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Competitive Equilibrium with Search Frictions: Arrow-Debreu meets Diamond-Mortensen-Pisarides*

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Abstract

When the trading process is characterized by search frictions, traders may be rationed so markets need not clear. We argue that rationing can be part of general equilibrium, even if it is outside its normal interpretation. We build a general equilibrium model where the uncertainty arising from rationing is incorporated in the definition of a commodity, in the spirit of the Arrow-Debreu theory. Prices of commodities then depend not only on their physical characteristics, but also on the probability that their trade is rationed. The standard definition of a competitive equilibrium is extended by replacing market clearing with a matching condition. This condition relates the traders' rationing probabilities to the measures of buyers and sellers in the market via an exogenous matching function, as in the search models of Diamond (1982a, 1982b), Mortensen (1982a, 1982b) and Pissarides (1984, 1985). When search frictions vanish (so matching is frictionless) our model is equivalent to the competitive assignment model of Gretsky, Ostroy and Zame (1992, 1999). We adopt their linear programming approach to derive the welfare and existence theorems in our environment.

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Key Words: search frictions; competitive (price-taking) equilibrium; matching function; linear programming; duality.

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1 Introduction

General equilibrium theory is the cornerstone for the analysis of competitive markets. In the theory of Arrow (1964) and Debreu (1959), trade is represented as a costless process. Any agent seeking to buy or sell a good at a given point in time can do so at the equilibrium market price. Trade involves no further costs in terms of time and resources. Search theory on the other hand highlights the costly nature of the trading process. Since the seminal work of Diamond (1981, 1982b), Mortensen (1982a, 1982b) and Pissarides (1984, 1985), this theory has become the dominant paradigm to study labor markets. There workers usually take time and spend resources in order to find a suitable employer and vice versa, and rationing arises in the form of unemployment. The key assumption of the Diamond-Mortensen-Pisarides model is that workers and firms must search for trading opportunities, and the outcome of their search is uncertain. Search frictions are modeled via an exogenous matching function which describes a random bilateral meeting process between workers and firms.[1] This random process implies that at any point in time some agents will manage to trade and others will not. Hence, unlike in the Arrow-Debreu model, agents may be rationed in equilibrium (so markets need not clear), and in general it will take time to trade.

In recent years the general equilibrium literature and the search literature have grown apart, suggesting that the two approaches cannot be reconciled. This paper aims at building a bridge between both literatures. Our view is that, while their description of the trading process is markedly different, the actual gap is narrower than the reading of the literature suggests. Specifically, we argue that rationing can be part of general equilibrium, even if it is outside its normal interpretation. The lesson is that the definition of "market clearance" has to be expanded to incorporate the existence of a trading technology that is not frictionless.

We study a prototypical class of search economies. In the spirit of the Arrow-Debreu theory, the uncertainty arising from rationing is incorporated in the definition of a commodity.[2] Prices of commodities then depend not only on their physical characteristics, but also on the probability that their trade is rationed. In a competitive equilibrium, agents take prices as given because they are infinitesimal relative to the size of the economy. Agents also take as given rationing probabilities, which are part of the description of a commodity. Markets are anonymous, so prices and rationing

[1]The matching function gives the measure of bilateral meetings between a worker and a firm as a function of the aggregate measures of firms and workers in the market, and possibly other variables such as the agents' search efforts.

[2]Markets are incomplete because agents cannot insure against this rationing uncertainty.

probabilities do not depend on the identities of the traders. The departure from the standard Arrow-Debreu definition of a competitive equilibrium is that market clearing is replaced with a matching condition which describes a trading technology that is not frictionless. The matching condition relates the trading probabilities of buyers and sellers to the aggregate measures of buyers and sellers in each market via an exogenous matching function, as in the Diamond-Mortensen-Pisarides search model. In equilibrium, prices of commodities adjust so that the optimal decisions of the agents are consistent with the matching condition.

The matching function captures the presence of external congestion effects in the trading process which arise from search frictions. Intuitively, as more buyers seek to trade a given good, the probability that each of them actually trades falls while the probability that a seller trades increases. Similarly for sellers. In other words, agents seeking to trade impose a negative congestion externality on traders on the same side of the market and a positive externality on traders on the other side of the market.

Critically, however, the above external effects are internalized in a competitive equilibrium, so the equilibrium allocation is constrained efficient.[3] Because rationing probabilities are specified in the definition of a commodity, they are explicitly priced. Hence, when agents choose to trade in a given market, they pay a price which depends not only on the physical characteristics of the good but also the probability that its trade is rationed. Suppose, for instance, that the same physical good trades in two locations. Suppose also that there are fewer buyers per seller in the first location in equilibrium. If buyers and sellers are expected utility maximizers who are free to choose the location where they trade, then the good should trade at a higher price in the first location (because there the probability of trading is higher for buyers and lower for sellers). The fact that the price of the good differs across locations is not surprising from the general equilibrium perspective since the objects traded are formally two different commodities (described by their physical characteristics and the level of "market tightness" or "congestion" at the trading location).

When search frictions vanish (so matching is frictionless) our model reduces to the competitive assignment model of Gretsky, Ostroy and Zame (1992, 1999). These authors develop a competitive formulation of the assignment game of Shapley (1955) and Shapley and Shubik (1972) where agents are infinitesimal relative to the size of the economy. We adopt their linear programming approach

[3]The definition of constrained efficiency takes into account the fact that the social planner (just like the market) is restricted by the exogenous matching technology.

to derive the welfare and existence theorems in our environment.[4] To the best of our knowledge, the linear programming approach has not been used before to study economies with search frictions.

1.1 Related Literature

We are not the first ones to argue that rationing can be incorporated in general equilibrium. Gale (1996) uses a general equilibrium approach to study the efficiency and existence properties of competitive markets in the presence of adverse selection. In his model, rationing arises from incomplete asymmetric (there are no search frictions). Peters (1997) adopts a similar approach to study a class of economies with search frictions similar to the one in this paper, so his work is closely related to ours. There is however a key modeling difference with respect to Gale (1996) and Peters (1997). In those papers, the objects of trade are contracts (involving exactly one buyer and one seller) which provide a complete specification of the terms of trade, *including prices.* Prices then are no longer the variable that adjusts to ensure that individual decisions are consistent with the underlying matching process. Instead it is trading probabilities that adjust.

In Gale (1996) agents take as given the probability of trading in each of the contract markets, as well as a belief about the unobservable type of the traders they meet randomly in these markets. In equilibrium, trading probabilities adjust so the long side of the market is rationed when there is an imbalance of buyers and sellers. Also, beliefs are rational in all active markets, and in inactive markets (e.g. for contracts not traded in equilibrium) they are pinned down using refinements analogous to those used in games of incomplete information.

In Peters (1997), as in our paper, the matching process is exogenous. Given this process, traders form beliefs about the probability of trading in each of the contract markets. As in Gale (1996), the condition that beliefs are rational in all active markets replaces the standard market clearing condition. Beliefs in inactive markets are assumed to be common for all traders and they are always well-defined. Specifically, traders believe that there many buyers and sellers in all markets, even if they are inactive in equilibrium. For the case of homogenous buyers and heterogeneous sellers, Peters (1997) shows that an equilibrium exists and is constrained efficient for a general class of matching functions under standard assumptions on preferences. More recently, Eeckhout and

[4]See Makowski and Ostroy (1996, 2003) for other general equilibrium formulations which use the linear programming approach. Jerez (2003, 2005), Rahman (2005) and Song (2006) adopt this approach to study the general equilibrium economies with asymmetric information.

Kircher (2008) show that these results hold also in environments with two-sided heterogeneity if the equilibrium allocation entails either positive or negative assortative matching.[5]

The approach in this paper is different in that the objects of trade specify rationing probabilities and it is prices that adjust so individual decisions are consistent with the aggregate trading technology. The two approaches are equivalent as they yield identical equilibrium allocations.[6] Thus they should be viewed as two sides of the same coin. For instance, the equivalent of the assumption that all traders have the same beliefs in our model is that all traders face the same prices (prices are anonymous). The equivalent of the assumption that beliefs in all markets (including inactive ones) are well-defined is that all commodities are priced even if they are not traded in equilibrium. The latter assumption is standard in general equilibrium models with a continuum of commodities, like ours (e.g. see Mas-Colell and Zame 1991). Finally, the matching condition in our model is the equivalent of the *generalized market clearing condition* in Peters (1997).

Our formulation is perhaps closer in spirit to standard Arrow-Debreu theory in that it fleshes out the allocating role of prices.[7] "Flipping things over" and laying out the model in this more standard form brings to light several key insights. The first is the connection with the competitive assignment model of Gretsky, Ostroy and Zame (1992, 1999) as the frictionless version of the model. This implies a crucial methodological advantage because it allows us to adopt their linear programming approach to establish our results. In particular, it allows us to characterize a competitive equilibrium and derive the welfare and existence theorems for a general class of search economies with heterogeneous buyers and sellers regardless of the pattern of matching displayed by the equilibrium allocation. In these sense, our results generalize those of Peters (1997, 2000) and Eeckhout and Kircher (2008). Interestingly, the linear programming methodology opens the door to the use of existing algorithms for computing competitive equilibria in search economies, which should be particularly interesting in macroeconomic and industrial-organization applications.

Our work is also related to the strategic (game theoretic) formulations of competitive equilibrium in search environments by Montgomery (1991), Peters (1991, 1997, 2000), Moen (1997), Shimer (1996), Acemoglu and Shimer (1999), Burdett, Shi, and Wright (2001), Shi (2001), Mortensen and

[5]Essentially, the match value function is either root-supermodular or it is weakly submodular.

[6]The welfare theorems derived in Peters (2000) and Eeckhout and Kircher (2008) and in this paper imply that the set of equilibrium allocations in the two approaches coincides with the set of constrained efficient allocations.

[7]In this sense, our formulation is closer to Gale (1992), which is a variation of Gale's (1996) model where agents choose outcomes taking as given both prices and trading probabilities, and in equilibrium prices and trading probabilities are such that individual decisions are consistent at the aggregate level.

Wright (2002) and Eeckhout and Kircher (2010), among others. Like Peters (1997) and Eeckhout and Kircher (2008), our model should be viewed as a reduced form of these strategic equilibrium notions. In particular, Peters (1991, 1997) provides game-theoretic foundations for the general equilibrium model. He considers a two-stage game of direct competition with potentially heterogeneous sellers and homogeneous buyers. Sellers first simultaneously and publicly offer contracts to buyers, and buyers then choose among these contracts. To capture the coordination problem among buyers, it is assumed that all buyers play identical strategies. Peters (1997) shows that, when the number of traders gets large, the equilibrium allocation of the game coincides with the competitive equilibrium allocation of a large search economy with an urn-ball matching function. Also, one set of beliefs that supports the competitive equilibrium allocation coincides with the sellers' conjectures in the limit game (who correctly anticipate the effects on buyers' behavior of making an off-the-equilibrium-path offer). These results provide a rationale for the general equilibrium formulation in large search economies, and for assuming an urn-ball matching function in that formulation.

The literature to which this paper contributes deviates from the original Diamond-Mortensen-Pisarides model in that prices are determined by a competitive process, and a sufficiently rich market structure is assumed that allows to internalize the search externalities. The Diamond-Mortensen-Pisarides model, and a large part of the search literature, assumes instead that each transaction between a buyer and a seller constitutes a bilateral monopoly. As a result, search equilibria are typically constrained inefficient in those models.[8]

2 The Economy

There is a continuum of buyers and sellers of finitely many types, indexed by $b \in B$ and $s \in S$, respectively. The set of agent types is then $I = B \cup S$, and the population is formally described by a Borel measure $\xi \in M_+(I)$ with full support.[9]

There is a finite number of indivisible goods, indexed by $h \in H$. Think of $h \in H$ as a list

[8]Unless the surplus is divided according to the bargaining rule that internalizes the search externalities (see Diamond 1982b and Hosios 1990).

[9]Since I is finite, we may write $\xi = \sum_{i \in I} \alpha_i \delta_i$, where $\alpha_i \in \Re_{++}$ represents the mass of type-i agents in the population and $\delta_i \in M_+(I)$ represents the Dirac (mass point) measure on i. The total measures of buyers and sellers are then $\xi(B) = \sum_{i \in B} \alpha_i$ and $\xi(S) = \sum_{i \in S} \alpha_i$.

of observable characteristics describing a good. In the labor market example, h is a list of features describing a job (e.g. sector, hours worked, tasks performed, location,...). In a housing market example, h is a list of features describing a housing unit (e.g. location, size, number of bedrooms/bathrooms, neighborhood, nearby transportation, year of construction...). Goods are indivisible because agents cannot "perform half of job h" or "buy half of housing unit h". They either perform the job (buy the unit) or not. It is possible though that the description of job h' is identical to that of job h except that h' requires half of the hours worked. In our formulation, h and h' are two *different* jobs (say "full time" and "part time"). Finally, it is convenient to include the autarky choice h_0 in H (e.g. the choice to remain unemployed).

In our model, each buyer wants to consume– and each seller can supply– at most one unit of a good. Agents have von Neumann-Morgernstern preferences. Utility is transferable, so there is also a divisible numeraire good (available in positive and negative amounts) which yields equal constant marginal utility to all agents.[10] Each buyer type b is characterized by a utility function $v_b : H \to \Re_+$; i.e., $v_b(h)$ is the value a type-b buyer assigns to a unit of good h. Each seller type s is described by a cost (or reservation utility) function $v_s : H \to \Re_+$. If seller s does not supply good h, the convention is to set $v_s(h)$ (close) to ∞.[11] In the labor market example, if a type-b firm hires a type-s worker to perform job h, then $v_b(h)$ is firm's productivity and $v_s(h)$ is the worker's disutility at that job. In the housing market example, if a type-b buyer purchases housing unit h from a type-s seller, then $v_b(h)$ is the buyer's valuation and $v_s(h)$ is the seller's reservation value for that unit. Valuations are normalized so $v_i(h_0) = 0$ for all $i \in I$.

So far our environment is identical to the continuous assignment model of Gretsky, Ostroy and Zame (1999). The key difference with respect to that model is that here the trading technology is not frictionless. In the labor market example, not all workers searching for jobs will find one, and the same is true for firms searching for workers. Also, a particular job h may be harder to find/fill that another job h' (just like certain housing units, say larger ones, may be harder to sell than others). In general, the probability that buyers and sellers are rationed will depend on the goods they seek to trade. Rationing probabilities, however, do not depend on the identity (type) of the traders since markets are assumed anonymous.

[10]These assumptions are standard in the search literature.

[11]We keep the range of all valuation functions v_i in $\Re_+$ so we can represent each v_i as a vector in the Euclidean space, $v_i \in \Re_+^{cardH}$. Alternatively, we could use the formulation in Gretsky, Ostroy, and Zame (1999) and identify seller types with the goods they supply: $H \equiv S$. The cost of a type-h seller is then represented by a positive scalar $v_h \in \Re_+$. We have chosen the first formulation because it simplifies the description of the environment and the proofs.

Buyers and sellers seeking to trade a given good meet bilaterally and at random, as in Diamond (1981,1982a, 1982b), Mortensen (1982a,1982b) and Pissarides (1984,1985). The random meeting process is described by an exogenous matching function. To ease notation, we assume that the matching function is the same for all goods.[12] Suppose a measure β of buyers and a measure σ of sellers seek to trade a given good. The matching function $\mathcal{M}(\beta,\sigma)$ determines the total measure of bilateral matches as a function of β and σ. It is standard to assume that $\mathcal{M} : R_+^2 \to R_+$ is continuous, strictly increasing, concave and homogeneous of degree one. Also, since the total number of matches cannot exceed the number of traders in the short side of the market, $\mathcal{M}(\beta,\sigma) \leq \min\{\beta,\sigma\}$. In particular, $\mathcal{M}(0,\sigma) = \mathcal{M}(\beta,0) = 0$. Also, we assume that the Law of Large Numbers holds, so the probability that a seller meets a buyer is

$$\pi(k) = \frac{\mathcal{M}(\beta,\sigma)}{\sigma} = \mathcal{M}(k,1) \tag{2.1}$$

where $k = \frac{\beta}{\sigma} \in \Re_+$ is the buyer-seller ratio. Likewise, the probability that a buyer meets a seller is

$$\alpha(k) = \frac{\mathcal{M}(\beta,\sigma)}{\beta} = \mathcal{M}(1,k^{-1}) = \pi(k)k^{-1}, \tag{2.2}$$

Note that $\alpha(k)$ and $\pi(k)$ also represent the fractions of buyers and sellers who meet a trading partner. Conversely, $1-\alpha(k)$ and $1-\pi(k)$ are the fractions of buyers and sellers who are rationed.

In the search literature it is common to refer to k as the level of market "tightness". Note that the function $\pi(k)$ is continuous, strictly increasing and concave, with $\pi(0) = 0$ and $\lim_{k\to\infty}\pi(k) = 1$. On the other hand, $\alpha(k)$ is continuous and decreasing, with $\lim_{k\to 0}\alpha(k) = 1$ and $\lim_{k\to\infty}\alpha(k) = 0$. Intuitively, the higher the buyer-seller ratio then the easier it is for sellers to meet buyers and the harder it is for buyers to meet sellers. As k goes to infinity (zero) the probability that a seller meets a buyer goes to one (zero) and the probability that a buyer meets a seller goes to zero (one).

3 The General Equilibrium Model

In this section, we define the commodity space and describe the set of feasible allocations.

Commodities

[12] Our results extend directly to the case where the matching function differs across goods. Just make the matching function specified below contingent on each $h \in H$, and denote it by $\mathcal{M}_h$.

In the spirit of the Arrow-Debreu treatment of uncertainty, we describe commodities both by their physical characteristics and the uncertainty involved in their trade. Formally, a commodity is a pair $(h, k) \in (H|h_0) \times \Re_+$, where h describes the physical good and k is the buyer-seller ratio in a trading post for good h.[13] Unlike in the Arrow-Debreu model, markets are incomplete because agents cannot insure against this trading uncertainty (e.g. by trading state-contingent claims).

Allocations

An allocation is an assignment of agents to commodities or to the autarky choice. Consider the space $M_c(I \times H \times \Re_+)$ of compactly supported Borel measures on $I \times H \times \Re_+$, endowed with the weak-star topology. Following the measure-theoretic description in Hart, Hildenbrand, and Kohlberg (1974), an allocation is formally described by a positive a measure $\mu \in M_{c+}(I \times H \times \Re_+)$. This means that, under allocation μ, a measure $\mu(A, F)$ of agents with types i in A are assigned to a commodity (h, k) in F for arbitrary Borel subsets $A \subset I$ and $F \subset H \times \Re_+$. It is also useful to define the marginals (or projections) of μ on the set I of agent types and on the set $H \times \Re_+$ of commodities, denoted by $\mu_I \in M_+(I)$ and $\mu_{H\times\Re_+} \in M_{c+}(H \times \Re_+)$ respectively. Here $M(I)$ denotes the set of Borel measures on I. Since I is finite, $M_c(I) = M(I)$, and $M(I)$ is isomorphic to the Euclidean space (see the Appendix). The commodities exchanged under allocation μ are the elements (h, k) in the support of $\mu_{H\times\Re_+}$ with $h \neq h_0$. We denote this support by $\text{supp}\mu_{H\times\Re_+}$.

Feasible allocations must be consistent with respect to the population ξ. This means that the total measure of agents of a given type who are assigned to the different commodities or to the autarky choice must be equal to the measure of such types who are present in the population (see also Gretsky, Ostroy, and Zame 1999 and Peters 1997):

$$\mu_I = \xi. \tag{3.1}$$

Equivalently, $\mu_I(A) = \xi(A)$ for all Borel subsets $A \subset I$.

Feasible allocations must also be consistent with respect to the matching technology. This requires that the measure of buyers who find a seller must be equal to the measure of sellers who find a buyer for (almost) all commodities exchanged under allocation μ:

$$\alpha(k)\mu(B, h, k) = \pi(k)\mu(S, h, k) \text{ for almost all } (h, k) \in \text{supp}\mu_{H\times\Re_+} \text{ with } h \neq h_0. \tag{3.2}$$

[13] As is standard, the numeraire good is not made explicit in the definition of a commodity.

For each such commodity (h, k), there is a measure $\mu(B, h, k)$ of potential buyers, a fraction $\alpha(k)$ of whom find a seller. Hence, the measure of buyers who find a seller is $\alpha(k)\mu(B, h, k)$. Similarly, the measure of sellers who find a buyer is $\pi(k)\mu(S, h, k)$. Condition (3.2) says that these two measures are equal.

4 Competitive equilibrium

In this section, we define a competitive equilibrium for the search economy. We then derive the welfare and existence theorems. We begin by describing the price space.

Prices

A price system is a non-negative continuous linear function on the set of commodities:[14]

$$p \in C_{+}(H \times \Re_{+}), \tag{4.1}$$

where $C_{+}(H \times \Re_{+})$ is endowed with the topology of uniform convergence on compact sets. That is, $p(h, k) \in \Re_{+}$ is the price of commodity (h, k); i.e., the price at which good h is traded in a trading post where the buyer-seller ratio is equal to k. Prices are normalized so $p(h_0, k) = 0$ for all k; i.e., the price associated to the no-trade choice is zero.

Prices then depend both on the physical characteristics of the goods and their associated rationing probabilities. This is intuitive. If the same physical good is trades in two markets where the ratio k of buyers to sellers is different, the price of the good will be different in the two markets. Specifically, the price will be higher in the market where k is lower. Formally, (h, k) and (h, k') are *two different commodities* whose prices will in general be different; i.e., $p(h, k) \neq p(h, k')$ if $k \neq k'$. On the other hand, since p is continuous, "similar commodities" have similar prices (see Mas-Colell 1975). That is, if the same physical good trades in two markets with a similar buyer-seller ratio k, the price of the good will be similar price in the two markets.

Competitive equilibrium

We are now ready to define a competitive equilibrium.

[14] Remember that the set of commodities $H \times \Re_{+}$ is an infinite set. We follow Mas-Colell's (1975) description of the price system for economies with a continuum of differentiated commodities. The difference is that, whereas in his model agents can trade any integer number of units of a given good, here agents trade at most one unit.

The key assumption is that agents have rational expectations about k, and hence about the probability with which they will be rationed when they choose to trade commodity (h, k). For a type-b buyer, the expected utility from commodity $(h, k) \in H \times \Re_+$ is

$$u_b(h, k; p) \equiv \alpha(k)[v_b(h) - p(h, k)]. \tag{4.2}$$

If $h \neq h_0$ the buyer trades with probability $\alpha(k)$, in which case she gets her valuation $v_b(h)$ net of the market price $p(h, k)$. The buyer's expected utility then increases with the trading probability (decreases with k) and decreases with the price. Similarly, for a type-s seller, the expected utility from commodity $(h, k) \in H \times \Re_+$ is

$$u_s(h, k; p) \equiv \pi(k)[p(h, k) - v_s(h)], \tag{4.3}$$

so it increases with both k and the price $p(h, k)$ for $h \neq h_0$. By assumption, the autarky choice yields zero expected utility to any agent; i.e., $u_i(h_0, k; p) = 0$ for all $i \in I$.

Definition 1. *A competitive (price-taking) equilibrium for the search economy is an allocation* $\mu^* \in M_{c+}(I \times H \times \Re_+)$ *and a price system* $p^* \in C_+(H \times \Re_+)$ *such that:*

(i) Agents choose $(h, k) \in H \times \Re_+$ *to maximize their expected utility taking* p^* *as given:*

$$v_i^*(p^*) \equiv \sup_{(h,k) \in H \times \Re_+} u_i(h, k; p^*) \; = u_i(h^*, k^*; p^*). \tag{4.4}$$

for almost all $(i, h^*, k^*) \in \text{supp}\mu^*$*.*

(ii) μ^* *is consistent with the population:*

$$\mu_I^* = \xi. \tag{4.5}$$

(iii) μ^* *is consistent with the matching technology:*

$$\alpha(k)\mu^*(B, h, k) = \pi(k)\mu^*(S, h, k) \text{ for almost all } (h, k) \in \text{supp}\mu^*_{H \times \Re_+} \text{ with } h \neq h_0. \tag{4.6}$$

Condition (i) requires that (almost) all buyers and sellers choose to trade a commodity that maximizes their expected utility at the equilibrium prices p^*. The expected indirect utility attained by type-i agents in equilibrium is denoted by $v_i^*(p^*)$. Condition (ii) ensures that the allocation is consistent with the population. The difference with respect to the standard definition of a competitive equilibrium is that, in condition (iii), market clearing has been replaced with the matching condition (3.2).

In the absence of search frictions, market clearing would require that the measures of buyers and sellers who trade in each market be equal:

$$\mu^*(B, h, k) = \mu^*(S, h, k) \text{ for almost all } (h, k) \in \text{supp}\mu^*_{H\times\Re_+} \text{ with } h \neq h_0. \tag{4.7}$$

This implies that $k = 1$ for all goods which are traded in equilibrium (there is no rationing). Indeed, when the matching condition in (iii) is replaced by (3.2), our definition of a competitive equilibrium is equivalent to that in Gretsky, Ostroy, and Zame (1999).

The matching condition in (iii) can also be expressed as

$$\mu^*(B, h, k) = k\mu^*(S, h, k) \text{ for almost all } (h, k) \in \text{supp}\mu_{H\times\Re_+} \text{ with } h \neq h_0. \tag{4.8}$$

using (2.2). Equation (4.8) says the measures of buyers and sellers who choose to trade in each market in equilibrium generate the exact buyer-seller ratio k that these traders take as given when they choose to participate in that market. So, as noted above, traders' beliefs are rational. This is Peter's (1997) *generalized market clearing condition.* This condition is also the parallel of Gale's (1996) condition of *fulfilled expectations in active markets.*

Remark. As in the standard Arrow-Debreu model, the key postulate is that *all agents take prices as given.* Moreover, here *agents also take as given* the buyer-seller ratio in each market, and hence the associated *rationing probabilities.* Both the price-taking postulate and the assumption that agents take rationing probabilities as given are natural in our continuum model. Since the seminal contributions of Ostroy (1980) and Makowski (1980) an important line of research has emerged which characterizes competitive economies as those where traders cannot affect prices, even if they try to.[15] Only in these perfectly competitive environments is the price-taking assumption justified. Gretsky, Ostroy, and Zame (1999) provide a characterization of perfect competition for

[15] For an insightful survey see Makowski and Ostroy (2001).

the assignment model. They show that the inability of agents to affect prices is only generic in environments where agents are infinitesimal. However, having a large economy need not be sufficient for perfect competition. It is also necessary that there is enough substitutability among agents present in the economy (for each agent type in the economy, there is another type which is "sufficiently similar"). In this paper, we have assumed a finite set of buyer and seller types and a positive mass of each type as this guarantees perfect substitutability among agents of the same type. Nevertheless, our results can be extended to an environment with a continuum of agent types and a continuum of goods, as long as B, S and H are compact sets. There the price-taking should be justified under similar conditions as in Gretsky, Ostroy, and Zame (1999). We have chosen to make our point in a general environment with finitely many goods and agent types, particularly because most search models in macroeconomic and industrial-organization applications belong to this class. This choice also simplifies our proofs.

Welfare Theorems and Existence

In this economy, the planner is restricted by the exogenous matching technology that brings buyers and sellers together in the market. That is, the planner can choose an assignment μ of agents to commodities, but unlike in Gretsky, Ostroy and Zame (1992,1999) it cannot choose an assignment of buyers to sellers of these commodities. The appropriate notion of efficiency is then that of constrained efficiency.

The total gains from trade from a given allocation $\mu \in M_{c+}(I \times H \times \Re_+)$:

$$\int_{(b,h,k)\in B\times H\times\Re_+} v^b(h)\alpha(k)d\mu(b,h,k) - \int_{(s,h,k)\in S\times H\times\Re_+} v^s(h)\pi(k)d\mu(s,h,k). \qquad (4.9)$$

The first term is the sum of the buyers' valuations for the physical goods assigned to them under allocation μ. Remember that $\mu(b,h,k)$ is the measure of type-b buyers who are assigned to commodity (h,k). A fraction $\alpha(k)$ of these buyers manage to trade so they receive their valuation $v^b(h)$, while the rest are rationed. Similarly, the second term is the sum of the sellers' costs (or reservation utilities) for the goods sold under allocation μ. A measure $\mu(s,h,k)$ of type-s sellers are assigned to commodity (h,k). Yet only a fraction $\pi(k)$ of them trade the good, at cost $v^s(h)$. The difference between the first and second term in (4.9) gives the total gains from trade.

The planner's problem is to choose an allocation that maximizes (4.9) subject to the aggregate feasibility constraints (3.1) and (3.2).[16] Although the aggregate feasibility constraint (3.2) is different from that in Gretsky, Ostroy, and Zame (1999), as in their model, the planner's problem is a linear programming problem. That is, the objective function and the constraints are linear in μ (i.e., they are just integrals with respect to μ). In the Appendix, we exploit this linear structure to prove all the results that follow. The first is the existence of constrained efficient allocations.

Theorem 1. *A solution to the planner's problem exists.*

The second result is the First Welfare Theorem, and can be proved using a slight variation of the standard argument.

Theorem 2. *(First Welfare Theorem) A competitive equilibrium allocation μ^* is constrained efficient.*

Proof. μ^* satisfies conditions (ii) and (iii) in the definition of a competitive equilibrium, so it is a feasible solution for the planner's problem. Moreover, condition (i) implies that, for almost all (i, h^*, k^*) in the support of μ^*,

$$u_i(h^*, k^*; p^*) \geq u_i(h, k; p^*) \text{ for all } (h, k) \in H \times \Re_+.$$

But then, for any other feasible allocation μ,

$$\int_{(i,h,k)\in I\times H\times\Re_+} u_i(h, k; p^*) d\mu^*(i, h, k) \geq \int_{(i,h,k)\in I\times H\times\Re_+} u_i(h, k; p^*) d\mu(i, h, k).$$

In words, total expected utility is at least as high under allocation μ^* than under allocation μ. Substituting (4.2), (4.3), and the feasibility condition (3.2) with respect to the matching technology above –and noting that as prices are just transfers which cancel out–yields

$$\begin{aligned} & \int_{(b,h,k)\in B\times H\times\Re_+} \alpha(k) v^b(h) d\mu^*(b, h, k) - \int_{(s,h,k)\in S\times H\times\Re_+} \pi(k) v^s(h) d\mu^*(s, h, k) \\ \geq & \int_{(b,h,k)\in B\times H\times\Re_+} \alpha(k) v^b(h) d\mu(b, h, k) - \int_{(s,h,k)\in S\times H\times\Re_+} \pi(k) v^s(h) d\mu(s, h, k). \end{aligned}$$

Hence, μ^* solves the planner's problem. □

[16] Note that the planner will only choose allocations where sellers are assigned to the commodities they supply. If a positive mass of sellers were assigned to a commodity they do not supply (with $v^s(h)$ close to ∞), the gains from trade will be close to $-\infty$. This is cannot be optimal since the autarky choice implies zero gains from trade.

The Second Welfare Theorem also holds, so any optimal solution μ to the planner's problem can be attained as a competitive equilibrium.

Theorem 3. *(Second Welfare Theorem) Let μ be an optimal solution to the planner's problem. Then there is a price system $p \in C_+(H \times \Re_+)$ such that (μ, p) is a competitive equilibrium.*

Finally, we establish the existence of a competitive equilibrium.

Theorem 4. *A competitive equilibrium exists.*

As shown in the Appendix, the proofs of Theorems 1, 3 and 4 rely on the fact that the matching function is continuous (i.e., the functions $\alpha(k)$ and $\pi(k)$ are continuous). Since B, S and H are finite sets, the functions we are integrating over in the planner's objective function and in the matching condition (3.2) are continuous functions. This continuity, combined with the fact that the trading probabilities are monotone in k and bounded above by one, is essentially all we need to prove the theorems. The same continuity property would arise if the matching function differed across goods as long as all matching functions are continuous.

5 Relation with Game Theoretic Notions of Competitive Equilibrium with Search Frictions

The Walrasian equilibrium notion in this paper is a reduced form of the strategic competitive equilibrium notions in the search literature. In the former, agents maximize their expected utility taking prices as given, and in equilibrium prices adjust so as to satisfy the matching condition. In the latter, by contrast, some agents (e.g. sellers) compete by simultaneously posting and committing to price offers, and other agents (e.g. buyers) observe all the posted offers and direct their search to the most attractive ones. Agents then behave strategically trying to exploit existing arbitrage opportunities. In equilibrium, there are no gains from arbitrage.[17] The connection between the two

[17] The equilibrium notion in Mortensen and Wright (2002) is related to ours in that it brings to light the allocating role of prices. They consider an economy with one good (labor) which may trade in different "submarkets" with different levels of market tightness which are explicitly priced. In equilibrium agents select the most preferred submarket taking as given the set of existing submarket and the prices in these submarkets. Prices adjust so that the actual level of market tightness in each submarket equals the level that agents take as given. Finally, it is not possible to open additional submarkets that would attract positive measures of buyers and sellers. The interpretation is that there are third-party market makers with a profit motive who exploit any arbitrage opportunities by setting up submarkets.

approaches is most clearly seen by looking at example economies typically used in the literature. The vast majority of these examples assume that agents on one side of the market are homogeneous, while agents on the other side are potentially heterogeneous. The connection best understood in the context of a static environment.

Example 1. Homogenous workers and homogenous firms. Take the simplest labor market example where workers and firms are homogeneous and all jobs are alike. There is a measure u of unemployed workers and a measure ν of vacancies, so $\hat{k} = \nu/u$ is the economy-wide labor market tightness (or buyer-seller ratio). Suppose for simplicity that ν is fixed (though it is trivial to incorporate free entry). Any match between a worker and a firm produces output y. The workers' disutility of labor is denoted by $b < y$. The payoffs for workers and firms when they do not find a match are normalized to zero. The wage ω specifies the division of the match surplus, and is determined endogenously as a result of the agents' strategic interaction (see below). Since workers and firms have always to option of not trading, $\omega \in [b, y]$. In addition to the properties described in Section 2, we assume that the matching function $\mathcal{M}$ is continuously differentiable and $\alpha(k)$ is convex.

In modeling the competitive game, we follow the approach in Peters (2000), Shimer (1996) and Acemoglu and Shimer (1999) (but see Moen (1997) for an alternative yet equivalent approach).[18] The timing of the game is as follows. In the first stage, each firm simultaneously announces and commits to a wage offer. In the second stage, workers observe all the announced offers and direct their search to the most attractive offer (possibly randomizing if they are indifferent). The set of firms who post a given wage and the set of workers who apply for jobs paying that wage then meet randomly according to the matching function $\mathcal{M}$. This means that the expected payoff to a firm who offers wage ω depends on the level of market tightness k that the offer generates, and is given by $\alpha(k)(y - \omega)$. Similarly, the expected payoff to a worker who applies for jobs paying ω is $\pi(k)(\omega - b)$. The intuition is that a high wage offer attracts more workers, increasing the firm's contact probability and decreasing the worker's contact probability. All agents take their decisions optimally based on their common beliefs about the relationship between each potential wage offer and the market tightness that the offer will generate. Moreover, these common beliefs, denoted by $k(\omega)$, are rational.

The equilibrium of the game is characterized as follows. Let Ω^* denote the set of equilibrium wage announcements. Since workers are ex ante identical, in equilibrium they must get a common

[18]See also the survey by Rogerson, Shimer, and Wright (2005).

expected payoff $U^{w*} \geq 0$; i.e., for all $\omega^* \in \Omega^*$,

$$\pi(k^*)(\omega^* - b) = U^{w*}, \tag{5.1}$$

where k^* is the market tightness level generated by ω^*. The traders' equilibrium beliefs $k^*(\omega)$ then satisfy (5.1) for $\omega \in \Omega^*$. The fundamental issue is how beliefs are determined off the equilibrium path (i.e., for wages not announced in equilibrium). The key assumption is that firms take the workers' level of market utility U^{w*} as given, since they are infinitesimal relative to the size of the market. Using Peters (2000)'s terminology, firms' payoffs have the *market utility property.* If a negligible mass of firms deviates and announces a wage offer $\omega' \notin \Omega^*$, they will assume that the deviation has no impact on the workers' ex ante expected payoff. In particular, their beliefs about $k(\omega')$ would be based on the assumption that buyers always modify their search strategies in a way that ensures that they are indifferent between the deviating offer ω' and the equilibrium offers $\omega^* \in \Omega^*$. In other words, the equilibrium beliefs $k^*(\omega)$ satisfy equation (5.1) *for all wage offers*, even those not announced in equilibrium. Hence, $k^*(\omega)$ coincides with the workers' indifference curve associated to U^{w*}.

All the above implies that any equilibrium wage announcement must maximize the firms' expected payoff subject to the constraint that workers get the market level of utility U^{w*}. That is, each wage $\omega^* \in \Omega^*$ and the market tightness k^* generated by that wage must solve

$$(\omega^*, k^*) = \underset{\omega \in [b,y],\, k \in R_+}{\arg\max} \; \alpha(k)(y - \omega) \quad \text{ s.t. } \pi(k)(\omega - b) = U^{w*}. \tag{5.2}$$

The properties of the matching function imply that this convex problem has a unique solution. Hence, all firms then announce the same wage ω^* and all workers seek to find a job at that wage. Since beliefs are rational, the market tightness generated by ω^* is then equal to the economy-wide level of market tightness: $k^* = \hat{k}$. As shown in Figure 1, an interior solution is characterized by the tangency between the indifference curves of the worker and the firm on the (ω, k) space:

$$\frac{\pi(k)}{\pi'(k)(w - b)} = -\frac{\alpha(k)}{\alpha'(k)(y - w)}. \tag{5.3}$$

In this case, the equilibrium value of ω^* solves (5.3) when $k = \hat{k}$. Substituting ω^* and $k = \hat{k}$ into (5.1) we obtain the equilibrium value of U^{w*}.[19]

[19] If $\omega^* \notin [b, y]$ problem (5.2) has a corner solution which is also easily characterized (either the workers or the firms

Figure 1 also makes it clear why any $\omega' \neq \omega^*$ cannot be an equilibrium outcome. Suppose it was, and let k' be the market tightness generated by ω'. It is easy to find a deviating offer ω'' with associated market tightness k'' satisfying the workers' indifference condition (5.1), such that the deviating firms get a higher expected payoff than non-deviating firms. Figure 1 depicts the case where $\omega' > \omega^*$. Note that $\omega'' = \omega^*$ constitutes a profitable deviation.

A direct consequence of this equilibrium notion is that the equilibrium outcome (ω^*, k^*) lies in the contract curve, so it is constrained efficient (see Moen 1997). Indeed, substituting $\alpha(k) = \pi(k)/k$ into (5.3) and rearranging yields the well-known Hosios (1990) condition:

$$\frac{y-w}{w-b} = \frac{\eta(k)}{1-\eta(k)} \text{ where } \eta(k) = \frac{\pi'(k)k}{\pi(k)} \tag{5.4}$$

This condition says that, at a constrained efficient allocation, the firms' share of the surplus is equal to the elasticity of the workers' trading probability $\pi(k)$.

It is now easy to see that this constrained efficient allocation is attained in a Walrasian equilibrium. In this economy, there is a single physical good (labor). A commodity is described by a level of market tightness $k \in \Re_+$, and a price system is a continuous wage function $\omega(k)$. (The price associated to the autarky choice is again zero). Flipping Figure 1 around we obtain Figure 2. There we can see that any continuous wage function lying between the indifference curves of the firm and the worker which is tangent to both indifference curves at $(\hat{k}, \omega^*)$ supports the constrained efficient allocation. Unlike in the standard Arrow-Debreu model, here there are many supporting price systems.[20] As noted in the introduction, this is a standard feature of models with a continuum of commodities, where the prices of commodities that are not traded in equilibrium are indeterminate. A standard selection rule is to take the supremum over the set of supporting price systems (see Gretsky, Ostroy, and Zame 1999). The selected equilibrium wage function $\omega^*(k)$ then coincides with the worker's indifference curve, so it is the inverse of the function $k^*(\omega)$ specifying the equilibrium beliefs of the strategic game.

Example 2. Homogeneous workers and heterogeneous firms. Suppose now that there are two firm types, $j = 1, 2$, and workers are again homogeneous. Firm types differ in their productivity y_j with $y_1 > y_2 > b$. Let ν_j denote the measure of vacancies at type-j firms.

get a zero expected payoff).

[20] A related issue arises in Peters' (1997) general equilibrium model where equilibrium beliefs are indeterminate.

Consider the interesting case where both firm types trade in equilibrium. An argument analogous to the one used above implies that any wage ω_j^* offered by a type-j firm and the market tightness k_j^* generated by that wage must solve

$$(\omega_j^*, k_j^*) = \underset{\omega \in [b,y],\, k \in R_+}{\arg\max}\ \alpha(k)(y_j - \omega) \text{ s.t. } \pi(k)(\omega - b) = U^{w*}, \ \text{ for } j = 1, 2, \tag{5.5}$$

where U^{w*} is again the workers' equilibrium expected payoff. All type-j firms then announce the same wage ω_j^*. The difference is that now type-1 and type-2 firms may announce different wages.

Given the properties of the matching function, again a unique, constrained efficient equilibrium exists (e.g. see Moen 1997 and Peters 2000). As shown in Figure 3, the indifference curves of the two firm types represented on the space (k, ω) satisfy the single crossing property (that of type 1 being steeper than that of type 2). Suppose again that ω_j^* lies in the interior of $[b, y_j]$ for each j. Let $x^* \in [0, u]$ denote the measure of workers who search for a job paying ω_1^*. By construction, workers are indifferent between the wages announced by both firm types. In equilibrium the indifference curve of type-1 firms is tangent to the worker's indifference curve associated to U^{w*} at $\left(\frac{\nu_1}{x^*}, \omega_1^*\right)$, while that of type-2 firms is tangent to the worker's indifference curve at $\left(\frac{\nu_2}{u-x^*}, \omega_2^*\right)$. That is, the equilibrium values of ω_1^*, ω_2^*, x^* and U^{w*} solve the following system of equations:

$$\alpha(\nu_1/x^*)(\omega_1^* - b) = U^{w*}, \tag{5.6}$$

$$\alpha(\nu_2/(u - x^*))(\omega_2^* - b) = U^{w*}, \tag{5.7}$$

$$\frac{y_1 - w_1^*}{w_1^* - b} = \frac{\eta(\nu_1/x^*)}{1 - \eta(\nu_1/x^*)} \tag{5.8}$$

$$\frac{y_2 - w_2^*}{w_2^* - b} = \frac{\eta(\nu_2/u - x^*)}{1 - \eta(\nu_2/(u - x^*))} \tag{5.9}$$

Again, the above is a Walrasian equilibrium allocation. Figure 3 shows that any continuous function $\omega(k)$ lying between the indifference curves of the worker and the two firm types which is tangent to the indifference curves of the worker and type-1 firms at $(\frac{\nu_1}{x^*}, \omega_1^*)$, and to the indifference curves of the worker and type-2 firms at $(\frac{\nu_2}{u-x^*}, \omega_2^*)$ supports the allocation. Again, there are many such functions (i.e., $\omega(k)$ is indeterminate for $k \notin \{\frac{\nu_1}{x^*}, \frac{\nu_2}{u-x^*}\}$). Using the selection rule in Example 1, $\omega^*(k)$ again coincides with the worker's indifference curve (i.e., the inverse of the equilibrium belief function of the strategic game).

Example 3. Heterogenous workers and heterogeneous firms. Suppose now that workers are also of two types, $i = 1, 2$, with $b_1 < b_2$. The indifference curves of the two worker types satisfy the single crossing property (that of type 1 being steeper than that of type 2). Suppose that $y_1 > y_2 > b_2 > b_1$, so all potential matches between a worker and a firm are profitable. Let μ_i be the measure of type-i workers. For simplicity, assume that $\nu_1 = \nu_2 = \nu$.

This third example is more complex because one needs to characterize the matching pattern that emerges in equilibrium; i.e., which type of worker is assigned to which type of firm (see Peters 1997, Shi 2001, Mortensen and Wright 2002 and Eeckhout and Kircher 2010). Potentially a given worker type may be assigned to both types of firms and vice versa.[21] Once we know the matching pattern, the appropriate tangency (e.g. Hosios) condition for each worker-firm match together with the rationality of equilibrium beliefs will determine the equilibrium outcome essentially as above. As noted by Shi 2001 and Eeckhout and Kircher 2010, the Hosios condition is only necessary but not sufficient to characterize the equilibrium of the game with two-sided heterogeneity.[22]

The single crossing property satisfied by the indifference curves of both workers and firms implies that each equilibrium wage offer $\omega^* \in \Omega^*$ attracts a single type of worker (so if two firms of the same type match with two different worker types they must be posting different wages). Figure 4 illustrates an equilibrium where type-1 firms trade only with type-2 workers and type-2 firms trade only with type-1 workers. Wages in these two type of matches satisfy the corresponding tangency condition. Since beliefs are rational, these wages generate market tightness levels $k_1^* = \nu/u_2$ and $k_2^* = \nu/u_1$, respectively. In this example, $u_1 < u_2$, and so $k_1^* < k_2^*$. For this matching pattern to emerge in equilibrium it is necessary that type-2 workers are not attracted by the offer ω_2^* of type-2 firms. That is, these workers do not prefer (k_2^*, ω_2^*) to (k_1^*, ω_1^*) (so in particular $\omega_1^* > \omega_2^*$). Similarly, type-1 workers should not be attracted by the offer of type-1 firms.

Again, this outcome can be decentralized as a Walrasian equilibrium allocation. With our price

[21] See Shi (2001) for an environment which rules this out.

[22] We have chosen an example with no complementarities; i.e., the value of matching a type i worker and a type-j firm is $f_{ij} = y_j - b_i$ and so $f_{11} + f_{22} = f_{12} + f_{21}$ (see also Mortensen and Wright 2002). See Eeckhout and Kircher 2008 (Figure 2) for a slightly more complicated example with complementarities. Shi 2001 and Eeckhout and Kircher 2010 show that in the presence of complementarities an interesting trade-off that arises between the gains from higher match values and the losses due to rationing. In particular, Eeckhout and Kircher 2010 consider a general class of economies with a continuum of buyer and seller types. They show that an equilibrium of the strategic game exists and is constrained efficient when (i) the match value function is either $\bar{n}$-root-supermodular where $\bar{n}$ is determined by the upper bound of the elasticity of the matching function $\mathcal{M}$, or (ii) it is nowhere $\underline{n}$-root-supermodular where $\underline{n}$ is determined by the lower bound of the elasticity of $\mathcal{M}$. In case (i) the equilibrium allocation displays positive assortative matching, and in case (ii) it displays negative assortative matching.

selection rule, the equilibrium wage function is given by the lower envelop of the workers' indifference curves. The first welfare theorem in this paper implies that this allocation is constrained efficient.

In principle, many possible matching patterns could arise in equilibrium. In our simple example, the equilibrium matching pattern could be identified through trial and error. For a given matching pattern, there is a candidate equilibrium outcome (i.e., satisfying the Hosios and the rational beliefs conditions). We just need to check whether there exist prices supporting this outcome. In more complex environments, the linear programming approach used in this paper will identify the equilibrium matching pattern. As shown in the Appendix, the equilibrium price associated to any commodity traded in equilibrium is just the shadow price of the matching condition associated to that commodity in the planner's problem. Since the planner' problem is linear, looking for equilibrium prices amounts to looking for shadow prices that satisfy the Complementary Slackness Theorem of Linear Programming. The complementary slackness conditions are the conditions that an efficient matching pattern must satisfy in addition to feasibility. In applications, the existing linear programming algorithms will calculate the equilibrium allocation.

A Appendix

In this section, we follow the linear programming approach in Gretsky, Ostroy and Zame (1992,1999) to prove Theorems 1 and 3. An alternative proof of Theorem 2 is also provided as a byproduct. We begin with some necessary notation.

A.1 Notation

For an arbitrary set Z, $C(Z)$ denotes the vector space of continuous real-valued functions on Z, endowed with the topology of uniform convergence on compact sets. The topological dual of $C(Z)$ is the space of signed Borel measures on Z which have compact support and are finite on compact sets (see Hewitt 1959). This space is denoted by $M_c(Z)$, and is endowed with the weak-star topology. Then $C(Z)$ is also the dual of $M_c(Z)$. We write $C_+(Z)$ and $M_{c+}(Z)$ for the respective positive cones of these spaces.

Let $C(Z)$ be paired in duality with $M_c(Z)$ with the standard bilinear form:

$$\langle f, \gamma \rangle = \int_{z \in Z} f(z) d\gamma(z), \quad f \in C(Z),\ \gamma \in M_c(Z).$$

In the special case where Z is compact, the topological dual of $C(Z)$ is the space of signed Borel measures on Z, denoted by $M(Z)$. If Z is finite, both $C(Z)$ and $M(Z)$ are isomorphic to the Euclidean space, so the integral above is replaced by a finite sum.

For any integer n, the product spaces $\prod_{j=1,\ldots,n} C(Z_j)$ and $\prod_{j=1,\ldots,n} M_c(Z_j)$ (endowed with the corresponding product topology) are also paired in duality with bilinear form:

$$\sum_{j=1}^{n} \langle f_j, \gamma_j \rangle, \quad (f_1, f_2, \ldots, f_n) \in \prod_{j=1,\ldots,n} C(Z_j),\ (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \prod_{j=1,\ldots,n} M_c(Z_j).$$

Denote the support of an arbitrary measure $\gamma_j \in M_c(Z_j)$ by $\text{supp}\gamma_j$.

A.2 The linear programming problems

We may write the planner's problem in Section 4 in a equivalent (more convenient) form.

We first decompose the measure $\xi \in M_+(I)$ describing the population into a pair measures $(\xi^{\mathcal{B}}, \xi^{\mathcal{S}}) \in M_+(B) \times M_+(S)$ which describe the respective subpopulations of buyers and sellers. Similarly, we decompose an allocation $\mu \in M_{c+}(I \times H \times \Re_+)$ into a pair of measures $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_{c+}(B \times H \times \Re_+) \times M_{c+}(S \times H \times \Re_+)$, where $\mu^{\mathcal{B}}$ describes the assignment of buyers to commodities and $\mu^{\mathcal{S}}$ describes the corresponding assignment for sellers. The corresponding marginals (or projections) on B and S are denoted by $\mu^{\mathcal{B}}_B \in M_+(B)$ and $\mu^{\mathcal{S}}_S \in M_+(S)$ respectively. The marginals on the set of commodities $H \times \Re_+$ are denoted by $\mu^{\mathcal{B}}_{H\times\Re_+}, \mu^{\mathcal{S}}_{H\times\Re_+} \in M_{c+}(H \times \Re_+)$. With this equivalent description of an allocation, the measure of buyers assigned to commodity (h, k) is $\mu^{\mathcal{B}}_{H\times\Re_+}(h, k) = \mu^{\mathcal{B}}(B, h, k)$. Similarly, the measure of sellers assigned to commodity (h, k) is $\mu^{\mathcal{S}}_{H\times\Re_+}(h, k) = \mu^{\mathcal{S}}(S, h, k)$. The matching condition (3.2) can then be expressed in terms of the marginals $\mu^{\mathcal{B}}_{H\times\Re_+}$ and $\mu^{\mathcal{S}}_{H\times\Re_+}$:

$$\alpha(k)\mu^{\mathcal{B}}_{H\times\Re_+}(h,k) = \pi(k)\mu^{\mathcal{S}}_{H\times\Re_+}(h,k) \text{ for almost all } (h,k) \in \{H|h_0\} \times \Re_+. \tag{A.1}$$

It is convenient to extend (A.1) to include the autarky choice by defining

$$\hat{\alpha}(h,k) = \begin{cases} \alpha(k) & \text{if } h \neq h_0 \\ 0 & \text{if } h = h_0 \end{cases} \quad \text{and} \tag{A.2}$$

$$\hat{\pi}(h,k) = \begin{cases} \pi(k) & \text{if } h \neq h_0 \\ 0 & \text{if } h = h_0 \end{cases} \tag{A.3}$$

Then (A.1) may be written as

$$\hat{\alpha}(h,k)\mu^{\mathcal{B}}_{H\times\Re_+}(h,k) = \hat{\pi}(h,k)\mu^{\mathcal{S}}_{H\times\Re_+}(h,k) \text{ for almost all } (h,k) \in H \times \Re_+. \tag{A.4}$$

With this equivalent description of the population and of an allocation, the problem of the planner is to find $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_c(B \times H \times \Re_+) \times M_c(S \times H \times \Re_+)$ to solve

$$(P) \qquad \sup \quad \int_{B\times H\times\Re_+} \alpha(k)v^b(h)d\mu^{\mathcal{B}}(b,h,k) - \int_{S\times H\times\Re_+} \pi(k)v^s(h)d\mu^{\mathcal{S}}(s,h,k)$$

s.t.

$$\begin{aligned} \mu^{\mathcal{B}}_B &= \xi^{\mathcal{B}}, && \text{(A.5)} \\ \mu^{\mathcal{S}}_S &= \xi^{\mathcal{S}}, && \text{(A.6)} \\ \hat{\alpha}(h,k)\mu^{\mathcal{B}}_{H\times\Re_+}(h,k) &= \hat{\pi}(h,k)\mu^{\mathcal{S}}_{H\times\Re_+}(h,k) \text{ for almost all } (h,k) \in H \times \Re_+, && \text{(A.7)} \\ \mu^{\mathcal{B}}, \mu^{\mathcal{S}} &\geq 0. && \text{(A.8)} \end{aligned}$$

Because the matching function is continuous (i.e., $\alpha(k)$ and $\pi(k)$ are continuous), the functions we are integrating over in the objective function are continuous. Similarly, in the constraint system (A.7), $\hat{\alpha}(h,k)$ and $\hat{\pi}(h,k)$ are continuous. This continuity property is key in the arguments that follow. All the arguments extend directly to an economy where the matching function differs across goods provided all matching functions are continuous.

The objective function and the constraint systems in problem (P) are linear on $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$. (Remember that the marginals are just integrals of the corresponding measures). Formally, problem (P) is an equality-constrained linear program (see Anderson and Nash 1987). Standard results in linear programming theory show that the *dual* problem for (P) is also a linear program. Whereas (P) is a maximization problem, the dual is a minimization problem. The primal and dual problems are related because, under certain regularity conditions, the dual variables are also the shadow prices of the primal constraints and vice versa.

Denote the Lagrange multiplier (or dual variable) associated with constraint (A.5) by $q^{\mathcal{B}} \in C(B)$, and that associated with (A.6) by $q^{\mathcal{S}} \in C(S)$. Denote the Lagrange multiplier associated with constraint (A.7) by $\lambda \in C(H \times \Re_+)$. The Lagrangian associated with problem (P) is

$$\begin{aligned}\mathcal{L} = &\int_{(b,h,k)\in B\times H\times\Re_+} \alpha(k)v^b(h)d\mu^{\mathcal{B}}(b,h,k) - \int_{(s,h,k)\in S\times H\times\Re_+} \pi(k)v^s(h)d\mu^{\mathcal{S}}(s,h,k)\\ &+\sum_{b\in B} q^{\mathcal{B}}(b)[\xi^{\mathcal{B}}(b) - \mu_B^{\mathcal{B}}(b)] + \sum_{s\in S} q^{\mathcal{S}}(s)[\xi^{\mathcal{S}}(s) - \mu_S^{\mathcal{S}}(s)]\\ &+\int_{(h,k)\in H\times\Re_+} \lambda(h,k)[\hat{\pi}(h,k)d\mu^{\mathcal{S}}_{H\times\Re_+}(h,k) - \hat{\alpha}(h,k)d\mu^{\mathcal{B}}_{H\times\Re_+}(h,k)].\end{aligned}$$

Since $\hat{\alpha}(h_0,k) = \hat{\pi}(h_0,k) = 0$, without loss of generality $\lambda(h_0,k) = 0$ for all $k \in \Re_+$.

Since $v^i(h_0) = 0$ for all $i \in B \cup S$, using (A.2)-(A.3) and rearranging yields

$$\begin{aligned}\mathcal{L} = &\sum_{b\in B} q^{\mathcal{B}}(b)d\xi^{\mathcal{B}}(b) + \sum_{s\in S} q^{\mathcal{S}}(s)d\xi^{\mathcal{S}}(s)\\ &-\int_{(b,h,k)\in B\times H\times\Re_+} [q^{\mathcal{B}}(b) - \hat{\alpha}(h,k)(v^b(h) - \lambda(h,k))]d\mu^{\mathcal{B}}(b,h,k)\\ &-\int_{(s,h,k)\in S\times H\times\Re_+} [q^{\mathcal{S}}(s) - \hat{\pi}(h,k)(\lambda(h,k) - v^s(h))]d\mu^{\mathcal{S}}(s,h,k).\end{aligned}$$

The dual problem (D) then consists of finding multipliers $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda) \in C(B)\times C(S)\times C(H\times\Re_+)$ to solve

$$(D) \qquad \inf \quad \sum_{b\in B} q^{\mathcal{B}}(b)d\xi^{\mathcal{B}}(b) + \sum_{s\in S} q^{\mathcal{S}}(s)d\xi^{\mathcal{S}}(s)$$

s.t.

$$q^{\mathcal{B}}(b) \geq \hat{\alpha}(h,k)(v^b(h) - \lambda(h,k)) \quad \forall (b,h,k) \in B \times H \times \Re_+, \tag{A.9}$$
$$q^{\mathcal{S}}(s) \geq \hat{\pi}(h,k)(\lambda(h,k) - v^s(h)) \quad \forall (s,h,k) \in S \times H \times \Re_+, \tag{A.10}$$

where again $\lambda(h_0,k) = 0$ for all $k \in \Re_+$ without loss of generality.[23]

[23] Using a more compact notation, the *primal* problem is to find $x = (\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_c(B\times H\times\Re_+)\times M_c(S\times H\times\Re_+)$ to solve

$$\begin{aligned}(P)\quad &\sup \quad \langle x, c\rangle\\ &\text{s.t.} \quad Ax = b,\\ &\qquad\quad x \geq 0.\end{aligned}$$

Here $c = (c^{\mathcal{B}}, c^{\mathcal{S}}) \in C(B \times H \times \Re_+) \times C(S \times H \times \Re_+)$ is given by $c^{\mathcal{B}}(b,h,k) = \hat{\alpha}(h,k)v^b(h)$ and $c^{\mathcal{S}}(s,h,k) = -\hat{\pi}(h,k)v^s(h)$, and $b = (\xi^{\mathcal{B}}, \xi^{\mathcal{S}}, 0) \in M(B)\times M(S)\times M_c(H\times\Re_+)$. Finally, $A : M_c(B\times H\times\Re_+)\times M_c(S\times H\times\Re_+) \to M(B)\times M(S)\times M_c(H\times\Re_+)$ is a continuous linear map defined by $A(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) = (\mu^{\mathcal{B}}_B, \mu^{\mathcal{S}}_S, \hat{\alpha}\mu^{\mathcal{B}}_{H\times\Re_+} - \hat{\pi}\mu^{\mathcal{S}}_{H\times\Re_+})$.

The *dual* problem is to find $y = (q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda) \in C(B)\times C(S)\times C(H\times\Re_+)$ to solve

$$\begin{aligned}(D)\quad &\inf \quad \langle b, y\rangle\\ &\text{s.t.} \quad A^*y \geq c,\end{aligned}$$

where $A^* : C(B)\times C(S)\times C(H\times\Re_+) \to C(B\times H\times\Re_+)\times C(S\times H\times\Re_+)$ is the adjoint of A. That is, A^* is defined by the relation $\langle x, (A^*y)\rangle = \langle Ax, y\rangle$, for all $x \in M_c(B\times H\times\Re_+)\times M_c(S\times H\times\Re_+)$ and all $y \in C(B)\times C(S)\times C(H\times\Re_+)$.

A.3 Existence of optimal solutions and absence of a duality gap

Denote the optimal values for problems (P) and (D) by $\nu(P)$ and $\nu(D)$, respectively. We first show that both problems are consistent (i.e. their feasible sets are not empty) and bounded (i.e. $\nu(P)$ and $\nu(D)$ are finite).

Lemma A. 1. *Problems (P) and (D) are consistent and bounded.*

Proof. The autarky allocation μ_0 where no one trades is a feasible solution for problem (P). There $\mu^{\mathcal{B}}_{0H\times\Re_+}$ and $\mu^{\mathcal{S}}_{0H\times\Re_+}$ are degenerate at $(h_0, \frac{\xi^{\mathcal{B}}(B)}{\xi^{\mathcal{S}}(S)})$, and $\mu^{\mathcal{B}}_{0B} = \xi^{\mathcal{B}}, \mu^{\mathcal{S}}_{0B} = \xi^{\mathcal{S}}$. Thus problem (P) is consistent. Also, since the gains from trade under autarky are zero, $\nu(P) \geq 0$.

In problem (D), set $\lambda = \lambda_0 \in C(H \times \Re_+)$ where $\lambda_0(h,k) = 0$ for all $(h,k) \in H \times \Re_+$. In the constraint system (A.9) and (A.10), $\hat{\alpha}(h,k)$ and $\pi(\hat{h,k})$ are bounded above by one. One then can find a feasible solution where $\lambda = \lambda_0$ by choosing $q^{\mathcal{B}}_0 \in C(B)$ and $q^{\mathcal{S}}_0 \in C(S)$ so that

$$\begin{aligned} q^{\mathcal{B}}_0(b) &= \sup_{h\in H}\{v^b(h)\} + \epsilon, \quad b \in B, \\ q^{\mathcal{S}}_0(s) &= \sup_{h\in H}\{-v^s(h)\} + \epsilon, \quad s \in S, \end{aligned}$$

for $\epsilon > 0$ small, since the sets H, B and S are finite. Thus problem (D) is consistent. Moreover,

$$\nu(D) \leq \sum_{b\in B} q^{\mathcal{B}}_0(b) d\xi^{\mathcal{B}}(b) + \sum_{s\in S} q^{\mathcal{S}}_0(s) d\xi^{\mathcal{S}}(s) < \infty.$$

Finally, by the weak duality theorem (Anderson and Nash 1987, Theorem 2.1), $\nu(P) \leq \nu(D)$, so both problems are bounded:

$$0 \leq \nu(P) \leq \nu(D) \leq \sum_{b\in B} q^{\mathcal{B}}_0(b) d\xi^{\mathcal{B}}(b) + \sum_{s\in S} q^{\mathcal{S}}_0(s) d\xi^{\mathcal{S}}(s) < \infty. \qquad \square$$

Unlike a finite linear program, a bounded infinite linear program need not have optimal solutions. Moreover, the primal and dual values need not coincide as a "positive duality gap" may occur ($\nu(P) < \nu(D)$). Below we show that the linear programs in this paper are solvable and have the same optimal value ($\nu(P) = \nu(D)$). This is all we need to prove the Second Welfare Theorem and the existence of a competitive equilibrium (see also Gretsky, Ostroy and Zame (1992, 1999)). Our proof follows closely the proof of Theorem 5.2 in Anderson and Nash (1987) on the Monge-Kantorovich mass-transfer problem (see also Gabriel, López-Martínez, and Hernández-Lerma 2001).[24]

We first show that problem (P) is solvable. This establishes Theorem 1.

Theorem A. 1. *Problem (P) has optimal solutions.*

[24] The proof is slightly different from that in Gretsky, Ostroy and Zame (1992, 1999). One reason is that, unlike in their paper, the measures describing an allocation are defined over a non-compact space (e.g. $B \times H \times \Re_+$). The method of proof is similar to that in Jerez (2003), where the same issue arises. The difference is that there the planner's problem is a linear semi-infinite programs (e.g. with finitely many constraints), which allows to restrict to measures with finite support. Proving existence of optimal dual solutions is slightly more involved here.

Proof. The feasible set of problem (P) is bounded, and the constraint map and objective function are weak-star continuous, so the result follows from Theorem 3.20 in Anderson and Nash (1987). $\square$

Next, we show that problem (D) satisfies the well-known Slater regularity condition (e.g. Krabs 1979, Section II.3.3). This implies that there is no duality gap.

Theorem A. 2. *There is no duality gap: $\nu(P) = \nu(D)$.*

Proof. The positive cone of $C(B \times H \times \Re_+) \times C(S \times H \times \Re_+)$ has a non-empty interior, denoted by Y_0. Also, $(q_0^{\mathcal{B}}, q_0^{\mathcal{S}}, \lambda_0) \in C_+(B) \times C_+(S) \times C_+(H \times \Re_+)$ in the proof of Lemma A.1 is a Slater point in the feasible set of problem (D). Since $\nu(D)$ is finite, Theorem 3.13 in Anderson and Nash (1987) implies that $\nu(P) = \nu(D)$. $\square$

By Theorem A.2, the Complementary Slackness Theorem (Anderson and Nash 1987, Theorem 3.2) may be applied to characterize optimal solutions for problems (P) and (D).

Theorem A. 3. *(Complementary Slackness Theorem) Feasible solutions $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ and $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda)$ for problems (P) and (D) are optimal if and only if they satisfy the complementary slackness conditions:*

$$q^{\mathcal{B}}(b) = \hat{\alpha}(h,k)(v^b(h) - \lambda(h,k)) \text{ for all } (b,h,k) \in \text{supp}\mu^{\mathcal{B}}, \quad \text{(A.11)}$$

$$q^{\mathcal{S}}(s) = \hat{\pi}(h,k)(v^s(h) - \lambda(h,k)) \text{ for all } (s,h,k) \in \text{supp}\mu^{\mathcal{S}}. \quad \text{(A.12)}$$

As noted by Anderson and Nash (1987), the solvability of problem (D) cannot be settled using an argument similar to that in Theorem A.1 because $C(H \times K)$ is not the dual of any normed space. We follow their approach and repose problem (D) in a different space, with the required compactness properties, and then appeal to the continuity of the functions $\hat{\alpha}$ and $\hat{\pi}$ to show that an optimal solution in the enlarged space lies in the original space. The proof involves three steps. Lemma A.2 first shows that the set of feasible dual solutions can be taken to be bounded without loss of generality. The proof uses the fact the sets B, S and H are finite. Lemma A.3 then shows that the buyer-seller ratio k in all markets can be restricted without loss of generality to lie on a compact subset K of $\Re_+$ (e.g. to be bounded above). The proof again appeals to the finiteness of B, S and H, and also uses the fact that the matching probabilities $\alpha(k)$ and $\pi(k)$ are continuous, $\pi(k)$ is strictly increasing, and $\lim_{k\to\infty} \alpha(k) = 0$. Finally, Theorem A.4 uses the results in Lemma A.2 and A.3 to establish the existence of optimal dual solutions.

Lemma A. 2. *The set of feasible dual solutions can be taken to be bounded without loss of generality.*

Proof. For $h = h_0$, the dual constraint systems (A.9)-(A.10) imply $q^{\mathcal{B}} \geq 0$, $q^{\mathcal{S}} \geq 0$. Also, since $V(D)$ is finite, there is no loss of generality in assuming that $q^{\mathcal{B}}$ and $q^{\mathcal{S}}$ are bounded above.

For any given $(h,k) \in \{H|h_0\} \times \Re_+$, if an optimal primal solution satisfies $\mu^{\mathcal{B}}(b,h,k) = 0$ for all $b \in B$ then $\mu^{\mathcal{B}}_{H\times\Re_+}(h,k) = 0$. That is, commodity (h,k) is not traded. The primal constraint (A.8) then implies that the shadow price $\lambda(h,k)$ associated to commodity (h,k) can be chosen arbitrarily. On the other hand, if $\mu^{\mathcal{B}}(\tilde{b},h,k) > 0$ for some $\tilde{b} \in B$ then $\mu^{\mathcal{B}}_{H\times\Re_+}(h,k) > 0$. Substituting (A.2) and (A.3) into (A.8) implies $\pi(k)\mu^{\mathcal{S}}_{H\times\Re_+}(h,k) > 0$ because $\alpha(k) > 0$ for all $k \in \Re_+$. This rules out $k = 0$ because $\pi(k) = 0$. It also implies that $\mu^{\mathcal{S}}(\tilde{s}, h, k) > 0$ for some $\tilde{s} \in S$. By Theorem A.3, optimal dual solutions then satisfy

$$q^{\mathcal{B}}(\tilde{b}) = \alpha(k)(v^{\tilde{b}}(h) - \lambda(h,k)), \tag{A.13}$$
$$q^{\mathcal{S}}(\tilde{s}) = \pi(k)(\lambda(h,k) - v^{\tilde{s}}(h)). \tag{A.14}$$

using (A.2)-(A.3). Since $q^{\mathcal{B}}(\tilde{b}), q^{\mathcal{S}}(\tilde{s}) \geq 0$ and $\alpha(k), \pi(k) > 0$, $v^{\tilde{s}}(h) \leq \lambda(h,k)) \leq v^{\tilde{b}}(h)$. Thus we may assume

$$\inf_{s\in S} v^s(h) \leq \lambda(h,k) \leq \sup_{b\in B} v^b(h), \quad (h,k) \in H \times \Re_+, \tag{A.15}$$

without loss of generality. Since B and S are finite sets, the infimum and the supremum in (A.15) are attained. □

Lemma A. 3. *If problem (D) is solvable then there exists a compact subset $K \subset \Re_+$ such that, if all the constraints which are associated with elements $(i,h,k) \in I \times H \times (\Re_+|K)$ are eliminated from problem (D), the set of optimal dual solutions does not change.*

Proof. Let $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda)$ be a feasible dual solution. Defining

$$q_1^{\mathcal{B}}(b) = \sup_{H\times\Re_+} \{\hat{\alpha}(h,k)(v^b(h) - \lambda(h,k))\}, \tag{A.16}$$
$$q_1^{\mathcal{S}}(s) = \sup_{H\times\Re_+} \{\hat{\pi}(h,k)(\lambda(h,k) - v^s(h))\}, \tag{A.17}$$

yields another feasible solution $(q_1^{\mathcal{B}}, q_1^{\mathcal{S}}, \lambda)$ (see (A.9)-(A.10)). Suppose that

$$q_1^{\mathcal{B}}(b) > \lim_{k\to\infty} \hat{\alpha}(h,k)(v^b(h) - \lambda(h,k)), \quad \forall h \in H, \forall b \in B, \tag{A.18}$$
$$q_1^{\mathcal{S}}(s) > \lim_{k\to\infty} \hat{\pi}(h,k)(\lambda(h,k) - v^s(h)), \quad \forall h \in H, \forall s \in S. \tag{A.19}$$

Since $\hat{\alpha}(h,k)$, $\hat{\pi}(h,k)$ and $\lambda(h,k)$ are continuous functions and B, S and H are finite sets, there exists $\bar{k}$ sufficiently large such that $K = [0, \bar{k}]$ satisfies the statement in the Lemma A.3 (the constraints associated with elements $(i,h,k) \in I \times H \times (\Re_+|K)$ do not bind). Suppose the statement in Lemma A.3 were not true. Then either (i) there is $\tilde{b} \in B$ and $\tilde{h} \in H|h_0$ such that (A.18) holds with equality, or (ii) there is $\tilde{s} \in S$ and $\tilde{h} \in H|h_0$ such that (A.19) holds with equality. The dual constraint systems associated to h_0 can be ignored once we assume $q^{\mathcal{B}}, q^{\mathcal{S}} \geq 0$ (see the proof of Lemma A.2).

Take case (i):

$$q_1^{\mathcal{B}}(\tilde{b}) = \lim_{k\to\infty} \hat{\alpha}(\tilde{h},k)[v^{\tilde{b}}(\tilde{h}) - \lambda(\tilde{h},k)] = 0 \tag{A.20}$$

since $\lim_{k\to\infty} \hat{\alpha}(\tilde{h},k) \to 0$ and $\lambda(\tilde{h},k)$ is bounded by Lemma A.2. Also, (A.9) implies $\lambda(h,k) \geq v^{\tilde{b}}(h)$ for all h and all $k \in \Re_+$ since $\hat{\alpha}(h,k) > 0$. On the other hand, $q_1^{\mathcal{B}}(\tilde{b})$ is attained at (h_0,k) for any $k \in \Re_+$ by definition. The first possibility is that $q_1^{\mathcal{B}}(\tilde{b})$ is not be attained for $\tilde{h}$ and any $k \in \Re_+$. But then the dual constraints associated to $(\tilde{b}, \tilde{h}, k)$ do not bind for $k \in \Re_+$ and can be ignored without loss of generality. Suppose the opposite. There are two cases. The first is that there exists $\bar{k}(\tilde{b}, \tilde{h})$ sufficiently large so either $q_1^{\mathcal{B}}(\tilde{b})$ is not attained for $k > \bar{k}(\tilde{b}, \tilde{h})$, or $q_1^{\mathcal{B}}(\tilde{b})$ is attained for $k > \bar{k}(\tilde{b}, \tilde{h})$ but $\mu^{\mathcal{B}}(\tilde{b}, \tilde{h}, k) = 0$. Theorem A.3 implies that in this case the dual constraints associated to $(\tilde{b}, \tilde{h}, k)$ can be ignored without loss of generality for all $k > \bar{k}(\tilde{b}, \tilde{h})$. The second case is that such an upper bound cannot exist. Take an increasing sequence $\bar{k}_j \to \infty$. For all j there then exists $\hat{k}_j > \bar{k}$ such that $(\tilde{b}, \tilde{h}, \hat{k}_j)$ lies in the support of $\mu^{\mathcal{B}}$. This means that the support of $\mu^{\mathcal{B}}$ the contains the sequence $\{(\tilde{b}, \tilde{h}, \hat{k}_j)\}$ where $\lim \hat{k}_j \to \infty$. But this is impossible since this support is compact by definition and hence bounded in $\Re_+$. We thus conclude that there exist $\bar{k}(\tilde{b}, \tilde{h})$ such that the constraints associated to $(\tilde{b}, \tilde{h}, k)$ can be ignored without loss of generality for all $k > \bar{k}(\tilde{b}, \tilde{h})$. For case (ii) a similar argument implies that there exist $\bar{k}(\tilde{s}, \tilde{h})$ such that the constraints associated to $(\tilde{s}, \tilde{h}, k)$ can be ignored without loss of generality for all $k > \bar{k}(\tilde{s}, \tilde{h})$. This completes our proof since, again, B and S are finite sets. □

Theorem A. 4. *Problem (D) has optimal solutions.*

Proof. Let us repose problem (D) with λ in $L^\infty(H \times K)$ (the dual of $L^1(H \times K)$). As before $q^{\mathcal{B}} \in C(B)$, $q^{\mathcal{S}} \in C(S)$ (since $C(B)$ and $C(S)$ are isomorphic to the Euclidean space). The new dual problem is solvable by Theorem 3.20 in Anderson and Nash (1987) since its feasible set is bounded (by an argument identical to that in Lemma A.2).

We now show that there exists an optimal solution of the above problem where λ is continuous. Suppose $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda)$ is optimal. Feasibility implies

$$\alpha(k)\lambda(h,k) \geq \alpha(k)v^b(h) - q^{\mathcal{B}}(b), \tag{A.21}$$

for all $b \in B$, $s \in S$, and $(h,k) \in (H|h_0) \times K$. Similarly, since $\pi(k) = \alpha(k)k$,

$$q^{\mathcal{S}}(s)/k + \alpha(k)v^s(h) \geq \alpha(k)\lambda(h,k), \tag{A.22}$$

for all $b \in B$, $s \in S$, and $(h,k) \in (H|h_0) \times (K|\{0\})$. Defining $\lambda_1 \in B(H \times K)$ so

$$\alpha(k)\lambda_1(h,k) = \max_{b \in B} \left\{ \alpha(k)v^b(h) - q^{\mathcal{B}}(b) \right\}, \quad (h,k) \in H \times K, \tag{A.23}$$

then yields a new optimal solution $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda_1)$. We now show that $\lambda_1(h,k)$ is continuous. For given (h,k), define

$$b_1 = \arg\max_{b \in B} \left\{ \alpha(k)v^b(h) - q^{\mathcal{B}}(b) \right\}. \tag{A.24}$$

Since K is compact and $\alpha(k)$ is continuous, $\alpha(k)$ is uniformly continuous on K. For every $\epsilon > 0$ there then exist $\delta > 0$ such that

$$|\alpha(k)v^{b_1}(h) - \alpha(k')v^{b_1}(h)| < \epsilon \tag{A.25}$$

whenever $|k' - k| < \delta$. Equations (A.21), (A.23), (A.24) and (A.25) then imply

$$\alpha(k')\lambda_1(h,k') \geq \alpha(k')v^{b_1}(h) - q^{\mathcal{B}}(b_1) > \alpha(k)v^{b_1}(h) - q^{\mathcal{B}}(b_1) - \epsilon = \alpha(k)\lambda_1(h,k) - \epsilon, \tag{A.26}$$

for any such k'. Likewise, defining

$$b_1' = \arg\max_{b \in B} \left\{ \alpha(k')v^b(h) - q^{\mathcal{B}}(b) \right\}, \tag{A.27}$$

there is $\delta' > 0$ such that

$$\alpha(k)\lambda_1(h,k) \geq \alpha(k')v^{b_1'}(h) - q^{\mathcal{B}}(b_1') - \epsilon = \alpha(k')\lambda_1(h,k') - \epsilon. \tag{A.28}$$

whenever $|k' - k| < \delta'$. Hence,

$$|\alpha(k)\lambda_1(h,k) - \alpha(k')\lambda_1(h,k')| < \epsilon \tag{A.29}$$

whenever $|k' - k| < \delta_0 = \min\{\delta, \delta'\}$. Because both $\alpha(k)\lambda_1(h,k)$ and $\alpha(k)$ are continuous, $\lambda_1(h,k)$ (the quotient of two continuous functions) is continuous. □

A.4 Welfare Theorems

As in Gretsky, Ostroy and Zame (1992,1999), the welfare theorems follow from Theorem A.3. The theorem implies a direct equivalence between the optimal solutions to problems (P) and (D) on the one hand, and competitive equilibrium allocation, prices and indirect utilities on the other hand. This equivalence implies that the existence of a competitive equilibrium is equivalent to existence of solutions to problems (P) and (D).

Theorem A. 5. *(Welfare theorems)*

(I) Let $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*}, p^*)$ be a competitive equilibrium. Also, define $q^{\mathcal{B}*}(b) = v_b^*(p^*)$ for each $b \in B$ and $q^{\mathcal{S}*}(s) = v_s^*(p^*)$ for each $s \in S$, where $v_i^*(p^*)$ is the equilibrium indirect utility of type-i agents. Then $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*})$ solves problem (P), and $(q^{\mathcal{B}*}, q^{\mathcal{S}*}, p^*)$ solves problem (D).

(II) Suppose $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ and $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda)$ are optimal solutions for problems (P) and (D). Then $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}, \lambda)$ is a competitive equilibrium. Moreover, $q^{\mathcal{B}}(b)$ gives the indirect utility of type-b buyers for each $b \in B$, and $q^{\mathcal{S}}(s)$ gives the indirect utility of type-s sellers for each $s \in S$ in equilibrium.

Proof. (I) Let $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*}, p^*)$ be a competitive equilibrium. Condition (ii) in Definition 1 is equivalent to (A.5) and (A.6), and condition (iii) is equivalent to (A.7). Also, $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*})$ trivially satisfies (A.8). Hence, $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*})$ is feasible for problem (P).

Define $q^{\mathcal{B}*}(b) = v_b^*(p^*)$ for each $b \in B$, and $q^{\mathcal{S}*}(s) = v_s^*(p^*)$ for each $s \in S$. Since B and S are finite sets, $q^{\mathcal{B}*} \in C(B)$ and $q^{\mathcal{B}*} \in C(S)$. By condition (i) in Definition 1,

$$\begin{aligned} q^{\mathcal{B}*}(b) &\geq \alpha(k)[v_b(h) - p^*(h,k)] \text{ for all } (b,h,k) \in B \times H \times \Re_+, \\ \text{with equality if } (b,h,k) &\in \text{supp}\mu^{\mathcal{B}*}, \text{ and} \\ q^{\mathcal{S}*}(s) &\geq \pi(k)(p^*(h,k) - v^s(h)) \quad \text{for all } (s,h,k) \in S \times H \times \Re_+, \\ \text{with equality if } (s,h,k) &\in \text{supp}\mu^{\mathcal{S}*}. \end{aligned} \tag{A.30}$$

Remember that $v^i(h_0) = 0$ for all $i \in B \cup S$ and p^* is normalized so $p^*(h_0,k) = 0$ for all $k \in \Re_+$. Hence, (A.30), combined with the definitions in (A.2)-(A.3), implies that $(q^{\mathcal{B}*}, q^{\mathcal{S}*}, p^*)$ is feasible for problem (D) and satisfies the dual complementary slackness conditions (A.11) and (A.12). By Theorem A.3, $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*})$ and $(q^{\mathcal{B}*}, q^{\mathcal{S}*}, p^*)$ are then optimal solutions for problems (D) and (P).

(II) Let $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ and $(q^{\mathcal{B}}, q^{\mathcal{S}}, \lambda)$ be optimal solutions for problems (P) and (D). By Theorem A.3, these solutions are feasible and satisfy the complementary slackness conditions. As noted above, the primal feasibility conditions (A.5) and (A.6) are equivalent to condition (ii) in Definition 1, while the primal feasibility condition (A.7) is equivalent to condition (iii).

Let $p \in C(H \times \Re_+)$ be given by $p = \lambda$. Since $p(h_0, k) = \lambda(h_0, k) = 0$, p satisfies our price normalization. The dual feasibility conditions (A.9)-(A.10) and associated complementary slackness conditions (A.11)-(A.12) imply that condition (i) (or equivalently (A.30)) holds for this choice of the price system. This is just the reverse of the argument in part (I). Moreover, $q^{\mathcal{B}}(b)$ gives the expected utility of type-b buyers for each $b \in B$, and $q^{\mathcal{S}}(s)$ gives the expected utility of type-s sellers for each $s \in S$ at these prices. Thus p decentralizes $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ as a competitive equilibrium. $\square$

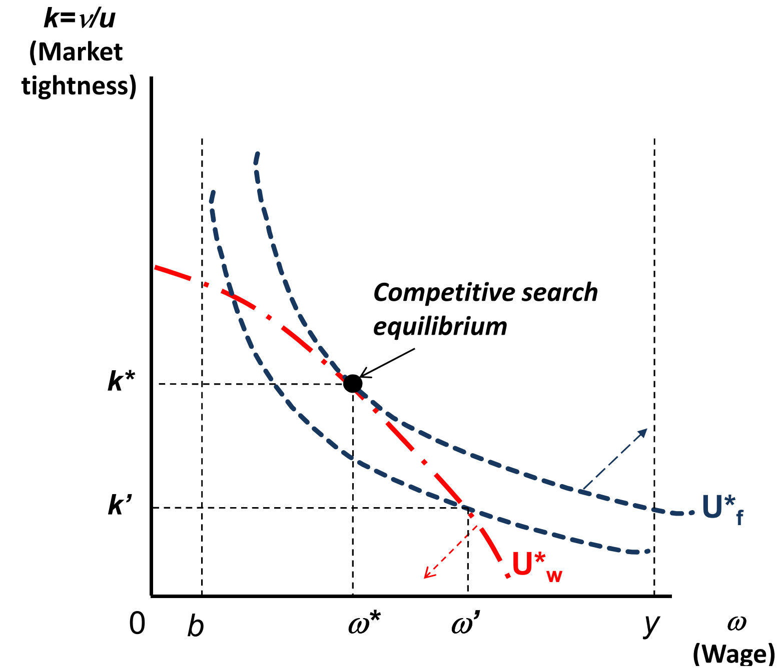

Figure 1: Equilibrium of the competitive game with homogeneous workers and firms.

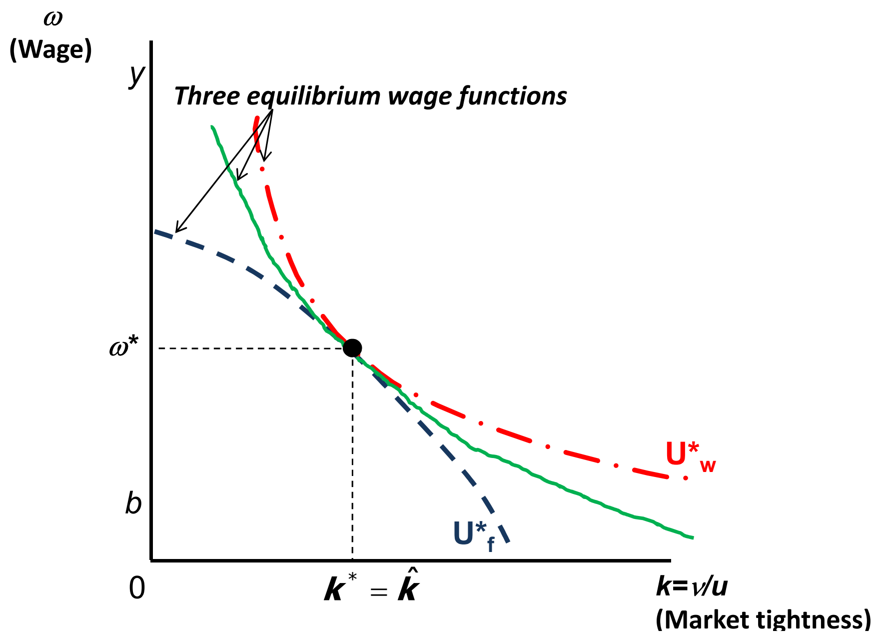

Figure 2: Walrasian equilibrium with homogeneous workers and firms.

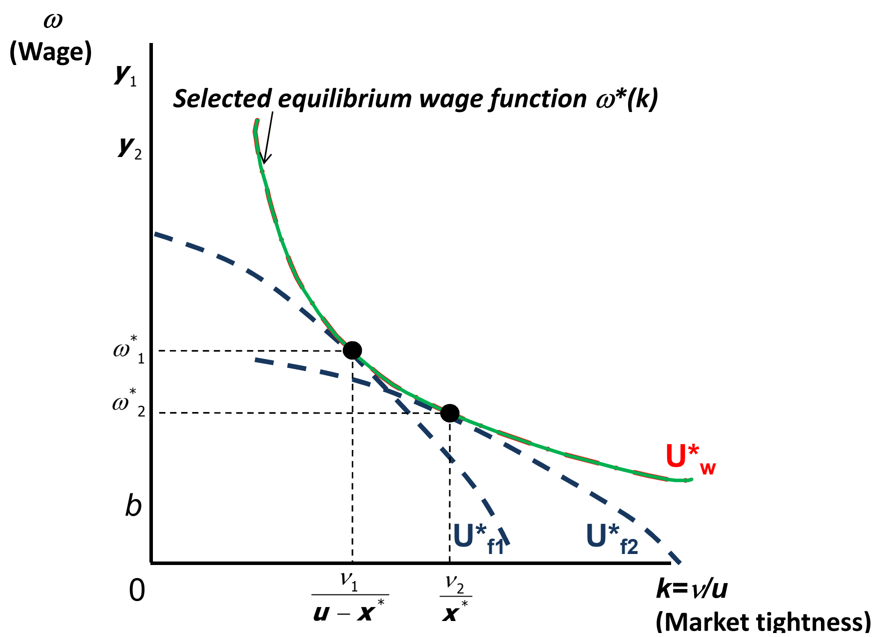

Figure 3: An example with two firm types.

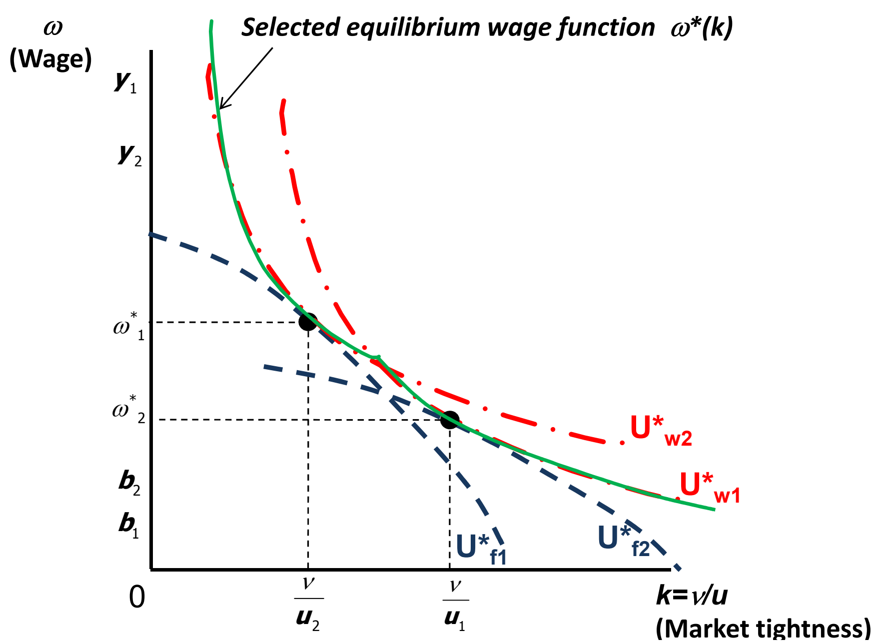

Figure 4: An example with two-sided heterogeneity.

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