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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

Comment on: "Auctions with a buy price: The case of reference-dependent preferences"*

Ángel Hernando-Veciana†
Universidad Carlos III de Madrid

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Shunda (2009) studies an auction model with reserve prices and buy prices, and with reference-dependent preferences as in Rosenkranz and Schmitz (2007) in which the reference point is a convex combination of the reserve price and the buy price. In this short note, I point out the existence of a mistake in his main result, Theorem 1. It should be amended to read as follows:

Theorem 1. *A risk-neutral seller maximizes her expected revenue by setting some* $v^* = \overline{v}$ *and thus by posting a buy price* $B(\overline{v}, r)$*, i.e. a buy price that none of the bidder types less than* $\overline{v}$ *would exercise with positive probability in equilibrium.*

Consequently, it is also incorrect his argument that (page 653, last paragraph):

> A seller would set [...] a buy price that some bidder type would accept with positive probability in equilibrium. This result clearly contrasts with the finding that a risk-neutral seller facing risk-neutral bidders would set a buy price so high that no bidder would accept it with positive probability in equilibrium.

The reason why Shunda's proof of Theorem 1 is wrong is that he mistakenly took the differential of $B(v_i, r)$ with respect to v_i at $v_i = \overline{v}$ to be zero (see for instance, page 660) for any r. It is easy to see in the numerical examples in Shunda's Fig. 1 and 2 that the left differential at $v_1 = \overline{v}$ is strictly positive. This can also be checked in the equation at the bottom of page 652 in Shunda (2009) and in Equation (6) in page 657. Note that although the right differential is equal to zero, the left differential is the relevant differential in Shunda's proof.

Naturally, his intuitive explanation is also wrong. He claims that the seller does not find it optimal to fix a buy price so high that it is not exercised by any bidder because (page 654, line 14):

> [...] one effect of increasing the auction's buy price is to attract low valuation bidders to the auction who would not have participated otherwise, and this has an adverse effect on the seller's revenue.

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†Department of Economics. Universidad Carlos III de Madrid. c/ Madrid, 126. 28903 Getafe (Madrid) SPAIN. Email: angel.hernando@uc3m.es URL: http://www.eco.uc3m.es/ahernando/

The mistake here is that a seller can increase her revenue by encompassing the increase in the buy price with an increase in her reserve price so that the minimum of the bidders' types that enters the auction remains the same. This is actually the main argument in the proof of my version of Theorem 1.

In the remaining of this note, I prove my version of Theorem 1. I shall not follow Shunda's approach. He finds that there is a monotone relationship between the buy price B^* and the threshold v^* for any given reserve price r. Thus, instead of solving for the optimal buy price B^* and reserve price r, he solves for the optimal threshold v^* and reserve price r. I find more natural to solve directly for the optimal buy price B^* and reserve price r. My approach also avoids the complex computations in which Shunda made his mistake.

The expected revenue of the seller with a reserve price r and buy price $B^* \geq r$ is (see the details in Appendix A):

$$\int_{\underline{\underline{v}}}^{v^*} \left(\beta(y) - \frac{1-F(y)}{f(y)(1+\epsilon)}\right) dF(y)^n + \beta(v^*)(1-F(v^*)^n), \tag{1}$$

where $\underline{\underline{v}}$, v^* and $\beta(y)$ denote the minimum type that participates in the auction, the minimum type that exercises the buy price in equilibrium, and the maximum willingness to pay of a bidder with type y for some given reserve price r and buy price B^*, respectively.[1] That is:[2]

$$\underline{\underline{v}} \equiv \max\{\underline{v}, \min\{\overline{v}, (1+\epsilon(1-\lambda))r - \epsilon(1-\lambda)B^*\}\}. \tag{2}$$

v^* is equal to $\underline{\underline{v}}$ in case $B^* < \beta(\underline{\underline{v}})$,[3] and it is defined in case $B^* \geq \beta(\underline{\underline{v}})$ as the unique solution in $[\underline{\underline{v}}, \overline{v})$ to:

$$\int_{\underline{\underline{v}}}^{v^*} F(y)^{n-1} dy = \left(\frac{1-F(v^*)^n}{n(1-F(v^*))}\right)(\beta(v^*) - B^*)(1+\epsilon), \tag{3}$$

if,

$$\int_{\underline{\underline{v}}}^{\overline{v}} F(y)^{n-1} dy < (\beta(\overline{v}) - B^*)(1+\epsilon),$$

and $v^* \equiv \overline{v}$ otherwise. Finally,

$$\beta(y) \equiv \frac{y + \epsilon(\lambda r + (1-\lambda)B^*)}{1+\epsilon}. \tag{4}$$

The seller chooses (r, B^*) to maximize Equation (1) subject to the constraint that $B^* \in [r, \gamma(r)]$ where $\gamma(r)$ is a function implicitly defined by the unique solution to:

$$\int_{\max\{\underline{v}, \min\{\overline{v}, (1+\epsilon(1-\lambda))r - \epsilon(1-\lambda)\gamma\}\}}^{\overline{v}} F(y)^{n-1} dy = \left(\frac{\overline{v} + \epsilon(\lambda r + (1-\lambda)\gamma)}{1+\epsilon} - \gamma\right)(1+\epsilon). \tag{5}$$

[1] As Shunda (2009), I do not make explicit the dependence of $\underline{\underline{v}}$, v^* and $\beta(y)$ on r and B^* to shorten the notation.

[2] These definitions differ from Shunda's (2009) in that I make explicit the boundary cases. Although they are not essential for my arguments, I include them for the sake of completeness. In any case, boundary cases are trivial but in the case $B^* < \beta(\underline{\underline{v}})$ that I explain more carefully in Footnote 3.

[3] To see why, note first that the constraint that $r \leq B^*$ means that $B^* < \beta(\underline{\underline{v}})$ can only occur when $r < \beta(\underline{v})$, and hence when $\underline{\underline{v}} = \underline{v}$. Second, $\underline{\underline{v}} = \underline{v}$ and $B^* < \beta(\underline{\underline{v}})$ means that $B^* < \beta(\underline{v})$ and hence a bidder with type $\underline{v}$ makes strictly positive profits exercising the buy price. Thus, it is optimal for her to do so since both the auction and stay out give her zero payoffs, and as a consequence $v^* = \underline{v}$.

For any given reserve price r, $\gamma(r)$ is an equilibrium threshold value such that buy prices greater than $\gamma(r)$ imply that $v^* = \overline{v}$, and hence no bidders' types exercise the buy price, and buy prices less than $\gamma(r)$ imply that $v^* < \overline{v}$, and hence an open set of bidders' types exercise the buy price.

The constraint $B^* \geq r$ is explicit in Shunda (2009). The constraint $B^* \leq \gamma(r)$ is without loss of generality under Shunda's assumption in the first full paragraph in page 655 of his paper. To see why, note that Shunda's assumption implies that the auctioneer does not find it profitable to fix a buy price strictly larger than $B(\overline{v}, r)$, and $B(\overline{v}, r) = \gamma(r)$.

The following proposition implies Theorem 1:

Proposition 1. *Any reserve price and buy price (r, B^*) such that $B^* < \gamma(r)$, i.e. auction configurations in which some open set of bidders' types exercise the buy price in equilibrium, is suboptimal for a risk neutral seller. Moreover, the seller's expected revenue is maximized by setting (r, B^*) such that $B^* = \gamma(r)$, i.e. an auction configuration in which there is no open set of bidders' types that exercise the buy price in equilibrium maximizes the seller's expected revenue.*

Proof. The first claim in Proposition 1 is obviously true when (r, B^*) induces $\underline{\underline{v}} = \overline{v}$. This is because $v^* \geq \underline{\underline{v}}$ and $\underline{\underline{v}} = \overline{v}$ imply that $v^* = \overline{v}$ and hence the seller gets zero revenue.[4]

In the case in which (r, B^*) induces a threshold $\underline{\underline{v}} < \overline{v}$, we shall show that there exists a reserve price $\tilde{r} \geq r$ and a buy price $\tilde{B}^* = \gamma(\tilde{r})$ that (i) induce the same threshold $\underline{\underline{v}}$ as (r, B^*) and (ii), give strictly greater expected revenue to the seller.

Claim (i) is obviously true if (r, B^*) induces $\underline{\underline{v}} = \underline{v}$, as in this case $(\tilde{r}, \tilde{B}^*) = (r, \gamma(r))$ also induces $\underline{\underline{v}} = \underline{v}$. Otherwise, claim (i) can be derived from the following three facts. First, $B^* < \gamma(r)$; second, $\gamma'(\hat{r}) \leq 1$ for $\hat{r} \geq r$, since by the implicit function theorem applied to Equation (5),[5]

$$\gamma'(\hat{r}) = \frac{\epsilon\lambda + (1 + \epsilon(1-\lambda))F(\tilde{\underline{\underline{v}}})^{n-1}}{1 + \epsilon\lambda + \epsilon(1-\lambda)F(\tilde{\underline{\underline{v}}})^{n-1}} = 1 - \frac{1 - F(\tilde{\underline{\underline{v}}})^{n-1}}{1 + \epsilon\lambda + \epsilon(1-\lambda)F(\tilde{\underline{\underline{v}}})^{n-1}} \leq 1,$$

for $\tilde{\underline{\underline{v}}} = \max\{\underline{v}, (1 + \epsilon(1-\lambda))\hat{r} - \epsilon(1-\lambda)\gamma(\hat{r})\}$; and third, the set of reserve prices and buy prices that induce the same threshold $\underline{\underline{v}}$ as (r, B^*), see Equation (2), is a straight line with slope:

$$\frac{1 + \epsilon(1-\lambda)}{\epsilon(1-\lambda)} = 1 + \frac{1}{\epsilon(1-\lambda)} > 1.$$

These three facts imply that the function γ crosses with the set of reserve prices and buy prices that induce the same threshold $\underline{\underline{v}}$ as (r, B^*) to the right and above of (r^*, B^*) as illustrated in Figure 1.

Claim (ii) follows from claim (i), Equation (1) and the following four facts. First, by definition of γ, a reserve price $\tilde{r}$ and a buy price $\tilde{B}^* = \gamma(\tilde{r})$ induce a threshold $v^* = \overline{v}$. Second, Equation (1) is increasing in v^*, since it is concave in v^* with a zero differential at $v^* = \overline{v}$ (see the details in Appendix B.) Third, $\tilde{B}^* = \gamma(\tilde{r}) > B^*$ since γ is increasing and

[4] The inequality $v^* \geq \underline{\underline{v}}$ is a direct consequence of $B^* \geq r$. See the definition of v^* and Footnote 3.

[5] The expression above is correct only up to the point in which $\tilde{\underline{\underline{v}}} = \overline{v}$. For higher values of $\hat{r}$, $\gamma'(\hat{r}) = \frac{\lambda\epsilon}{1+\lambda\epsilon}$. This is irrelevant for our arguments because the graph of γ crosses with the set of reserve prices and buy prices that induce the same threshold $\underline{\underline{v}}$ as (r, B^*) before this point is reached, and in any case the property that $\gamma'(\hat{r}) \leq 1$ is verified.

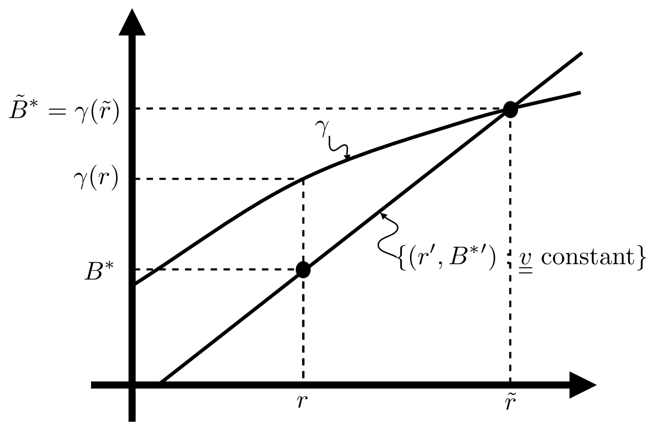

Figure 1: Illustration of the proof of claim (i) in the proof of Proposition 1.

$\gamma(r) > B^*$. Four, β increases with B^* and r as can be easily deduced from its definition in Equation (4).

To complete the proof of the proposition requires to show that the seller's optimization problem has a solution. Since the seller's expected revenue is continuous in r and B^*, it is sufficient to show that the constraint set $\{(r, B^*) \in \mathbb{R}_+^2 : B^* \in [r, \gamma(r)]\}$ is compact. To prove this, it is sufficient to show that the constraint set is contained in $[0, \overline{v}]^2$. That this last claim is true can be deduced from the fact that the definition of γ implies[6] that $\gamma(r) < r$ if $r > \overline{v}$. ■

Appendix A

In this appendix, I show that the seller's expected revenue when she posts a reserve price r and a buy price $B^* \geq r$ is equal in equilibrium to Equation (1). Shunda (2009) shows, see the last line of page 654, that this expected revenue is equal to:

$$rnF(\underline{\underline{v}})^{n-1}(F(v^*) - F(\underline{\underline{v}})) + \int_{\underline{\underline{v}}}^{v^*} \beta(y)n(n-1)F(y)^{n-2}f(y)(F(v^*) - F(y))dy + B^*(1 - F(v^*)^n). \quad (6)$$

This makes the cases $\underline{\underline{v}} = v^* = \underline{v}$ and $\underline{\underline{v}} = v^* = \overline{v}$ direct. In the remaining cases, note

[6] The left hand side of Equation (5) is non negative and as a consequence we can deduce from the right hand side that $\gamma(r) \leq \frac{\overline{v}+\lambda\epsilon r}{1+\lambda\epsilon}$. This is, $\gamma(r)$ is less or equal than a convex combination between $\overline{v}$ and r. Hence $\gamma(r) < r$ if $r > \overline{v}$.

that integrating by parts the integral and using that $\beta(\underline{\underline{v}}) = r$ if $\underline{\underline{v}} \in (\underline{v}, \overline{v})$, one can show that the first two terms are equal to:

$$\int_{\underline{\underline{v}}}^{v^*} \beta(y) dF(y)^n - \int_{\underline{\underline{v}}}^{v^*} n \frac{F(v^*) - F(y)}{1+\epsilon} F(y)^{n-1} dy =$$
$$\int_{\underline{\underline{v}}}^{v^*} \left(\beta(y) - \frac{1 - F(y)}{f(y)(1+\epsilon)} \right) dF(y)^n + n \frac{1 - F(v^*)}{1+\epsilon} \int_{\underline{\underline{v}}}^{v^*} F(y)^{n-1} dy.$$

This completes the proof if $v^* = \overline{v}$. Otherwise, note that $v^* \in (\underline{v}, \overline{v})$ verifies Equation (3) from which one can deduce that the last term in Equation (6) is equal to:

$$\beta(v^*)(1 - F(v^*)^n) - \frac{n(1 - F(v^*))}{1+\epsilon} \int_{\underline{\underline{v}}}^{v^*} F(y)^{n-1} dy,$$

as desired.

Appendix B

The first derivative of Equation (1) with respect to v^* is equal to:

$$\left(\beta(v^*) - \frac{1 - F(v^*)}{f(v^*)(1+\epsilon)} \right) nF(v^*)^{n-1} f(v^*) - \beta(v^*) nF(v^*)^{n-1} f(v^*) + \frac{1}{1+\epsilon}(1 - F(v^*)^n)$$
$$= \frac{1}{1+\epsilon} \left(1 + (n-1)F(v^*)^n - nF(v^*)^{n-1} \right),$$

that is equal to zero when $v^* = \overline{v}$. The second derivative is equal to:

$$-\frac{1}{1+\epsilon} n(n-1) F(v^*)^{n-2} f(v^*) \left(1 - F(v^*)\right),$$

that is negative.

References

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