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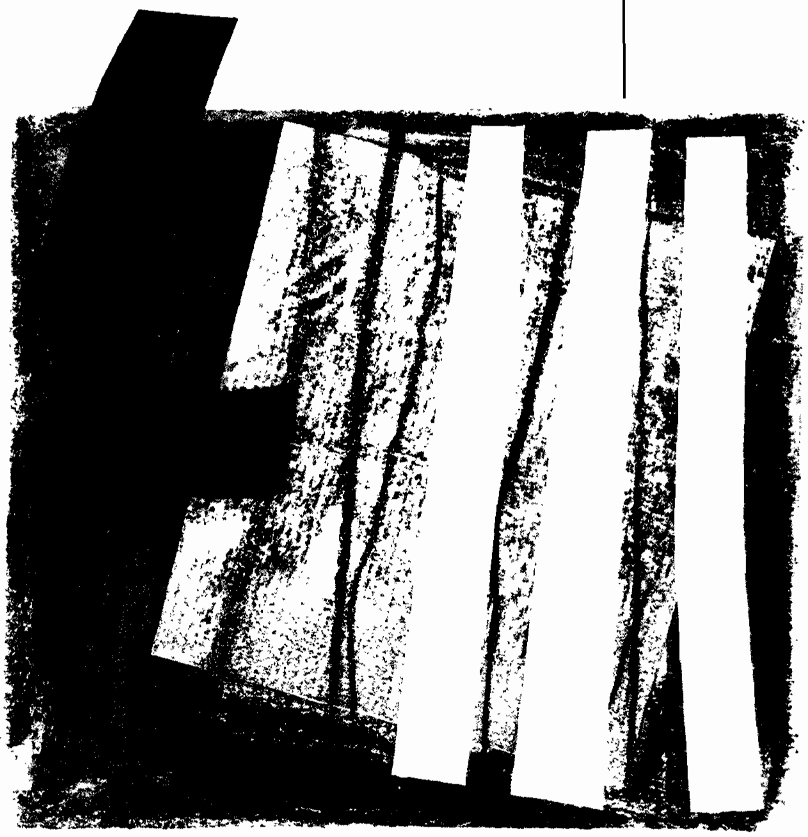

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Abstract

This paper proposes new measures providing us with the level of sequential arbitrage in a bond market. Each measure generates a concrete proxy for the Term Structure of Interest Rates. The set of proxies allows us to compute the exact market price of any bond, may measure the tax effect, may measure the credit risk when dealing with non-default free bonds, and may solve the usual puzzle when dealing with extendible or callable bonds. Finally, an empirical test of our findings is implemented in the Spanish market.

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Envelopes for the term structure of interest rates

ALEJANDRO BALBÁS*AND SUSANA LÓPEZ†

ABSTRACT. This paper proposes new measures providing us with the level of sequential arbitrage in a bond market. Each measure generates a concrete proxy for the Term Structure of Interest Rates. The set of proxies allows us to compute the exact market price of any bond, may measure the tax effect, may measure the credit risk when dealing with non-default free bonds, and may solve the usual puzzle when dealing with extendible or callable bonds. Finally, an empirical test of our findings is implemented in the Spanish market.

1. INTRODUCTION

The estimation of state price densities implicit in financial asset prices is becoming more and more important in financial literature. This is justified because the state price densities provide us with pricing rules and enable the price of new securities to be determined. Although parametric technics play a crucial role (see for instance [11]) non-parametric estimations are also becoming an useful tool (see for instance [1]) since they do not have to impose any special form.

Pricing rules are related to the term structure of interest rates when dealing with bonds markets, and both parametric and non-parametric technics are being introduced also in this case (for instance, [10] and [14] present interesting examples of both kinds of methods).

The existence of state prices in a general market or the existence of a term structure of interest rates in a bond market is the necessary and sufficient condition to guarantee that the market is arbitrage free (sequential arbitrage free in the case of bonds markets), and furthermore, the arbitrage absence is always assumed in every theoretical approach concerning asset pricing or asset allocation. However, the empirical evidence seams to reveal that the arbitrage may occur in practice (see for instance [17], [15], [7], [16] and [5]) and this fact has motivated several authors to introduce new measures providing us with the level of arbitrage in real markets (see [7] and [6]). This paper follows the approach of [6] in order to define some measures of

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the degree of sequential arbitrage in bonds markets.[1] These measures yield practical tools useful to traders, and permit us to analyze the existence of sequential arbitrage.

All the measures are defined by means of optimization problems that yield the maximum relative (with respect to the price of the purchased or sold bonds) income or profit generated by a sequential arbitrage portfolio. Then, the measures indicate how much money may be obtained and, therefore, the effect of market imperfections may be discounted. Besides, dual problems also lead to the measures and generate proxies for the term structure.[2] Both dual and primal problems indicate how prices must increase (decrease) in order to prevent the sequential arbitrage. [3] [4] Since the measures maximize the total income of sequential arbitrage, they also minimize the real price modifications that lead to a sequential arbitrage-free model.

The discussion above shows that our measures may apply to analyze the effect of market imperfections. So, the arbitrage may be caused by the tax effect, for instance, that makes some prices grow and other prices decrease. The measures also allow us to consider all the available bonds in order to determine the term structure. It is known that the information contained in default-free and option-free bonds is often incomplete and does not generate accurate expressions for the term structure. So, the negative option prices puzzle appears because bonds with embedded options are excluded when computing the term structure and the obtained pricing rule (term structure) provides negative prices for some options. Our measures solve this caveat and they indicate how the price or non-default-free or non-option-free bonds may be modified when testing the term structure.[5]

The article is organized as follows. Section 2 introduces the basic assumptions and notations. Section 3 presents some optimization problems that lead to several measures of sequential arbitrage, and the main properties and interpretations of these measures are also studied. Section 4 presents a dual approach, new measures and the set of envelopes for the term structure. Section 5 illustrates how the theory may help to deal with some usual caveats of this literature, Section 6 extends the main theoretical findings and Section 7 reports the results of an empirical test implemented

[1] The analysis introduced in [6] applies for a general market and measures the level of arbitrage. Here we adapt the study so that one can also involve sequential arbitrage strategies. This extension makes things more difficult and this is the reason why we will develope the theory once more. Nevertheless, some proofs are quite similar to those presented in [6] and, consequently, they will be omitted in this paper.

[2] Recall that an exact term structure cannot exist if the sequential arbitrage occurs.

[3] Each proxy for the term structure leads to some "extreme" prices for the bonds, so we decided to use the term "envelope" to refer to these proxies.

[4] [14] also presents some primal and dual optimization problems that allow us to analyze the existence of sequential arbitrage, but in this paper we study the relationships among the solutions of the optimization problems (the measures), the real market prices and the term structure envelopes.

[5] *i.e.*, the measures give bounds for the risk premium or the option price.

in the Spanish market. The test involves all the topics addressed in the paper. So, the existence of sequential arbitrage is analyzed, as well as the credit risk and the price of non-option-free bonds. Furthermore, some relationships among the envelopes of the term structure and the empirical term structure obtained in the previous paper [20] are briefly illustrated.[6]

The last section summarizes and concludes the article.

2. Preliminaries and notations

Consider n arbitrary bonds B_j, $j = 1, 2, ..., n$, available in the market, and denote by $P = (p_1, p_2, ..., p_n)$, $p_j > 0$, $j = 1, 2, ..., n$, the vector whose components are the current prices. Suppose that $T = \{t_1, t_2, ..., t_m\}$ represents the set of future dates in which bondholders will receive the corresponding payoff and denote by $a_{ij} \geq 0$ the amount of money paid by B_j at t_i, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$. In order to avoid some mathematical difficulties we will impose the following weak inequality whose economic interpretation is obvious

$$\sum_{i=1}^{m} a_{ij} > p_j \tag{1}$$

for every $j = 1, 2, ..., n$. Consider finally that A represents the $m \times n$ matrix whose rows are $A_i = (a_{i,1}, a_{i,2}, ..., a_{i,n})$, $i = 1, 2, ..., m$, and $\tilde{A} = \begin{pmatrix} A_0 \\ A \end{pmatrix}$ represents the $(m+1) \times n$ matrix obtained by adding A plus a first row equal to $A_0 = -P$. If $X = (x_1, x_2, ..., x_n)$ represents the portfolio composed of x_j units of B_j, $j = 1, 2, ..., n$, then PX^T equals the current price of X, AX^T equals its future payoffs and $\tilde{A}X^T$ equals the whole set of cash flows of X.[7] The following matrices, whose dimensions are $(m+1) \times (m+1)$ and $m \times (m+1)$ respectively, will also play an important role in the analysis

$$I^*_{m+1} = \begin{pmatrix} 1,0,0,...,0 \\ 1,1,0,...,0 \\ .... \\ 1,1,1,...,1 \end{pmatrix}, \qquad I^{**} = \begin{pmatrix} 1,1,0,...,0 \\ 1,1,1,...,0 \\ .... \\ 1,1,1,...,1 \end{pmatrix}$$

and I^*_m will be similar to I^*_{m+1} but with m rows and columns.

We follow the previous literature in order to introduce the concepts of arbitrage and sequential arbitrage.

[6] [20] proposes and applies some modifications of [10] in order to estimate the term structure in the Spanish case.

[7] As usual, given an arbitrary matrix M, the transpose of M will be denoted by M^T.

Definition 1. *X is said to be an arbitrage portfolio if $\tilde{A}X^T \neq 0$ and $\tilde{A}X^T \geq 0$.*[8]

X is said to be a sequential arbitrage portfolio if $I^*_{m+1}\tilde{A}X^T \neq 0$ and $I^*_{m+1}\tilde{A}X^T \geq 0$.[9]

It is known that the (sequential) arbitrage absence may be characterized by the existence of discount factors μ_i, $i = 1, 2, ..., m$, with adequate properties, or equivalently, by the existence of a Term Structure of Interest Rates. The statement below clarifies this idea.

Theorem 2. *The model is arbitrage free (respectively, sequential arbitrage free) if and only if there exist $\mu = (\mu_1, \mu_2, ...\mu_m)$ such that $\mu_i > 0$, $i = 1, 2, ..., m$, and $\mu A = P$ (respectively, $1 > \mu_1 > \mu_2 > ... > \mu_m > 0$ and $\mu A = P$).* ■

In order to measure the level of sequential arbitrage we will also require some extensions of Definition 1. So, we adopt the concept "*arbitrage of the second type*" in the line of [12], Chapter 2.

Definition 3. *X is said to be a sequential arbitrage portfolio of the second type if $PX^T < 0$ and $I^{**}\tilde{A}X^T \geq 0$.*

X is said to be a strong sequential arbitrage portfolio if $PX^T < 0$ and $I^*_m AX^T \geq 0$.

The absence of strong and second type sequential arbitrage will be characterized in future sections. Obviously, strong sequential arbitrage portfolios are also sequential arbitrage portfolios of the second type, but the converse does not necessarily hold.

3. Sequential arbitrage measurement

This section is devoted to introduce two general measures providing us with the level of sequential arbitrage of the second type. Both measures will be derived from optimization problems that maximize the difference in price of "two similar strategies". So, in a first step let us consider two portfolios $h = (h_1, h_2, ..., h_n)$ and $k = (k_1, k_2, ..., k_n)$ with $h_j, k_j \geq 0$, $j = 1, 2, ..., n$, and problems below

$$Max \ -PX \quad \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ x_j \geq -h_j, \ j = 1, 2, ..., n \end{cases} \tag{2}$$

and

$$Max \ -PX \quad \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ x_j \leq k_j, \ j = 1, 2, ..., n \end{cases} \tag{3}$$

[8]As usual, given an arbitrary matrix M, the inequality $M \geq 0$ means that M does not contain any negative element. Similar notations will appear in similar cases.

[9]*i.e.*, an arbitrage portfolio has non-negative cash flows and will pay a positive amount at one date at least. On the contrary, a sequential arbitrage portfolio might imply negative cash flows but every negative cash flow is overcome by the amount of money previously received.

that maximize the current income provided by a portfolio bounded from below (above) by $-h$ (k) and whose cash flows do not imply any liability.

Proposition 4. *(2) and (3) are bounded and solvable for every h and k with non-negative components.*

Proof. (2) is bounded and solvable because the zero-portfolio is feasible and, consequently, the constraint $PX \leq 0$ may be added without affecting the solvability and the optimal value. Besides, this new constraint leads to

$$x_j \leq \frac{\sum_{r\neq j} -x_r p_r}{p_j} \leq \frac{\sum_{r=1}^{n} h_r p_r}{p_j} \tag{4}$$

$j = 1, 2, ..., n$, and the feasible set becomes bounded and compact.

In order to prove that (3) is solvable, it is sufficient to show that its feasible set is also bounded and compact. Denote by

$$C_j = \sum_{i=0}^{m} a_{ij} \tag{5}$$

$j = 1, 2, ..., n$. Then (1) shows that $C_j > 0$, $j = 1, 2, ..., n$. Moreover $I^{**}\tilde{A}X^T \geq 0$ leads to $(A_0 + A_1 + ... + A_m)X^T \geq 0$ or, equivalently, $\sum_{i=0}^{m} C_j x_j \geq 0$. Thus, $\sum_{i=0}^{m} C_j x_j^+ \geq \sum_{i=0}^{m} C_j x_j^-$,[10] from where $x_j^- \leq \dfrac{\sum_{r=0}^{m} C_r x_r^+}{C_j}$ and, therefore,

$$x_j^- \leq \frac{\sum_{r=0}^{m} C_r k_r}{C_j} \tag{6}$$

holds for every $j = 1, 2, ..., n$. Hence, the feasible set is bounded and the proof is completed. ■

Let us introduce the functions $\varphi(h)$ and $\psi(k)$ by means of the optimal values of (2) and (3). They yield the maximum current income provided by a sequential arbitrage portfolio when short (long) positions are bounded by h (k). The following result summarizes their properties. The proof is quite simple and omitted.

Proposition 5. *(a) $\varphi(h + h') \geq \varphi(h) + \varphi(h')$ for every pair of portfolios h and h' without short positions*

(b) $\varphi(\alpha h) = \alpha\varphi(h)$ for every portfolio h without short positions and every $\alpha \in \mathbb{R}$, $\alpha \geq 0$.

[10] As usual, $\alpha^+ = Sup\{\alpha, 0\}$, $\alpha^- = Sup\{0, -\alpha\}$ and $\alpha = \alpha^+ - \alpha^-$ for every $\alpha \in \mathbb{R}$. Similar notations will also be used for vectors and matrices.

(c) φ *is an increasing concave function such that* $0 \leq \varphi(h) \leq \sum_{j=1}^{n} h_j p_j$ *for every portfolio* h *without short positions.*

(d) $\psi(k + k') \geq \psi(k) + \psi(k')$ *for every pair of portfolios* k *and* k' *without short positions.*

(e) $\psi(\alpha k) = \alpha\psi(k)$ *for every portfolio* k *without short positions and every* $\alpha \in \mathbb{R}$, $\alpha \geq 0$.

(f) ψ *is an increasing concave function such that* $0 \leq \psi(k) \leq \sum_{j=1}^{n} k_j C_j$ *for every portfolio* k *without short positions.* ■

Proposition 6. *The following assertions are equivalent:*

(a) There are no sequential arbitrage opportunities of the second type.

(b) $\varphi(h) = 0$ *for every portfolio* h.

(c) $\psi(k) = 0$ *for every portfolio* k.

Proof. It follows from Definition 3 that the model is second type sequential arbitrage-free if and only if $PX^T \geq 0$ for every X verifying the constraints of (2) or (3). Thus the arbitrage absence holds if and only if $X = 0$ solves (2) or (3). ■

Proposition 7. φ *and* ψ *are piecewise linear and continuous.*

Proof. The dual problems of (2) and (3) are (see, for instance, [2], [19] or [4])

$$Min\ \lambda h^T \quad \begin{cases} \mu A + \lambda = P(1 + \mu_1) \\ \mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0 \\ \lambda_j \geq 0,\ j = 1, 2, \ldots, n \end{cases} \tag{7}$$

and

$$Min\ \lambda k^T \quad \begin{cases} \mu A - \lambda = P(1 + \mu_1) \\ \mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0 \\ \lambda_j \geq 0,\ j = 1, 2, \ldots, n \end{cases} \tag{8}$$

respectively, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ being the decision variables. They are linear and bounded since so are (2) and (3). Moreover, their optimal values are $\varphi(h)$ and $\psi(k)$ and their feasible sets do not depend on h or k and have a finite number of extreme points. Since the dual solution must be attained at some extreme point, φ and ψ become the minimum of a finite number of linear functions and, therefore, piecewise linear and continuous. ■

Next we will introduce two new optimizations problems and, from them, two measures providing us with the level of sequential arbitrage of the second type. So, consider

$$Max\ \varphi(h) \quad \begin{cases} \sum_{r=j}^{n} h_j p_j \leq 1 \\ h_j \geq 0,\ j = 1, 2, \ldots, n \end{cases} \tag{9}$$

and

$$Max\ \psi(k)\ \begin{cases} \sum_{r=j}^{n} k_j p_j \leq 1 \\ k_j \geq 0,\ j = 1, 2, ..., n \end{cases} \tag{10}$$

Remark 1. *Clearly, both problems are solvable since their objectives are continuous and their feasible sets are compact. We will denote by h_* and k^* their solutions, by $£_*$ and $£^*$ their optimal values, and by X_* and X^* the solutions of (2) and (3) when $h = h_*$ and $k = k^*$. $£_*$ and $£^*$ may be considered as measures of sequential arbitrage, in the sense that they provide the maximum income generated by riskless portfolios whose short or long positions are not greater than one dollar. If these measures increase then the income increases and the level of sequential arbitrage increases too. $£_*$ and $£^*$ are relative measures since the upper bound for the price of h or k must be imposed. Otherwise, Propositions 5b and 5e imply that (9) and (10) are not bounded unless their optimal values equal zero. Propositions 5c and 5f guarantee that $0 \leq £_* \leq 1$[11] and $0 \leq £^* \leq \sum_{j=1}^{n} \frac{C_j}{p_j}$, where C_j is given by (5). Finally, Proposition 6 clearly implies the following result:*

Theorem 8. *The second type sequential arbitrage absence holds if and only if $£_* = 0$ or, equivalently, $£^* = 0$.* ■

Although both measures have been introduced in two steps (first we must compute φ and ψ, and later we must solve (9) and (10)), it is clear that they can also be computed in practice by solving following problems

$$Max\ -PX\ \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ x_j + h_j \geq 0,\ j = 1, 2, ..., n \\ \sum_{r=j}^{n} h_j p_j \leq 1 \\ h_j \geq 0,\ j = 1, 2, ..., n \end{cases} \tag{11}$$

and

$$Max\ -PX\ \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ x_j - k_j \leq 0,\ j = 1, 2, ..., n \\ \sum_{r=j}^{n} k_j p_j \leq 1 \\ k_j \geq 0,\ j = 1, 2, ..., n \end{cases} \tag{12}$$

[11]It may be proved that $£_* < 1$. In fact, if $£_* = 1$ it trivially follows from (2) that $X_* = -h_*$. Thus, the first constraint of (2) leads to $\sum_{j=1}^{n} C_j h_{*j} \leq 0$, where C_j is given by (5). Since (1) shows that $C_j > 0$, $h_{*j} \geq 0$ and $h_* \neq 0$ (because $£_* = 1$), we have a contradiction.

whose decision variables are (X,h) and (X,k) respectively. Their dual problems are (see for instance [4])

$$Min\ \theta \begin{cases} \mu A+\lambda=P(1+\mu_1) \\ \lambda_j \leq \theta p_j,\ j=1,2,...,n \\ \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \\ \lambda_j \geq 0,\ j=1,2,...,n \end{cases} \tag{13}$$

and

$$Min\ \theta \begin{cases} \mu A-\lambda=P(1+\mu_1) \\ \lambda_j \leq \theta p_j,\ j=1,2,...,n \\ \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \\ \lambda_j \geq 0,\ j=1,2,...,n \end{cases} \tag{14}$$

whose decision variables are $\theta \in \mathbb{R}$, $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$ and $\mu=(\mu_1,\mu_2,...,\mu_m)$. They yield $\pounds_*$ and $\pounds^*$ too.

Hence, primal problems provide the measures and optimal arbitrage portfolios, while dual problems provide the measures and some proxies for the term structure. The dual solutions main properties will be addressed in future sections.

The last part of this section is devoted to establish some significant relationships between both measures. These relationships will permit us to prove many properties concerning dual solutions. They also illustrates that our measures yield the maximum difference in price between two "similar" portfolios, since X_* and X^* are quite close to k^*-h_*.

Lemma 9. *Assume that $\pounds_*>0$. Let $j\in\{1,2,...,n\}$ and assume that $h_{*j}>0$ (respectively, $k_j^*>0$). Then, $x_{*j}=-h_{*j}$ (respectively, $x_j^*=k_j^*$).*

Proof. Both properties are analogous, so let us prove the first one. It is sufficient to show that $X_*^-=h_*$. Since $X_*^-\leq h_*$ (X_* must be feasible) then $X_*^-\neq h_*$ would imply $\sum_{j=1}^n x_{*j}^-p_j<\sum_{j=1}^n h_{*j}p_j$. Therefore, there would exist $\lambda>1$ such that $\lambda\sum_{j=1}^n x_{*j}^-p_j\leq 1$. Hence, λX_*^- would be (9)-feasible, and λX_* would be (2)-feasible with $h=\lambda X_*^-$. Furthermore, $\varphi(\lambda X_*^-)\geq -\lambda PX_*^T>-PX_*^T=\varphi(h_*)$ and h_* would not be the solution of (9). ■

Theorem 10. *Assume that $\pounds_*>0$. The following assertions are fulfilled:*

(a) $\pounds^*=\dfrac{\pounds_*}{1-\pounds_*}$, $\pounds_*=\dfrac{\pounds^*}{1+\pounds^*}$ and $\pounds_*\leq\pounds^*$

(b) $X^*=k^*-(1+\pounds^*)h_*$ and $X_*=(1-\pounds_*)k^*-h_*$

(c) X^{*+} and X_*^+ are proportional to k^* and X^{*-} and X_*^- are proportional to h^*.

(d) $X^*=(1+\pounds^*)X_*$

Proof. Consider the problems

$$Max\ f(X) = -\frac{\sum_{j=1}^{n} p_i x_i}{\sum_{j=1}^{n} p_i x_i^-} \quad \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ X^+,\ X^- \neq 0 \end{cases}$$

and

$$Max\ g(X) = -\frac{\sum_{j=1}^{n} p_i x_i}{\sum_{j=1}^{n} p_i x_i^+} \quad \begin{cases} I^{**}\tilde{A}X^T \geq 0 \\ X^+,\ X^- \neq 0 \end{cases}$$

that provide us with the optimal ratio between sequential arbitrage earnings and the price of the sold and purchased bonds respectively.[12] The relationship $g(X) = \dfrac{f(X)}{1-f(X)}$ may be easily proved. Then, bearing in mind that $[0,1) \ni t \rightarrow \dfrac{t}{1-t} \in [0,\infty)$ is a one-to-one increasing function, both ratios must attain the maximum value at the same portfolio, and following [6], they are maximized at X_* and X^*, and achieve the values $\pounds_*$ and $\pounds^*$ respectively. Therefore, $\pounds^* = \dfrac{\pounds_*}{1-\pounds_*}$ and $\pounds_* = \dfrac{\pounds^*}{1+\pounds^*}$ are clear and $\pounds_* \leq \pounds^*$ follows from $t \leq \dfrac{t}{1-t}$ for every $t \in [0,1)$. Now, the remainder of the theorem trivially follows from the previous lemma. ■

4. Term structure envelopes and basic bounds for market prices

Dual problems provide a proxy for the term structure of interest rates. In fact, the constraint $\mu A \pm \lambda = P(1+\mu_1)$ may be written as $\tilde{\mu} A \pm \tilde{\lambda} = P$, where $\tilde{\mu} = \dfrac{\mu}{1+\mu_1}$ and $\tilde{\lambda} = \dfrac{\lambda}{1+\mu_1}$. Then, if $(\mu_*, \lambda_*, \pounds_*)$ and $(\mu^*, \lambda^*, \pounds^*)$ solve (13) and (14) respectively, $\tilde{\mu}^*$ and $\tilde{\mu}_*$ may be interpreted as proxies for the term structure that lead to theoretical prices given by $P_* = \tilde{\mu}_* A$ and $P^* = \tilde{\mu}^* A$.

Latter ideas may be summarized as follows:

Theorem 11. *There are no second type sequential arbitrage portfolios if and only if there exists $\tilde{\mu}_*$ such that $P = \tilde{\mu}_* A$ and $1 > \tilde{\mu}_{*1} \geq \tilde{\mu}_{*2} \geq \ldots \geq \tilde{\mu}_{*m} \geq 0$.*

Proof. Theorem 8 ensures that the model is second type sequential arbitrage free if and only if $\pounds_* = 0$. Problem (13) shows that this equality holds if and only if $\mu_* A = P(1+\mu_{*1})$. Therefore, it only remains to show that $\mu_{*1} > 0$. If $\mu_{*1} = 0$ then $\mu_* = 0$, from where $\lambda_* = P$ and the optimal value of (13) equals one, against Footnote 11. ■

[12] Notice that X^+ and X^- do not vanish if X is a sequential arbitrage portfolio of the second type. If fact, if $X^+ = 0$ then X^- may be normalized so that its total price can equal one and, consequently, $\varphi(X^-) = 1$ against Footnote 11. On the other hand $X^- = 0$ leads to $PX^T = PX^{+T} \geq 0$, and X cannot be a sequantial arbitrage portfolio of the second type.

$\tilde{\lambda}^*$ and $\tilde{\lambda}_*$ may be interpreted as the "market error" in the sense that they must vanish to guarantee the absence of sequential arbitrage of the second type. Since $\tilde{\lambda}^*$ and $\tilde{\lambda}_*$ have nonnegative components, P_* and P^* are respectively lower and upper bounds for the vector P of market prices, and the statement above shows that P_* and P^* prevent the sequential arbitrage of the second type.

In order to ensure that μ_*, μ^*, λ^* and λ_* may be also computed from (7) and (8), we will prove the following result:

Proposition 12. *If* $(\mu_*, \lambda_*, \pounds_*)$ *and* $(\mu^*, \lambda^*, \pounds^*)$ *solve (13) and (14) respectively, then* (μ_*, λ_*) *and* (μ^*, λ^*) *solve (7) and (8) with* $h = h_*$ *and* $k = k^*$.

Proof. If $(\mu_*, \lambda_*, \pounds_*)$ solves (13) then (μ_*, λ_*) verifies the complementary slackness conditions for (11) and (13), and therefore, (μ_*, λ_*) verifies the complementary slackness conditions for (7) and (8).

The remainder of the proof is absolutely similar. ■

Let us consider now

$$\tilde{\pounds}_* = \frac{\pounds_*}{1 + \mu_{*1}} \tag{15}$$

and

$$\tilde{\pounds}^* = \frac{\pounds^*}{1 + \mu_1^*} \tag{16}$$

The following result allows $\tilde{\pounds}_*$ and $\tilde{\pounds}^*$ to be interpreted as new measures of the second type sequential arbitrage degree, in the sense that they are percentages of the committed error when pricing the bonds. So, to prevent the second type sequential arbitrage, the price of h_* must decrease in the percentage $\tilde{\pounds}_*$ or the price oh k^* must increase in the percentage $\tilde{\pounds}^*$. Moreover, $\tilde{\mu}_*$ and $\tilde{\mu}^*$ may be understood as envelopes for the term structure, in sense that real market prices will always lie within the spread provided by both term structures and will equal a extreme of this spread when dealing with bonds included in h_* or k^*. To be precise, $\tilde{\mu}^*$ matches the price of those bonds included in h_*, while $\tilde{\mu}_*$ matches the price of k^*.

Theorem 13. *(a)* $p_{*j} \leq p_j \leq p_j^*$, $j = 1, 2, ..., n$.

(b) If $k_j^* > 0$, *then* $p_{*j} = p_j = p_j^* \dfrac{1}{1 + \tilde{\pounds}^*}$

(c) If $h_{*j} > 0$, *then* $p_{*j} \dfrac{1}{1 - \tilde{\pounds}_*} = p_j = p_j^*$.

(d) $P_* X^T \leq P X^T \leq P^* X^T$ *for every portfolio* X *without short positions. Furthermore, the first (second) expression holds in terms of equality if* X *is composed of*

those bonds included in k^* (h_*).[13] [14]

Proof. (a) is obvious and (d) trivially follows from (a), (b) and (c). Besides, (b) and (c) are analogous, so let us prove (b). Assume that $k_j^* > 0$. Then Lemma 9 ensures that $x_j^* > 0$. Consequently, Theorem 10d shows that $x_{*j} > 0$. The complementary slackness conditions between (2) and (7) show that $\lambda_{*j} = 0$, and therefore, $p_{*j} = p_j$.

Besides, since $\pounds^*$ is the optimal value of (3),

$$\pounds^* = \sum_{j=1}^{n} \lambda_j^* \, k_j^* = [\mu^* A - P(1+\mu_1^*)]k^{*T} = \mu^* A k^{*T} - P(1+\mu_1^*)k^{*T}$$

and, bearing in mind that the price Pk^{*T} equals 1,

$$\pounds^* = \mu^* A k^{*T} - (1+\mu_1^*)$$

from where

$$\tilde{\pounds}^* = \tilde{\mu}^* A k^{*T} - 1 \tag{17}$$

Furthermore, the constraints of (14) guarantee that

$$\lambda_j^* \leq \pounds^* p_j$$

from where

$$\tilde{\lambda}_j^* \leq \tilde{\pounds}^* p_j$$

Thus,

$$p_j^* - p_j \leq \tilde{\pounds}^* p_j$$

and therefore

$$p_j^* \leq p_j(1+ \tilde{\pounds}^*) \tag{18}$$

for $j = 1, 2, ..., n$. Suppose that $p_{j_0}^* < p_{j_0}(1+ \tilde{\pounds}^*)$ holds for some j_0 with $k_{j_0}^* > 0$. Then, (18) leads to $\sum_{j=1}^n p_j^* k_j^* < (1+ \tilde{\pounds}_*) \sum_{j=1}^n p_j k_j^* = (1+ \tilde{\pounds}^*)$, contradicting (17). ■

Theorems 10 and 13 give the intuition underlying the measures $\pounds_*$, $\pounds^*$, $\tilde{\pounds}_*$ and $\tilde{\pounds}^*$. The first and second one represent the maximum relative income available from

[13] Notice that statements (b) and (c) guarantee that $\tilde{\pounds}_*$ and $\tilde{\pounds}^*$ are correctly defined in (15) and (16), in the sense that they do not depend on μ_{*1} or μ_1^*. Thus, if the model is not second type sequantial arbitrage free, the first components of all the possible solutions of (13) and (14) coincide.

[14] Statements (b) and (c) show that the exact price of those bonds included in h_* and k^* may be computed by using both $\tilde{\mu}_{*1}$ or $\tilde{\mu}_1^*$. Thus, both envelopes apply to match the exact price of any portfolio unless it contains a bond neither included in h_* nor included in k^*, in which case, (d) still provides useful bounds on the market price of this portfolio.

a second type sequential arbitrage strategy,[15] while the rest are the committed error (in percentage) when pricing the bonds. Thus, these measures and the corresponding optimization problems may be an useful tool to analyze in practice the existence of arbitrage in bonds markets. At the end of this paper we will report the results of our empirical test in the Spanish markets.

5. APPLICATIONS: CREDIT RISK MEASUREMENT AND BONDS WITH EMBEDDED OPTIONS

The developed theory may be useful to study several issues concerning bonds markets. For instance, the tax effect may imply the existence of sequential arbitrage to be apparent but not real.[16] So, those bonds for which the tax effect is most negative should compose the portfolio k^* and their prices would grow from p_j to $p_j^* = p_j(1+\tilde{\mathcal{L}}^*)$ (see Theorem 13) if taxes were homogeneous for all the available bonds. Thus, $\tilde{\mathcal{L}}^*$ may be interpreted as the tax effect on the bonds of k^*.

More interesting applications are related to the credit risk measurement and the classical puzzle concerning bonds with embedded options. Regarding the first topic, suppose that a first group $\{B_1, B_2, ..., B_r\}$ is composed of default-free bonds ($r < n$) but the price of B_j, $j > r$, incorporates a risk premium. In such a case, the term structure should be estimated without considering the information provided by $p_{r+1}, ..., p_n$. Nevertheless, the measures $\mathcal{L}_*$, $\mathcal{L}^*$, $\tilde{\mathcal{L}}_*$ and $\tilde{\mathcal{L}}^*$ allows us to incorporate the information of these bonds. In fact, suppose that $\tilde{\mathcal{L}}_* = 0$ if we only deal with $\{B_1, B_2, ..., B_r\}$ and denote by $\tilde{\mathcal{L}}_*(j)$ the value of $\tilde{\mathcal{L}}_*$ when dealing with $\{B_1, B_2, ..., B_r, B_j\}$, $j = r+1, r+2, ..., n$. Then, h_* should be composed of B_j, and, according to Theorem 13, $p_j \tilde{\mathcal{L}}_*(j)$ may be understood as the risk premium associated to B_j.[17] The method may also apply by computing $\tilde{\mathcal{L}}_*(r+1, ..., n)$, the measure value after considering the whole set of bonds. In such a case $\tilde{\mathcal{L}}_*(r+1, ..., n)$ will provide the greatest risk premium in percentage, and corresponds to those bonds included in h_*.

The procedure above allows us to compute the term structure by drawing on the information provided by non default-free bonds. Consequently, the final term structure is more accurate, in the sense that it also prices risky bonds. Furthermore, the procedure is coherent since a negative risk premium will never appear.

The interest of incorporating the information contained in all the available bonds is more clear when dealing with extendible or callable bonds. In such a case, the

[15] Or the maximum difference in prices associated to similar portfolios, since strategies proportional to h_* and k^* provide quite analogous payoffs and we are measuring the difference between their prices.

[16] Interesting analyses of the tax effect may be found in [8] or [9], where the authors also point out the existence of several term structures when dealing with non-homogeneous bonds. However, the approach provided by these references and our analysis are quite different.

[17] *i.e.*, owing to the risk premium the price of B_j must raise in the percentage indicated by $\tilde{\mathcal{L}}_*(j)$.

empirical evidence shows that it is not convenient to eliminate these bonds because the term structure provided by the pure default-free and option-free bonds may lead to negative prices for the embedded options. This fact has been pointed out by [18] for the *U.S.* markets and by [3] for the Canadian market, amongst others.

This caveat has been addressed in [13] by applying which the authors call "an implied norm approach". We propose here an alternative procedure based on the developed measures. So, if we assume again that the set $\{B_1, B_2, ..., B_r\}$ is composed of default-free and option-free bonds and B_j, $j > r$, is an extendible (respectively, callable) bond, then $\tilde{\pounds}_* = 0$ and $\tilde{\pounds}_*(j) \geq 0$ must hold [18] and h_* must contain the $j-th$ bond B_j (respectively, $\tilde{\pounds}^* = 0$, $\tilde{\pounds}^*(j) \geq 0$ and k^* must contain the $j-th$ bond). Whence, according to Theorem 13, $\tilde{\pounds}_*(j)p_j$ (respectively, $\tilde{\pounds}^*(j)p_j$) provides us with the price of the embedded put (respectively, call). As in the previous case, it is clear that the whole set of bonds may be used in order to determine the term structure. In such a case, $\tilde{\pounds}_*$ and $\tilde{\pounds}^*$ yield maximum variations in bonds prices, that are motivated by the options.

As said above, the procedure allows us to incorporate all the information available in the market when estimating the term structure. This makes the estimation more accurate and avoids some of the caveats previously arisen in the literature. Moreover, the technic is purely non-parametric and just consists in modifying market prices so that the maximum difference in prices between two similar strategies (or the maximum income provided by a second type sequential arbitrage portfolio) becomes null.

6. Strong sequential arbitrage

The analysis presented in Sections 3 and 4 becomes easier if we focus on the strong sequential arbitrage. In this case the arbitrage measurement developed in [6] only requires minor modifications to reflect the special properties of a bond market, since the arbitrage income may be considered an arbitrage profit and, accordingly, it is not necessary to distinguish between "the maximum difference in price of similar portfolios" and "the minimum growth-fall of prices that prevents the arbitrage".[19]

As a consequence, we will just introduce the measures and present a synopsis of their main properties. The proofs are quite similar to those provided in [6] or in Sections 3 and 4.

Given two portfolios h and k without short positions we will consider the problems

$$Max \ -PX \quad \begin{cases} I_m^* A X^T \geq 0 \\ x_j \geq -h_j,\ j = 1, 2, ..., n \end{cases} \tag{19}$$

[18] We draw on the same notation as in the previous case of credit risk measurement.

[19] *i.e.*, according to Definition 3, the income provided by a strong sequential arbitrage protfolio is not required to overcome future loses (or negative cash flows), and therefore it is a real arbitrage profit. As a consequence, $\pounds_*$ and $\tilde{\pounds}_*$ (or $\pounds^*$ and $\tilde{\pounds}^*$) will be identical when focusing on strong sequential arbitrage.

and

$$Max \ -PX \ \begin{cases} I_m^* AX^T \geq 0 \\ x_j \leq k_j, \ j = 1, 2, ..., n \end{cases} \tag{20}$$

that generate the functions $\Phi(h)$ and $\Psi(k)$. These are well-defined piecewise linear and continuous functions and an analogous result to Proposition 5 also holds. The market is strong sequential arbitrage free if and only if $\Phi(h) = 0$ for every portfolio h without short positions (or $\Psi(k) = 0$ for every portfolio k without short positions) and the optimal values of following problems may be interpreted as measures of the level of strong sequential arbitrage

$$Max \ \Phi(h) \ \begin{cases} \sum_{r=j}^{n} h_j p_j \leq 1 \\ h_j \geq 0, \ j = 1, 2, ..., n \end{cases}$$

and

$$Max \ \Psi(k) \ \begin{cases} \sum_{r=j}^{n} k_j p_j \leq 1 \\ k_j \geq 0, \ j = 1, 2, ..., n \end{cases}$$

Denoting by h_{**} and k^{**} their solutions, by X_{**} and X^{**} the corresponding solutions of (19) and (20) with $h = h_{**}$ and $k = k^{**}$, and by $\pounds_{**}$ and $\pounds^{**}$ their optimal values, it is clear that we are measuring the relative (per dollar in short-long position) strong arbitrage profit available in the market. The degree of arbitrage increases if $\pounds_{**}$ and $\pounds^{**}$ increase, and the arbitrage absence holds when both measures vanish. Moreover, both measures may be determined without previously computing the functions $\Phi(h)$ and $\Psi(k)$. In fact, it is sufficient to solve

$$Max \ -PX \ \begin{cases} I_m^* AX^T \geq 0 \\ x_j + h_j \geq 0, \ j = 1, 2, ..., n \\ \sum_{r=j}^{n} h_j p_j \leq 1 \\ h_j \geq 0, \ j = 1, 2, ..., n \end{cases} \tag{21}$$

and

$$Max \ -PX \ \begin{cases} I_m^* AX^T \geq 0 \\ x_j - k_j \leq 0, \ j = 1, 2, ..., n \\ \sum_{r=j}^{n} k_j p_j \leq 1 \\ k_j \geq 0, \ j = 1, 2, ..., n \end{cases} \tag{22}$$

whose decision variables are (X, h) and (X, k) respectively, or their dual problems

$$Min \ \theta \ \begin{cases} \mu A + \lambda = P \\ \lambda_j \leq \theta p_j, \ j = 1, 2, ..., n \\ \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \\ \lambda_j \geq 0, \ j = 1, 2, ..., n \end{cases}$$

and

$$Min\ \theta \begin{cases} \mu A - \lambda = P \\ \lambda_j \le \theta p_j,\ j = 1,2,...,n \\ \mu_1 \ge \mu_2 \ge ... \ge \mu_m \ge 0 \\ \lambda_j \ge 0,\ j = 1,2,...,n \end{cases}$$

whose decision variables are $\theta \in \mathbb{R}$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\mu = (\mu_1, \mu_2, ..., \mu_m)$. Since the dual constraint is now $\mu A \pm \lambda = P$ instead of $\mu A \pm \lambda = P(1+\mu_1)$ (see (13) or (14)), dual solutions directly yield a proxy for the term structure, without previously dividing by $(1+\mu_1)$. As a consequence, denoting by (μ_{**}, λ_{**}) and (μ^{**}, λ^{**}) the dual solutions and by $P_{**} = \mu_{**}A$ and $P^{**} = \mu^{**}A$, Theorems 10, 11 and 13 may be slightly modified and lead to the following results [20]

Theorem 14. *Assume that* $\pounds_{**} > 0$. *The following assertions are fulfilled:*

(*a*) $\pounds^{**} = \dfrac{\pounds_{**}}{1-\pounds_{**}}$, $\pounds_{**} = \dfrac{\pounds^{**}}{1+\pounds^{**}}$ *and* $\pounds_{**} \le \pounds^{**}$

(*b*) $X^{**} = k^{**} - (1+\pounds^{**})h_{**}$ *and* $X_{**} = (1-\pounds_{**})k^{**} - h_{**}$

(*c*) X^{**+} *and* X_{**}^{+} *are proportional to* k^{**} *and* X^{**-} *and* X_{**}^{-} *are proportional to* h^{**}.

(*d*) $X^{**} = (1+\pounds^{**})X_{**}$ ■

Theorem 15. *There are no strong sequential arbitrage portfolios if and only if there exists* μ_{**} *such that* $P = \mu_{**}A$ *and* $\mu_{**1} \ge \mu_{**2} \ge ... \ge \mu_{**m} \ge 0$.

Theorem 16. (*a*) $p_{**j} \le p_j \le p_j^{**}$, $j = 1,2,...,n$.

(*b*) *If* $k_j^{**} > 0$, *then* $p_{**j} = p_j = p_j^{**}\dfrac{1}{1+\pounds^{**}}$

(*c*) *If* $h_{**j} > 0$, *then* $p_{**j}\dfrac{1}{1-\pounds_{**}} = p_j = p_j^{**}$.

(*d*) $P_{**}X^T \le PX^T \le P^{**}X^T$ *for every portfolio* X *without short positions. Furthermore, the first (second) expression holds in terms of equality if* X *is composed of those bonds included in* k^{**} (h_{**}). ■

The latter theorems point out the advantage of each approach. So, Theorem 15 shows that the term structure envelopes μ_{**} and μ^{**} may contain larger discount factors than one, while Theorem 2 clearly establishes that a real term structure is composed of lower discount factors than one.[21] However, $\pounds_{**}$ and $\pounds^{**}$ are relative

[20] The interpretation of this result is as in Theorems 10 and 13 and does not need to be repeated.

[21] Anyway, recall that a real term structure is not compatible with the sequential arbitrage and does not exist if $\pounds_{**} > 0$. Besides, though μ_{**} and μ^{**} may contain larger discount factors than one they match the price of the available bonds and lead to the sequential arbitrage profits (and price errors) $\pounds_{**}$ and $\pounds^{**}$.

arbitrage gains and therefore these measures appropriately reflect the level of arbitrage. If they are large then the arbitrage profits are large, and prices must be modified according to large percentages. On the contrary, $\mathcal{L}_*$ and $\mathcal{L}^*$ reflect discrepancies in prices of similar portfolios but not profits. They may be large even if $\tilde{\mathcal{L}}_*$ and $\tilde{\mathcal{L}}^*$ achieve small values and slight modifications of prices eliminate the presence of arbitrage.

Another pair of measures may be introduced by means of the pair of problems

$$Min\ \theta \quad \begin{cases} \mu A \pm \lambda = P \\ \lambda_j \leq \theta p_j,\ j = 1, 2, ..., n \\ 1 \geq \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \\ \lambda_j \geq 0,\ j = 1, 2, ..., n \end{cases} \tag{23}$$

whose decision variables are $\theta \in \mathbb{R}$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\mu = (\mu_1, \mu_2, ..., \mu_m)$. (23) also provides proxies for the term structure of interest rates whose discount factors are never larger than one. Nevertheless, the dual problem of (23) is not related to any sort of income provided by a sequential arbitrage portfolio. Consequently, it seams complicated to divide the set of bonds in order to interpret which prices are matched by each term structure. Moreover, there is no link among the term structure envelopes, the measures of sequential arbitrage and the sequential arbitrage strategies available in the market.

7. Empirical results

The existence of sequential arbitrage has been tested in the Spanish market. We have used the database of daily prices provided by the Bank of Spain, that contains the price of real transactions corresponding to default free bonds issued by the Spanish government, bonds issued by regional governments of several Spanish communities and bonds with embedded options also issued by regional governments. We have focused on the year 1994.[22] Regarding the existence of strong or second type sequential arbitrage portfolios composed of default free and option free bonds, Table 1 summarizes the results. It is clear that the sequential arbitrage existence cannot be globally rejected. The existence of strong sequential arbitrage (66 days in a sample composed of 245 days) and second type sequential arbitrage (67 days) are closely related, and the values of $\mathcal{L}_{**}$ and $\tilde{\mathcal{L}}_*$ are almost similar. Moreover, bearing in mind the interest rates level in the tested year, it is rather difficult to accept that tax effects may imply a value of $\mathcal{L}_{**}$ greater than 25 basic points.[23] Then, the lack of synchronized

[22]There is not any special reason to choose the year 1994. We just tried to test an arbitrary year and made a random decision.

[23]Besides, the best effect of taxes corresponds to the so called "*Letras del Tesoro*" (zero coupon bonds whose maturity equals one year) but the strategy X_{**} usually incorporates some "*Letras del Tesoro*" in long position and-or bonds with larger maturity in short position.

data [24] is the unique reason that may explain these empirical results, but a value $\pounds_{**} \geq 0.0025$ implies that some investors have obtained prices significantly different to those provided by a sequential arbitrage free market. With regard to the term structure estimation, μ_{**} is usually quite close to the proxy provided in [20], while μ^{**} presents slight discrepancies that allows us to match the price of those bonds included in h_{**}.

Table 2 presents the empirical results when dealing with bonds of the Spanish government and the Spanish communities. These type of bonds were traded in 212 days, the strong sequential arbitrage appeared in 141 days and the second type sequential arbitrage appeared in 142 days. Once again, $\pounds_{**}$ and $\tilde{\pounds}_{*}$ are almost similar. The number of days without sequential arbitrage is big enough so that one can assume that the credit risk is almost zero. Thus, local governments also merit the investors' favorable opinion. Bonds issued by private companies would probably lead to larger values for the measures, but this test is left for future research.

Anyway, we have a greater number of days for which sequential arbitrage profits were available. Once more, illiquidity and the lack of synchronized data may explain these results but it is interesting to point out that there are important differences among prices provided by any term structure and real transaction prices.[25] The strategy X_{**} usually incorporates almost all the available bonds, μ_{**} and μ^{**} show a significant variation with respect to their value when dealing only with bonds issued by the government, and both term structures match almost all the available prices.[26]

The last test also incorporates extendible and-or callable bonds issued by local governments. Table 3 summarizes the results. The sample period consists in 216 days and the strong and second type arbitrage occur in 149 and 151 days respectively. $\pounds_{**}$ and $\tilde{\pounds}_{*}$ are almost similar once more. The high number of days for which $\tilde{\pounds}_{*}$ vanishes seems to reveal inefficiency in the market unless the embedded option prices must almost always equal zero. On the other hand, the results are quite close to those provided in Table 2 what again suggests some inefficiencies in the market. For a few number of days it is possible to price the embedded option according to the criteria developed in Section 5, but the option price could very often imply arbitrage because it almost vanishes.[27] Anyway, the term structure envelopes allows us to match the price of any bond and a negative option price is never obtained which seem to reveal some advantages with respect to the approaches provided in previous literature. Moreover,

[24] It was not possible for us to obtain perfectly synchronized prices.

[25] *i.e.*, some investors obtain prices significantly better (worse) than those generated by any term structure.

[26] This seems to be an interesting advantage provided by the use of envelopes instead of a unique term structure.

[27] This conclusion may be suppoted by another arguments. For instance, there are several dates for which the arbitrage strategy X_{**} sells callable bonds and-or buys extendible bonds.

the term structure envelopes incorporate all the information available in the market, and no bonds have to be excluded.

Summarizing, our results seem to reveal some inefficiencies in the market and, therefore, the existence of sequential arbitrage cannot be rejected, along with the existence of arbitrage strategies that contain bonds with embedded options.[28] The credit risk associated to the local governments of the Spanish communities seems to vanish. The term structure or the term structure envelopes (if the sequential arbitrage occurs) always matches the price of all the available bonds and a negative risk premium or option price is never obtained.

8. Conclusions

The term structure of interest rates is usually estimated by only drawing on default free and option free bonds. This is a clear limitation in a context in which the estimation of state-price densities implicit in financial asset prices is becoming more and more important. Furthermore, the common methods to estimate the term structure have led to several puzzles since the information contained in many market prices has to be excluded.

This article has shown that the term structure may be approximated by using all the available securities, in which case, the "real" term structure may be substituted by a family of envelopes. The family of envelopes allows us to compute the exact market price on any bond and related securities, can measure interesting effects and solves some classical caveats. Moreover, it provides practical procedures useful to traders and researchers, since they detect the presence of sequential arbitrage and analyze the level of efficiency in bonds markets.

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Table 1. Default-free and option-free bonds.

(245 tested days. 67 days reflect second type sequential arbitrage. 66 days reflect strong sequential arbitrage)

Value of $\tilde{\mathcal{L}}_*$	Number of days	Value of $\mathcal{L}_{**}$	Number of days
$\tilde{\mathcal{L}}_* \leq 0.001$	17	$\mathcal{L}_{**} \leq 0.001$	17
$0.001 < \tilde{\mathcal{L}}_* \leq 0.01$	32	$0.001 < \mathcal{L}_{**} \leq 0.01$	32
$0.01 < \tilde{\mathcal{L}}_* \leq 0.02$	10	$0.01 < \mathcal{L}_{**} \leq 0.02$	9
$0.02 < \tilde{\mathcal{L}}_* \leq 0.05$	3	$0.02 < \mathcal{L}_{**} \leq 0.05$	3
$0.05 < \tilde{\mathcal{L}}_* \leq 0.1$	5	$0.05 < \mathcal{L}_{**} \leq 0.1$	5

Table 2. Option-free bonds with risk premium.

(212 tested days. 142 days reflect second type sequential arbitrage. 141 days reflect strong sequential arbitrage)

Value of $\tilde{\mathcal{L}}_*$	Number of days	Value of $\mathcal{L}_{**}$	Number of days
$\tilde{\mathcal{L}}_* \leq 0.001$	15	$\mathcal{L}_{**} \leq 0.001$	15
$0.001 < \tilde{\mathcal{L}}_* \leq 0.01$	62	$0.001 < \mathcal{L}_{**} \leq 0.01$	61
$0.01 < \tilde{\mathcal{L}}_* \leq 0.02$	42	$0.01 < \mathcal{L}_{**} \leq 0.02$	42
$0.02 < \tilde{\mathcal{L}}_* \leq 0.05$	17	$0.02 < \mathcal{L}_{**} \leq 0.05$	17
$0.05 < \tilde{\mathcal{L}}_* \leq 0.1$	6	$0.05 < \mathcal{L}_{**} \leq 0.1$	6

Table 3. Bonds with risk premium and embedded option.

(216 tested days. 151 days reflect second type sequential arbitrage. 149 days reflect strong sequential arbitrage)

Value of $\tilde{\mathcal{L}}_*$	Number of days	Value of $\mathcal{L}_{**}$	Number of days
$\tilde{\mathcal{L}}_* \leq 0.001$	14	$\mathcal{L}_{**} \leq 0.001$	14
$0.001 < \tilde{\mathcal{L}}_* \leq 0.01$	59	$0.001 < \mathcal{L}_{**} \leq 0.01$	57
$0.01 < \tilde{\mathcal{L}}_* \leq 0.02$	43	$0.01 < \mathcal{L}_{**} \leq 0.02$	43
$0.02 < \tilde{\mathcal{L}}_* \leq 0.05$	25	$0.02 < \mathcal{L}_{**} \leq 0.05$	25
$0.05 < \tilde{\mathcal{L}}_* \leq 0.1$	6	$0.05 < \mathcal{L}_{**} \leq 0.1$	10
$0.1 < \tilde{\mathcal{L}}_*$	4	$0.1 < \mathcal{L}_{**}$	0