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TWO-SIDED REFLECTION PROBLEM FOR THE MARKOV MODULATED BROWNIAN MOTION

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Abstract

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Keywords: Two-sided Skorokhod reflection; Markov Modulated Brownian Motion

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Two-sided reflection problem for the Markov Modulated Brownian Motion

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Abstract

In this paper we consider the two-sided reflection of a Markov modulated Brownian motion by analyzing the spectral properties of the matrix polynomial associated with the generator of the free process. We show how to compute for the general case the Laplace transform of the stationary distribution and the average loss rates at both barriers for the reflected process. This work extends previous partial results by broadening and completing the analysis for the special cases when the spectrum of the generator is non-semi-simple, and for the delicate case where the asymptotic drift of the process is zero.

1. Introduction

Fluid models for queues have been extensively studied in the literature and they are especially useful in the performance analysis of telecommunication systems, such as the ATM (Asynchronous Transfer Mode) systems, where the packet sizes are so small compared to the traffic load that the fluid assumption looks natural.

The original model assumed that the transmission rates where constant over exponential periods of time and that they changed according to an underlying Markov chain, see the basic references [1, 6, 7, 17, 19] to give just a short list.

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Subsequent research showed that the model with deterministic transmission rates could be usefully extended to include an additive stochastic Brownian component. This modification led to the development of the theory of the Markov-modulated Brownian queues that is the main topic of present work. The seminal papers in this area are the works of Rogers [18] and Asmussen [2] that in parallel looked at the one side reflection of the Markov-modulated Brownian motion (MMBM).

An important property of the MMBM is that the analysis of the two-sided reflection for this process can approximate the one of a general Markov Additive Process (MAP), as it was explicitly shown in [4], by expanding the modulating state space and introducing additional dummy states with no diffusion component to mimic jumps with phase-type distributions. As a result the knowledge of the stationary distribution for this model is very useful in application since the MMBM can be adapted to mimic a large variety of specific models.

In [18] the author already looked at the two-sided reflection of the MMBM and he showed how to derive the stationary distribution under the restrictive assumption that the diffusive part is not subject to modulation. In [4] by using a martingale technique, the authors expressed the Laplace transform of the stationary distribution as a function of the average loss rates at the two boundaries. Moreover, they showed how to compute the average loss rates under restrictive assumptions on the spectrum of the matrix polynomial associated to the generator of the process that in the sequel will be referred to as the *matrix exponent*.

The contribution of this work is to extend the investigation done in [4] by a complete understanding and deployment of the structure of the spectrum of the matrix exponent. In particular we use the methodology of the *Jordan chains* that have been introduced for the first time in [5] as a tool to analyze the first passage times of the completely-asymmetric MAP processes, i.e. MAP processes admitting jumps of only one sign. We show that for the general case it is possible to derive a set of independent linear equations whose solution gives the average loss rates required to completely compute the Laplace transform of the stationary distribution.

Since throwout all paper we make extensive use of matrix notation we shortly list here some typographic conventions that we are going to adopt. Bold symbols will denote column vectors and in particular, $\mathbf{1}$ and $\mathbf{0}$ are the vectors of 1's and 0's respectively, $\boldsymbol{e}_i$ is a vector with ith element being 1 and all others being 0. The identical and the null matrices are denoted by $\mathbb{I}$ and $\mathbb{O}$ while $\Delta_{\boldsymbol{x}} = \mathrm{diag}[x_1, \ldots, x_n]$ denotes the diagonal matrix whose diagonal vector is equal to $\boldsymbol{x}$. Given a column vector $\boldsymbol{x}$ we denote by $\boldsymbol{x}^{\mathrm{t}}$ the transposed row vector. With the notation $\mathrm{col}\big[A_i\big]_{i=0}^{n} = \mathrm{col}\big[A_1, \ldots, A_n\big]$, we denote a block matrix built by allocating in one column all the blocks A_i that are assumed to have all the same number of columns. A λ-Jordan block is a square matrix having the main diagonal with all elements equal to λ, the first upper diagonal having all elements equal to 1 and the remaining elements all equal to 0. A Jordan matrix is a block diagonal matrix whose diagonal elements are all given by Jordan blocks.

2. The model

The Markov modulated Brownian motion (MMBM) is a bivariate continuous Markov process $(X(t), J(t))$ defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E = \{1, \dots, N\}$, transition rate matrix $Q = (q_{ij})$ and a (unique) stationary distribution $\boldsymbol{\pi}$. Let $B(t)$ be a standard Brownian motion and denote by $\boldsymbol{a}$ and $\boldsymbol{\sigma} \geq \mathbf{0}$ the constant drift and diffusion vectors of size N. We define the modulated diffusion component $X(t)$ as the process starting at zero and evolving according to the stochastic differential equation

$$dX(t) = a_{J(t)}\,dt + \sigma_{J(t)}\,dB(t). \tag{1}$$

The MMBM $(X(t), J(t))$ can be also characterized as the subclass of Markov additive processes (MAP) with a.s. continuous sample paths in the component $X(t)$. In particular it can be proved that its moment generating function is well defined for any $\alpha \in \mathbb{C}$ and is given by

$$\mathbb{E}_i[e^{\alpha X(t)} 1\{J(t) = j\}] = (e^{F(\alpha)t})_{ij}, \tag{2}$$

cf. [3, Prop. XI.2.2], where $\mathbb{E}_i(\cdot)$ denotes expectation given that $J(0) = i$.

The matrix term, $F(\alpha)$, that appears in the right side of (2) is a second order matrix polynomial given by the following expression

$$F(\alpha) = \frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\alpha^2 + \Delta_{\boldsymbol{a}}\,\alpha + Q. \tag{3}$$

It is usually referred to as the *matrix exponent* of $(X(t), J(t))$, see also [5], and it represents the multidimensional analog of the Laplace exponent.

An important quantity associated to the process $(X(t), J(t))$ is the *asymptotic drift*:

$$\kappa = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_i X(t) = \sum_i \pi_i a_i, \tag{4}$$

which does not depend on the initial state i of $J(t)$ [3, Cor. XI.2.7].

2.1. The two-sided Skorokhod reflection

The double sided reflection of a MMBM $(X(t), J(t))$, at the barriers 0 and b is defined as the Markov process $(W(t), J(t))$ such that the continuous part, $W(t) \in [0, b]$, $t \geq 0$, can be expressed in the following way

$$W(t) = X(t) + L(t) - U(t),$$

where $L(t)$ and $U(t)$ are called the *lower-regulator* and *upper-regulator* processes, i.e. two non-negative monotone non-decreasing processes that can increase only when $W(t) = 0$ and $W(t) = b$ respectively. This last property usually is expressed by the following integral relations

$$\int_0^t (W(s) - 0)\, dL(s) = 0 \quad \text{and} \quad \int_0^t (b - W(s))\, dU(s) = 0, \quad t \geq 0.$$

In the literature there are many proofs that such a process exists and is unique, see for example [20] and we denote by (W, J) a random vector that is distributed according to its (unique) stationary distribution.

It is shown in [4] that

$$\mathbb{E}\left[e^{\alpha W} \boldsymbol{e}_J^{\mathrm{t}}\right] \cdot F(\alpha) = \alpha(e^{\alpha b} \boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}}), \tag{5}$$

where $\boldsymbol{u}$ and $\boldsymbol{\ell}$ are column vectors of unknown constants having the following sample-path interpretation

$$\boldsymbol{\ell} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \boldsymbol{e}_{J(s)}\, L(ds) \quad \text{and} \quad \boldsymbol{u} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \boldsymbol{e}_{J(s)}\, U(ds). \tag{6}$$

In section 5, we show that it is always possible to compute these two constant vectors as the solution of a system of linear equations and characterize the stationary distribution of (W, J) by its Laplace transform through Eqn. (5). This result completes the investigation on MMBM contained in [4] and extends the previous works [12, 13, 18].

2.2. First Passage Process

The study of the two-sided reflected process is closely related with the analysis of the exit times for the process $X(t)$ from an interval, see for example [3, Prop. XIV.3.7]. To this aim we define, for $x \geq 0$, the first passage times above the level $+x$ (resp. below level $-x$) for the process $X(t)$ as

$$\tau^{\pm}(x) = \inf\{t > 0 : \pm X(t) > x\} \tag{7}$$

with the usual assumption that $\inf\{\emptyset\} = \infty$.

It is known that on $\{J(\tau^{\pm}(x)) = i\}$, the process $(X(t+\tau^{\pm}(x)) - X(\tau_x, J(t+\tau^{\pm}(x)))), t \geq 0$ is independent from $(X(t), J(t)), t \in [0, \tau^{\pm}(x)]$ and has the same law as the original process under $\mathbb{P}_i$, that is the law of the process $(X(t), J(t))$ given that $J(0) = i$. It follows that the time-changed processes $J(\tau^{\pm}(x))$, $x \geq 0$ are time-homogeneous Markov chains.

According to the values of the diffusion and the drift coefficients, the process $X(t)$ is allowed to reach a new minimum or a new maximum only when $J(t)$ belongs to some subset of the state space E. We denote by E_+ and E_-, the set of states where the process $X(t)$ can respectively increase and decrease, and by $E_\uparrow$ and $E_\downarrow$ the states where the process $X(t)$ has respectively monotone non-decreasing and non-increasing paths. We have that

$$\begin{aligned} E_+ &= \{j \in E; \sigma_j > 0 \text{ or } a_j > 0\}, \quad E_\uparrow = \{j \in E; \sigma_j = 0 \text{ and } a_j \geq 0\}, \\ E_- &= \{j \in E; \sigma_j > 0 \text{ or } a_j < 0\}, \quad E_\downarrow = \{j \in E; \sigma_j = 0 \text{ and } a_j \leq 0\}, \end{aligned}$$

and we define $N_\pm = |E_\pm|$.

Letting $\{\partial\}$ be an absorbing state corresponding to $J(\infty)$, we note that $J(\tau^\pm(x))$, $x \geq 0$ lives on $E_\pm \cup \{\partial\}$, because $X(t)$ can not hit new maximum (resp. minimum) when $J(t)$ is in a state belonging to $E_\downarrow$ (resp. $E_\uparrow$), see also [16]. Let $\Lambda^\pm$ be the $N_\pm \times N_\pm$ dimensional transition rate matrices of $J(\tau^\pm(x))$ restricted to $E_\pm$, and define the starting distribution matrices $\Pi^\pm$ as the $N \times N_\pm$ matrix given

$$\Pi^\pm{}_{ij} = \mathbb{P}_i(J(\tau^\pm(0)) = j), \text{ where } i \in E \text{ and } j \in E_\pm, \tag{8}$$

we obtain that

$$\mathbb{P}_i(J(\tau^\pm(x)) = j) = (\Pi^\pm\, e^{\Lambda^\pm\, x})_{ij}, \quad x \geq 0. \tag{9}$$

The restriction of the matrices $\Pi^\pm$ to the rows in $E_\pm$ are the identity matrices as $\mathbb{P}_i\{\tau^\pm(x) = 0\} = 1$ when $i \in E_\pm$ [15, Thm. 6.5].

In the following, for a given a matrix A whose rows are indexed by the set of states in E, we denote by $A_\pm$ the submatrices made by the only rows indexed by the subsets of states in $E_\pm$. In addition by $\mathcal{M}$ we denote the set of all square irreducible transition rate matrices and by $\mathcal{M}_0$ and $\mathcal{M}_1$ its subsets made of, respectively, all transient and recurrent matrices.

According to the notation above we have that $\Lambda^\pm \in \mathcal{M}_{1\{\pm\kappa\geq 0\}}$ and that, for $\pm\kappa \geq 0$, $\Pi^\pm \mathbf{1}_\pm = \mathbf{1}$.

Noticing that the process $(-X(t), J(t))$ is a MMBM with same diffusion vector $\boldsymbol{\sigma}$ and drift vector $-\boldsymbol{a}$, the following result holds, see [5, Cor. 2].

Proposition 1. *If $(X(t), J(t))$ is a MMBM then $(\Pi^\pm, \Lambda^\pm)$ are the unique pairs (P, M), with $P_\pm = \mathbb{I}^\pm$ and $M \in \mathcal{M}_{1\{\pm\kappa\geq 0\}}$, which satisfy the following matrix equations*

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\, P\, M^2 \mp \Delta_{\boldsymbol{a}}\, P\, M + Q\, P = \mathbb{O}^\pm. \tag{10}$$

The result above has been rediscovered in different degrees of generality in various previous works, such as [3, 18, 14].

3. Matrix Polynomials

In this section we review some basic facts from matrix polynomial theory and for a complete reference we suggest the reader to look at the book [8]. Let $A(z)$ be a complex matrix polynomial ($N \times N$ dimensional) of order l, i.e.

$$A(z) = \sum_{j=0}^{l} A_j\, z^j$$

and associate to it the complex function $\alpha(z) = \det(A(z))$ that we assume is not identically equal to 0 for $z \in \mathbb{C}$. A complex number λ is an *eigenvalue* of $A(z)$ of multiplicity n_λ if $\alpha(z) = (z - \lambda)^{n_\lambda}\, \alpha^*(z)$ with $\alpha^*(z) \neq 0$. Given a sequence of vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$, with $\boldsymbol{v}_0 \neq \mathbf{0}$, we say that it is a *Jordan chain* of size $n > 0$ associated to the eigenvalue λ if

$$\sum_{j=0}^{k} \frac{1}{j!} A^{(j)}(\lambda)\, \boldsymbol{v}_{k-j} = \mathbf{0}, \qquad k = 0, \ldots, n-1, \tag{11}$$

and it can be shown it always holds that $n \leq n_\lambda$.

If we define the matrix $V = [\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}]$ and denote by Γ the λ-Jordan block of size n it is easy to see that Eqn. (11) is equivalent to the following matrix relation, (see Prop. 1.10 and Thm. 7.3 in [8])

$$\sum_{j=0}^{l} A_j V\, \Gamma^j = 0 \tag{12}$$

and that the condition $\boldsymbol{v}_0 \neq \mathbf{0}$ ensures that $\mathrm{rank}(\mathrm{col}[V\,\Gamma^j]_{j=0}^{l-1}) = n$, where $\mathrm{rank}(A)$ denotes the rank of the matrix A. By the definition above it follows that $\boldsymbol{v}_0$ is a regular eigenvector associated to the eigenvalue 0 of the matrix $A(\lambda)$, while the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ are generally referred to as *generalized eigenvectors*. If m_λ is the geometric multiplicity of the eigenvalue λ, i.e. the dimension of the subspace $\mathrm{Ker}(A(\lambda))$, by collecting the longest Jordan chains for each eigenvector of λ, $(V_1, \Gamma_1), \ldots (V_{m_\lambda}, \Gamma_{m_\lambda})$ and defining the pair of matrices (V, Γ) with $V = [V_1, \ldots V_{m_\lambda}]$ and $\Gamma = \mathrm{diag}[\Gamma_1, \ldots \Gamma_{m_\lambda}]$, we have that Eqn. (12) still holds and in addition $\mathrm{rank}(\mathrm{col}[V\,\Gamma^j]_{j=0}^{l-1}) = n_\lambda$. This allows for the following definition

Definition 2. *A* Jordan pair *associated to the eigenvalue λ of a matrix polynomial $A(z)$ is defined as a pair of matrices (V, Γ), with Γ of Jordan type and whose sizes are respectively $N \times n_\lambda$ and $n_\lambda \times n_\lambda$, satisfying Eqn. (12) and the additional condition*

$$\mathrm{rank}(\mathrm{col}[V\,\Gamma^j]_{j=0}^{l-1}) = n_\lambda. \tag{13}$$

where n_λ is the algebraic multiplicity of λ as a zero of $\alpha(z)$.

The set of Jordan chains forming the matrix V in the definition above are usually also referred to as a *canonical set of Jordan chains* corresponding to λ.

The important property of the Jordan pairs is that they can provide a base in the space $\mathbb{C}^{n_A}$, where n_A is the degree of $\alpha(z)$. Indeed the following property of linear independence between Jordan pairs corresponding to different eigenvalues holds true (see Thm. 7.3 in in [8]).

Proposition 3. *Let $(V_i, \Gamma_i)_i$, with $i = 1, \ldots, K$ be a set of Jordan pairs associated to distinct eigenvalues λ_i and with respectively lengths n_i, then defining $V = [V_1, \ldots V_K]$ and $\Gamma = \mathrm{diag}[\Gamma_1, \ldots, \Gamma_K]$ we have that the pair (V, J) satisfies Eqn. (12) and in addition*

$$\mathrm{rank}(\mathrm{col}[V\,\Gamma^j]_{j=0}^{l-1}) = n, \tag{14}$$

with $n = \sum_{i=1}^{K} n_i$.

By collecting all the Jordan pairs corresponding to each and every eigenvalue of $A(z)$ we can characterize the finite spectrum of the matrix polynomial, i.e. the set of all roots of $\alpha(z)$ together with their multiplicities. This leads to the definition of the concept of *finite Jordan pair*, that extends to the whole matrix polynomial the concept of Jordan pair corresponding to a specific eigenvalue.

Definition 4. *A* finite Jordan pair *for a matrix polynomial $A(z)$ is defined as a pair of matrices (V, Γ), with Γ of Jordan type and whose sizes are respectively $N \times n_A$ and $n_A \times n_A$, satisfying Eqn. (12) and the additional condition*

$$\operatorname{rank}(\operatorname{col}[V\,\Gamma^j]_{j=0}^{l-1}) = n_A. \tag{15}$$

where n_A is the degree of $\alpha(z)$.

Remark 1. If the pair of matrix (V, Γ) satisfies the assumptions of Definition 4 with Γ not Jordan, it is referred to as a *finite standard pair* of $A(z)$.

It is easy to realize that the finite Jordan (standard) pairs are not uniquely defined, however it is known they all belong to the same equivalent class with respect to the following similarity property (see Thm. 1.25 in [8]).

Proposition 5. *Given two finite Jordan (standard) pairs (V', Γ') and (V'', Γ''), it exists an invertible $n_A \times n_A$ square matrix D such that*

$$V'' = V'\,D \quad \text{and} \quad \Gamma'' = D^{-1}\,\Gamma'\,D. \tag{16}$$

Finally we state the following technical result that will be used in the next sections.

Proposition 6. *Let (V, Γ) be a Jordan pair of $A(z)$ and $\boldsymbol{c}^t(z)$ an N dimensional row vector $n-1$ times differentiable on $\mathbb{C}$. Assuming that the product $\boldsymbol{b}^t(z) = \boldsymbol{c}^t(z)A(z)$ can be expressed as a linear combination $\boldsymbol{b}^t(z) = \sum_{i=1}^K f_i(z)\,\boldsymbol{u}_i^t$, of some given vectors $\boldsymbol{u}_i^t$ and weight functions $f_i(z)$ with $i = 1, \dots, K$, then the following equation holds*

$$\sum_{i=1}^K \boldsymbol{u}_i^t\,V\,f_i(\Gamma) = \boldsymbol{0}^t \tag{17}$$

where for an analytic function $f(z)$, its value at the square matrix A is defined as $f(A) = \sum_{j=0}^\infty \frac{1}{j!} f^{(j)}(0) A^j$.

Proof. In [9] it is shown that given a Jordan chain (V_0, Γ_0) corresponding to the eigenvalue λ_0 of $A(z)$, where $V_0 = [\boldsymbol{v}_0, \dots, \boldsymbol{v}_{n-1}]$, the following relation holds true

$$\sum_{j=0}^k \frac{1}{j!} (\boldsymbol{b}^t)^{(j)}(\lambda_0)\,\boldsymbol{v}_{k-j} \quad k = 0, \dots, n-1. \tag{18}$$

and writing explicitly the expression of $\boldsymbol{b}^t(z)$ we obtain, for $k = 0, \dots, n-1$

$$\sum_{j=0}^k \frac{1}{j!} \sum_{i=1}^K f_i^{(j)}(\lambda_0)\,\boldsymbol{u}_i^t\,\boldsymbol{v}_{k-j} = \sum_{i=1}^K \boldsymbol{u}_i^t \sum_{j=0}^{n-1} \boldsymbol{v}_j \frac{1}{(k-j)!} f_i^{(j)}(\lambda_0) 1\{j \le k\}. \tag{19}$$

The result follows by noticing that $\frac{1}{(k-j)!} f_i^{(j)}(\lambda_0) 1\{j \le k\}$ for $k = 0, \ldots, n-1$ is equal to the j component of the k column of the matrix $f_i(\Gamma_0)$. The general result follows by considering separately all the Jordan chains contained in (V, Γ).

4. The Matrix Exponent

The matrix exponent $F(\alpha)$ defined in (3) is a second order matrix polynomial and in this section we are going to determine a finite Jordan pair of its in terms of the matrices $\Lambda^{\pm}$ and $\Pi^{\pm}$ that characterize the processes $J(\tau^{\pm}(x))$ according to Eqn. (9). We use the Jordan decomposition of the matrices $\Lambda^{\pm}$

$$\Lambda^{+} = -R^{+}\,\Gamma^{+}(R^{+})^{-1} \quad \text{and} \quad \Lambda^{-} = +R^{-}\,\Gamma^{-}(R^{-})^{-1} \tag{20}$$

where we assume that the columns of the matrices $R^{\pm}$ and $\Gamma^{\pm}$ are arranged in the following form

$$R^{\pm} = \begin{cases} [\mathbf{1}, \hat{R}^{\pm}] & \text{if } \pm\kappa \ge 0 \\ \hat{R}^{\pm} & \text{if } \pm\kappa < 0 \end{cases} \; ; \quad \Gamma^{\pm} = \begin{cases} \mathrm{diag}[(0), \hat{\Gamma}^{\pm}] & \text{if } \pm\kappa \ge 0 \\ \hat{\Gamma}^{\pm} & \text{if } \pm\kappa < 0 \end{cases} \tag{21}$$

In (21), $\hat{\Gamma}^{+}$ and $\hat{\Gamma}^{-}$ denote the Jordan matrices with all eigenvalues having real parts respectively strictly negative and strictly positive and $\hat{R}^{+}$ and $\hat{R}^{-}$ refer to the corresponding submatrices in R^{+} and R^{-}. We define the matrices

$$V^{\pm} = \Pi^{\pm} R^{\pm} \quad \text{and} \quad \hat{V}^{\pm} = \Pi^{+} \hat{R}^{\pm} \tag{22}$$

and according with the adopted notation we have that $V^{\pm}_{\pm} = R^{\pm}$ and $\hat{V}^{\pm}_{\pm} = \hat{R}^{\pm}$ because the restrictions of the matrices $\Pi^{\pm}$ to the rows in $E_{\pm}$ are equal to the identity matrices.

From Prop. 1 and the decompositions (20) we immediately see that $\det(F(\alpha))$ has at least $N_{+} + N_{-} - (1 + 1\{\kappa = 0\})$ non zero roots and one zero root. Since the total number of roots counting multiplicities is $N_{+} + N_{-}$, applying the following lemma we deduce that 0 is a simple root when $\kappa \neq 0$ and a double root when $\kappa = 0$.

Lemma 7. *Assuming that $\kappa = 0$, then 0 is a root of $\det(F(\alpha))$ with multiplicity greater than 1.*

PROOF. It is enough to prove that there exists a vector $\boldsymbol{h}$ such that the pair $(\mathbf{1}, \boldsymbol{h})$ forms a Jordan chain of size 2 associated to the eigenvalue 0. According to (11) the following equation as to be satisfied

$$F(0)\boldsymbol{h} + F'(0)\mathbf{1} = Q\boldsymbol{h} + \boldsymbol{a} = \mathbf{0}. \tag{23}$$

By multiplying last equation on the left by $\boldsymbol{\pi}^{t}$ we see that $\kappa = 0$ is a necessary condition to have the multiplicity of the 0 root greater than 1, we now prove that it is also sufficient. Since $\boldsymbol{\pi}^{t}\,[Q,\,\boldsymbol{a}] = \mathbf{0}$ and $\boldsymbol{\pi} > \mathbf{0}$, we have that the row rank of the matrix $[Q,\,\boldsymbol{a}]$ is at most $N-1$, therefore so it is the column rank.

Since we know that Q contains a subset of $N-1$ independent vectors the rank of the matrix is equal to $N-1$. It means that the vector $-\boldsymbol{a}$ can be written as a non-null linear combination of the other vectors and the coefficients of this linear combination provide the required vector $\boldsymbol{h}$.

We define the pair (V_0, Γ_0) as the Jordan pair associated with the null eigenvalue and it is given by

$$V_0 = \begin{cases} [\mathbf{1}] & \text{if } \kappa \neq 0 \\ [\mathbf{1}, \boldsymbol{h}] & \text{if } \kappa = 0 \end{cases} \quad \text{and} \quad \Gamma_0 = \begin{cases} (0) & \text{if } \kappa \neq 0 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } \kappa = 0 \end{cases} \tag{24}$$

where the vector $\boldsymbol{h}$ was identified in Lemma 7 as a solution of Eqn. (23).

We finally construct a finite Jordan pair of $F(\alpha)$ as the pair of matrices (V, Γ) given by

$$V = [V_0, \hat{V}^+, \hat{V}^-] \quad \text{and} \quad \Gamma = \text{diag}[\Gamma_0, \hat{\Gamma}^+, \hat{\Gamma}^-]. \tag{25}$$

The pair (V, Γ) satisfies Eqn. (12) by Prop. 1 and the construction of the pair (V_0, Γ_0). Therefore to verify that (V, Γ) is indeed a finite Jordan pair of $F(\alpha)$ we need only to verify that condition (15) holds, but this follows immediately by Prop. 3. We resume this result in the following theorem.

Theorem 8. *The pair of matrices (V, Γ) defined according to (25) is a finite Jordan pair for the matrix polynomial $F(\alpha)$.*

It is worth to notice that given a general Jordan pair for $F(\alpha)$ it is always possible to recover the pairs $(\Pi^\pm, \Lambda^\pm)$ that characterize the processes $J(\tau^\pm(x))$ according to Eqn. (9). Assume that (V, Γ) is a Jordan pair, then after separating the eigenvectors according to the sign of the real part of the corresponding eigenvalues we can always decompose the matrices V and Γ as in (25). By letting the matrices $\hat{R}^\pm = \hat{V}^\pm_\pm$, we can reconstruct the matrices $R^\pm$ and $\Lambda^\pm$ according to Eqns. (20) and (21) together with the matrices $\Pi^\pm = V^\pm (R^\pm)^{-1}$. The fact that the matrices $R^\pm$ are invertible follows by the definition (20) and the fact that according to Prop. 5 all possible Jordan pairs are similar to each other.

Finally, by applying Prop. 5 for the case $\kappa \neq 0$, the following neat result holds true.

Corollary 9. *Assume that $\pm\kappa > 0$ then a finite standard pair of $F(\alpha)$ is given by*

$$([\Pi^+, \Pi^-], \text{diag}[\mp\Lambda^+, \pm\Lambda^-]) \tag{26}$$

5. Computing the upper and lower loss-rate vectors

In ths section we show that it is always possible to find the value of the constant vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$ as a solution of a linear system that can be determined by a given Jordan pair of the matrix polynomial $F(\alpha)$.

Before stating our main result we prove the following technical lemma that is useful to take care of the special case when the asymptotic drift $\kappa = 0$.

Lemma 10. *If $\kappa = 0$, then $V_0 = [\mathbf{1}, \boldsymbol{h}]$ and*

$$b\boldsymbol{u}^{t}\mathbf{1} + (\boldsymbol{u}^{t} - \boldsymbol{\ell}^{t})\boldsymbol{h} = \boldsymbol{\pi}^{t}(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}) \neq 0. \tag{27}$$

Proof. Differentiating Eqn. (5) at 0 and right multiplying by $\boldsymbol{h}$, we obtain the identity

$$(\boldsymbol{u}^{t} - \boldsymbol{\ell}^{t})\boldsymbol{h} = \mathbb{E}[W\boldsymbol{e}^{t}_J]Q\boldsymbol{h} + \mathbb{E}[\boldsymbol{e}^{t}_J]\Delta_{\boldsymbol{a}}\boldsymbol{h} = -\mathbb{E}[W\boldsymbol{e}^{t}_J]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}^{t}_J]\Delta_{\boldsymbol{a}}\boldsymbol{h}, \tag{28}$$

where the second equality follows from (23). Differentiating Eqn. (5) twice at 0 and multiplying by $\mathbf{1}$, we find

$$b\,\boldsymbol{u}^{t}\mathbf{1} = \mathbb{E}[W\boldsymbol{e}^{t}_J]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}^{t}_J]\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1}, \tag{29}$$

which summed with the previous equation gives (27). We conclude the proof by showing that the resulting expression cannot equal 0.

Since the maximum length of a Jordan chain cannot exceed the algebraic multiplicity of the associated eigenvalue, we have that for any vector $\boldsymbol{v}$ it holds that

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h} + Q\boldsymbol{v} \neq \mathbf{0}, \tag{30}$$

because otherwise $(\mathbf{1}, \boldsymbol{h}, \boldsymbol{v})$ would be a Jordan chain associated with λ_0, which has multiplicity 2. This implies that $\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h}$ is not in the column space of Q, which is known to be of dimension $N-1$, because Q is irreducible. Moreover, $\boldsymbol{\pi}^{t}Q = \mathbf{0}^{t}$, thus $\boldsymbol{\pi}^{t}(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h}) \neq 0$.

The following theorem states the main result of the paper.

Theorem 11. *The unknown vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$ in Eqn. (5) can be uniquely identified in the following way: $\boldsymbol{u}_{\downarrow} = \mathbf{0}$ and $\boldsymbol{\ell}_{\uparrow} = \mathbf{0}$ while the vectors $\boldsymbol{u}_{+}$ and $\boldsymbol{\ell}_{-}$ are the solutions of the system of equations*

$$(\boldsymbol{u}^{t}_{+}, -\boldsymbol{\ell}^{t}_{-}) \begin{pmatrix} V_{+}\,e^{b\,\Gamma} \\ -V_{-} \end{pmatrix} = (\boldsymbol{k}^{t}, 0, \ldots, 0), \tag{31}$$

where

$$\boldsymbol{k}^{t} = \boldsymbol{\pi}^{t}(\Delta_{\boldsymbol{a}}, \frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}) \begin{pmatrix} V_0 \\ V_0\,\Gamma_0 \end{pmatrix} \tag{32}$$

is a vector with dimension equal to the multiplicity of the 0 root of $\det(F(\alpha))$.

(V, Γ) is any finite Jordan pair of $F(\alpha)$ having the Jordan chain (V_0, Γ_0) associated to the eigenvalue 0 in the first position. An expression for the matrices V_0 and Γ_0 is given in (24).

Remark 2. Since $\boldsymbol{\pi}^{\mathrm{t}}\Delta_{\boldsymbol{a}}\mathbf{1}$ equals the asymptotic drift κ, Eqn. (32) reduces to

$$\boldsymbol{k}^{\mathrm{t}} = \begin{cases} \kappa, & \text{if } \kappa \neq 0 \\ (0, \boldsymbol{\pi}^{\mathrm{t}}(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h})), & \text{if } \kappa = 0. \end{cases} \tag{33}$$

Proof. The proofs of the facts that $\boldsymbol{u}_{\downarrow} = \mathbf{0}$ and $\boldsymbol{\ell}_{\uparrow} = \mathbf{0}$ and, moreover, $(\boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}})\mathbf{1} = \boldsymbol{\pi}^{\mathrm{t}}\Delta_{\boldsymbol{a}}\mathbf{1} = \kappa$ were already given in [4] and follows directly from (6). The rest of the proof is split into two steps. First we show that $(\boldsymbol{u}_+^{\mathrm{t}}, \boldsymbol{\ell}_-^{\mathrm{t}})$ solves (31), and then we show that the solution is unique.

Step 1: Applying Prop. 6 with $\boldsymbol{c}^{\mathrm{t}}(z) = \mathbb{E}\left[e^{zW}\boldsymbol{e}_J^{\mathrm{t}}\right]$ and according to (5) with $\boldsymbol{b}^{\mathrm{t}}(z) = \alpha(e^{\alpha b}\boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}})$, we have that

$$\boldsymbol{u}^{\mathrm{t}}Ve^{b\Gamma}\Gamma - \boldsymbol{\ell}^{\mathrm{t}}V\Gamma = \mathbf{0}^{\mathrm{t}}. \tag{34}$$

Let $\hat{\Gamma}$ be the matrix Γ with Jordan block Γ_0 replaced with $\mathbb{I}$. Suppose first that $\kappa \neq 0$. Then $(V_0, \Gamma_0) = ([\mathbf{1}], (0))$ and so (34) can be rewritten as

$$\boldsymbol{u}^{\mathrm{t}}Ve^{b\Gamma}\hat{\Gamma} - \boldsymbol{\ell}^{\mathrm{t}}V\hat{\Gamma} = ((\boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}})\mathbf{1}, 0, \ldots, 0). \tag{35}$$

Multiply by $\hat{\Gamma}^{-1}$ from the right to obtain (31). Supposing that $\kappa = 0$, we have that adding

$$\hat{\boldsymbol{k}}^{\mathrm{t}} = \boldsymbol{u}^{\mathrm{t}}V_0e^{b\Gamma_0}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \boldsymbol{\ell}^{\mathrm{t}}V_0\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{36}$$

to the first two elements of the vectors appearing on the both sides of (34), leads to

$$\boldsymbol{u}^{\mathrm{t}}Ve^{b\Gamma}\hat{\Gamma} - \boldsymbol{\ell}^{\mathrm{t}}V\hat{\Gamma} = (\hat{\boldsymbol{k}}^{\mathrm{t}}, 0, \ldots, 0). \tag{37}$$

To complete Step 1 it is now enough to show that $\hat{\boldsymbol{k}} = \boldsymbol{k}$. A simple computation reveals that $\hat{\boldsymbol{k}}^{\mathrm{t}} = \boldsymbol{u}^{\mathrm{t}}(\mathbf{1}, (b-1)\mathbf{1} + \boldsymbol{h}) - \boldsymbol{\ell}^{\mathrm{t}}(\mathbf{1}, -\mathbf{1} + \boldsymbol{h}) = (0, b\boldsymbol{u}^{\mathrm{t}}\mathbf{1} + (\boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}})\boldsymbol{h})$, where we used that $(\boldsymbol{u}^{\mathrm{t}} - \boldsymbol{\ell}^{\mathrm{t}})\mathbf{1} = \kappa = 0$. Use Lemma 10 and (33) to see that $\hat{\boldsymbol{k}} = \boldsymbol{k}$.

Step 2 (uniqueness): Without loss of generality we assume that $\kappa \geq 0$. It is easy to see that $(\boldsymbol{u}_+^{\mathrm{t}}, \boldsymbol{\ell}_-^{\mathrm{t}})$ solves (31) if and only if

$$(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}})\begin{pmatrix} V_+^+e^{b\Gamma^+} & V_+^-e^{b\Gamma^-} \\ V_-^+ & V_-^- \end{pmatrix} = \kappa\boldsymbol{e}_1^{\mathrm{t}} \tag{38}$$

and, in addition when $\kappa = 0$ Eqn. (27) holds true because of the construction of the matrices $V^{\pm}$ and $\Gamma^{\pm}$. Note also that we can right multiply both sides of the above display by the same matrix to obtain

$$(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}})\begin{pmatrix} V_+^+ & V_+^-e^{b\Gamma^-} \\ V_-^+e^{-b\Gamma^+} & V_-^- \end{pmatrix} = \kappa\boldsymbol{e}_1^{\mathrm{t}}\begin{pmatrix} e^{-b\Gamma^+} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}. \tag{39}$$

Noticing that

$$\begin{aligned} V_-^+ &= \Pi_-^+V_+^+, \qquad & V_+^+e^{-b\Gamma^+} &= e^{b\Lambda^+}V_+^+, \\ V_+^- &= \Pi_+^-V_-^-, \qquad & V_-^-e^{+b\Gamma^-} &= e^{b\Lambda^-}V_-^-, \end{aligned} \tag{40}$$

we obtain

$$\begin{pmatrix} V_+^+ & V_+^- e^{b\Gamma^-} \\ V_-^+ e^{-b\Gamma^+} & V_-^- \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} \begin{pmatrix} V_+^+ & \mathbb{O} \\ \mathbb{O} & V_-^- \end{pmatrix}. \tag{41}$$

By definition we know that V_+^+ and V_-^- are invertible matrices. Moreover, $\Pi_-^+ e^{b\Lambda^+}$ and $\Pi_+^- e^{b\Lambda^-}$ are irreducible transition probability matrices, so the first matrix on the right hand side of (41), call it M, is an irreducible non-negative matrix, which is non-strictly diagonally dominant. If $\kappa > 0$, then $\Pi_+^- e^{b\Lambda^-}$ is sub-stochastic, which implies that M is irreducibly diagonally dominant and hence invertible [10]. If $\kappa = 0$, then M has a simple eigenvalue at 0 by Perron-Frobenius, so

$$(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} = \mathbf{0}^{\mathrm{t}} \tag{42}$$

determines the vector $(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}})$ up to a scalar, which is then identified using (27):

$$(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}}) \begin{pmatrix} b\mathbf{1} + \boldsymbol{h} \\ \boldsymbol{h} \end{pmatrix} = \boldsymbol{\pi}^{\mathrm{t}}(\Delta_{\boldsymbol{a}} \boldsymbol{h} + \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 \mathbf{1}), \tag{43}$$

which is non-zero by Lemma 10.

Finally, we state a corollary, which identifies (in the case of non-zero asymptotic drift) vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ in terms of matrices $\Lambda^\pm$ and $\Pi^\pm$. We believe that there should exist a direct probabilistic argument leading to this identity. Moreover, it is interesting to investigate if such a result holds in the case of countably infinite state space E.

Corollary 12. *It holds that*

$$(\boldsymbol{u}_+^{t}, -\boldsymbol{\ell}_-^{t}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} = \begin{cases} \kappa(\boldsymbol{\pi}_{\Lambda^+}^{t}, \mathbf{0}_-^{t}), & \text{if } \kappa > 0 \\ \mathbf{0}^{t}, & \text{if } \kappa = 0 \\ \kappa(\mathbf{0}_+^{t}, \boldsymbol{\pi}_{\Lambda^-}^{t}), & \text{if } \kappa < 0 \end{cases}, \tag{44}$$

where $\boldsymbol{\pi}_{\Lambda^\pm}$ is the unique stationary distribution of $\Lambda^\pm$, which is well-defined if $\kappa \neq 0$.

Proof. Assume that $\kappa > 0$. From the above proof we know that

$$(\boldsymbol{u}_+^{\mathrm{t}}, -\boldsymbol{\ell}_-^{\mathrm{t}}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} \begin{pmatrix} V_+^+ & \mathbb{O} \\ \mathbb{O} & V_-^- \end{pmatrix} = \kappa(\boldsymbol{e}_1^{\mathrm{t}}, \mathbf{0}_-^{\mathrm{t}}). \tag{45}$$

Hence it is enough to check that $\boldsymbol{\pi}_{\Lambda^+} = \boldsymbol{e}_1^{\mathrm{t}}(V_+^+)^{-1} = \boldsymbol{e}_1^{\mathrm{t}}(R^+)^{-1}$, which is immediate in view of the defining equations (20) and (22). The case of $\kappa < 0$ is symmetric, and the case of $\kappa = 0$ is trivial.

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