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Strategic Profit Sharing Leads to Collusion in Bertrand Oligopolies[1]

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Abstract

One simple way to endogenize the degree of cross ownership in an industry is that firms give away part of their profits. We show that this possibility of unilaterally giving profits away to the rival previous to Bertrand competition opens the door to multiple equilibria. In the symmetric duopoly with constant marginal costs any price between the cost and the monopolistic price can be sustained in a subgame perfect equilibrium. Thus, tacit collusion in the one shot game can be achieved. Further, any market share can also be sustained for any equilibrium price. These results are extended to more than two firms and to asymmetric costs.

JEL classification: L12.

Key words: profit sharing; oligopoly; collusion; cross ownership; Bertrand.

1 Introduction

The present paper studies the consequences of introducing a new strategy in a Bertrand oligopoly with homogenous goods that consists on unilaterally and voluntarily giving a part of own profits to rivals previous to the Bertrand game. Thus, in a first stage, firms decide simultaneously what part of their profits to give away to their rival and then, in a second stage, they choose a price in a standard Bertrand competition. Notice that the action of giving away profits is decided unilaterally and unconditionally in a non-cooperative framework. After the decision to give profits is done, the giving firm is commited to it.

Ferreira and Waddles (2010) explore this strategy in a general setting, and apply it to both Cournot and Bertrand competition with heterogenous goods. They show that the strategy is more likely to allow for tacit collusion when choice variables are strategic complements and when monopolistic profits are big compared to the non-cooperative equilibrium in the oligopoly. In fact, they find that for Bertrand competition with goods with low heterogeneity some degree of tacit collusion is possible. However, their methodology relies on the differentiability of the profits functions, and cannot be applied to Bertrand competition with homogenous goods. The special characteristics of the Bertrand model, with a discontinuous profits function, and the qualitatively different results we find, call for the separate analysis presented in this work.

By adding a stage of profit sharing to the Bertrand model, we show that firms may be able to support prices between the marginal cost and the monopoly price, thus obtaining positive profits in almost all of the equilibria. This remarkable result, that resembles a Folk theorem, is robust to the number of firms and to cost asymmetries. Furthermore, for a given equilibrium price, any share of the market can also be supported in a subgame perfect equilibrium. For the duopoly case we completely characterize the set of pure strategy equilibria. However, in the extension to more than two firms, and due to the increasing complication in the multiplicity of equilibria, we only show the existence of the equilibria in the subgames that is sufficient to support the desired equilibrium price.

One may be tempted to argue that the result is not surprising. To put it in Reitman's words (Reitman, 94):

> "To take the simplest example, suppose symmetric Cournot duopolists each own 50% of the profits from its competitor's product. In choosing its own strategy, each firm's objective will be to maximize the sum of the two firms' profits, and will choose the collusive

output level in equilibrium."

However, this intuition is misleading, as one has to check that the individual incentives make this situation, the choice of sharing a 50% of the rival's profits, an equilibrium. In Ferreira and Waddles (2010) we already showed that this is the case only in some scenarios.

The effects of cross ownership on competition has been explored at least since the work by Reynolds and Snapp (1986), where it is shown that cross ownership serves to internalize free rider problems associated with policing collusion. Since then, other authors have worked out many related issues. For example, Farrell and Shapiro (1990) study a one-way cross ownership model where a big firm wants to acquire assets from another firm. Throughout a single-period Cournot oligopoly model, they show that, as the degree of cross ownership among rivals increases, the equilibrium in the market becomes less competitive.

In a dynamic setting, Malueg (1992) shows that, if firms interact repeatedly, increasing cross ownership may reduce the likelihood of collusion. A high level of cross ownership may even entail a lower likelihood of collusion than no-cross-ownership would. Gilo *et al.* (2006) explore this issue with more detail to show that, in general, the incentives to tacitly collude depend in a complex way on the entire partial cross ownership.

More related to our work, Reitman (1994) considers an oligopolistic model with conjectural variations in which firms buy claims to profits of other firms, and find that in the more rivalrous competition (i.e., more than Cournot), firms are willing to form partial ownership agreements to take advantage of a reduced competition. Later on, Alley (1997) develops a conjectural variation model that allows for partial ownership arrangements to the Japanese and US automobile industries to study the degrees of competitiveness and collusion in both countries.

Other recent works include Jackson and Wilkie (2005), that characterize the outcomes of games when players can make binding offers of strategy-contingent side payments before the game is played.

All these works, except Reitman (1994), have in common the exogeneity of the degree of cross-ownership. In our work, by contrast, this is endogenously determined in the model. In Reitman (1994), the cross ownership is decided in a mechanism in which firms buy claims to other firms' profits. In practice, this price need not be a cash payment, but any sort of contribution to production, marketing, etc., that does not affect the variable costs of production. There are two main differences with our model. First, we show that there is no need for this cash payment, and, second, that the choice of how much of the profits to share does not need to be part of an agreement betwenn two firms, as we only need that the level of profit sharing be decided by the giving firm. Remarkably, we observe the same kind of results (the willingness to share profits is higher if the competition is stronger) with these differences in the model in our companion work (Ferreira and Waddles, 2010). This willingness takes the highest level in the Bertrand case, analized in the present work.

This way, our work also contributes to the literature that views the Bertrand model as paradoxical, as it predicts perfect competition when there are only two firms in the market. Some authors have strongly criticized the Bertrand model pointing out its lack of realism. For instance, they think that it could be improved by relaxing some of its crucial assumptions like the timing of the game or the perfect substitutability of products. Others have attempted to find out a solution that fits to the real world. For example, Edgeworth (1897) solved it by introducing the elegant idea of capacity constraints, by which firms cannot sell more than they are able to produce. Later, Kreps and Scheinkman (1983) treated capacities as endogenous decisions previous to price competition, and showed that the new model leads to Cournot outcomes.

One of the problems in Kreps and Scheinkman is that, for some capacity choices, the only equilibria are in mixed strategies, which are not uniformly accepted as a satisfactory explanation of pricing behavior by oligopoly firms[1]. After all, in a mixed strategy equilibrium, firms can regret ex post their decisions, and, since prices can easily be changed, the stability of the equilibrium may be called into question[2]. In our model, every price in the range is obtained in a pure strategy equilibria and, thus, is immune to this criticism.

The infinite repetition of the Bertrand competition offers another way out of the paradox, as the Folk Theorem states that, for sufficiently high discount rates, any price between the cost and the monopoly price can be attained in a subgame perfect equilibrium and also that any market share among the different firms can be obtained. We obtain the same result without the need of repetition.

We proceed as follows. In Section 2 we present the analysis for the standard Bertrand model with equal marginal costs. Section 3 modifies the model to allow firms to have different marginal costs. Section 4 generalizes the previous models to n firms with equal marginal costs and to n firms with asymmetric costs. Section 5 concludes.

[1] See Shapiro (1989) and Maggi (1996).

[2] In a more recent work Moreno and Ubeda (2006) are able to provide a more elegant model in which the equilibria exist in pure strategies. Their model uses the capacity and price choices of each firm to construct a supply function game, in which, again, the Cournot outcome is obtained in equilibrium.

2 Profit sharing in a Bertrand duopoly

Consider two firms, 1 and 2, that compete *a la* Bertrand in a homogeneous market, and that each firm incurs a cost $c \in [0, 1)$ per unit of production. Let the market demand function be $q = D(p) = 1 - p$, and assume that firms do not have capacity constraints, and always supply the demand they face. The (before sharing) profit function for Firm i is:

$$\Pi_i = \begin{cases} (p_i - c)(1 - p_i) & if \quad p_i < p_j \\ \frac{1}{2}(p_i - c)(1 - p_i) & if \quad p_i = p_j \quad i, j = 1, 2 \ (i \neq j) \\ 0 & if \quad p_i > p_j \end{cases}$$

Consider now the following two-stage game of profit-sharing. In the first stage Firm i $(i = 1, 2)$chooses α_i, the proportion of profits Π_i that Firm i gives to Firm j $(\neq i)$. In the second stage, Firm i selects price p_i after observing the choices in the previous stage. Profits after sharing are given by $P_i = (1 - \alpha_i)\Pi_i + \alpha_j \Pi_j$. We will be interested in finding the set of prices that can be supported in a subgame perfect Nash equilibrium in this market.

So far, the profit-giving strategy has been performed directly. However, in real life, this strategy may be hidden behind a more complicated relation. To see this, consider the following simple case of a joint venture.

Let $\beta_i \in [0, 1]$ denote the part of its own profits that Firm i is willing to invest in a joint venture along with Firm j. The total investment in the joint venture is, then, given by $\beta_1\Pi_1 + \beta_2\Pi_2$. We will assume a simple joint venture activity with net profits given by $F = k(\beta_1\Pi_1 + \beta_2\Pi_2)$ where $k > 0$. Finally, we assume that each firm receives $s_i F$, where $s_i = \frac{\beta_i}{\beta_i + \beta_j}$. Consequently, we can write the new profit function of each firm as $P_i = (1 - \beta_i)\Pi_i + s_i k(\beta_i \Pi_i + \beta_j \Pi_j)$ or $P_i = [1 - (1 - s_i k)\beta_i]\Pi_i + s_i k \beta_j \Pi_j$.

It is now straightforward to see that $(1 - s_i k)\beta_i$ plays the role of α_i, and $s_i k \beta_j$ the role of α_j in the previous model, and that conditions on α_i and α_i can be translated as conditions on β_i, β_j and on k for the profit-giving strategy to be profitable in equilibrium[3]. As far as we know, only Reynolds and Snapp (1986) make a connection between cross ownership and joint ventures, although, in their case, the connection is made to show an incentive to investing in entry deterrence strategies.

2.1 The second stage

To find a subgame perfect Nash equilibrium (SPNE), we begin by finding the set of pure strategies Nash equilibria in the second-stage. Although for

[3]For instance, condition $\alpha_1 + \alpha_2$ below is satisfied for $\beta_1 = \beta_2 = \frac{1}{4}$ and any k.

the purposes of finding equilibria that supports the different prices it is not necessary to find all the equilibria in these subgames, we do it for its own interest and for the sake of completeness. In a series of lemmata, we find all equilibria in pure strategies for all subgames in the second stage. Then we summarize our findings in Proposition 5. In what follows, denote the monopoly price by p^m.

Lemma 1 *The price configuration* (p_1, p_2) *s.t.* $p_1 = p_2 = c$, *is an equilibrium for any* (α_1, α_2).

Proof. The proof is straightforward, as unilateral deviations result in zero or negative profits. ■

Lemma 2 *The price configuration* (p_1, p_2) *s.t.* $c < p_1 = p_2 = p \leq p^m$ *is never an equilibrium for any* (α_1, α_2) *such that* $\alpha_1 + \alpha_2 \neq 1$. *In the case* $p > p^m$ *there is no equilibrium with* $p_1 = p_2$ *if* $\alpha_1 + \alpha_2 < 1$.

Proof. The expressions for profits before and after sharing take the forms $\Pi_i = \frac{1}{2}(p-c)(1-p)$, and $P_i = \frac{1}{2}(1-\alpha_i+\alpha_j)(p-c)(1-p)$.

First, if $\alpha_1 + \alpha_2 > 1$, Player i has an incentive to deviate from $p_i = p$ to $p_i' > p$, leaving all the market to Firm j. Profits after this deviation are, then, $P_i' = \alpha_j (p-c)(1-p)$. Clearly, $P_i' > P_i$ as long as $\alpha_j > \frac{1}{2}(1-\alpha_i+\alpha_j)$, or, equivalently, $\alpha_1 + \alpha_2 > 1$.

Second, if $\alpha_1 + \alpha_2 < 1$, Player i has an incentive to deviate from $p_i = p$ to $p_i' < p - \varepsilon$ if $p \leq p^m$, and to $p_i' = p^m$ if $p > p^m$. In both cases, if takes the whole market to itself.

If $p \leq p^m$, profits after the deviation are given by $P_i' = (1-\alpha_i)(p_i'-c)(1-p_i')$, which can be made arbitrarily close to

$$\sup_{p_i' < p} P_i' = (1-\alpha_i)(p-c)(1-p) = P_i^{\sup}.$$

One can see that $P_i^{\sup} > P_i$ as long as $1 - \alpha_i > \frac{1}{2}(1-\alpha_i+\alpha_j)$, or $\alpha_1 + \alpha_2 < 1$.

If $p > p^m$ Firm i can deviate to $p_i = p^m$ and obtain $P_i^m = (1-\alpha_i)(p^m-c)(1-p^m)$. Clearly, $P_i^m > P_i^{\sup} > P_i$ if $\alpha_1 + \alpha_2 < 1$. ■

The next lemma is the key to our results. It shows that, for some configuration of profit shares, any price between the cost and the monopoly price may be sustained in an equilibrium where both firms set the same price, and thus share the market equally. Since they also share profits, the incentives are conflicting. The perspectives after deviating to a lower price must balance the increase in $(1-\alpha_i)\Pi_i$ with the decrease in $\alpha_j \Pi_j$. Similarly, a deviation to a higher price must balance the decrease in $(1-\alpha_i)\Pi_i$ with the increase in $\alpha_j \Pi_j$. These balances make the existence of an equilibrium with $p > c$ possible if $\alpha_1 + \alpha_2 = 1$.

Lemma 3 *Prices (p_1, p_2) s.t. $p_1 = p_2 = p \in [c, p^m]$ are equilibria for (α_1, α_2) such that $\alpha_1 + \alpha_2 = 1$.*

Proof. Lemma 1 proves the case for $p_1 = p_2 = c$. Suppose, then, that $p \in (c,\, p^m]$, and that Firm i sets a price p_i' below p and above c, then profits after the deviation are $\Pi_i' = (1-p_i')\,(p_i' - c) \geq 0$, $\Pi_j' = 0$, and

$$P_i' = (1-\alpha_i)\,\Pi_i' = (1-\alpha_i)\,(1-p_i')\,(p_i'-c)\,.$$

The $\sup_{p_i'} P_i' = P_i^{\text{sup}} = (1-\alpha_i)\,(1-p_i')\,(p_i'-c)$, the best deviation, is achieved at $p_i' = p$ as long as $p \leq p^m$. To avoid a profitable deviation, we need $P_i^{\text{sup}} \leq P_i$, or $(1-\alpha_i) \leq \frac{1}{2}\,(1-\alpha_i+\alpha_j)$, which gives

$$\alpha_1 + \alpha_2 \geq 1. \tag{1}$$

Similarly, for any price $p_i'' > p$, we have $\Pi_i'' = 0$ and $\Pi_j'' = (1-p_j)\,(p_j - c) >$
0

$$P_i'' = \alpha_j \Pi_j'' = \alpha_j\,(1-p_j)\,(p_j-c) = \alpha_j\,(1-p)\,(p-c)\,.$$

To avoid a profitable deviation, we need $P_i'' \leq P_i$, or $\alpha_j \leq \frac{1}{2}\,(1-\alpha_i+\alpha_j)$, which implies

$$\alpha_1 + \alpha_2 \leq 1. \tag{2}$$

Inequalities (1) and (2) represent the non-deviation conditions, and both are satisfied when $\alpha_1 + \alpha_2 = 1$. ■

The next lemma completes the set of equilibria that one can find using pure strategies. It shows the possibility of equilibria with different prices. The following notation will be useful for the next statements: $\Pi(p) = (1-p)(p-c)$. I.e., $\Pi(p)$ denotes the total profits in the market if firms set price p.

Lemma 4 *The only cases in which (p_1, p_2) s.t. $p_i \neq p_j$ constitute an equilibrium are given by the conditions $\alpha_1 + \alpha_2 \geq 1$ and $p_i = p^m < \bar{p} \leq p_j$ for $\bar{p}_j$ satisfying $\frac{\Pi(p^m)}{\Pi(\bar{p})} > \frac{\alpha_j}{1-\alpha_i}$.*

The strategy of the proof is similar to that in Lemma 3. However the conditions are more complicated. It should be clear that, if $p_i < p_j \leq p^m$, Firm i could set a price closer to p_j and increase $(1-\alpha_i)\Pi_i$ without changing $\alpha_j \Pi_j = 0$, thus increasing P_i. If $p_i = p^m < p_j$, things are more complicated. Certainly, the deviation to set p_i closer to p_j, but still under it, will not work. However, it could be the case that Firm i wants Firm j to have all the market (or part of it) to take advantage of the increase in $\alpha_j \Pi_j$. We need to find the conditions to ensure that this deviation does not work. The details of the proof are left to the Appendix.

The next proposition summarizes the Nash equilibria in pure strategies that can be found in the subgames after the choice of (α_1, α_2).

Proposition 5 *The Bertrand game with a profit-sharing previous stage has the following pure strategy Nash equilibria in the subgames:*

(a) If $\alpha_1 + \alpha_2 = 1$*, then the equilibria are* (p_1, p_2) *s.t.* $p_1 = p_2 = p \in [c, p^m]$ *or* $p_i = p^m < \bar{p} \leq p_j$ *for* $\bar{p}$ *satisfying* $\frac{\Pi(p^m)}{\Pi(\bar{p})} = \frac{\alpha_j}{1-\alpha_i}$.

(b) If $\alpha_1 + \alpha_2 > 1$*, then the equilibria are* (p_1, p_2) *s.t.* $p_1 = p_2 = c$ *or* $p_i = p^m < \bar{p}_j \leq p_j$ *for* $\bar{p}_j$ *satisfying* $\frac{\Pi(p^m)}{\Pi(\bar{p})} > \frac{\alpha_j}{1-\alpha_i}$.

(c) If $\alpha_1 + \alpha_2 < 1$*, then the equilibrium is* (p_1, p_2) *s.t.* $p_1 = p_2 = c$.

2.2 The whole game

The next proposition shows that any price between perfect competition and monopoly can be achieved, yielding positive profits to the industry.

Proposition 6 *Any price pair* (p_1^*, p_2^*) *such that* $c \leq p_1^* = p_2^* = p^* \leq p_m$ *can be sustained in a SPNE. Further, for the cases* $c < p_1^* = p_2^* = p^* \leq p_m$ *the SPNE implies that, in the first stage the shares* (α_1^*, α_2^*) *satisfy* $\alpha_1^* + \alpha_2^* = 1$.

Proof. Consider the following strategy. In the first step players play (α_1^*, α_2^*) such that $\alpha_1^* + \alpha_2^* = 1$. In the second stage they play (p_1^*, p_2^*) such that $c \leq p_1^* = p_2^* = p^* \leq p_m$ if $\alpha_1 + \alpha_2 = 1$, and $p_1 = p_2 = c$ otherwise.

Since the prices constitute Nash equilibria in the respective subgames, it remains to check that there are no profitable deviations for the firms in the first stage. This is straightforward, as any deviation in the first period has the consequence that $\alpha_1 + \alpha_2 \neq 1$ and, therefore, in the second period, the equilibrium implies $p_1 = p_2 = c$ and zero profits. Thus, no deviation is profitable. ■

Proposition 6 has the flavor of a Folk Theorem, even if it deals with a one shot game. If firms follow the tacit agreement to share profits in a certain way in the first stage, they get a high price in the second. If they do not, they get zero.

Intuitively, it does not seem surprising that $\alpha_1 = \alpha_2 = \frac{1}{2}$ results in monopolistic profits, as both firms control half of the profits of the rival, and that each one of them has the same objective function as a monopolist. However, this intuition does not take into account the entire story. On the one hand, having half the profits of the rival, and giving away half of the own, provides the incentive not to undercut the rival, but this also occurs at any other price. On the other hand, a price below p^m does not provide incentives to unilaterally increase the price, as it would if the firm behaved as a monopolist. This is because this action will not affect the total market profits, as the other firm gets the whole market.

The other interesting aspect of Proposition 6 comes from the fact that this same argument applies whenever $\alpha_1 + \alpha_2 = 1$. For instance, if Firm 1 gives as little as a 10% of its profits to Firm 2, then, it needs to receive 90% of Firm 2's profits in the equilibrium. This results in Firm 1 having 90% of total market profits. Thus, Proposition 6 not only says that any price may be sustained in a SPNE, but also that any final market share can be sustained.

3 Asymmetric costs

We consider the same model as in the previous section except that we allow Firm 1 and Firm 2 to have different marginal costs (assume that $c_1 < c_2 < p_1^m$, where p_1^m is the monopolist price of Firm 1), and that we let Firm 1 supply the entire market whenever its price is less or equal to Firm 2's price. This last condition is the natural one to impose due to the effects of the discontinuity of the profits function on the equilibrium if we do otherwise. If the firm with the lowest cost does not get all the market when both set price $p_1 = p_2 = c_2$, Firm 1 will set a price slightly below c_2, but since the best response is not well defined, the situation $(p_1 = c_2 - \varepsilon, p_2 = c_2)$ could not be an equilibrium. However, those prices constitute a perfectly good equilibrium if all quantities must be multiples of ε. To impose the rule that the firm with the lowest cost gets all the market if prices are the same saves the equilibrium in the continuous case. Then, the profit of each firm is:

$$\Pi_1 = \begin{cases} (p_1 - c_1)(1 - p_1) & if \quad p_1 \leq p_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi_2 = \begin{cases} (p_2 - c_2)(1 - p_2) & if \quad p_2 < p_1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 7 *The price configuration* (p_1, p_2) *s.t.* $p_1 = p_2 = c_2$*, is an equilibrium for any* (α_1, α_2)*.*

Proof. The proof is straightforward, as a deviation to set a smaller price by Firm 1 (resp., Firm 2) clearly implies smaller profits for Firm 1 (resp., losses for Firm 2). Likewise, if Firm 1 sets a higher price, Firm 2 takes all the market, it which it makes zero profits. Firm 2 does not change anything by increasing its price. ■

Lemma 8 next is the counterpart of Lemma 3, and it shows the possibility of multiple equilibria. As it was the case in Lemma 3, the key is to find conditions in (α_1, α_2) to balance the changes in both $(1-\alpha_i)\Pi_i$ and $\alpha_j\Pi_j$ after a deviation in order for that deviation not to be profitable. The most interesting part of the lemma is the fact that the conditions in (α_1, α_2) are less restrictive than in the case of symmetric firms.

In the sequel, p_i^m will denote the monopoly price when costs are c_i.

Lemma 8 *If* $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$ *and* $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$*, then* $p_1 = p_2 = p \in (c_2, p_1^m]$ *are equilibrium prices.*

If $\frac{\alpha_2}{1-\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$ *or* $\frac{1-\alpha_2}{\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$*, then the following prices constitute an equilibrium in the subgame,* $p_1 = p_2 = p \in (c_2, p_1^m]$ *such that* $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2}$ *and* $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2}$*.*

Proof. Consider a situation in which $p_1 = p_2 = p$, where profits are given by $P_1 = (1-\alpha_1)(p-c_1)(1-p)$ and $P_2 = \alpha_1 (p-c_1)(1-p)$

i) Suppose that Firm 1 sets a price p_1' above p_2 (by setting $p_1' < p_2$ we have that Π_1 decreases while Π_2 remains unchanged, so that no improvement in payoffs can be expected), then $\Pi_1' = 0$ and $\Pi_2 = (p-c_2)(1-p) \geq 0$, and

$$P_1' = \alpha_2 \Pi_2 = \alpha_2 (p-c_2)(1-p)$$

To avoid a profitable deviation, we need $P_1' \leq P_1$, or $\alpha_2 (p-c_2) \leq (1-\alpha_1)(p-c_1)$, which gives

$$\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2} \tag{3}$$

Let us study the deviation for Firm 2

ii) Suppose that Firm 2 sets a price p_2' below p_1 (setting $p_2' > p_1$ changes nothing), then $\Pi_1 = 0$ and $\Pi_2' = (p_2' - c_2)(1-p_2') \geq 0$, and

$$P_2' = (1-\alpha_2)\Pi_2' = (1-\alpha_2)(p_2'-c_2)(1-p_2')$$

The $\sup_{p_2'} P_2' = P_2^{\text{sup}} = (1-\alpha_2)(p_2' - c_2)(1-p_2')$ (the best deviation) is achieved at $p_2' = p$ as long as $p \leq p_2^m$ (p_2^m is the Firm 2 monopoly price). To avoid a profitable deviation, we need $P_2^{\text{sup}} \leq P_2$, or $(1-\alpha_2)(p-c_2) \leq \alpha_1 (p-c_1)$, which gives

$$\frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2} \tag{4}$$

Inequalities (3) and (4) represent the non-deviation conditions, and both are satisfied when

$$\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2} \text{ and } \frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2}.$$

The statements of the proposition follow once one realizes that $\frac{p_1^m - c_1}{p_1^m - c_2} \leq \frac{p-c_1}{p-c_2} < \infty$ for all $p \in (c_2, p_1^m]$. ■

Lemma 9 *There is no equilibrium with $p_1 = p_2 > p_1^m$.*

Proof. Consider (p_1, p_2) s.t. $p_1 = p_2 = p > p_1^m$. In this case, $P_1 = (1-\alpha_1)(p-c_1)(1-p)$, which can be increased with deviation $p_1' = p^m$. ■

Finally, Lemma 10 completes the search for equilibria in pure strategies showing the equilibria in which firms set different prices. The proof is left to the Appendix.

Lemma 10 *The necessary and sufficient condition for an equilibrium with different prices is $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, and the possible equilibria are given by the pairs (p_1, p_2) s.t. $c_2 < p_1 = p_1^m < \bar{p}_2 \leq p_2$ for $\bar{p}_2$ satisfying $\frac{\Pi(p_1^m)}{\Pi(\bar{p}_2)} > \frac{\alpha_2}{1-\alpha_1}$.*

The next proposition summarizes our findings about equilibria in the second stage of the two asymmetric firms case.

Proposition 11 *The Bertrand game with a profit-sharing previous stage and costs $c_1 < c_2$ has the following pure strategy Nash equilibria in the subgames:*

(a) If $\frac{\alpha_2}{1-\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$ or $\frac{1-\alpha_2}{\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with equal prices are (p_1, p_2) s.t. $p_1 = p_2 = p \in [c_2, p_1^m]$ and $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2}$ and $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2}$.

(b) If $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$ and $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with equal prices are (p_1, p_2) s.t. $p_1 = p_2 = p \in [c_2, p_1^m]$.

(c) If $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with different prices are (p_1, p_2) s.t. $c_2 < p_1 = p_1^m < \bar{p}_2 \leq p_2$ for $\bar{p}_2$ satisfying $\frac{\Pi(p_1^m)}{\Pi(\bar{p}_2)} > \frac{\alpha_2}{1-\alpha_1}$. If $\frac{1-\alpha_2}{\alpha_1} > \frac{p_1^m - c_1}{p_1^m - c_2}$ there are no equilibria with different prices.

The next proposition is the counterpart of Proposition 6 and shows that, with different costs, it is also possible to support multiple market prices in a subgame perfect Nash equilibrium. In this case, any price between the cost of the least competitive firm and the monopoly price of the most competitive one can be found in a SPNE.

Proposition 12 *Any price pair* (p_1^*, p_2^*) *such that* $c_2 \leq p_1^* = p_2^* = p^* \leq p_1^m$ *can be sustained in a SPNE.*

Proof. The proposition is trivial for $p^* = c_2$, so we only provide the proof for the other cases.

Consider the following strategy. In the first step players play (α_1^*, α_2^*) such that $\frac{\alpha_1^*}{1-\alpha_2^*} \leq \frac{p^*-c_1}{p^*-c_2}$, $\frac{1-\alpha_2^*}{\alpha_1^*} \leq \frac{p^*-c_1}{p^*-c_2}$ and $\alpha_1^* < 1 - \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$. In the second stage they play $(p_1, p_2) = (p_1^*, p_2^*)$ if in the first stage $(\alpha_1, \alpha_2) = (\alpha_1^*, \alpha_2^*)$, and $p_1 = p_2 = c_2$ otherwise. Notice that the conditions are satisfied for $\alpha_1^* + \alpha_2^* = 1$ and any $p^* \in (c_2, p_1^m]$.

Since the prices constitute Nash equilibria in the respective subgames, we need to check that there are no profitable deviations for the firms in the first stage Profits in equilibrium are $P_1 = (1 - \alpha_1^*)(p^* - c_1)(1 - p^*)$ and $P_2 = \alpha_1 (p^* - c_1)(1 - p^*)$.

If Firm 1 deviates to $\alpha_1' \neq \alpha_1^*$ it triggers a subgame in which it gets $P_1' = (1 - \alpha_1')(c_2 - c_1)(1 - c_2)$. Given this continuation, the best deviation is $\alpha_1' = 0$, with $P_1'(\alpha_1' = 0) = (c_2 - c_1)(1 - c_2)$. The deviation is not profitable if $P_1'(\alpha_1' = 0) \leq P_1$, something that must be the case for $\alpha_1^* < 1 - \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$ as $(p^* - c_1)(1 - p^*) > (c_2 - c_1)(1 - c_2)$ whenever $c_2 < p^* \leq p_1^m$.

Finally, deviations by Firm 2 can only give this firm smaller profits, as price in out of equilibrium subgames is $p = c_2$, which implies $P_2' = \alpha_1^* (c_2 - c_1)(1 - c_2) \leq P_2$. ■

Let $\frac{p^*-c_1}{p^*-c_2} = t$. Clearly $t > 1$, thus conditions $\frac{\alpha_1^*}{1-\alpha_2^*} \leq \frac{p^*-c_1}{p^*-c_2}$ and $\frac{\alpha_2^*}{1-\alpha_1^*} \leq \frac{p^*-c_1}{p^*-c_2}$ are satisfied if $\frac{1}{t}\alpha_1^* + \alpha_2^* \leq 1$ and $\alpha_1^* + \frac{1}{t}\alpha_2^* \leq 1$, which are satisfied for all $\alpha_1^* + \alpha_2^* \leq 1$ and for some values such that $\alpha_1^* + \alpha_2^* > 1$. Also, notice that the restriction that α_1^* be small enough is more restrictive the lower the equilibrium price one wants to support. In particular it is enough that $1 - \alpha_1^* \geq \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$. Thus, the margins for a profit sharing equilibrium are quite broad.

The fact that we do not have the equivalent to the symmetric case condition $\alpha_1^* + \alpha_2^* \geq 1$ is due to the specific tie breaking rule, that favors the firm with the lowest cost with the whole market, and thus deviations by the firm with the highest cost are not particularly important. In fact, one can prove that the rule that divides the market equally if prices are the same cannot support this kind of equilibria. However, tie breaking rules that assign a proportion of the market higher than $\frac{1}{2}$ but less than 1 to the low cost firm can still support prices between c_2 and p_1^m.

4 Extension to n firms

4.1 Equal costs

In this section, we consider n firms indexed by $i \in N = \{1, 2, ...n\}$ in a homogeneous market. We suppose that each firm incurs a cost c per unit of production. The market demand function is $q = D(p) = 1 - p$. We assume that firms do not have capacity constraints and always supply the demand they face. For a given vector of price choices $(p_1, ..., p_n)$, consider $p_{\min} = \min\{p_1, ..., p_n\}$, and define N' as $N' = \{i \in N : p_i = p_{\min}\}$, and let $n' = cardN'$. Therefore, the profit function of Firm i can be written as:

$$\Pi_i = \begin{cases} (p_i - c)(1 - p_i) & if \quad p_i < p_j \quad \forall j \neq i \\ \frac{1}{n'}(p_i - c)(1 - p_i) & if \quad p_i = \min\{p_1, ..., p_n\} \\ 0 & \quad\quad otherwise \end{cases}$$

Let β_{ij} denote the part of the profit that Firm i shares with firm j. We suppose that $\beta_{ij} \in (0, 1)$ and $\sum_{j=1}^{n} \beta_{ij} = 1$. Consequently, we can write the new profit function P_i of each firm as:

$$P_i = \beta_{ii}\Pi_i + \sum_{j=1(j \neq i)}^{n} \beta_{ji}\Pi_j.$$

As before, we consider a two-stage game whose sequences are thus defined. In the first stage of the game, Firm i chooses $(\beta_{i1}, ..., \beta_{in})$, while in the second stage of the game, it selects p_i.

To find the complete set of equilibria in all subgames becomes a complicated, tedious exercise as the number of firms increase. However, we can still prove the existence of a SPNE to support any price between c and the monopoly price.

Proposition 13 *Any price vector $(p_1^*, ..., p_n^*)$ such that $c \leq p_i^* = p^* \leq p_1^m$ for all $i \in N$ can be sustained in a SPNE. Further, the equilibrium requires that firms share profits to satisfy*

$$\sum_{j=1(j \neq i)}^{n} \beta_{ij} + \frac{1}{n-1} \sum_{j=1(j \neq i)}^{n} \beta_{ji} = 1 \text{ for all } i \in N. \quad (5)$$

The proof is similar to that in propositions 5 and 6 and can be found in the Appendix. Condition (5) is the generalization of the condition $\alpha_1^*+\alpha_2^*=1$ for the case of two symmetric firms. Notice that it satisfied for $\beta_{ij}=\frac{1}{n}$ for all $i,j\in N$.

4.2 Different costs

Order firms according to costs, so that $c_1\leq c_2\leq ...\leq c_n$, let N_1 be the set of the firms with the lowest cost, i.e., $N_1=\{i\in N: c_i=c_1\}$, and denote by n_1 the cardinal of N_1. Now the profit function is:

$$\Pi_i=\begin{cases}(p_i-c_i)(1-p_i) & if \quad p_i<p_j \quad \forall j\neq i\\ \frac{1}{n'}(p_i-c_i)(1-p_i) & if \quad p_i=\min\{p_1,...,p_n\}\\ 0 & \quad otherwise\end{cases}$$

Now we can state our last proposition for the general case of n firms and asymmetric costs.

Proposition 14 *Any price p s.t. $\overline{c}\leq p\leq p_1^m$, where $\overline{c}=c_1$ if $n_1>1$, and $\overline{c}=c_2$ if $n_1=1$, and where p_1^m is the monopoly price of Firm 1, can be supported in a subgame perfect equilibrium in pure strategies.*

The formal proof is given in the Appendix, but a sketch can be presented as follows. If $n_1>1$, the idea is to share the market among the firms with the lowest cost (firms with $c_i=c_1$) in a similar fashion as in Proposition 13. The other firms choose not to share profits and set price equal their marginal costs. If $n_1=1$, however, the market will be shared among Firm 1 and the firms with the next-to-lowest costs (firms with $c_i=c_2$) in a way similar to that in Proposition 12, with the added feature that now we can have more than one firm with a higher cost. Thus, conditions in Proposition 12 must be adapted to this possibility. The general arguments, however, still hold.

5 Conclusion

We considered Bertrand oligopolies with homogeneous goods, linear demand and constant marginal costs, and found that it is possible to support equilibrium prices above marginal costs by introducing a previous stage of profit sharing. In this stage, firms voluntarily and independently of each other decide how much of their profits they will give away to rivals.

The range of prices that can be supported in a subgame perfect equilibrium varies between the second lowest marginal cost and the monopoly price of the lowest cost firm. Further, there is also a great range of the final market shares (after counting the effects of profit sharing) that can be supported in equilibrium, thus making the range of possible payoffs similar to those provided by the Folk theorem in a repeated game.

Ferreira and Waddle (2010) analyzed the strategic-profit-sharing strategy in oligopolistic scenarios with differentiable payoff functions. In that work it was shown that the strategy was more likely to facilitate some degree of tacit collusion the higher the strategic complementarity of second stage game, and the higher the differences between monopoly and oligopoly equilibria. Both conditions are satisfied in the present model. However, as nothing could be shown in general about the extend of the degree of collusion, if any, a separate analysis was necessary.

References

Alley, W. (1997), "Partial Ownership Arrangements and Collusion in the Automobile Industry," *Journal of Industrial Economics* 45, 191-205.

Edgeworth, F. (1897) "La teoria pura del Monopolio" *Giornale degli Economisti* 40: pp. 13-31. Reprinted in English as "The Pure Theory of Monopoly," in F. Edgeworth, *Papers Relating to Political Economy* 1, 111-142. London: MacMillan & Co., Ltd., 1925.

Farrell, J., and C. Shapiro (1990), "Asset ownership and market structure in oligopoly," *RAND Journal of Economics* 21, 275-292.

Ferreira, J.L, Waddle, R. (2010), "Strategic profit sharing between firms," *International Journal of Economic Theory*, forthcoming.

Gilo, D., Moshe, Y., and Spiegel Y. (2006), "Partial Ownership and Tacit Collusion," *RAND Journal of Economics* 37(1), 81-99.

Jackson, M. O., and S. Wilkie (2005), "Endogenous games and mechanisms: Side payments among players," *Review of Economics Studies* 72, 543-566.

Kreps, D., Scheinkman, J. (1983), "Quantity precommitment and Bertrand competition yield to Cournot outcomes," *Bell Journal of Economics* 14, 326-337.

Maggi, G. (1996), "Strategic trade policies with endogenous mode of competition," *American Economic Review* 86, 237-258.

Malueg, D. (1992), "Collusive behavior and partial ownership of rivals," *International Journal of Industrial Organization* 10, 27-34.

Moreno, D., Ubeda, L. (2006), "Capacity precommitment and price competition yield to the Cournot outcome," *Games and Economic Behavior* 56, 323-332.

Reitman, D. (1994), "Partial Ownership Arrangements and Potential for Collusion," *Journal of Industrial Economics* 42, 313-322.

Reynolds. R.J., and Snapp, B.R. (1986), "The competitive effects of partial equity interests and joint ventures," *International Journal of Industrial Organization* 4, 141-153.

Shapiro, C. (1989), "Theories of oligopoly behavior," Chapter 6 in: Schmalensee, R., Willig, R.D. (Eds.), *Handbook of Industrial Organization* 1, 329-410 North-Holland, Amsterdam.

Appendix

Proofs.

Lemma 4.

Proof. Without lost of generality, let (p_i, p_j) be such that $c \leq p_i < p_j$. Then

$P_i = (1-\alpha_i)\,\Pi_i = (1-\alpha_i)\,(p_i - c)\,(1-p_i)$, and

$P_j = \alpha_i \Pi_i = \alpha_i\,(p_i - c)\,(1-p_i)$.

Since prices p_i and p_j are different, we have to study separately the deviations for each firm. Let us check first Firm j. We will consider $p_i \leq p_m$ since, otherwise, Firm i will deviate to a lower price. If $c = p_i$ profits by Firm i can increase by increasing p_i. Consider, then that $c < p_i$ and first check for deviations by Firm j.

i) If Firm j sets a price p_j' below p_i, its new profits are given by $P_j' = (1-\alpha_j)\,\Pi_j' = (1-\alpha_j)\,(p_j' - c)(1-p_j')$. The $\sup P_j'$ is achieved at $p_j' = p_i$, with $P_j'(p_j' = p_i) = (1-\alpha_j)\,\Pi_i$. Thus, to avoid a profitable deviation, we need $P_j = \alpha_i \Pi_i \geq P_j' = (1-\alpha_j)\,\Pi_i$, which implies $\alpha_1 + \alpha_2 \geq 1$.

ii) If Firm j sets a price p_j'' equal to p_i, then $\Pi_i'' = \Pi_j'' = \frac{1}{2}(p_j'' - c)(1-p_j'') = \frac{1}{2}\Pi_i > 0$, with $P_j'' = \frac{1}{2}\,(1-\alpha_j+\alpha_i)\,\Pi_i$. To avoid a profitable deviation, we need $P_j = \alpha_i \Pi_i \geq P_j'' = \frac{1}{2}\,(1-\alpha_j+\alpha_i)\,\Pi_i$, which implies $\alpha_1 + \alpha_2 \geq 1$, as before.

Now, let us check deviations for Firm i.

i') Suppose that Firm i deviates to set a price p_i' above p_i and below p_j, then $P_i' = (1-\alpha_i)\,(p_i' - c)\,(1-p_i') = (1-\alpha_i)\,\Pi_i'$. This kind of deviation is profitable as long as $p_i < p^m$. Thus, a necessary condition for Firm i not to be willing to deviate is $p_i \geq p^m$. If $p_i > p^m$, a deviation with p_i' slightly below p_i will be profitable. Hence, the equilibrium requires $p_i = p^m$.

ii') Suppose, then, that $p_i = p^m$, and that Firm i considers a deviation to set its price $p_i'' > p_j$ (any other deviation below p_j is clearly unprofitable), then $\Pi_i'' = 0$ and $\Pi_j'' = (p_j - c)\,(1-p_j)$, with $P_i'' = \alpha_j \Pi_j''$. For the deviation not to be profitable we need $P_i = (1-\alpha_i)\,\Pi_m \geq P_i'' = \alpha_j \Pi_j''$, which implies

$$\frac{\Pi\,(p^m)}{\Pi\,(p_j)} \geq \frac{\alpha_j}{1-\alpha_i}. \tag{6}$$

iii') Suppose that $p_i = p^m$, and that Firm i considers a deviation to set its price $p_i''' = p_j$. Now $\Pi_i''' = \Pi_j''' = \frac{1}{2}\,(p_j - c)\,(1-p_j)$, and $P_i''' =$

$\frac{1}{2}(1-\alpha_j+\alpha_i)\,\Pi(p_j)$. To avoid a profitable deviation, we need $P_i=(1-\alpha_i)\,\Pi_m \geq P_i''' = \frac{1}{2}(1-\alpha_i+\alpha_j)\,\Pi(p_j)$, which implies

$$\frac{\Pi_m}{\Pi(p_j)} \geq \frac{1-\alpha_i+\alpha_j}{2(1-\alpha_i)}. \tag{7}$$

A deviation to a price $p_i^{iv} < p_j$ is clearly unproductive. Inequalities (6) and (7) are satisfied for all $p_j \geq \bar{p}$ if they are satisfied for $\bar{p}$. Of them, (6) is more restrictive and, therefore, both are satisfied for all $p_j \geq \bar{p}$ such that

$$\frac{\Pi(p^m)}{\Pi(\bar{p})} \geq \frac{\alpha_j}{1-\alpha_i}$$

is satisfied, which provides the condition for the equilibrium. ■

Lemma 10.

Proof. Let (p_1, p_2) such that $c_2 \leq p_1 < p_2$. Then $\Pi_1 = (p_1-c_1)(1-p_1) > 0$ and $\Pi_2 = 0$,with

$P_1 = (1-\alpha_1)\,\Pi_1 = (1-\alpha_1)(p_1-c_1)(1-p_1)$, and

$P_2 = \alpha_1 \Pi_1 = \alpha_1 (p_1-c_1)(1-p_1)$

Since prices p_1 and p_2 are different, we have to study separately the deviation for each firm. Let us check first Firm 2. We will consider $p_1 \leq p_1^m$ since, otherwise, Firm 1 will deviate to a lower price.

i) If Firm 2 sets a price p_2' below p_1 (other deviations are trivially shown not to provide higher profits), its new profits are given by $P_2' = (1-\alpha_2)\,\Pi_2' = (1-\alpha_2)(p_2'-c_2)(1-p_2')$. The sup P_2' is achieved at $p_2' = p_1$, with $P_2'(p_2' = p_1) = (1-\alpha_2)(p_1-c_2)(1-p_1)$. Thus, to avoid a profitable deviation, we need $P_2 = \alpha_1(p_1-c_1)(1-p_1) \geq P_2' = (1-\alpha_2)(p_1-c_2)(1-p_1)$, which implies

$$\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1-c_1}{p_1-c_2}. \tag{8}$$

Now, let us check for Firm 1.

i) Suppose that Firm 1 deviates to set a price p_1' above p_1 and below p_2, then $P_1' = (1-\alpha_1)(p_1'-c)(1-p_1') = (1-\alpha_1)\,\Pi_1'$. This kind of deviation is profitable as long as $p_1 < p_1^m$. Thus, a necessary condition for Firm 1 not to be willing to deviate is $p_1 \geq p_1^m$. If $p_1 > p_1^m$, a deviation with p_1' slightly below p_1 will be profitable. Hence, the equilibrium requires $p_1 = p_1^m$, and condition (8) becomes

$$\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}.$$

ii) Suppose, then, that $p_1 = p_1^m$, and that Firm i considers a deviation to set its price p_1'' above p_2 (any other deviation below p_2 is clearly unprofitable), then $\Pi_1'' = 0$ and $\Pi_2 = (p_2 - c_2)(1 - p_2)$, with $P_1'' = \alpha_2 \Pi_2$. For the deviation not to be profitable we need $P_1 = (1-\alpha_1)\Pi(p_1^m) \geq P_1'' = \alpha_1 \Pi_2$, which implies

$$\frac{\Pi(p_1^m)}{\Pi(\bar{p_2})} \geq \frac{\alpha_2}{1-\alpha_1} \tag{9}$$

The above inequality provides the condition for the equilibrium. It will be satisfied if p_2 is set so high that $D(p_2) = 0$.

Now, it remains to show that there are no other equilibrium prices if $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$. To that effect, let us consider (p_1, p_2) such that $p_1 < p_2$ and $p_1 < p_1^m$. In this case, Firm 1 would set a price p_1''' greater than p_1, but lower than p_2 to get a profit $P_1''' = (1-\alpha_1)\Pi_1'''(p_1''', p_2)$, which is greater than $(1-\alpha_1)\Pi_1(p_1, p_2)$ as long as $p_1''' < p_1^m$. If $p_1 > p_1^m$, the profitable deviation occurs with a price $p_1''' < p_1$. ■

Proposition 13.

Proof. First we start by specifying the equilibria in the subgames. It is straightforward to see that, in every subgame, the price vector $(p_1, ..., p_n)$ s.t. $p_1 = ...p_n = c$ constitutes an equilibrium. Next we find equilibrium prices above the cost. The profits of Firm i $(i = 1, ..., n)$ in a situation in which $c < p_1 = ... = p_n = p \leq p^m$ are $\Pi_i = \frac{1}{n}(p-c)(1-p)$ and $P_i = \left(1 - \sum_{j=1(j\neq i)}^n \beta_{ij}\right)\Pi_i + \sum_{j=1(j\neq i)}^n \beta_{ji}\Pi_j$ or

$$P_i = \frac{1}{n}\left(1 - \sum_{j=1(j\neq i)}^n \beta_{ij} + \sum_{j=1(j\neq i)}^n \beta_{ji}\right)(p-c)(1-p)$$

Let us study possible deviations by Firm i.

i) Suppose that Firm i deviates to a price $p_i' < p$, then $\Pi_i' = (1-p_i')(p_i' - c) > 0$ and $\Pi_j = 0$, and

$$P_i' = (1 - \sum_{j=1(j\neq i)}^n \beta_{ij}\Pi_i') = \left(1 - \sum_{j=1(j\neq i)}^n \beta_{ij}\right)(1-p_i')(p_i' - c)$$

Since $p \leq p^m$, the best deviation provides at most

$$\sup_{p_i' < p} P_i' = P_i^{\sup} = \left(1 - \sum_{j=1(j\neq i)}^n \beta_{ij}\right)(1-p)(p-c).$$

Therefore, to avoid a possible deviation we need $P_i^{\sup} \leq P_i$, which implies $\left(1-\sum_{j=1(j\neq i)}^{n}\beta_{ij}\right) \leq \frac{1}{n}\left(1-\sum_{j=1(j\neq i)}^{n}\beta_{ij}+\sum_{j=1(j\neq i)}^{n}\beta_{ji}\right)$, and that is satisfied if

$$(n-1)\sum_{j=1(j\neq i)}^{n}\beta_{ij}+\sum_{j=1(j\neq i)}^{n}\beta_{ji} \geq n-1 \text{ for all } i \in N \tag{10}$$

ii) Suppose that Firm i sets a price $p_i'' > p$ then $\Pi_j = \frac{1}{n-1}(1-p)(p-c) > 0$ and $\Pi_i'' = 0$, with

$P_i'' = \sum_{j=1(j\neq i)}^{n}\beta_{ji}\Pi_j = \frac{1}{n-1}\sum_{j=1(j\neq i)}^{n}\beta_{ji}(1-p)(p-c)$.

To avoid a possible deviation we need $P_i'' \leq P_i$, implying $\frac{1}{n-1}\sum_{j=1(j\neq i)}^{n}\beta_{ji} \leq \frac{1}{n}\left(1-\sum_{j=1(j\neq i)}^{n}\beta_{ij}+\sum_{j=1(j\neq i)}^{n}\beta_{ji}\right)$, which is satisfied if

$$(n-1)\sum_{j=1(j\neq i)}^{n}\beta_{ij}+\sum_{j=1(j\neq i)}^{n}\beta_{ji} \leq n-1 \tag{11}$$

Inequalities (10) and (11) represent the non-deviation conditions and both are satisfied when

$$\sum_{j=1(j\neq i)}^{n}\beta_{ij}+\frac{1}{n-1}\sum_{j=1(j\neq i)}^{n}\beta_{ji} = 1. \tag{12}$$

The proof is complete by making firms choose $\beta_{ij} = \beta_{ij}^*$, where $(\beta_{ij}^*)_{i,j}$ satisfies (12), in the first stage, and $(p_1^*, ..., p_n^*)$ such that $c \leq p_i^* = p^* \leq p_1^m$ if $(\beta_{ij}^*)_{i,j}$ was indeed chosen in the first stage, and $p_i^* = c$ otherwise. ■

Proposition 14.

Proof. First we start by finding subgames in which the different prices are equilibria. To this end, start by finding the expression for P_i, in a situation in which $c_n < p_i = p \leq p_1^m$ for all $i \in N$. Profits in this situation are given by

$\Pi_i = \frac{1}{n_1}(p-c_1)(1-p)$ for all $i \in N_1$,

$\Pi_i = 0$ for all $i \notin N_1$,

$P_i = \left(1-\sum_{j=1(j\neq i)}^{n}\beta_{ij}\right)\Pi_i + \sum_{j=1(j\neq i)}^{n_1}\beta_{ji}\Pi_j$

$= \frac{1}{n_1}(1 - \sum_{j=1(j\neq i)}^{n} \beta_{ij} + \sum_{j=1(j\neq i)}^{n_1} \beta_{ji}) (p - c_1) (1 - p)$ for all $i \in N_1$,

and

$P_i = \frac{1}{n_1}(\sum_{j=1(j\neq i)}^{n_1} \beta_{ji}) (p - c_1) (1 - p)$ for all $i \notin N_1$.

Now we can consider deviations from this point.

(i) Case 1: $n_1 > 1$, $i \in n_1$, deviation to $p_i' < p$.

The consequence is $\Pi_i' = (p - c_1) (1 - p)$, while $\Pi_j' = 0$ for all $j \neq i$. Since $p \leq p_1^m = p_i$, its maximum new profits P_i' will be computed as

$\sup_{p_i'<p} P_i' = P_i^{\sup} = \left(1 - \sum_{j=1(j\neq i)}^{n} \beta_{ij}\right) (p - c_1) (1 - p)$.

To get $P_i^{\sup} \leq P_i$ we need

$1 - \sum_{j=1(j\neq i)}^{n} \beta_{ij} \leq \frac{1}{n_1}\left((1 - \sum_{j=1(j\neq i)}^{n} \beta_{ij}) + (\sum_{j=1(j\neq i)}^{n_1} \beta_{ji})\right)$, or

$$\sum_{j=1(j\neq i)}^{n} \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j\neq i)}^{n_1} \beta_{ji} \geq 1. \tag{13}$$

(ii) Case 2: $n_1 > 1$, $i \in N_1$, deviation to $p_i'' > p$.

The new profits are $\Pi_i'' = 0$, $\Pi_j'' = \frac{1}{n_1}(p - c_1) (1 - p)$ for all $j \in N_1 \setminus \{i\}$, and $\Pi_j'' = 0$ for all $j \notin N_1$. This gives

$P_i'' = \frac{1}{n_1 - 1}(\sum_{j=1(j\neq i)}^{n'} \beta_{ji}) (p - c_1) (1 - p)$.

In order for the deviation not to be profitable we need $P_i'' \leq P_i$:

$\frac{1}{n_1 - 1} \sum_{j=1(j\neq i)}^{n_1} \beta_{ji} \leq \frac{1}{n_1}\left(1 - \sum_{j=1(j\neq i)}^{n} \beta_{ij} + \sum_{j=1(j\neq i)}^{n_1} \beta_{ji}\right)$, or

$$\sum_{j=1(j\neq i)}^{n} \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j\neq i)}^{n_1} \beta_{ji} \leq 1. \tag{14}$$

Conditions (13) and (14) are satisfied when

$$\sum_{j=1(j\neq i)}^{n} \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j\neq i)}^{n_1} \beta_{ji} = 1 \text{ for all } i \in n_1. \tag{15}$$

(iii) Case 3: $n_1 = 1$, in which case $N_1 = \{1\}$, and the deviation of $i \in N_1$ is the deviation of Firm 1. In this case $P_1 = (1 - \sum_{j=2}^{n} \beta_{1j}) (p - c_1) (1 - p)$. Consider $p_1' < p$.

With this deviation $\Pi_1' \leq \Pi_1$, and $\Pi_j' = \Pi_j = 0$ for all $j \neq i$, with $P_1' = (1 - \sum_{j=2}^{n} \beta_{ij}) (p_1' - c_1) (1 - p_1')$. The consequence is $P_1' \leq P_1$ as long as $p \leq p_1^m$.

(iv) Case 4: $n_1 = 1$, deviation by Firm 1 to $p_1' > p$.

After this deviation, the market will be shared among the firms with costs at the level of Firm 2, c_2. Denote this set of firms by N_2; i.e., $N_2 = \{i \in N : c_i = c_2\}$, n_2 will denote the cardinal of N_2 Profits are $\Pi_i' = \frac{1}{n_2} (p - c_2) (1 - p)$ for all $i \in N_2$, and $\Pi_i' = 0$ for all $i \notin N_2$. Profits of Firm 1 are

$P_1' = \frac{1}{n_2} (\sum_{j=2}^{n_2+1} \beta_{j1}) (p - c_2) (1 - p)$.

The deviation is not profitable if

$\frac{1}{n_2} (\sum_{j=2}^{n_2+1} \beta_{j1}) (p - c_2) (1 - p) \leq (1 - \sum_{j=2}^{n} \beta_{1j}) (p - c_1) (1 - p)$, or

$$\frac{\frac{1}{n_2} (\sum_{j=2}^{n_2+1} \beta_{j1})}{1 - \sum_{j=2}^{n} \beta_{1j}} \leq \frac{p - c_1}{p - c_2}. \tag{16}$$

(v) Case 5: $n_1 = 1$, deviations by $i \neq 1$.

A deviation to a higher price changes nothing, so consider $p_i' < p$, with the effect that $\Pi_i' = (p_i' - c_i) (1 - p_i')$, and $\Pi_j' = \Pi_j = 0$ for all $j \neq 1$. Then

$\sup_{p_i' < p} P_i' = P_i^{\sup} = \left(1 - \sum_{j=1(j \neq i)}^{n} \beta_{ij}\right) (p - c_i) (1 - p)$.

The deviation is not profitable if $P_i^{\sup} \leq P_i = \beta_{1i} (p - c_1) (1 - p)$

$\left(1 - \sum_{j=1(j \neq i)}^{n} \beta_{ij}\right) (p - c_i) (1 - p) \leq \beta_{1i} (p - c_1) (1 - p)$, or

$$\frac{1 - \sum_{j=1(j \neq i)}^{n} \beta_{ij}}{\beta_{1i}} \leq \frac{p - c_1}{p - c_i} \text{ for all } i \neq 1. \tag{17}$$

Condition (17) is the counterpart of condition (12) for the case of n firms with equal costs, and is satisfied if $\beta_{ij} = \frac{1}{n}$ for all $i, j \in N$, while conditions (15) and (16) are the counterpart of conditions (3) and (4) for two firms with different costs, and they are satisfied whenever $\frac{1}{n_2} (\sum_{j=2}^{n_2+1} \beta_{j1}) + \sum_{j=2}^{n} \beta_{1j} \leq 1$, and $\sum_{j=1(j \neq i)}^{n} \beta_{ij} + \beta_{1i} \leq 1$ for all $i \neq 1$, as $\frac{p - c_1}{p - c_i} \geq 1$, which in turn are satisfied if β_{ij} are small enough.

To support a price $p \in (c_i, c_{i+1}]$ with $c_i < c_{i+1}$, one can set $p_j = p$ for all j such that $c_j \leq c_{i+1}$, and $p_j = c_j$. In this case, firms with a cost higher than c_{i+1} do not receive or give any profits. This way, the new game is as the one before with firms with the highest costs are out of it, and we can the argument above.

The price $p = c_1$ if $n_1 > 1$ or $p = c_2$ if $n_1 = 1$ is supported straightforward.

Finally, to show the SPNE in the whole game, proceed as follows. Choose a price p s.t. $\overline{c} \leq p \leq p_1^m$, find the conditions on $(\beta_{ij}(p))_{ij}$ for which the price p can sustained in equilibrium. We just made sure that such conditions exist. Now the equilibrium is as follows

(a) in stage 1, choose $(\beta_{ij}(p))_{ij}$,

(b) if $n_1 > 1$, in stage 2 choose $p_i = p$ if $(\beta_{ij})_{ij} = (\beta_{ij}(p))_{ij}$, and $p_i = c_i$ otherwise for all $i \in N$ or

(c) if $n_1 = 1$, in stage 2 choose $p_i = p$ if $(\beta_{ij})_{ij} = (\beta_{ij}(p))_{ij}$, and $p_i = \max\{c_2, c_i\}$ for all $i \in N$

If $n_1 > 1$ Firms in N_1 do not deviate to a different β_{ij} as that will imply zero profits in the second stage.

If $n_1 = 1$, Firm 1 will not be willing to deviate if β_{1j} is small enough. The best deviation implies not to share profits with any rival, thus getting $P_1' = (c_2 - c_1)(1 - c_2)$ in the second stage,which is lower than $P_1 = (1 - \sum_{j=2}^{n} \beta_{1j})(p - c_1)(1 - p)$ if β_{1j} are small enough as $p > c_2$. ■