Spectral Lattices of $\overline{\mathbb{R}}_{\max,+}$-Formal Contexts

Francisco J. Valverde-Albacete and Carmen Peláez-Moreno *

Dpto. de Teoría de la Señal y de las Comunicaciones.
Universidad Carlos III de Madrid
Avda. de la Universidad, 30. Leganés 28911. Spain
fva,carmen@tsc.uc3m.es

Abstract. In [13] a generalisation of Formal Concept Analysis was introduced with data mining applications in mind, $\mathcal{K}$-Formal Concept Analysis, where incidences take values in certain kinds of semirings, instead of the standard Boolean carrier set. Subsequently, the structural lattice of such generalised contexts was introduced in [15], to provide a limited equivalent to the main theorem of $\mathcal{K}$-Formal Concept Analysis, resting on a crucial parameter, the degree of existence of the object-attribute pairs φ. In this paper we introduce the spectral lattice of a concrete instance of $\mathcal{K}$-Formal Concept Analysis, as a further means to clarify the structural and the $\mathcal{K}$-Concept Lattices and the choice of φ. Specifically, we develop techniques to obtain the join- and meet-irreducibles of a $\overline{\mathbb{R}}_{\max,+}$-Concept Lattice independently of φ and try to clarify its relation to the corresponding structural lattice.

1 Motivation: the Analysis of Confusion Matrices with $\mathcal{K}$-Formal Concept Analysis

Consider sets of *entities* G and *patterns* M with $|G| = g \in \mathbb{N},\ |M| = m \in \mathbb{N}$ and a device called a *classifier* accepting a *characterisation* of an entity $i,\ 0 \leq i \leq g$, normally a vector of features, and returning the index of a pattern $j,\ 0 \leq j \leq m$.

A *confusion matrix* or *contingency table* $C \in \mathbb{N}^{g\times n}$ tries to capture at a glance the performance of such classifier: for each classification act we increase C_{ij} by one, tallying classification hits and errors, which makes C a *semiring-valued matrix*. With the aim of better understanding the performance of the classifier we would like to find a way to analyse the geometry of the spaces associated to matrices with properties similar to those of C .

For that purpose, in [13] a generalisation of Formal Concept Analysis was introduced that allows incidences to take values in *dioids*, or idempotent semirings: for $g, m \in \mathbb{N}$, given two sets of objects $G = \{g_i\}_{i=1}^{g}$, and attributes $M = \{m_j\}_{j=1}^{m}$, let $\mathcal{K}$, be a complete, idempotent semifield [2,13], and a $\mathcal{K}$-valued matrix, $R \in K^{g\times m}$, the triple $(G, M, R)_{\mathcal{K}}$ is called a $\mathcal{K}$*-formal context*. We interpret $R_{ij} = \lambda$ as "object g_i has attribute m_j in degree λ" or, dually, "attribute m_j is manifested in object g_i to degree λ".

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$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{Y} \arrow[rr, shift right, "\cdot^{R}_{\varphi}"'] \arrow[d, shift right, "\gamma_{\mathcal{Y}}"'] \arrow[drr, "\cdot^{R}_{\varphi}"' near start] & & \mathcal{X} \arrow[ll, shift right, "{}^{R}_{\varphi}\cdot"'] \arrow[d, shift right, "\gamma_{\mathcal{X}}"'] \arrow[dll, "{}^{R}_{\varphi}\cdot" near start] \\
\overline{\mathcal{Y}} = {}^{R}_{\varphi}(\mathcal{X}) \arrow[u, shift right, "\hookrightarrow_{\mathcal{Y}}"'] \arrow[rr, shift right, "\cdot^{R}_{\varphi}"'] & & \overline{\mathcal{X}} = (\mathcal{Y})^{R}_{\varphi} \arrow[u, shift right, "\hookrightarrow_{\mathcal{X}}"'] \arrow[ll, shift right, "{}^{R}_{\varphi}\cdot"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
\tilde{\mathcal{Y}} \arrow[rr, shift right, "\cdot^{R}_{\rho_{\max}}"'] \arrow[d, shift right, "\gamma_{\tilde{\mathcal{Y}}}"'] \arrow[drr, "\cdot^{R}_{\rho_{\max}}"' near start] & & \tilde{\mathcal{X}} \arrow[ll, shift right, "{}^{R}_{\rho_{\max}}\cdot"'] \arrow[d, shift right, "\gamma_{\tilde{\mathcal{X}}}"'] \arrow[dll, "{}^{R}_{\rho_{\max}}\cdot" near start] \\
\overline{\tilde{\mathcal{Y}}} = {}^{R}_{\rho_{\max}}\left(\tilde{\mathcal{X}}\right) \arrow[u, shift right, "\hookrightarrow_{\tilde{\mathcal{Y}}}"'] \arrow[rr, shift right, "\cdot^{R}_{\rho_{\max}}"'] & & \overline{\tilde{\mathcal{X}}} = \left(\tilde{\mathcal{Y}}\right)^{R}_{\rho_{\max}} \arrow[u, shift right, "\hookrightarrow_{\tilde{\mathcal{X}}}"'] \arrow[ll, shift right, "{}^{R}_{\rho_{\max}}\cdot"']
\end{tikzcd}
$$

$(\cdot^{R}_{\varphi}, {}^{R}_{\varphi}\cdot) : \mathcal{Y} \leftrightharpoons \mathcal{X}$ $\qquad$ $((\cdot)^{R}_{\rho_{\max}}, {}^{R}_{\rho_{\max}}(\cdot)) : \tilde{\mathcal{Y}} \leftrightharpoons \tilde{\mathcal{X}}$

Fig. 1. Diagrams depicting the structures in the Galois connection of Eq. 1 (left), and Corollary 10.1, (right).

Now, for each (multiplicatively) invertible $\varphi \in K$, call $(\mathcal{K}, \varphi)$ a *reflexive idempotent semiring* if the following maps define a Galois connection, with $Y = K^{1\times g}, X = K^{m\times 1}$ and the bracket $\langle \cdot \mid \cdot \rangle : Y \times X \to K, (y, x) \mapsto \langle y \mid x \rangle = yRx$, ([3,13] and §2.1 below):

$$
\begin{aligned}
\cdot^{R}_{\varphi} &: Y \to X & \qquad y^{R}_{\varphi} &= \bigvee \{\, x \in X \mid \langle y \mid x \rangle \leq \varphi \,\} & \qquad (1)\\
{}^{R}_{\varphi}\cdot &: X \to Y & {}^{R}_{\varphi}x &= \bigvee \{\, y \in Y \mid \langle y \mid x \rangle \leq \varphi \,\} \,.
\end{aligned}
$$

in which case we call them the *φ-polars* of the $\mathcal{K}$-formal context $(G, M, R)_{\mathcal{K}}$. Under such conditions:

1. The images $\overline{\mathcal{Y}} = {}^{R}_{\varphi}(\mathcal{X})$ and $\overline{\mathcal{X}} = (\mathcal{Y})^{R}_{\varphi}$ are dually inverse complete subsemimodules of $\mathcal{Y}$ and $\mathcal{X}$, respectively. They are obtained from the original semimodules by the closure operators: $\gamma_{\mathcal{Y}} : Y \to Y, y \mapsto \gamma_{\mathcal{Y}}(y) = {}^{R}_{\varphi}((y)^{R}_{\varphi})$ and $\gamma_{\mathcal{X}} : X \to X, x \mapsto \gamma_{\mathcal{X}}(x) = ({}^{R}_{\varphi}(x))^{R}_{\varphi}$.
2. A (formal) φ-concept of the formal context $(G, M, R)_{\mathcal{K}}$ is a pair $(a, b) \in Y \times X$ such that $a^{R}_{\varphi} = b$ and ${}^{R}_{\varphi}b = a$. We call a the *φ-extent* and b the *φ-intent* of the concept (a, b), and φ its *(maximum) degree of existence.*
3. If $(a_1, b_1), (a_2, b_2)$ are φ-concepts of a context, they are ordered by the relation $(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq a_2 \iff b_1 \overset{op}{\leq} b_2$, called the *hierarchical order*. The set of all concepts ordered in this way is called the *φ-concept lattice, $\underline{\mathfrak{B}}^{\varphi}(G, M, R)_{\mathcal{K}}$, of the $\mathcal{K}$-valued context $(G, M, R)_{\mathcal{K}}$* .

In [14] a preliminary application of $\mathcal{K}$-Concept Lattices to data mining was described to characterise the behaviour of n-class classifiers. Two sources of complexity associated to trying to understand such lattices were detected therein: first, the potentially vast size of $\mathcal{K}$-Concept Lattices, and second, the need to sweep over parameter $\varphi \in K$ to find all possible lattices which prove only slightly different for similar φ's, pairwise considered.

To overcome the first difficulty the *structural lattice* of a $\mathcal{K}$-formal context was introduced in [15] as a sort of skeleton for it. This had the supplementary benefit of providing a (limited) second half for the fundamental theorem of $\mathcal{K}$-Concept Lattices.

Discouragingly, nothing conclusive was found in the same paper with regard to the role of φ in the series of lattices generated by sweeping over the parameter apart from a non-monotone relationship on the number of concepts.

Therefore, in this paper we introduce another lattice related to a $\mathcal{K}$-Formal Context which we obtain independently of any φ, the *spectral lattice of a $\mathcal{K}$-Formal Context*. Furthemore, for gaining a more concrete understanding of the problem, we apply the results stated so far to the well-known maxplus $\mathbb{R}_{\max,+}$ and minplus $\mathbb{R}_{\min,+}$ semirings [1]. This not only provides concrete examples for all abstract notions we have been manipulating so far but also enables to leverage powerful techniques specially developed for such semirings.

For that purpose, we introduce these concrete algebras and a notation to be able to handle expressions mixing them in Section 2.1. With this toolkit we easily introduce in Section 2.2 the notion of the spectrum of any square, $\mathbb{R}_{\max,+}$-valued matrix. As the main contribution of this paper, in Section 3 we find dually order isomorphic lattices related to the spectra of the projectors onto the image subsemimodules of the Galois connection, $\overline{\mathcal{Y}}$ and $\overline{\mathcal{X}}$, implied in the main theorem. Finally, in Section 4 we show a more involved application to the analysis of confusion matrices.

2 $\overline{\mathbb{R}}_{\max,+}$ Spectral Theory

2.1 $\overline{\mathbb{R}}_{\max,+}$ and $\overline{\mathbb{R}}_{\min,+}$ Algebra

Idempotent semirings. A *semiring*[1] $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e\rangle$ is an algebra whose additive structure, $\langle S, \oplus, \epsilon\rangle$, is a commutative monoid and whose multiplicative structure, $\langle S\backslash\{\epsilon\}, \otimes, e\rangle$, is a monoid whose multiplication distributes over addition from right and left and whose neutral element w.r.t. $\oplus$ is absorbing for $\otimes$, i.e. $\forall a \in S,\ \epsilon \otimes a = \epsilon$. On any semiring $\mathcal{S}$ left and right multiplications can be defined: $\mathrm{L}_a : S \to S, b \mapsto \mathrm{L}_a(b) = ab$, and $\mathrm{R}_a : S \to S, b \mapsto \mathrm{R}_a(b) = ba$. A *commutative semiring* is one whose multiplicative structure is commutative.

A *semifield* $\mathcal{K}$ is a semiring whose multiplicative structure $\langle K\backslash\{\epsilon\}, \otimes\rangle$ is a group, that is, there is an operation, $\cdot^{-1} : K\backslash\{\epsilon\} \to S\backslash\{\epsilon\}$ such that $\forall a \in K, a \otimes a^{-1} = a^{-1} \otimes a = e$. For commutative semifields whose multiplicative structure is a commutative group we have $(a \otimes b)^{-1} = a^{-1} \otimes b^{-1}$.

An *idempotent semiring or dioid* (for double monoid) $\mathcal{D}$ is a semiring whose addition is idempotent, $\forall a \in D, a \oplus a = a$, that is, whose additive structure $\langle D, \oplus, \epsilon\rangle$ is an *idempotent semigroup*. Compared to a ring, an idempotent semiring crucially lacks additive inverses. All idempotent commutative monoids $\langle D, \oplus, \epsilon\rangle$ are endowed with a *natural order*, $\forall a, b \in D, a \preceq b \iff a \oplus b = b$, which turns them into $\vee$-semilattices with least upper bound defined as $a \vee b = a \oplus b$. Moreover, the neutral element for the additive structure of semiring $\mathcal{D}$ is the infimum for this natural order, $\epsilon = \bot$. Hence all dioids are sup-semilattices

[1] Henceforth $\mathcal{S}$ will be a generic semiring, $\mathcal{K}$ a semifield, and $\mathcal{D}$ an idempotent semiring.

$\langle D, \preceq \rangle$ with a bottom element. A dioid whose multiplicative structure is a group is an *idempotent semifield.* The formula for the infimum of two elements in such case was already given by Dedekind [4]: the meet law is: $a \wedge b = a \otimes (a \oplus b)^{-1} \otimes b$, hence idempotent semifields are already lattices.

A semiring $\mathcal{S}$ is *complete*, if for any index set I including the empty set, and any $\{a_i\}_{i\in I} \subseteq \mathcal{S}$ the (possibly infinite) summations $\bigoplus_{i\in I} a_i$ are defined and the distributivity conditions: $\left(\bigoplus_{i\in I} a_i\right) \otimes c = \bigoplus_{i\in I} (a_i \otimes c)$ and $c \otimes \left(\bigoplus_{i\in I} a_i\right) = \bigoplus_{i\in I} (c \otimes a_i)$, are satisfied. Note that for $c = e$ the above demand that infinite sums have a result. In complete semirings one can define the *Kleene star* of an element, $a \in S$, $a^* = \sum_{i=0}^{\infty} a^i$, and also $a^+ = \sum_{i=1}^{\infty} a^i$, with $a^+ = a \otimes a^*$ and $a^* = e \oplus a^+$.

A dioid $\mathcal{D}$ is *complete*, if it is complete as a naturally ordered set $\langle D, \preceq \rangle$ and left (L_a) and right (R_a) multiplications are lower semicontinuous, that is join-preserving.

Example 1 (The Maxplus and Minplus semifields).

1. *The* ***Maxplus semifield****,* $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{ -\infty \}, \max, +, -\infty, 0 \rangle$ *with inverse* $\cdot^{-1} := -\cdot$ *is an idempotent commutative semifield. It is incomplete because its bottom has no inverse:* $\forall a \in \mathbb{R} \cup \{ -\infty \},\ a + (-\infty) = -\infty \neq 0$.
2. *The* ***Minplus semifield****,* $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{ \infty \}, \min, +, \infty, 0 \rangle$ *is an idempotent commutative semifield, with the same inverse as the previous example. It is incomplete for a similar reason:* $\forall a \in \mathbb{R} \cup \{ \infty \},\ a + \infty = \infty \neq 0$.

Top Completion of idempotent semifields ([8,10,11,12]) . A non-trivial idempotent semifield $\mathcal{D} \neq \{\epsilon, e\}$ (that is, non-isomorphic to $\mathcal{B}$) cannot contain a top element, $\top$, hence it cannot be a complete dioid. In [14] a procedure is described whereby one can obtain from any (incomplete) idempotent semiring $\mathcal{D}$ a completion as follows.

For any lattice-ordered group $\mathcal{G} = \langle G, \preceq, \otimes \rangle$: adjoin two elements $\bot$ and $\top$ to G to obtain $\overline{G} = G \cup \{\bot, \top\}$ and extend the order to $\overline{G}$ as $\bot \preceq a \preceq \top, \forall a \in \bar{G}$. Then extend the product to two different operations, *upper,* $\dot{\otimes}$ *, and lower,* $\underset{\cdot}{\otimes}$ *, multiplications*:

$$a \underset{\cdot}{\otimes} b = \begin{cases} \bot & \text{if } a, b \in G \cup \{\bot, \top\}, \text{with } a = \bot, \text{ or } b = \bot \ . \\ \top & \text{if } a, b \in G \cup \{\top\}, \text{with } a = \top, \text{ or } b = \top \ . \\ a \otimes b & \text{if } a, b \in G \ . \end{cases} \tag{2}$$

$$a \dot{\otimes} b = \begin{cases} \top & \text{if } a, b \in G \cup \{\bot, \top\}, \text{with } a = \top, \text{ or } b = \top \ . \\ \bot & \text{if } a, b \in G \cup \{\bot\}, \text{with } a = \bot, \text{ or } b = \bot \ . \\ a \otimes b & \text{if } a, b \in G \ . \end{cases} \tag{3}$$

to obtain the structure $\overline{\mathcal{G}} = \langle \overline{G}, \preceq, \dot{\otimes}, \underset{\cdot}{\otimes} \rangle$, known as the *canonical enlargement of* $\mathcal{G} = \langle G, \preceq, \otimes \rangle$. In this structure, $\underset{\cdot}{\otimes}$ and $\dot{\otimes}$ are associative and commutative over $\overline{G}$, as the original $\otimes$ was over G, and the isotony of the product with respect to the natural order extends to $\overline{\mathcal{G}}$. Furthermore, if e is the unit element of $\langle G, \otimes \rangle$, it is similarly the unit of $\langle \overline{G}, \overset{\cdot}{\otimes} \rangle$ and $\langle \overline{G}, \underset{\cdot}{\otimes} \rangle$. The *top completion* of a dioid $\mathcal{D}$ is another dioid $\overline{\mathcal{D}} = \langle \overline{D}, \underset{\cdot}{\oplus}, \underset{\cdot}{\otimes}, \epsilon, e \rangle$ where: $\overline{D} = D \cup \{\top\}$ and in which $\underset{\cdot}{\otimes}$ coincides with its definition above when $\mathcal{D}$ is considered as bearing a lattice-ordered (multiplicative semi-)group, and we extend $\oplus$ with the extra top-element:

$$a \underset{\cdot}{\oplus} b = \begin{cases} \top & \text{if } a = \top \text{ or } b = \top \ . \\ a \oplus b, & \text{if } a, b \in D \ . \end{cases} \tag{4}$$

Given an (incomplete) idempotent semifield $\mathcal{D}$, on its top enlargement as above, $\overline{\mathcal{D}}$, we extend the *notation* for the inverse with the following conventions: $\epsilon^{-1} = \top, \ \top^{-1} = \epsilon$. In that way we have two related *completed idempotent semifield* structures:

- a complete lattice for the natural order $\langle \overline{D}, \preceq \rangle$, the one we have been focusing on, $\overline{\mathcal{D}} = \langle \overline{D}, \underset{\cdot}{\oplus} = \vee, \underset{\cdot}{\otimes}, \bot, e \rangle$, and
- a complete lattice for the *dual of the natural order*, $\langle \overline{D}, \preceq^{\mathrm{d}} \rangle = \langle \overline{D}, \succeq \rangle$, $\overline{\mathcal{D}}^{\mathrm{d}} = \langle \overline{D}, \overset{\cdot}{\oplus} = \wedge, \overset{\cdot}{\otimes}, \top, e \rangle$ where the meet is defined (on $\mathcal{D}$) by Dedekind's formula and the definition of $\overset{\cdot}{\otimes}$ follows equation (3).

Example 2. *Using the procedure above, we have that:*

- *The top completion of* $\mathbb{R}_{\max,+}$ *is* $\overline{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty \, \infty\}, \max, \underset{\cdot}{+}, -\infty, 0 \rangle$, *the* completed Maxplus semifield.
- *The top completion of* $\mathbb{R}_{\min,+}$ *is* $\overline{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \overset{\cdot}{+}, \infty, 0 \rangle$ *the* completed Minplus semifield .

Note that in this notation we have $-\infty \underset{\cdot}{+} \infty = -\infty$ *and* $-\infty \overset{\cdot}{+} \infty = \infty$*, which solves several issues in dealing with the separately completed dioids, as promised.*

In the completed structure, which we prefer to denote by $\overline{\mathcal{K}}$, we have the following De Morgan-like relations between the multiplications, their residuals and inversion:

Proposition 1 ([12], lemma 2.2). *In the top enlargement* $\overline{\mathcal{K}}$ *of any commutative semifield* $\mathcal{K}$ *we have:*

$$(a \underset{\cdot}{\oplus} b)^{-1} = a^{-1} \overset{\cdot}{\oplus} b^{-1} \qquad (a \overset{\cdot}{\oplus} b)^{-1} = a^{-1} \underset{\cdot}{\oplus} b^{-1} \tag{5}$$
$$(a \underset{\cdot}{\otimes} b)^{-1} = a^{-1} \overset{\cdot}{\otimes} b^{-1} \qquad (a \overset{\cdot}{\otimes} b)^{-1} = a^{-1} \underset{\cdot}{\otimes} b^{-1}$$

If $\overline{\mathcal{K}}$ is the completion of an idempotent *semifield, its upper and lower residuals:*

$$a \underset{\cdot}{\otimes} b \preceq c \Leftrightarrow b \preceq a \underset{\cdot}{\backslash} c \Leftrightarrow a \preceq c \underset{\cdot}{/} b \qquad a \overset{\cdot}{\otimes} b \preceq^{d} c \Leftrightarrow b \preceq^{d} a \overset{\cdot}{\backslash} c \Leftrightarrow a \preceq^{d} c \overset{\cdot}{/} b \tag{6}$$

can be expressed in terms of the multiplications and inversion as:

$$\begin{aligned} a \underset{\cdot}{\backslash} c &= a^{-1} \overset{\cdot}{\otimes} c = (a \underset{\cdot}{\otimes} c^{-1})^{-1} \qquad & c \underset{\cdot}{/} a &= c \overset{\cdot}{\otimes} a^{-1} = (c^{-1} \underset{\cdot}{\otimes} a)^{-1} \\ a \overset{\cdot}{\backslash} c &= a^{-1} \underset{\cdot}{\otimes} c = (a \overset{\cdot}{\otimes} c^{-1})^{-1} \qquad & c \overset{\cdot}{/} a &= c \underset{\cdot}{\otimes} a^{-1} = (c^{-1} \overset{\cdot}{\otimes} a)^{-1} \end{aligned} \tag{7}$$

Although associativity of $\underset{\cdot}{\oplus}$ with respect to $\overset{\cdot}{\oplus}$ would be desirable, the farther we can get is:

Proposition 2 ([8], proposition 3.c). *For all $x, y, z \in \overline{\mathcal{K}}$:*

$$(x \overset{\cdot}{\oplus} y) \underset{\cdot}{\oplus} z \leq x \overset{\cdot}{\oplus} (y \underset{\cdot}{\oplus} z) \tag{8}$$

Example 3 (Residuation in $\overline{\mathbb{R}}_{\max,+}$, $\overline{\mathbb{R}}_{\min,+}$). *The residuals of $\underset{\cdot}{+}$ and $\overset{\cdot}{+}$ are:*

$$\begin{aligned} a \underset{\cdot}{\backslash} c &:= (-a) \overset{\cdot}{+} c = -(a \underset{\cdot}{+} (-c)) \qquad & c \underset{\cdot}{/} a &:= c \overset{\cdot}{+} (-a) = -((-c) \underset{\cdot}{+} a) \\ a \overset{\cdot}{\backslash} c &:= (-a) \underset{\cdot}{+} c = -(a \overset{\cdot}{+} (-c)) \qquad & c \overset{\cdot}{/} a &:= c \underset{\cdot}{+} (-a) = -((-c) \overset{\cdot}{+} a) \end{aligned}$$

Idempotent semimodules of matrices. A semimodule over a semiring is defined in a similar way to a module over a ring [3,7,6]: a *left $\mathcal{S}$-semimodule*, $\mathcal{Y}$ over a semiring $\mathcal{S}$ is an additive commutative monoid $\langle Y, \oplus, \epsilon_{\mathcal{Y}} \rangle$ endowed with a map $(\lambda, y) \mapsto \lambda \odot y$ such that $\forall \lambda, \mu \in S, \quad y, z \in Y$. Following the convention of dropping the symbols for the scalar action and semiring multiplication we have:

$$\begin{aligned} (\lambda\mu)y &= \lambda(\mu y) \qquad & \epsilon_S \odot y &= \epsilon_{\mathcal{Y}} \\ \lambda(y \oplus z) &= \lambda y \oplus \lambda z \qquad & e_S \odot y &= y \end{aligned} \tag{9}$$

The definition of a *right $\mathcal{S}$-semimodule* $\mathcal{X}$ follows the same pattern with the help of a *right action*, $(\lambda, x) \mapsto x \odot \lambda$ and similar axioms to those of (9). A $(\mathcal{K}, \mathcal{S})$-*semimodule* is a set M endowed with left $\mathcal{K}$-semimodule and a right $\mathcal{S}$-semimodule structures, and a $(\mathcal{K}, \mathcal{S})$-*bisemimodule* a $(\mathcal{K}, \mathcal{S})$-semimodule such that the left and right multiplications commute. For a left $\mathcal{S}$-semimodule, $\mathcal{Y}$, the left and right multiplications are defined as: $\mathrm{L}^{\mathcal{S}}_{\lambda} : Y \to Y, y \mapsto \mathrm{L}^{\mathcal{S}}_{\lambda}(y) = \lambda y$, and $\mathrm{R}^{\mathcal{Y}}_{y} : S \to Y, \lambda \mapsto \mathrm{R}^{\mathcal{Y}}_{y}(\lambda) = \lambda y$. And similarly, for a right S-semimodule.

Example 4. *Each semiring, $\mathcal{K}$, is a left (right) semimodule over itself, with the semiring product as left (right) action. Therefore, it is a ($\mathcal{K}$, $\mathcal{K}$)-bisemimodule over itself, because both actions commute by associativity. Such is the case for the Boolean $(\mathcal{B}, \mathcal{B})$-bisemimodule, the Maxplus and the Minplus bisemimodules. These are all complete and idempotent.*

Example 5 (Finite matrix semirings and semimodules). *Let $\mathcal{S}$ be a semiring. $\mathcal{M}_n(\mathcal{S}) = \langle S^{n\times n}, \oplus, \otimes, \mathcal{E}, E\rangle$ is semiring of (square) matrices over $\mathcal{S}$ with $S^{n\times n}$ denoting the set of square matrices over the semiring, matrix operations $(A\oplus B)_{ij} = A_{ij}\oplus B_{ij}, 0 \leq i,j \leq n$ and $(A\otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}, 0 \leq i,j \leq n$, null element the matrix $\mathcal{E}$, $\mathcal{E}_{ij} = \epsilon, 0 \leq i,j \leq n$ and unit E, $E_{ii} = e, 0 \leq i \leq n$, $E_{ij} = \epsilon, 0 \leq i,j \leq n, i \neq j$. Such semirings are not commutative in general even if $\mathcal{S}$ is, except for $\mathcal{M}_1(\mathcal{S}) = \mathcal{S}$. They are complete and idempotent if $\mathcal{S}$ is, in which case, the Kleene star of a square matrix, $A \in \mathcal{M}_n(\mathcal{S})$, can be calculated efficiently: $A^* = \mathcal{E} \oplus A \oplus A^2 \dots A^n$.*

For $g, m \in \mathbb{N}$, the semimodule of finite matrices $\mathcal{M}_{g\times m}(\mathcal{S}) = \langle S^{g\times m}, \oplus, \mathcal{E}\rangle$ is a $(\mathcal{M}_g(\mathcal{S}), \mathcal{M}_m(\mathcal{S}))$-bisemimodule, with matrix multiplication-like left and right actions and componentwise addition. Special cases of it are:

- *the bisemimodules of column vectors $\mathcal{M}_{m\times 1}(\mathcal{S})$ and row vectors $\mathcal{M}_{1\times g}(\mathcal{S})$.*
- *the semiring of square matrices $\mathcal{M}_g(\mathcal{S})$ with $g = m$, also a bisemimodule.*

If $\mathcal{S} \equiv \mathcal{D}$ is idempotent (resp. complete), then all are idempotent (resp. complete) with the component-wise partial order as their natural order. If $\overline{\mathcal{D}}$ is a completed semifield, then matrix multiplications read for appropriate A, B and summations:

$$(A \underset{\cdot}{\otimes} B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \underset{\cdot}{\otimes} B_{kj} \qquad\qquad (A \dot{\otimes} B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \dot{\otimes} B_{kj}$$

For the completed semifields $\overline{\mathbb{R}}_{\max,+}$ and $\overline{\mathbb{R}}_{\min,+}$, we have:

$$(A \underset{\cdot}{\otimes} B)_{ij} := \max_{k=1}^{n}(A_{ik} \underset{\cdot}{+} B_{kj}) \qquad\qquad (C \dot{\otimes} D)_{ij} := \min_{k=1}^{n}(C_{ik} \dot{+} D_{kj})$$

Residuation in matrix semimodules. A left $\mathcal{D}$-semimodule $\mathcal{Y}$ over an idempotent semiring $\mathcal{D}$ inherits the idempotent law: $\forall v \in Y,\ v \oplus v = v$, which induces a *natural order* on the semimodule: $\forall v, w \in Y,\ v \leq w \iff v \oplus w = w$, whereby it becomes a $\vee$-semilattice, with $\epsilon_{\mathcal{Y}}$ its minimum. In the following we systematically equate left (respectively right) idempotent $\mathcal{D}$-semimodules and row (respectively column) semimodules over an idempotent semiring $\mathcal{D}$. When $\mathcal{D}$ is a complete idempotent semiring, a left $\mathcal{D}$-semimodule $\mathcal{Y}$ is *complete (in its natural order)* if it is complete as a naturally ordered set and its left and right multiplications are lower semicontinuous. Trivially, it is also a complete lattice, with join and meet operations given by: $v \leq w \iff v \vee w = w \iff v \wedge w = v$. This extends naturally to right- and bisemimodules.

As in the semiring case, because of the natural order structure, the actions of idempotent semimodules admit residuation: given a complete, idempotent left $\mathcal{D}$-semimodule, $\mathcal{Y}$, we define for all $y, z \in Y,\ \lambda \in D$ the residuals are: $\left(\mathrm{L}_\lambda^{\mathcal{D}}\right)^\# : Y \to Y, \left(\mathrm{L}_\lambda^{\mathcal{D}}\right)^\# (z) = \lambda \backslash z$ and $\left(\mathrm{R}_y^{\mathcal{Y}}\right)^\# : Y \to D, \left(\mathrm{R}_y^{\mathcal{Y}}\right)^\# (z) = z/y$ and likewise for a right semimodule.

If $\mathcal{D}$ is idempotent (resp. complete), then finite matrix semimodules are idempotent (resp. complete) with the componentwise partial order as their natural order. Therefore we can define residuated operations as ([2], p. 196): let $\mathcal{D}$ be a complete dioid in which $\wedge$ exists, and $A \in D^{m\times n}, B \in D^{m\times p}, C \in D^{n\times p}$, then their left, $A\backslash B$, and right B/C residuals are:

$$(A \backslash B)_{ij} = \bigwedge_{k=1}^{m} (A_{ki} \backslash B_{kj}) \qquad (B / C)_{ij} = \bigwedge_{k=1}^{p} (B_{ik} / C_{jk}) \tag{10}$$

For $\overline{\mathcal{K}}$ a completed idempotent semifield as in subsection 2.1, the left and right residuals of $\underset{\cdot}{\otimes}$ and $\dot{\otimes}$ are (with the appropriate summations):

$$(A \underset{\cdot}{\backslash} B)_{ij} = \bigoplus_{k=1}^{m} \left(A_{ki}^{-1} \dot{\otimes} B_{kj} \right) \qquad (A \overset{\cdot}{\backslash} B)_{ij} = \bigoplus_{k=1}^{m} \left(A_{ki}^{-1} \underset{\cdot}{\otimes} B_{kj} \right) \tag{11}$$
$$(B \underset{\cdot}{/} C)_{ij} = \bigoplus_{k=1}^{p} \left(B_{ik} \dot{\otimes} C_{jk}^{-1} \right) \qquad (B \overset{\cdot}{/} C)_{ij} = \bigoplus_{k=1}^{p} \left(B_{ik} \underset{\cdot}{\otimes} C_{jk}^{-1} \right)$$

To pave the way for some results in Section 3 we have:

Proposition 3 (Adapted from [5], §5.3.3 and 5.4). *For u, v, w in the appropriate $\mathcal{S}$-semimodules, $(u \underset{\cdot}{\backslash} v) \underset{\cdot}{\otimes} w \leq u \underset{\cdot}{\backslash}(v \underset{\cdot}{\otimes} w)$ and equality holds when $w \in S$ is invertible or $w \in \mathcal{M}_{g\times m}(\mathcal{S})$ has at least one finite component in every row and column.*

Definition 6 (Conjugations). *For $\mathcal{Y} \cong \mathcal{K}^{1\times n}, \mathcal{X} \cong \mathcal{K}^{n\times 1}$ left and right semimodules, respectively, over an idempotent reflexive semifield $(\mathcal{K}, \varphi)$ and bracket $\langle \cdot \mid \cdot \rangle : Y \times X \to K$, $\langle y \mid x \rangle = y \underset{\cdot}{\otimes} x$ [13] we define a* conjugation *to be the Galois connection obtained from the maps in eq. (1): $y^{\circledast} = y \underset{\cdot}{\backslash} e_{\mathcal{D}}$, ${}^{\circledast}x = e_{\mathcal{D}} \underset{\cdot}{/} x$, and we write simply: $(\cdot^{\circledast}, {}^{\circledast}\cdot) : \mathcal{Y} \rightharpoonup \mathcal{X}$. For any other invertible element $\varphi \in K$ we have the φ-*conjugations*: $y_{\varphi}^{\circledast} = y \underset{\cdot}{\backslash} \varphi = y \underset{\cdot}{\backslash}(e_{\mathcal{D}} \dot{\otimes} \varphi) = y^{\circledast} \dot{\otimes} \varphi$ and ${}^{\circledast}_{\varphi}x = \varphi \dot{\otimes} {}^{\circledast}x$.*

For instance, the conjugations in $\overline{\mathbb{R}}_{\max,+}$ are: $y^{\circledast} := -y^{\mathrm{t}}, {}^{\circledast}x := -x^{\mathrm{t}}$, where $\cdot^{\mathrm{t}} : Y \to Y$ stands for transposition. We also define without further ado: $y^{-1} = (y^{\mathrm{t}})^{\circledast} = (y^{\circledast})^{\mathrm{t}}$ and similarly for right semimodules.

For adequate invertible unitary matrices, $E_{\mathcal{M}_n(\mathcal{D})}, (\mathcal{M}_n(\mathcal{D}), E_{\mathcal{M}_n(\mathcal{D})})$ is reflexive hence the conjugations of Def. (6) exist for $R \in D^{g\times m}$:

$$R^{\circledast} = R \underset{\cdot}{\backslash} E_{\mathcal{M}_g(\mathcal{D})} \qquad {}^{\circledast}R = E_{\mathcal{M}_m(\mathcal{D})} \underset{\cdot}{/} R \tag{12}$$

and we can write analogues of Prop. 1 compactly:

Proposition 4. *In the top completion, $\overline{\mathcal{D}}$, of an idempotent semifield the following De Morgan-like laws hold:*

$$(A \underset{\cdot}{\oplus} B)^{\circledast} = A^{\circledast} \dot{\oplus} B^{\circledast} \qquad (A \dot{\oplus} B)^{\circledast} = A^{\circledast} \underset{\cdot}{\oplus} B^{\circledast} \tag{13}$$
$$(A \underset{\cdot}{\otimes} B)^{\circledast} = B^{\circledast} \dot{\otimes} A^{\circledast} \qquad (A \dot{\otimes} B)^{\circledast} = B^{\circledast} \underset{\cdot}{\otimes} A^{\circledast}$$

the following residuation laws hold:

$$A \underset{\cdot}{\setminus} C = A^{\circledast} \dot{\otimes} C = (C^{\circledast} \underset{\cdot}{\otimes} A)^{\circledast} \qquad A \dot{\setminus} C = A^{\circledast} \underset{\cdot}{\otimes} C = (C^{\circledast} \dot{\otimes} A)^{\circledast} \tag{14}$$

$$C \underset{\cdot}{/} A = C \dot{\otimes} A^{\circledast} = (A \underset{\cdot}{\otimes} C^{\circledast})^{\circledast} \qquad C \dot{/} A = C \underset{\cdot}{\otimes} A^{\circledast} = (A \dot{\otimes} C^{\circledast})^{\circledast}$$

and similarly for left conjugates.

2.2 Spectra of Reducible and Irreducible Matrices

Graphs related to a matrix. Consider a digraph $\Gamma = (V, E)$, with V a set of vertices and $E \subseteq V^2$ a set of edges. If there is a walk from a vertex i to a vertex j in Γ we say that i *has access to* j, $i \rightsquigarrow j$. This relation is transitive and reflexive. The *access equivalent classes* of Γ are the equivalence classes of the transitive, symmetric and reflexive closure of the access relation, $i \leftrightsquigarrow j \Leftrightarrow i \rightsquigarrow j \wedge j \rightsquigarrow i$. Γ is *strongly connected* if it only has one class. When $C, C' \in V/\leftrightsquigarrow$, we say that a class C *has access to a class* C', if some vertex of C has access to some vertex of C', and we say that it is *final* if it has only access to itself.

Now consider a matrix with values in a semiring, $A \in D^{n \times n}$. The *digraph* $\Gamma(A)$ *associated to this matrix* consists of the set of vertices $V = \{1, \ldots n\}$ and a set of edges, $E = \{(i, j) \mid A_{ij} \neq \epsilon_D\}$. The *classes of a matrix* A are the (access equivalent) classes of $\Gamma(A)$, hence we say that the matrix A is *irreducible* if $\Gamma(A)$ is strongly connected, and *reducible* otherwise.

A *walk* in $\Gamma(A)$ is a sequence of edges pairwise sharing an element $w = (v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$. The weight of a walk is $|w|_A = A_{v_1 v_2} \otimes A_{v_2 v_3} \otimes \ldots \otimes A_{v_{k-1} v_k}$, and its *length* is $|w| = k - 1$. Call a *cycle* a walk with $v_1 = v_k$ and its *cycle mean* the ratio of weight-to-length. Therefore the *maximal cycle mean*, $\rho_{\max}(A)$, is the maximum of the cycle means over all cycles of $\Gamma(A)$:

$$\rho_{\max}(A) = \max_{c \text{ cycle of } \Gamma(A)} \frac{|c|_A}{|c|} \tag{15}$$

A cycle that attains such a maximum is called a *critical cycle*. Call the union of the critical cycles the *critical digraph*, $\Gamma_c(A)$, and its vertices, the *critical vertices*, V_c . Also, call the (access equivalent) classes of the critical digraph $\Gamma_c(A)$ the *critical classes of A*.

Eigenvalues and eigenvectors in idempotent semimodules. Let $\overline{\mathcal{D}}$ be a completed dioid. An *eigenvector* of $A \in D^{n \times n}$ is a vector $x \in D^n \backslash \{\epsilon\}$ such that $A \underset{\cdot}{\otimes} x = \lambda \underset{\cdot}{\otimes} x$ for some $\lambda \in D$ which is called the *(geometric) eigenvalue corresponding to* x. If λ is an eigenvalue of A then the *eigenspace* of A for the eigenvalue λ is the set of vectors, $eig(A, \lambda) = \{x \in \overline{D}^n \mid A \underset{\cdot}{\otimes} x = \lambda \underset{\cdot}{\otimes} x\}$.

To put a concrete example, the $\overline{\mathbb{R}}_{\max,+}$ spectral theory shows notorious differences with normal spectral theory. For $\overline{\mathcal{D}} := \overline{\mathbb{R}}_{\max,+}$ the eigenvalue equation becomes:

$$\max_{1\le j\le n}\{A_{ij} + x_j\} = \lambda + x_i,\ \forall 1 \le i \le n \tag{16}$$

Now, if we define the *normalised matrix* as $\tilde{A} = \rho_{\max}(A)^{-1}A$ when D is a semifield, the following facts all refer to irreducible $A \in \mathbb{R}^{n\times n}_{\max,+}$ [1]:

Property 7 (Spectra of irreducible $\overline{\mathbb{R}}_{\max,+}$-matrices).

1. *For any matrix A, $\rho_{\max}(A)$ is an eigenvalue of A, and any eigenvalue of A is less than or equal to $\rho_{\max}(A)$.*
2. *An eigenvalue of A associated with an eigenvector in $\overline{\mathbb{R}}^{n}_{\max,+}$ must be equal to $\rho_{\max}(A)$.*
3. *If A is irreducible, then $\rho_{\max}(A) > \epsilon$ and it is the only eigenvalue of A .*
4. *For all critical vertices $i \in V_c(A)$, the column $\tilde{A}^*_{\cdot i}$ is an eigenvector of A for the eigenvalue $\rho_{\max}(A)$.*
5. *If i and j belong to the same critical class, then $\tilde{A}^*_{\cdot i} = \tilde{A}^*_{\cdot j} \otimes \tilde{A}^*_{ji}$.*
6. *(Eigenspace for the eigenvalue $\rho_{\max}(A)$). Let $\{C_t\}_{t=1}^{s}$ be the set of critical classes of A. Arbitrarily select one vertex i_t from each class. The columns $\tilde{A}^*_{\cdot i_s}, t = 1\ldots s$ span the eigenspace of A for the maximal cycle mean $\rho_{\max}(A)$, $eig(A, \rho_{\max}(A)) = span\left(\{\tilde{A}^*_{\cdot i_s}\}_{t=1}^{s}\right)$.*

The most notable difference here is the existence of a single eigenvalue $\rho_{\max}$ per irreducible matrix. In fact in such situations we drop the specification of the eigenvalue from the eigenspace notation $eig(A) = eig(A, \rho_{\max})$ thereby implying that A is irreducible.

Now, denote by $A[C, C]$ the submatrix of A selected by the vertices in class C and call a class C of A *basic* if $\rho_{\max}(A[C, C]) = \rho_{\max}(A)$. The following facts relate to reducible matrices[2]:

Property 8 (Spectra of reducible $\overline{\mathbb{R}}_{\max,+}$-matrices).

1. *A scalar $\lambda \ne \epsilon$ is an eigenvalue of A if and only if there is at least one class of A such that $\rho_{\max}(A[C, C]) = \lambda$ and $\rho_{\max}(A[C, C]) \ge \rho_{\max}(A[C', C'])$ for all classes C' that have access to D . The* spectrum *of A, $spec(A)$, is the set of such eigenvalues, which is essentially the union of the spectra of some of its irreducible blocks.*
2. *$A \in \mathbb{R}^{n\times n}$ has an eigenvector in $\mathbb{R}^n$ iff all its final classes are basic.*
3. *(Eigenspace for eigenvalue λ.) Let $\{C^k\}_{k=1}^{m}$ denote all the classes of A such that if $\rho_{\max}(A[C^k, C^k]) = \lambda_k$ then $\rho_{\max}(A[C', C']) \le \lambda_k$ for all classes C' that have access to C^k . For every $1 \le k \le m$, let $\{C^k_t\}_{t=1}^{s_k}$ denote the critical classes of the matrix $A[C^k, C^k]$. For each $1 \le k \le m, 1 \le t \le s_k$, choose an arbitrary $j_{k,t} \in C^k_t$. Then the columns of the λ-normalized columns $eig(A, \lambda) = span(\{(\lambda \setminus A)^*_{\cdot j_{k,t}} \mid 1 \le k \le m, 1 \le t \le s_k, j_{k,t} \in C^k_t\})$ span the eigenspace of A for λ and any spanning family of this eigenspace contains a scalar multiple of every one of these.*

---

[2] We mention in passing that there are algorithms for transforming a reducible matrix into an upper or lower block-triangular form.

Again the extra requisites on the spectral eigenvalues related to the order of the reachable classes is a deviation from "standard" spectral theory.

Calculating such spectra is specially easy in a certain kind of matrices:

Definition 9. *Let $A \in \mathcal{M}_n(\overline{\mathbb{R}}_{\max,+})$. After [9], we call A* definite *if its maximal cycle mean is $\rho_{\max}(A) = e$ and its diagonal entries equal $A_{ii} = e$.*

We have the following:

Proposition 5 ([9], prop. 7). *If A is a definite matrix, then:*

1. *It has a unique eigenvalue $\lambda = e = \rho_{\max}(A)$.*
2. *$eig(A) = span(A^*)$.*

The important thing about definite matrices is that the very complex eigenvalue-eigenvector calculation is reduced to the calculation of a star operation. We prove in passing the next result to be used later implying that the left and right residuals of any rectangular matrix are halfway to being a definite matrix:

Proposition 6. *Let $\overline{\mathcal{K}}$ be the top completion of an idempotent semifield. For $R \in \mathcal{M}_{g\times m}(\overline{\mathcal{K}})$, the diagonal entries of $R \setminus R \in \mathcal{M}_m(\overline{\mathcal{K}})$ and $R \underset{\cdot}{/} R \in \mathcal{M}_g(\overline{\mathcal{K}})$ equal e iff at least some row, or column of $\dot{R}$ is finite.*

Proof. Call $P = R \underset{\cdot}{\setminus} R \in \mathcal{M}_m(\overline{\mathcal{D}})$. Recall that $R \underset{\cdot}{\setminus} R = R^{\circledast} \dot{\otimes} R$, so for each $1 \leq i \leq m, P_{ii} = \bigoplus_{1\leq i\leq m} R^{\circledast}_{ij} \dot{\otimes} R_{ji}$. Now, $R^{\circledast}_{ij} = R^{-1}_{ji}$. Hence, for $R_{ji} \in \mathcal{D}$ this means $R^{\circledast}_{ij} \dot{\otimes} R_{ji} = R^{-1}_{ji} \dot{\otimes} R_{ji} = e$, and for $R_{ji} \in \{\bot, \top\}$, $R^{-1}_{ji} \dot{\otimes} R_{ji} = \top$. If at least one of the elements is finite, then the total sum, being an inf, becomes e . The proof for $R \underset{\cdot}{/} R$ is the same. □

3 The Spectral Lattice of an $\overline{\mathbb{R}}_{\max,+}$-Context

Consider the *right* semimodules $\mathcal{Y} \cong \overline{K}^{g\times 1}$, $\mathcal{X} \cong \overline{K}^{m\times 1}$ and the bracket $\langle y \mid x\rangle = y^{\mathrm{t}} \underset{\cdot}{\otimes} R \underset{\cdot}{\otimes} x$ where we have switched to consider columns as vectors as customary in data mining and signal processing applications[3]. We can give algebraic expressions for the φ-polars in the completed semifield:

Proposition 7. *The φ-polars have the algebraic form: $y^R_\varphi = R^{\circledast} \dot{\otimes} y^{-1} \dot{\otimes} \varphi$, ${}^R_\varphi x = \varphi \dot{\otimes} x^{\circledast} \dot{\otimes} R^{\circledast}$.*

Proof. This is straightforward using the maxplus/minplus algebra developed in section 2.1:

$$
\begin{aligned}
y^R_\varphi &= (y^{\mathrm{t}} \underset{\cdot}{\otimes} R) \underset{\cdot}{\setminus} \varphi & \qquad {}^R_\varphi x &= \varphi \underset{\cdot}{/} (R \underset{\cdot}{\otimes} x) \\
&= (y^{\mathrm{t}} \underset{\cdot}{\otimes} R)^{\circledast} \dot{\otimes} \varphi & &= \varphi \dot{\otimes} (R \underset{\cdot}{\otimes} x)^{\circledast} \\
&= R^{\circledast} \dot{\otimes} y^{-1} \dot{\otimes} \varphi & &= \varphi \dot{\otimes} x^{\circledast} \dot{\otimes} R^{\circledast}
\end{aligned}
\tag{17}
$$

[3] This will only entail minimal tinkering with the notation.

$\square$

This suggests that we call $\tilde{x} = x \dot{/} \varphi = x \underset{\cdot}{\otimes} \varphi^{\circledast}$ and $\tilde{y}^{\mathrm{t}} = \varphi \dot{\backslash} y^{\mathrm{t}} = \varphi^{\circledast} \underset{\cdot}{\otimes} y^{\mathrm{t}}$, equivalently $\tilde{y} = y \dot{/} \varphi^{\mathrm{t}} = y^{\mathrm{t}} \underset{\cdot}{\otimes} \varphi^{-1}$, so that the *normalised semimodules (wrt. φ)* are:

$$\tilde{\mathcal{Y}}^{\mathrm{t}} = \{\tilde{y}^{\mathrm{t}} \mid y^{\mathrm{t}} \in \mathcal{Y}^{\mathrm{t}}\} \qquad \tilde{\mathcal{Y}} = \{\tilde{y} \mid y \in \mathcal{Y}\} \qquad \tilde{\mathcal{X}} = \{\tilde{x} \mid x^{\mathrm{t}} \in \mathcal{X}\} \tag{18}$$

Then we have the following:

Proposition 8 (Decoupled eigenequations).

1. *With $P_{\mathcal{Y}^{\mathrm{t}}} = R \mathbin{\dot{\otimes}} R^{\circledast} \in \mathcal{M}_g(\overline{\mathbb{R}}_{\max,+})$, we have $\tilde{y}^{\mathrm{t}} \underset{\cdot}{\otimes} P_{\mathcal{Y}^{\mathrm{t}}} = \tilde{y}^{\mathrm{t}}$.*
2. *With $P_{\mathcal{X}} = R^{\circledast} \mathbin{\dot{\otimes}} R \in \mathcal{M}_m(\overline{\mathbb{R}}_{\max,+})$ we have $P_{\mathcal{X}} \underset{\cdot}{\otimes} \tilde{x} = \tilde{x}$.*
3. *With $P_{\mathcal{Y}} = (P_{\mathcal{Y}^{\mathrm{t}}})^{\mathrm{t}} = (R \mathbin{\dot{\otimes}} R^{\circledast})^{\mathrm{t}} = R^{-1} \mathbin{\dot{\otimes}} R^{\mathrm{t}}$ we have $P_{\mathcal{Y}} \underset{\cdot}{\otimes} \tilde{y} = \tilde{y}$.*

Proof. The equation for the concepts can be written as:

$$ {}^{R}_{\varphi}x = y^{\mathrm{t}} \qquad\qquad y^{R}_{\varphi} = x \tag{19}$$

Therefore, equating Eqs. (17) and (19):

$$\varphi \mathbin{\dot{\otimes}} x^{\circledast} \mathbin{\dot{\otimes}} R^{\circledast} = y^{\mathrm{t}} \qquad\qquad R^{\circledast} \mathbin{\dot{\otimes}} y^{-1} \mathbin{\dot{\otimes}} \varphi = x$$

hence from $x^{\circledast} = \varphi^{\circledast} \underset{\cdot}{\otimes} y^{\mathrm{t}} \underset{\cdot}{\otimes} R$ and $y^{-1} = R \underset{\cdot}{\otimes} x \underset{\cdot}{\otimes} \varphi^{\circledast}$ we get:

$$\varphi \mathbin{\dot{\otimes}} (\varphi^{\circledast} \underset{\cdot}{\otimes} y^{\mathrm{t}} \underset{\cdot}{\otimes} R) \mathbin{\dot{\otimes}} R^{\circledast} = y^{\mathrm{t}} \qquad\qquad R^{\circledast} \mathbin{\dot{\otimes}} (R \underset{\cdot}{\otimes} x \underset{\cdot}{\otimes} \varphi^{\circledast}) \mathbin{\dot{\otimes}} \varphi = x$$

whence, for invertible φ:

$$(\varphi^{\circledast} \underset{\cdot}{\otimes} y^{\mathrm{t}} \underset{\cdot}{\otimes} R) \mathbin{\dot{\otimes}} R^{\circledast} = \varphi \dot{\backslash} y^{\mathrm{t}} \qquad\qquad R^{\circledast} \mathbin{\dot{\otimes}} (R \underset{\cdot}{\otimes} x \underset{\cdot}{\otimes} \varphi^{\circledast}) = x \dot{/} \varphi$$

$$(\tilde{y}^{\mathrm{t}} \underset{\cdot}{\otimes} R) \mathbin{\dot{\otimes}} R^{\circledast} = \tilde{y}^{\mathrm{t}} \qquad\qquad R^{\circledast} \mathbin{\dot{\otimes}} (R \underset{\cdot}{\otimes} \tilde{x}) = \tilde{x}$$

Finally, by Prop. 3 we have:

$$\tilde{y}^{\mathrm{t}} \underset{\cdot}{\otimes} (R \mathbin{\dot{\otimes}} R^{\circledast}) = \tilde{y}^{\mathrm{t}} \qquad\qquad (R^{\circledast} \mathbin{\dot{\otimes}} R) \underset{\cdot}{\otimes} \tilde{x} = \tilde{x} \tag{20}$$

For the third proposition we write $P_{\mathcal{Y}} = (P_{\mathcal{Y}^{\mathrm{t}}})^{\mathrm{t}} = (R \mathbin{\dot{\otimes}} R^{\circledast})^{\mathrm{t}} = R^{-1} \mathbin{\dot{\otimes}} R^{\mathrm{t}}$ and then transpose the whole equation for $\tilde{y}^{\mathrm{t}}$ in Eq. (20). $\square$

Note that although it is apparently a major unbalance, the eigenvalue equation for $P_{\mathcal{Y}}$ allows us to write the very balanced:

$$\begin{bmatrix} R^{-1} \dot{\otimes} R^{\mathrm{t}} & 0_{g \times m} \\ 0_{m \times g} & R^{\circledast} \dot{\otimes} R \end{bmatrix} \underset{\cdot}{\otimes} \begin{bmatrix} \tilde{y} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \tilde{y} \\ \tilde{x} \end{bmatrix} \tag{21}$$

For $\varphi^{\circledast} = \varphi^{-1}$ and $\tilde{z} = [\tilde{y}^{\mathrm{t}}\ \tilde{x}^{\mathrm{t}}]^{\mathrm{t}} = z \underset{\cdot}{\otimes} \varphi^{\circledast}$ we can write $C \underset{\cdot}{\otimes} \tilde{z} = \tilde{z}$, and call it the *extended eigenvalue equation*, which shows that the normalised formal concepts are also the fixpoint of some sort of matrix operator.

Another practical advantage of using $P_{\mathcal{Y}}$ is to be able to refer all results to column semimodules. This is what we will do hence.

Consider now $\overline{\mathcal{K}} = \overline{\mathbb{R}}_{\max,+}$. With regard to the eigenspaces of these projections we have the following proposition:

Proposition 9. *$P_{\mathcal{Y}}$ and $P_{\mathcal{X}}$ are definite matrices.*

Proof. The proof that they are matrices with their diagonals set to $e_{\mathcal{D}}$ is in Prop. 6. Now consider any of the equations in Proposition 8. These are clearly equations for the eigenvalue $\lambda = e_{\mathcal{D}}$. From Property 7.2 this eigenvalue has to be $\rho_{\max}$. □

Proposition 10. *1. $P_{\mathcal{Y}}$ and $P_{\mathcal{X}}$ are closure operators in matrix form over their respective normalised semimodules.*
2. $P_{\mathcal{Y}} = P_{\mathcal{Y}}^{*}$ and $P_{\mathcal{X}} = P_{\mathcal{X}}^{*}$.

Proof. From Props. 2 and 2, $eig(\overline{\mathcal{Y}}^{\mathrm{t}}, \rho_{\max}) = span(\left(\overline{\mathcal{Y}}^{\mathrm{t}}\right)^{*}) = span(\overline{\mathcal{X}})$. □

Then we have the following easy corollaries:

Corollary 11 (The spectral Galois connection).

1. *$P_{\mathcal{Y}^{\mathrm{t}}}$ and $P_{\mathcal{X}}$ are the closure operators in matrix form of the Galois connection $((\cdot)^{R}_{\rho_{\max}}, {}^{R}_{\rho_{\max}}(\cdot)) : \tilde{\mathcal{Y}} \rightharpoonup \tilde{\mathcal{X}}$*
2. *The subsemimodule $\overline{\tilde{\mathcal{Y}}}$ is the eigenspace $eig(P_{\mathcal{Y}}) = span(P_{\mathcal{Y}})$ and the subsemimodule $\overline{\tilde{\mathcal{X}}}$ is the eigenspace $eig(P_{\mathcal{X}}) = span(P_{\mathcal{X}})$.*

Proof. For the first subproposition, consider the polars of the generic Galois connection and rewrite: $y^{R}_{\varphi} = R^{\circledast} \dot{\otimes} (\varphi^{\circledast} \underset{\cdot}{\otimes} y)^{\circledast} = (\tilde{y}^{\mathrm{t}} \underset{\cdot}{\otimes} R) \dot{\backslash} e = \tilde{y}^{R}_{e} = \tilde{y}^{R}_{\rho_{\max}}$, and similarly n ${}^{R}_{\varphi}x = {}^{R}_{\rho_{\max}}\tilde{x}$. By ([3], Th. 42) this is a Galois connection, whose closure operators by the proof of Proposition 8 are exactly the matrices pointed to above. For the second proposition, combine Propositions 5.2 and 10.2 to get the closure lattices, $\overline{\tilde{\mathcal{Y}}}$ and $\overline{\tilde{\mathcal{X}}}$. □

This suggests that $\varphi = \rho_{\max} = e$ for both matrices is a special choice, so we give it its right status:

Definition 10. *For a set G of objects and a set M of attributes, of widths $g, m \in \mathbb{R}$ respectively, with $R \in \mathcal{M}_{g\times m}(\overline{\mathbb{R}}_{\max,+})$ building the $\overline{\mathbb{R}}_{\max,+}$-formal context $\mathbb{K} = (G, M, R)_{\overline{\mathbb{R}}_{\max,+}}$ the* spectral lattice, *$\mathfrak{B}^{\rho_{\max}}(G, M, R)_{\overline{\mathbb{R}}_{\max,+}}$ is the lattice of $\rho_{\max}$-formal concepts of the connection $((\cdot)^{R}_{\rho_{\max}}, {}^{R}_{\rho_{\max}}(\cdot)) : \tilde{\mathcal{Y}} \rightharpoonup \tilde{\mathcal{X}}$*

Indeed, this is the Galois connection depicted to the right of Figure 1. The next proposition paves the way for a more familiar representation, the structural lattice:

Proposition 12. *1. The join irreducibles $\mathcal{J}\left(\mathfrak{B}^{\rho_{\max}}(G, M, R)_{\overline{\mathbb{R}}_{\max,+}}\right)$ are the pairs (a_i, b_i) such that i ranges over the columns of $P_{\mathcal{Y}}$ and $b_i = (a_i)^{R}_{\rho_{\max}}$.*

2. The meet irreducibles $\mathcal{M}\left(\mathfrak{B}^{\rho_{\max}}(G, M, R)_{\overline{\mathbb{R}}_{\max,+}}\right)$ are the pairs (a_j, b_j) such that j ranges over the columns of $P_{\mathcal{X}}$ and $a_j = {}^{R}_{\rho_{\max}}(b_j)$.

Proof. The Galois connection between the closure lattices $\overline{\tilde{\mathcal{Y}}}$ and $\overline{\tilde{\mathcal{X}}}$ ensures that the columns of each of the projectors are the basis of the eigenspaces, that is the join-irreducibles of each lattice. The join-irreducibles of $\overline{\tilde{\mathcal{Y}}}$ generate the join-irreducibles of $\mathfrak{B}^{\rho_{\max}}(G, M, R)_{\overline{\mathbb{R}}_{\max,+}}$ by applying the polars of the Galois connection. However, because of the inversion for the second domain, the join-irreducibles of $\overline{\tilde{\mathcal{X}}}$ generate the *meet-irreducibles* of $\mathfrak{B}^{\rho_{\max}}(G, M, R)_{\overline{\mathbb{R}}_{\max,+}}$. □

Once we have both meet- and join-irreducibles it is easy to obtain the structural (concept) lattice of the spectral lattice by the procedure described in [13].

4 Application: the Analysis of Confusion Matrices

To illustrate the calculations behind the spectral lattice we retake now the problem of analysing confusion matrices. Figure 2 illustrates one such matrices with the usual hypothesis in pattern recognition, $g = m$.

For simplification's sake, consider every row and column in C to have at least one non-null entry and call D_G and D_M those diagonal matrices such that their diagonal elements are the sums of rows and columns respectively, $(D_G)_{ii} = \sum_{j=1}^{m} C_{ij}$, $(D_M)_{jj} = \sum_{i=1}^{g} C_{ij}$, $(D_G)_{ij} = (D_M)_{ij} = 0, i \neq j$. Therefore, D_G, D_M are invertible so the matrix $R = \log[(D_G)^{-1} C (D_M)^{-1}]$ is defined and has entries in $\mathbb{R} \cup \{-\infty\}$. In this case, R happens to be irreducible.

$$M = \begin{bmatrix} 5 & 3 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 11 \end{bmatrix} \qquad R = \begin{bmatrix} 3.821457e & 2.852357 & \varepsilon \\ -0.7378621 & 2.272438 & -5.856696 \\ \varepsilon & -1.249387 & 2.796319 \end{bmatrix} \cdot 10^{-01}$$

Fig. 2. The confusion matrix, M, its version as a $\mathbb{R}_{\max,+}$ matrix, R.

Now consider $P_{\mathcal{Y}}$ and $P_{\mathcal{X}}$ as per the definitions in Eq. (8). These are both definite and irreducible hence their eigenvectors are all of their columns. The structural spectral Formal Context and its Concept Lattice are shown in Fig. 3.

	dog?	cat?	rabbit?
Dog	×	×	
Cat		×	
Rabbit			×

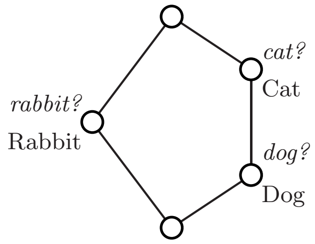

Fig. 3. Structural spectral context and Concept lattice for matrix R in Figure 2.

Interestingly, this trivial lattice already justifies the asymmetric treatment of real and recognised classifier tags. It questions Pattern Recognition approaches to confusion matrix analysis that impose a symmetrical structure on these.

As a further example we introduce the (abridged) analysis of the performance of an automatic speech recognizer for Spanish in figure (4). Its confusion matrix (and $\overline{\mathbb{R}}_{\max,+}$-Formal Context), illustrates the baseline performance for a certain type of recognition technology: most of the vowels can be adequately decoded, the less so nasals. However, approximants (soft /b/,/d/,/g/ in vocalic context) cannot be told apart in the spectral lattice at all, a weakness of this recognizer.

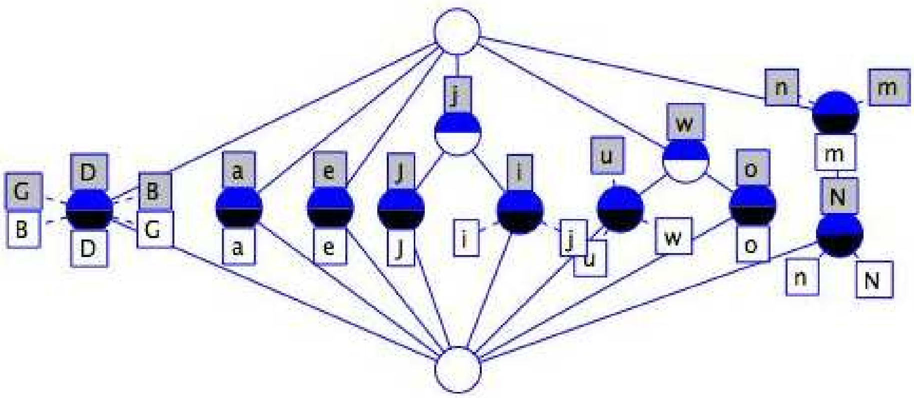

Fig. 4. Confusion matrix for an automatic speech recognizer for Spanish. Objects: real phonemes; attributes: recognised phonemes (SAMPA). /J/ stands for the phoneme of letter “ñ”.

5 Conclusion

In this paper, we have tried to justify the importance of a particular value of the exploration parameter φ to obtain structural lattices [15] for the concrete case of $\overline{\mathbb{R}}_{\max,+}$-Formal Concept Analysis, viz, the case where we consider it to be an eigenvalue of the projectors onto the closure lattices in the Galois connection. The latter can be readily obtained as the left and right residuals of the $\overline{\mathbb{R}}_{\max,+}$-valued incidence in the completed semiring, which makes the spectral estimation a very light process computationally speaking.

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