2 h_{t-1}, \quad (6a)$$

$$q_t = \gamma_0 + \gamma_1\eta_{t-1}^2 + \gamma_2 q_{t-1}. \quad (6b)$$

In this case, $\overline{\kappa}_\varepsilon = \frac{2\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\alpha_2-\alpha_2^2}$, $\overline{\kappa}_\eta = \frac{2\gamma_1^2}{1-3\gamma_1^2-2\gamma_1\gamma_2-\gamma_2^2}$, $\rho_1^{\varepsilon^2} = \frac{\alpha_1(1-\alpha_1\alpha_2-\alpha_2^2)}{1-2\alpha_1\alpha_2-\alpha_2^2}$ and $\rho_1^{\eta^2} = \frac{\gamma_1(1-\gamma_1\gamma_2-\gamma_2^2)}{1-2\gamma_1\gamma_2-\gamma_2^2}$. Consequently, the excess kurtosis and acf of squares of Δy_t are given by

$$\kappa_{\Delta y} = \frac{3}{(q+2)^2}\left[q^2\frac{2\gamma_1^2}{1-3\gamma_1^2-2\gamma_1\gamma_2-\gamma_2^2} + 4\frac{\alpha_1(1+\alpha_1-\alpha_1\alpha_2-\alpha_2^2)}{1-3\alpha_1^2-2\alpha_1\alpha_2-\alpha_2^2}\right]. \quad (7)$$

$$\rho_\tau^{(\Delta y)^2} = \begin{cases} \dfrac{q^2\rho_1^{\eta^2}(\kappa_\eta+2) + (\kappa_\varepsilon+2)(1+\rho_1^{\varepsilon^2}(2+\alpha_1+\alpha_2))}{(q+2)^2(\kappa_{\Delta y}+2)}, & \tau=1 \\ (\alpha_1+\alpha_2)\rho_{\tau-1}^{(\Delta y)^2} + \dfrac{(\gamma_1+\gamma_2-\alpha_1-\alpha_2)q^2(\gamma_1+\gamma_2)^{\tau-2}\rho_1^{\eta^2}(\kappa_\eta+2)}{(q+2)^2(\kappa_{\Delta y}+2)}, & \tau\geq 2. \end{cases} \quad (8)$$

Note that when the persistence of both noises is the same, i.e. $\gamma_1+\gamma_2=\alpha_1+\alpha_2$, or only one noise is heteroscedastic, the acf of squares has an exponential decay, as in a GARCH(p,q) process. However, in general, the decay of the autocorrelations in Eq. (8) is not exponential. It can be proved that the rate of decay of $\rho_\tau^{(\Delta y)^2}$ converges to max$(\alpha_1+\alpha_2; \gamma_1+\gamma_2)$ as τ increases. Therefore, when the persistence of the GARCH processes are close to each other, the rate of decay of $\rho_\tau^{(\Delta y)^2}$ will be approximately constant. Consequently, although the behavior of Δy_t is not GARCH, exponential structures implied by GARCH processes can be good approximations for its acf of squares. As an illustration, Fig. 1 plots the acf of squares for a particular specification of the disturbances, together with the corresponding rate of decay from the second lag (left and central columns).

3. Properties of the IMA noise

3.1. Excess kurtosis and acf of squares

The objective of this subsection is to derive the excess kurtosis and acf of squares of a_t in model (3) when ε_t and η_t are GARCH(1,1) noises. Consider the reduced form IMA(1,1) model given in Eq. (3). The excess kurtosis of Δy_t in this case is given by

$$\overline{\kappa}_{\Delta y} = \frac{\overline{\kappa}_a(1+\theta^4) + 6\theta^2\rho_1^{a^2}(\overline{\kappa}_a+2)}{(1+\theta^2)^2}, \quad (9)$$

where $\overline{\kappa}_\alpha$ and $\rho_1^{\alpha^2}$ are the excess kurtosis of a_t and the first order autocorrelation of a_t^2, respectively. The acf of Δy_t^2 is given by

$$\rho_\tau^{(\Delta y)^2} = \frac{\overline{\kappa}_a+2}{(1+\theta^2)^2(\overline{\kappa}_{\Delta y}+2)}[(1+\theta^4)\rho_\tau^{a^2} + \theta^2(\rho_{\tau-1}^{a^2}+\rho_{\tau+1}^{a^2})], \tau\geq 1. \quad (10)$$

The expressions of $\overline{\kappa}_\alpha$ and $\rho_\tau^{\alpha^2}$ are related to those of the unobserved component noises in Eq. (4) and (5) in a way that is not easy to derive analytically. However, one may find approximations of these moments by equalling the excess kurtosis of Δy_t given by Eq. (4) and (9), and the autocorrelations in Eq. (4) and (10)[3], as follows

$$(\overline{\kappa}_a+2)(1+\theta^4+6\theta^2\rho_1^{a^2}) \equiv (1+\theta)^4(\overline{\kappa}_\eta+2) - 8\theta(1+\theta)^2 + 2\theta^2(\overline{\kappa}_\varepsilon+2)(1+3\rho_1^{\varepsilon^2}), \quad (11a)$$

$$(\overline{\kappa}_a+2)[(1+\theta^4)\rho_\tau^{a^2} + \theta^2(\rho_{\tau-1}^{a^2}+\rho_{\tau+1}^{a^2})] \equiv \theta^2(\overline{\kappa}_\varepsilon+2)(\rho_{\tau-1}^{\varepsilon^2}+2\rho_\tau^{\varepsilon^2}+\rho_{\tau+1}^{\varepsilon^2}) + (1+\theta)^4(\overline{\kappa}_\eta+2)\rho_\tau^{\eta^2}, \tau\geq 1. \quad (11b)$$

When the local level disturbances are homoscedastic but non Gaussian, then $\rho_\tau^{\varepsilon^2}=\rho_\tau^{\eta^2}=0$ and $\overline{\kappa}_\varepsilon$ and $\overline{\kappa}_\eta$ are different from zero. In this case, though still uncorrelated, a_t is not independent as the autocorrelations of squares are different from zero; see (Breidt and Davis, 1992). As an illustration, Table 1 reports the acf of a_t^2 for several values of q, $\overline{\kappa}_\varepsilon$ and $\overline{\kappa}_\eta$, obtained from the resolution of Eq. (11a) and (11b). Note that these autocorrelations do not decay exponentially

[2] See (Broto and Ruiz, 2006) for the particular case of a LLM with GQARCH disturbances to account for asymmetries in volatility.

[3] To obtain Eq. (11a), recall that θ can be defined in terms of q, so that the following expressions result: $1+\theta^2=-\theta(q+2)$, $1+\theta^4=\theta^2(q^2+4q+2)$.

Table 1
Moments of the reduced form noise, a_t, for different models with non-Gaussian homoscedastic noises.

q	κ_ε	κ_η	θ	κ_{Δ_y}	$\rho_1^{(\Delta y)^2}$	κ_α	$\rho_1^{\alpha^2}$	$\rho_2^{\alpha^2}$	$\rho_3^{\alpha^2}$	$\rho_4^{\alpha^2}$	$\rho_5^{\alpha^2}$
0.5	0	3	0.5	0.120	0.151	0.273	0.030	0.008	0.002	0.001	0.000
$\sqrt{2}$	0	3	0.324	0.515	0.068	0.665	0.026	0.003	0.000	0.000	0.000
0.5	3	3	0.5	1.080	0.260	0.818	0.194	0.048	0.012	0.003	0.001
$\sqrt{2}$	3	3	0.324	1.029	0.142	1.120	0.063	0.007	0.001	0.000	0.000
0.5	3	0	0.5	0.960	0.270	0.546	0.241	0.060	0.015	0.004	0.001
$\sqrt{2}$	3	0	0.324	0.515	0.171	0.456	0.109	0.011	0.001	0.000	0.000

Table 2
Monte Carlo averages and standard deviations (in parenthesis) of the QML estimates of the IMA-GARCH parameters. The series are generated by a local level model with GARCH noises with parameters $\alpha_1=\gamma_1=0.15$, $\alpha_2=\gamma_2=0.8$ and $q=1$ ($\theta=-0.382$). $Q(\mathbf{10})$ reports the percentage of series in which the null of homoscedasticity is rejected at 5% when using the statistics proposed by (Rodriguez and Ruiz, 2005) at lag 10.

	Estimated IMA(1,1) on y_t				Estimated GARCH(1,1) on a_t		
	$T=200$	$T=1000$	$T=5000$		$T=200$	$T=1000$	$T=5000$
$\hat{\theta}=$	0.389(0.12)	0.386(0.06)	0.383(0.03)	$\hat{\delta}_1=$	0.094(0.06)	0.094(0.02)	0.094(0.01)
Q(10)=	52.4%	99.6%	100.0%	$\hat{\delta}_1+\hat{\delta}_2=$	0.841(0.21)	0.937(0.04)	0.945(0.01)

and, consequently, they do not reflect the presence of GARCH effects in the series. However, it is possible to reject the null of homoscedasticity when using tests based on the autocorrelations of squares.

When the noises are stationary, the right hand side of Eq. (11b) converges to zero as τ increases. Consequently, there exists a value of τ, say $\tau_{\max}$, large enough such that $\rho_\tau^{\alpha^2}\approx 0$ for $\tau>\tau_{\max}$. Therefore, solving the system backwards, we can find $\overline{\kappa}_\alpha$ and $\rho_\tau^{\alpha^2}$ for different specifications of the unobserved component noises. As an illustration, Fig. 1 plots the acf of a_t^2 when both noises are GARCH(1,1) (right column).

3.2. Heteroscedastic IMA models

We have seen that if ε_t or η_t are GARCH processes, a_t does not share all their properties. However, it still has excess kurtosis and positive autocorrelations of squares. On the other hand, when analyzing real time series, it is usual to fit GARCH processes to the residuals of ARIMA models whenever they show evidence of conditional heteroscedasticity. Therefore, we simulate data to analyze the effects of fitting IMA GARCH models to series with conditionally heteroscedastic stochastic levels. Consider that a_t is assumed to be a GARCH(1,1) model, given by $a_t = a_t^\dagger\sqrt{s_t}$, where $\alpha_t^\dagger$ is a Gaussian white noise process and

$$s_t = \delta_0 + \delta_1 a_{t-1}^2 + \delta_2 s_{t-1}. \tag{12}$$

We generate 1000 series by model 1 with GARCH disturbances with parameters $\alpha_1=\gamma_1=0.15$, $\gamma_1=\gamma_2=0.8$ and $q=1$ and sample sizes $T=200$, 1000 and 5000. The parameters of the IMA GARCH model are estimated by QML in two steps, estimating first the MA parameter, θ, and then fitting the GARCH model to the residuals. Furthermore, we also test for homoscedasticity in the residuals of the first step using the test proposed by (Rodriguez and Ruiz, 2005). Table 2, which reports the Monte Carlo means and standard deviations of the QML estimates together with the percentage of rejections of the homoscedasticity in the residuals, shows that the estimator of θ is unbiased even in moderate samples. However, when testing for homoscedasticity in the residuals, the null is not rejected in 47.6% of the series when $T=200$. This result is also reflected in the fact that the estimates of δ_1 are not significantly different from zero when $T=200$. Increasing the sample size leads to significant ARCH effects. Furthermore, the average of the ARCH parameter estimates is the same regardless of the sample size, in particular 0.094, while the average of the GARCH parameter estimates increases with the sample size. Consequently, the persistence, measured by $\hat{\delta}_1+\hat{\delta}_2$, increases. In particular, the persistence goes from 0.841 when $T=200$ to 0.945 when $T=5000$. Note that the common ARCH parameter of the original disturbances is 0.15 while the common persistence is 0.95. Therefore, although in large samples, δ_1 is underestimated with respect to the ARCH parameters in ε_t and η_t, the average persistence of the reduced form GARCH model is the same as the common persistence observed in the local level disturbances.

4. Conclusions

We show that the reduced form noise of unobserved component models with independent non Gaussian noises is uncorrelated although non independent. On the other hand, when the noises are GARCH, the reduced form noise is not a proper GARCH but it can be well approximated by it. However, taking differences in series with conditionally heteroscedastic stochastic trends weakens the strength of the heteroscedasticity. This result could be expected as the heteroscedasticity weakens under contemporaneous aggregation; see, for example, (Zaffaroni, 2007). Consequently, in small samples, often one cannot reject the null of homoscedasticity in series composed of one or more conditionally heteroscedastic components. This apparent homoscedasticity of the reduced form noise may have important implications when building prediction intervals for future values of the series of interest; see (Pellegrini et al., 2007).

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