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New Approach to Stochastic Optimal Control and Applications to Economics[1]

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Abstract

This paper provides new insights into the solution of optimal stochastic control problems by means of a system of partial differential equations, which characterize directly the optimal control. This new system is obtained by the application of the stochastic maximum principle at every initial condition, assuming that the optimal controls are smooth enough. The type of problems considered are those where the diffusion coefficient is independent of the control variables, which are supposed to be interior to the control region. The results obtained are applied to the study of the classical consumption–savings model.

Keywords: Optimal stochastic control, Itô's formula, Hamilton–Jacobi–Bellman equation, semilinear parabolic equation, consumption–savings model.

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1 Introduction

Three major approaches in stochastic optimal control can be differentiated: dynamic programming, duality and the maximum principle.

Dynamic programming obtains, by means of the optimality principle of Bellman, the Hamilton–Jacobi–Bellman equation, which characterizes the value function; see Refs. 1–5. Under some smoothness and regularity assumptions on the solution, it is possible to obtain, at least implicitly, the optimal control. This is the content of the so called *verification theorems* which appear in Fleming and Rishel (Ref. 1) or Fleming and Soner (Ref. 3). However, the problem of recovering the optimal control from the gradient of the value function by means of solving a static optimization problem remains, and this can be difficult to do.

Duality methods, also known in stochastic control theory as the Martingale approach, have become very popular in recent years because they provide powerful tools for the study of some classes of stochastic control problems. Martingale methods are particularly useful for problems appearing in finance, such as the model of Merton (Ref. 6). Duality reduces the original problem to one of finite dimension. The approach is based on the martingale representation theorem and the Girsanov transformation. We refer the reader to Bismut (Ref. 7), Bismut (Ref. 8) and the monograph by Duffie (Ref. 9) for an account of the theory and the references therein.

The stochastic maximum principle has been completely developed in recent years in Peng (Ref. 10) and Yong and Zhou (Ref. 5). It is the counterpart of the maximum principle for deterministic problems. The distinctive feature is the use of the concept of forward–backward stochastic differential equations, which naturally arise governing the evolution of the state variables and the corresponding adjoint variables. Antecedents of the maximum principle are found in Kushner (Ref. 11), Bismut[4] (Ref. 7) or Haussmann (Ref. 12). Other developments, applicable to problems with differential equations with random coefficients can be found in Marti (Ref. 13).

It is the aim of this paper to develop a new approach to stochastic control. The novelty comes from the fact that we obtain a system of PDEs that a smooth Markov

[4]The maximum principle, duality methods and the concept of forward–backward stochastic differential equations have its roots in the work of Bismut, who gave a very complete and rigorous theory regarding these topics.

control must satisfy and that also provides sufficient condition for optimality, in the spirit of the verification theorems. Although the system is obtained using classical methods—the maximum principle applied to every initial condition—the authors have not found any reference in the literature to the possibility of establishing a system of PDEs to characterize the optimal control directly. The equations of this new system are of a different type than the HJB. In the case considered in this paper, where the control does not affect the diffusion coefficient in the state equation, both the HJB and the equations of the new system are semilinear. There is an important difference, however, because the nonlinearities in the first order derivatives in the former equation can be very general, whereas in the latter they are always of quadratic type. This fact can be used to establish the existence and uniqueness of smooth optimal Markov controls as it will be shown in Section 5.

Our approach has the following limitations:

(i) we consider only problems where the diffusion coefficient is independent of the control variables;

(ii) the optimal control is interior to the control region;

(iii) controls are Markovian and of class $\mathcal{C}^{1,2}$;

(iv) the number of control variables is greater than or equal to the number of state variables.

It is worth noting that many control problems share these properties, specially some important models arising in Economics.

The idea to systematically obtain a system of PDEs for the optimal control date back to the paper by Bourdache–Siguerdidjane and Fliess (Ref. 14) for deterministic control problems. The method was later extended to differential games in Rincón–Zapatero et al. (Ref. 15) and Rincón–Zapatero (Ref. 16).

The paper is organized as follows. In Section 2 we present the control problem and the first hypotheses and notations. In Section 3 we find a system of partial differential equations that a vector of optimal controls of class $\mathcal{C}^{1,2}$ must satisfy. Section 4 is devoted to establishing sufficient conditions to guarantee that a vector of admissible controls satisfying the system is an optimal control of the problem. Hence sections 3 and 4 respectively, provide necessary and sufficient conditions for optimality. In Section 5 we present an application of our results to a classical consumption model. Concluding remarks are stated in Section 6.

2 Control Problem

In this section the framework for the stochastic control problem to be considered is presented. First we shall introduce some useful notation. The partial derivatives are indicated by subscripts and ∂_x stands for *total derivation*; the partial derivative of a scalar function with respect to a vector is a column vector; given a real vector function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a vector $z \in \mathbb{R}^n$, g_z is defined as the matrix $(\partial g^i / \partial z^j)_{i,j}$; for a matrix A, $A^{(i)}$ denotes the ith column and A^{ij} denotes the (i,j) element; vectors $v \in \mathbb{R}^n$ are column vectors and v^i is the ith component; finally, $^\top$ denotes transposition.

Let a time interval $[0,T]$ with $0 < T \leq \infty$ and let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. Assume that on this space a d–dimensional Brownian motion $\{w(t), \mathcal{F}_t\}_{t\in[0,T]}$ is defined with $\{\mathcal{F}_t\}_{t\in[0,T]}$ being the Brownian filtration. Let E denote expectation under the probability measure P.

The state space is $\mathbb{R}^n$ and the control region is some subset $U \subseteq \mathbb{R}^m$, with $m \geq n$. This assumption will be explained later, in Remark 3.3. A U–valued *control process* $\{(u(s), \mathcal{F}_s)\}$ defined on $[t,T] \times \Omega$ is an $\mathcal{F}_s$–progressively measurable map $(r,\omega) \to u(r,\omega)$ from $[t,s] \times \Omega$ into U, that is, $u(t,\omega)$ is $\mathcal{B}_s \times \mathcal{F}_s$–measurable for each $s \in [t,T]$, where $\mathcal{B}_s$ denotes the Borel σ–field in $[t,s]$. For simplicity, we will denote $u(t)$ to $u(t,\omega)$.

The state process $\xi \in \mathbb{R}^n$ obeys the system of controlled stochastic differential equations of the form

$$d\xi(s) = f(s, \xi(s), u(s))\, ds + \sigma(s, \xi(s))\, dw(s), \quad s \geq t, \tag{1}$$

with initial condition $\xi(t) = x$. ξ^u will sometimes be used to indicate the dependence of the state variable with respect to the control u. An important feature of the above system is that the noise coefficient, σ, is independent of the control variable, u. Here $\sigma = (\sigma_{ij})$ is an $n \times d$ matrix.

Definition 2.1 (Admissible Control). A control $\{(u(t), \mathcal{F}_t)\}_{t\in[0,T]}$ is called admissible if

(i) for every (t, x) the system of SDEs (1) with initial condition $\xi(t) = x$ admits a pathwise unique strong solution;

(ii) there exists some function $\phi : [0, T] \times \mathbb{R}^n \longrightarrow U$ of class $\mathcal{C}^{1,2}$ such that u is in relative feedback to ϕ, i.e. $u(s) = \phi(s, \xi(s))$ for every $s \in [0, T]$.

Let $\mathcal{U}(t, x)$ denote the set of admissible controls corresponding to the initial condition $(t, x) \in [0, T] \times \mathbb{R}^n$.

According to the definition, we are considering Markovian controls. If ϕ is time independent, the corresponding control will be called a stationary Markov control. u and ϕ will sometimes be identified in the notation.

Given initial data $(t, x) \in [0, T] \times \mathbb{R}^n$, the criterion to be maximized is

$$J(t, x; u) = \mathrm{E}_{tx} \left\{ \int_t^T L(s, \xi(s), u(s))\, ds + S(T, \xi(T)) \right\}, \tag{2}$$

where E_{tx} denotes conditional expectation with respect to the initial condition (t, x). In the following, the subscript will be eliminated if there is no confusion. The functions $f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times d}$, $L : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$, $S : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, are all assumed to be of class $\mathcal{C}^2$ with respect to (x, u) and of class $\mathcal{C}^1$ with respect to t. The assumptions established so far will be assumed to hold throughout the paper. Given that our aim is to solve the problem *for every* $(t, x) \in [0, T] \times \mathbb{R}^n$, $\mathcal{U}$ will often be written instead of $\mathcal{U}(t, x)$.

In the specification of the problem we have supposed $m \geq n$, that is, the dimension of the control variable is *greater than or equal to* the dimension of the state variable. This is a crucial assumption for the following developments. However, for the sake of simplicity, the case $m = n$ will be considered first and then we will show in Remark 3.3 that the case $m > n$ can be reduced to the equality situation.

With a view to applying the stochastic maximum principle as it is stated in Yong and Zhou (Ref. 5), an additional assumption will be imposed.

(A1) There exists a constant $C > 0$ and a modulus of continuity $\bar{\omega} : [0, \infty) \rightarrow [0, \infty)$

such that for $\psi = f,\ \sigma,\ L,\ S$, we have

$$\begin{aligned}
|\psi(t,x,u) - \psi(t,\widehat{x},\widehat{u})| &\le C\|x-\widehat{x}\| + \bar{\omega}(\|u-\widehat{u}\|),\\
|\psi_x(t,x,u) - \psi_x(t,\widehat{x},\widehat{u})| &\le C\|x-\widehat{x}\| + \bar{\omega}(\|u-\widehat{u}\|),\\
|\psi_{xx}(t,x,u) - \psi_{xx}(t,\widehat{x},\widehat{u})| &\le \bar{\omega}(\|x-\widehat{x}\| + \|u-\widehat{u}\|),\\
&\qquad \forall t \in [0,T],\ x,\widehat{x} \in \mathbb{R}^n,\ u,\widehat{u} \in U,\\
|\psi(t,0,u)| &\le C, \qquad \forall (t,u) \in [0,T]\times U.
\end{aligned}$$

Consider a control satisfying property (ii) of Definition 2.1. Then the Lipschitz and linear growth conditions on f and σ postulated in **(A1)** imply that the control also satisfies (i), that is, it is admissible; see Yong and Zhou (Ref. 5), p. 114. However, the assumptions are quite stringent and will only be used in the derivation of the quasilinear system as a necessary condition for optimality. Sufficiency conditions, which will be established in Section 4, do not make use of hypothesis **(A1)**.

The backward evolution operator associated with (1) is given by

$$\mathcal{A}^\phi W(t,x) = W_t(t,x) + W_x^\top(t,x) f(t,x,\phi(t,x)) + (1/2)\,\mathrm{Tr}\{(\sigma\sigma^\top W_{xx})(t,x)\},$$

with $W : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^{1,2}$ and where

$$\mathrm{Tr}\{\sigma\sigma^\top W_{xx}\} := \left(\mathrm{Tr}\{\sigma\sigma^\top W^1_{xx}\},\ldots,\mathrm{Tr}\{\sigma\sigma^\top W^n_{xx}\}\right)^\top.$$

The *value function* is defined as $V(t,x) = \sup_{u\in\mathcal{U}(t,x)} J(t,x;u)$. An admissible control $\widehat{u} \in \mathcal{U}$ is *optimal* if $V(t,x) = J(t,x;\widehat{u})$ for every initial condition (t,x).

The standard approach adopted in the literature to determine an optimal control is to solve the HJB equation

$$V_t(s,x) + \max_{u\in U}\left\{L(s,x,u) + V_x(s,x)^\top f(s,x,u) + (1/2)\,\mathrm{Tr}\left\{(\sigma\sigma^\top V_{xx})(s,x)\right\}\right\} = 0, \tag{3}$$

$$V(T,x) = S(T,x), \quad t \le s \le T, \quad x \in \mathbb{R}^n. \tag{4}$$

3 Necessary Conditions

Our purpose in this section is to find a system of PDEs that an optimal control must satisfy. Let $L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$ be the set of all processes $X(\cdot)$ with values in $\mathbb{R}^n$ adapted to filtration $\{\mathcal{F}_t\}_{t\ge 0}$ such that $\mathrm{E}\int_0^T \|X(t)\|^2\,dt < \infty$. As previously stated, hypothesis

(A1) allows us to apply the stochastic maximum principle, so that, if given the initial condition (t,x), the pair (ξ, u) is optimal, with $u \in \mathcal{U}(t,x)$, then there exist processes $p \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$, $q \in \left(L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)\right)^d$ satisfying for $s \in [t,T]$ the first order adjoint equations

$$dp(s) = -\Big(H_x(s,\xi(s),\phi(s,\xi(s)),p(s)) + \sum_{i=1}^{d} \sigma_x^{(i)}(s,\xi(s))^\top q^{(i)}(s)\Big)ds + q(s)dw(s), \tag{5}$$

$$p(T) = S_x(T,\xi(T)), \tag{6}$$

where $H(t,x,u,p) = L(t,x,u) + p^\top f(t,x,u)$ is the deterministic Hamiltonian function, corresponding to the associated deterministic problem, with $\sigma \equiv 0$. A more precise notation for the adjoint processes is $p(s;\,t,x)$ and $q(s;\,t,x)$ with $s \in [t,T]$, though in the following, we will suppress the dependence with respect to the initial condition (t,x).

Furthermore, the following maximization condition

$$H(s,\xi(s),\phi(s,\xi(s)),p(s)) = \max_{u\in U} H(s,\xi(s),u,p(s)) \tag{7}$$

holds for every $s \in [t,T]$, P a.s.

For the next result, which establishes a necessary condition of optimality in terms of a new system of PDEs, we define

$$\Gamma(t,x,u) := -f_u^{-\top} L_u(t,x,u), \tag{8}$$

and a "diagonal" matrix of suitable dimensions $\Sigma^\top := \operatorname{diag}(\sigma^\top \quad \sigma^\top \partial_x \Gamma \quad (\phi_x \sigma)^\top)$. $\widehat{H}_{\{\cdot\}}$ will denote $H_{\{\cdot\}}$ once (8) is substituted into it.

Theorem 3.1 (Necessary Conditions). Let assumption **(A1)** on the coefficient functions be satisfied. Let $\phi \in \mathcal{U}$ be an interior optimal Markov control such that $\det f_u(t,x,\phi) \neq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Then ϕ satisfies

$$0 = \widehat{H}_{ut} + \widehat{H}_{ux}^\top f + \widehat{H}_{up}^\top \Big(- \widehat{H}_x - \sum_{i=1}^{m} \sigma_x^{(i)} \big(\partial_x \Gamma\, \sigma\big)^{(i)} \Big) + \widehat{H}_{uu} \mathcal{A}^\phi \phi + \frac{1}{2} \operatorname{Tr}\{\Sigma\Sigma^\top \nabla^2 \widehat{H}_u\} \tag{9}$$

and the final condition

$$L_{u^i}(T,x,\phi(T,x)) + S_x(T,x)^\top f_{u^i}(T,x,\phi(T,x)) = 0, \quad i = 1,\dots,n. \tag{10}$$

Proof. Since that by assumption the maximizing argument is interior to U, (7) implies

$$H_{u^i}(s,\xi(s),\phi(s,\xi(s)),p(s)) = 0, \quad \forall s \in [t,T], \quad \text{P a.s.}, \tag{11}$$

for all $i = 1,\dots,n$. Assuming that f_u is invertible for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$, it is possible to obtain the unique solution of the above *linear system* in the adjoint variable p, $L_u + f_u^\top p = 0$, as

$$p = -f_u^{-\top} L_u. \tag{12}$$

An obvious consequence of (11) is $dH_{u^i}(s,\xi(s),u(s),p(s)) = 0$ a.s. For an admissible feedback ϕ, Itô's rule is applicable to $u(s) = \phi(s,\xi(s))$, hence omitting the arguments and in differential notation

$$du^i = d\phi^i = \phi^i_t dt + \phi^i_x d\xi + (1/2) d\xi^\top \phi^i_{xx} d\xi, \quad i = 1,\dots,n. \tag{13}$$

Applying again Itô's rule to H_{u^i} for $i = 1,\dots,n$, we have

$$0 = dH_{u^i} = H_{u^i t}\, dt + \nabla H_{u^i} \begin{pmatrix} d\xi \\ dp \\ d\phi \end{pmatrix} + \frac{1}{2} (d\xi^\top \; dp^\top \; d\phi^\top)\, \nabla^2 H_{u^i} \begin{pmatrix} d\xi \\ dp \\ d\phi \end{pmatrix}, \tag{14}$$

where ∇ and ∇^2 denote the gradient and the Hessian matrix operators respectively, with respect to the variables (x,p,u). Substituting (1), (5), (13) in the equality (14) and taking into account that $H_{u^i pp} = 0$ for all i, because the Hamiltonian is linear in p, the following system of stochastic differential equations holds along $(s,\xi(s),u(s),p(s))$ a.s.:

$$\begin{aligned} 0 = &\Big(H_{u^i t} + H_{u^i x}^\top f + H_{u^i p}^\top \big(- H_x - \sum_{i=1}^m \sigma_x^{(i)} q^{(i)} \big) + H_{u^i u}^\top \mathcal{A}^\phi \phi + \frac{1}{2} \operatorname{Tr}\{\Sigma\Sigma^\top \nabla^2 \widehat{H}_u\} \Big)\, ds \\ &+ \big(H_{u^i x}^\top \sigma + H_{u^i p}^\top q + H_{u^i u}^\top \phi_x \sigma \big)\, dw(s). \end{aligned} \tag{15}$$

Therefore, both the drift term and the diffusion coefficient of this system of SDEs must be identically null a.s. In order to obtain a system of PDEs for the optimal control, we must eliminate the adjoint vector p by means of (12). Equating the diffusion coefficient to zero we get $q = -f_u^{-\top}(\widehat{H}_{ux}\sigma + \widehat{H}_{uu}\phi_x\sigma)$ a.s., $s \geq t$. In fact, q can be expressed as

$$q = (\partial_x \Gamma)\, \sigma, \tag{16}$$

The possibility to write q as shown in (16) follows from the identities $H_{u^i p} = f_{u^i}$, $\Gamma_x = -f_u^{-\top} \widehat{H}_{ux}$ and $\Gamma_u = -f_u^{-\top} \widehat{H}_{uu}$. The drift term in (15) also vanishes a.e., hence after substitution of (12) and (16), the system of PDEs (9) characterizing an admissible optimal control is obtained. Note that (9) and (16) are valid a.s. along the optimal trajectory, but at $(t, \xi(t)) = (t, x)$, (9) holds with certainty.

The stochastic maximum principle also provides a boundary condition at time T for the system of PDEs, which is implicitly given by (10). This follows from (6) and (12) evaluated at $t = T$; we will suppose that it is possible to obtain $\phi(T, x) := \varphi(x)$ for a function φ sufficiently regular. For this is enough to check if the hypotheses of the Implicit Function Theorem are fulfilled. □

Some comments about the structure of the system and comparison with the HJB equation (3) are pertinent here. The system is semilinear because the terms involving the second order derivatives of ϕ are independent of the solution. Furthermore, assuming the invertibility of $\widehat{H}_{uu}$, the system is weakly coupled, that is, the second order derivatives of ϕ^i appear only in equation i. The first order derivatives are coupled and appear in a non–linear way derived from the quadratic–type terms

$$\mathrm{Tr}\{\phi_x \sigma (\phi_x \sigma)^\top \widehat{H}_{uuu}\} \qquad \text{and} \qquad \mathrm{Tr}\{\phi_x \sigma (\phi_x \sigma)^\top \widehat{H}_{upu}(-f_u^{-\top} \widehat{H}_{uu})\}.$$

This is a very interesting feature that will be used to show existence of an optimal policy function in the model explored in Section 5. Whereas the HJB equation is also of semilinear type, the non linearity with respect to ϕ_x can be much more general and not only of quadratic type. On the other hand, it must be pointed out that the HJB equation is a *single* equation, whereas we have obtained a *system* of n PDEs, but with a more simple structure.

Remark 3.1 It would be possible to replace the smoothness assumption on ϕ for a weaker one. Given that all that is needed is to apply Itô's rule, Theorem 3.1 is true if the class of Markov controls is $(W^{1,2}_{l,\,\mathrm{loc}}([0, T] \times \mathbb{R}^n))^n$, $l \geq 2$, the space of functions such that the weak partial derivatives of order 1 with respect to time and order 2 with respect to x are in $(L^l_{\mathrm{loc}}([0, T] \times \mathbb{R}^n))^n$; see Krylov (Ref. 2). Note that the hypotheses imposed imply that H_u belongs to $W^{1,2}_{l,\,\mathrm{loc}}([0, T] \times \mathbb{R}^n)$.

Remark 3.2 There is in the literature a different but closely related system of PDEs which characterize the vector of adjoint variables under some regularity assumptions, see equation (17) below. This system was obtained for the first time in Bismut (Ref. 7) and later in[5] Elliot (Ref. 17). It is important to note that the system below depends also on the optimal control and for this reason it appears with a simple structure. To obtain the system for the adjoint variables we can proceed as follows. Let us suppose the existence of a vector function γ of class $\mathcal{C}^{1,2}$ depending of the variables (s, y) and such that $p(s) = \gamma(s, \xi(s))$, where p is the adjoint variable of the problem with initial condition (t, x). Applying Itô's rule to $\gamma(s, \xi(s))$ we have

$$d\gamma = \big(\gamma_t + \gamma_x f + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top \gamma_{xx}\}\big)\, ds + \gamma_x \sigma\, dw. \tag{17}$$

Once the validity of the maximum principle is established, by the uniqueness of solutions of (5) we can match the diffusion terms and drift terms in expressions (5) and (17), to obtain $q = \gamma_x \sigma$ and

$$-\Big(H_x + \sum_{i=1}^{n} (\sigma_x^{(i)})^\top q^{(i)}\Big) = \gamma_t + \gamma_x f + \frac{1}{2}\operatorname{Tr}\left\{\sigma\sigma^\top \gamma_{xx}\right\}.$$

Of course, the first equality is nothing but (16). Therefore we find that the second identity is transformed into

$$\gamma_t + \gamma_x f + H_x + \sum_{i=1}^{m} (\sigma_x^{(i)})^\top \gamma_x \sigma^{(i)} + \frac{1}{2}\operatorname{Tr}\left\{\sigma\sigma^\top \gamma_{xx}\right\} = 0. \tag{18}$$

For the derivation of this identity, the *equal dimension* condition between the state and control variables is not needed. Furthermore, the equality $q = \gamma_x \sigma$ allows situations to be handled where the diffusion parameter σ also depends on the control variables, u. In this case the elimination of optimal control variables is not so straightforward. As already observed, the system (18) also depends on the unknown vector of optimal controls. Supposing it is possible to obtain a sufficiently regular function $\widetilde{u}(t, x, z)$ such that $\widehat{\phi}(t, x) = \widetilde{u}(t, x, \gamma(t, x))$, that is, $\widetilde{u}$ is the inverse function of Γ with respect to its third component for all (t, x), then by substituting in the previous system of equations, we obtain a system of PDEs that truly characterize the vector of adjoint variables. However, writing the system for the optimal control does not require the inverse function $\widetilde{u}$ to be found, which can be hard or impossible to do, even in scalar problems. Under the conditions contemplated in this paper it is only necessary to solve the *linear* system (12) to obtain $\gamma(t, x) = \Gamma(t, x, \phi(t, x))$ and by substituting in (18), to arrive at the desired PDE system for ϕ. Of course, this is simply system (9).

[5] In Elliot (Ref. 17) a misprint is registered making the equation shown different to that appearing in Bismut (Ref. 7) and in (18) of the present paper.

Remark 3.3 Case $m > n$. When the number of control variables is greater than the number of state variables, $m > n$, the linear system in p is over–determinate. Because the maximum principle holds, the existence of a solution to the over–determined system is assured. This solution can be obtained as follows. Suppose that f_u has rank n for all t, x, u; then $f_u^\top f_u$ has full rank n, hence from (11) $p = -(f_u f_u^\top)^{-1} f_u L_u$. Now the argument runs as shown above, obtaining a system of n PDEs for $m > n$ unknowns. These equations can be complemented with an algebraic relationship between the controls, which is obtained from the fact that the system $H_u = 0$ admits a solution in p. In this way, $m - n$ control variables can be formally expressed by means of n of them.

In the case $m < n$, elimination of p is not so straightforward. Now, the procedure to obtain a system of PDEs for the control would be to take $n - m + 1$ Itô differentials in the identity $H_u \equiv 0$. This leads to PDE equations for the optimal control of higher order, and of a very different nature than (9), hence this case will not be pursued in this paper.

4 Sufficient Conditions

The main objective in this section is to show that a solution of class $\mathcal{C}^{1,2}$ of (9)–(10), maximizing the Hamiltonian function for all (t, x) and satisfying some additional assumptions, is an optimal Markov control for problem (1)–(2). This result is, therefore, similar to the *verification theorems* in Fleming and Rishel (Ref. 1) or Fleming and Soner (Ref. 3).

The process ξ depends on the initial condition (t, x). In the following, $\xi^j_{x^i}$ will denote the partial derivative of ξ^j with respect to x^i.

We consider the following assumption:

(A2) $\mathrm{E}\left\{ \int_t^T \left(\gamma^j \sigma^{(j)}_{x^l} + q^{(j)}\right) \xi^j_{x^i}\, dw(s) \right\} = 0,$ *for every* $i, j, l = 1, \ldots, n.$

Remark 4.1 As is well known

$$\mathrm{E}\left\{ \int_t^T |\gamma^j \sigma^{jk}_{x^l} + q^{jk}|^2 |\xi^j_{x^i}|^2\, ds \right\} < \infty \qquad \forall i, j, l = 1, \ldots, n, \quad \forall k = 1, \ldots, d \tag{19}$$

implies **(A2)**. Another form to express (19) is writing $\partial_x(\gamma^\top \sigma)(s, \xi(s)) \in (L^2_{\mathcal{F}}([t, T]; \mathbb{R}^n)^d$. The following assumptions on the coefficient functions and on the Markov control $\widehat{\phi}$ guarantee the fulfillment of (19).

(i) For all $i, j, l = 1, \dots, n$,

$$|f^j_{x^l}(t,x,u)| + |f^j_{u^l}(t,x,u)| + |\sigma^{ij}_{x^l}(t,x)| \le C,$$
$$|f^j(t,x,u)| + |\sigma^{ij}(t,x)| \le C(1 + \|x\|),$$

for some constant $C \in \mathbb{R}$;

(ii) $\widehat{\phi}$ is admissible and $\mathrm{E} \int_t^T \|\widehat{\phi}(s, \xi(s))\|^l \, ds < \infty$ for all $l \in \mathbb{N}$.

(iii) For every $j, l = 1, \dots, n$

$$|\gamma^j_{x^l}(t,x)\sigma^{lk}(t,x)| + |\gamma^j(t,x)| \le C(1 + \|x\|^\kappa),$$

for some constants $C \in \mathbb{R}$, $\kappa \in \mathbb{N}$.

Standard estimates in the theory of SDEs show that (i) and (ii) imply $\mathrm{E}\{\|\xi\|^l\} < \infty$ for all $l \in \mathbb{N}$. Hypothesis (iii) then assures

$$\mathrm{E}\left\{\int_t^T |q^{jk}|^2 \, ds\right\} \le \sum_{l=1}^n \mathrm{E}\left\{\int_t^T |\gamma^j_{x^l}\sigma^{lk}|^2 \, ds\right\} \le C\mathrm{E}\left\{\int_t^T (1 + \|\xi\|^\kappa)^2 \, ds\right\} < \infty,$$

where we have used $q = \gamma_x \sigma$. On the other hand, (ii) and the boundedness of the functions $f^j_{x^l}$, $\sigma^{jk}_{x^l}$ give $\mathrm{E}\{\|\xi^j_{x^i}\|^l\} < \infty$. Combining all the above estimates (19) obviously holds.

The following result establishes that the adjoint process $p(s) = \gamma(s, \xi(s))$ is the gradient with respect to x of the objective functional. This result, of independent interest, is a previous step in the formulation of the sufficiency theorem that will be stated later.

Recall from the previous section that Γ denotes $f_u^{-\top} L_u$ and $\gamma(t,x)$ denotes $\Gamma(t, x, \widehat{\phi}(t,x))$, where $\widehat{\phi}$ is an admissible Markov control solving the semilinear system.

Proposition 4.1 (Shadow Price). Let $\widehat{\phi} \in \mathcal{U}$ be a solution of (9)–(10) such that assumption **(A2)** is satisfied and $f_u(t, x, \widehat{\phi})$ is invertible for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Then

$$J_x(t,x;\widehat{\phi}) = \Gamma(t,x,\widehat{\phi}(t,x)) = p(t),$$
$$J_{xx}(t,x;\widehat{\phi})\sigma(t,x) = q(t),$$

for every $(t,x) \in [0,T] \times \mathbb{R}^n$.

Proof. Applying Itô's rule to $p(s) = \Gamma(s, \xi(s), \widehat{\phi}(s, \xi(s)))$, $t \le s \le T$, we obtain

$$dp(s) = \big(\Gamma_t + \Gamma_x f + \Gamma_u \mathcal{A}^{\widehat{\phi}}\widehat{\phi} + \frac{1}{2}\operatorname{Tr}\left\{\sigma\sigma^\top \Gamma_{xx} + 2\widehat{\phi}_x \sigma\sigma^\top \Gamma_{xu} + \widehat{\phi}_x \sigma(\widehat{\phi}_x \sigma)^\top \Gamma_{uu}\right\}\big)\, ds$$
$$+ (\Gamma_x + \Gamma_u \widehat{\phi}_x)\sigma\, dw(s),$$

where the arguments have been eliminated to simplify the notation. Taking into account the identities

$$f_u^\top \Gamma_\eta = -\widehat{H}_{u\eta}, \quad f_u^\top \Gamma_{\eta\nu} = f_{u\nu}^\top f_u^{-\top} \widehat{H}_{u\eta} + f_{u\eta}^\top f_u^{-\top} \widehat{H}_{u\nu} - \widehat{H}_{u\eta\nu},$$

that hold for every $\eta, \nu \in \{u^1, \dots, u^n, x^1, \dots, x^n\}$, and using that $\widehat{\phi}$ solves (9), it is easy to check that p satisfies (5) if we choose $q = (\partial_x \Gamma)\sigma$ as in (16). Hence $q^{jk} = \sum_{l=1}^n \gamma^j_{x^l} \sigma^{lk}$.

Every $\xi^j_{x^i}$ satisfies the linear system of stochastic differential equations

$$d\xi^j_{x^i} = \sum_{l=1}^n f^j_{x^l} \xi^l_{x^i}\, ds + \sum_{k=1}^d \sum_{l=1}^n \sigma^{jk}_{x^l} \xi^l_{x^i} dw^k, \tag{20}$$

with $\xi^j_{x^i}(t) = \delta_{ij}$, see Gihman and Skorohod (Ref. 18), with δ_{ij} denoting Kronecker's delta. The product $\xi^j_{x^i} p^j$ satisfies the following stochastic differential equation

$$d(\xi^j_{x^i} p^j) = p^j d\xi^j_{x^i} + \xi^j_{x^i} dp^j + A^{ij}\, ds, \tag{21}$$

where $A^{ij} := \sum_{k=1}^d \sum_{l=1}^n \sigma^{jk}_{x^l} \xi^l_{x^i} q^{jk}$. Now by means of a simple calculation using (5), (20), (21) and (18) the following equality holds

$$\sum_{j=1}^n d(\xi^j_{x^i} p^j) = -\sum_{j=1}^n L_{x^j} \xi^j_{x^i}\, ds + \sum_{k=1}^d B^{ik}\, dw^k, \tag{22}$$

with $B^{ik} := \sum_{j=1}^n \sum_{l=1}^n (\sigma^{jk}_{x^l} \xi^j_{x^i} p^j + q^{jk} \xi^j_{x^i})$. Taking conditional expectations in (22) and considering hypothesis **(B)**, we obtain

$$\sum_{j=1}^n \mathrm{E}\left\{\xi^j_{x^i}(T) p^j(T)\right\} = \sum_{j=1}^n \mathrm{E}\left\{\xi^j_{x^i}(t) p^j(t)\right\} - \mathrm{E}\left\{\int_t^T \sum_{j=1}^n L_{x^j}\big(s, \xi(s), \widehat{\phi}(s, \xi(s))\big) \xi^j_{x^i}(s)\, ds\right\}. \tag{23}$$

Obviously, $\sum_{j=1}^n \mathrm{E}\left\{\xi^j_{x^i}(t) p^j(t)\right\} = p^i(t)$ and because $\widehat{\phi}$ verifies the final condition (10), the equality

$$\sum_{j=1}^n \mathrm{E}\left\{\xi^j_{x^i}(T) p^j(T)\right\} = \sum_{j=1}^n \mathrm{E}\left\{S_{x^j}(T, \xi(T)) \xi^j_{x^i}(T)\right\}$$

holds. The following step is to interchange the order of integration and derivation and also the expectation operator in (23) to obtain

$$\begin{aligned}\Gamma^i(t,x,\widehat{\phi}(t,x)) &= \Gamma^i(t,\xi(t),\widehat{\phi}(t,\xi(t))) \\ &= p^i(t) \\ &= \frac{\partial}{\partial x^i}\mathrm{E}\left\{\int_t^T L\big(s,\xi(s),\widehat{\phi}(s,\xi(s))\big)\,ds + S(T,\xi(T))\right\} \\ &= J_{x^i}(t,x;\widehat{\phi}),\end{aligned}$$

for all $i = 1, \ldots, n$.

Finally, note that $J_{xx}(t,x;\widehat{\phi})\sigma(t,x) = \partial_x\Gamma(t,x,\widehat{\phi}(t,x))\,\sigma(t,x) = q(t)$. □

Once we have identified the vector of adjoint variables with the gradient of the objective functional, the system (9) can be expressed in *conservative form*. Since γ is the gradient with respect to the variable x of the function $J(t,x;\widehat{\phi})$, which is of class $\mathcal{C}^3$, $\gamma^i_{x^j} = \gamma^j_{x^i}$ is satisfied for every $i,j = 1,\ldots,n$, because the crossed second order partial derivatives of the function J coincide. By the same argument, $\gamma^i_{x^r x^j} = \gamma^r_{x^j x^i}$ for all $i,j,r = 1,\ldots,n$. On the other hand, after some tedious calculations, we find

$$\big(\mathrm{Tr}\{\sigma\sigma^\top\gamma_x\}\big)_{x^r} = \mathrm{Tr}\{\sigma\sigma^\top\gamma^r_{xx}\} + 2\sum_{i=1}^m(\sigma^{(i)}_{x^r})^\top\Big(\sum_{j=1}^n\gamma^j_x\sigma^{ji}\Big)$$

and substituting this expression in (18), we obtain

$$\gamma_t + \partial_x\big(L + \gamma^\top f + \frac{1}{2}\,\mathrm{Tr}\{\sigma\sigma^\top\gamma_x\}\big) = 0, \tag{24}$$

where the fact that $H_u = 0$ holds at the optimal control has been used. It is interesting to compare the structure of (24) which is expressed in conservative form, with that of (18), which appears in non conservative form.

In terms of $\Gamma(t,x,\phi)$ (24) can be rewritten as

$$\partial_t\Gamma(t,x,\phi(t,x)) + \partial_x\Big(\mathcal{H}(t,x,\phi(t,x)) + \frac{1}{2}\,\mathrm{Tr}\left\{\sigma(t,x)\sigma(t,x)^\top\partial_x\Gamma(t,x,\phi(t,x))\right\}\Big) = 0, \tag{25}$$

with $\mathcal{H}(t,x,u) := H(t,x,u,\Gamma(t,x,u))$.

Taking total derivatives, a system of partial differential equations of second order arise, which is the same as (9). Expressing the system in conservative form is useful, because it allows us in the next theorem to establish a sufficient result of optimality. It also makes possible to obtain the value function from the control, as will be shown in the following section.

Now we are in position to establish the following sufficient condition for optimality.

Theorem 4.1 (Verification Theorem). Let $\widehat{\phi} \in \mathcal{U}$ be a solution of (9)–(10) such that assumption **(A2)** is satisfied and $f_u(t, x, \widehat{\phi})$ is invertible for all $(t, x) \in [0, T] \times \mathbb{R}^n$. Suppose further that the following maximization property holds for all $(t, x) \in [0, T] \times \mathbb{R}^n$, for every admissible Markov control u,

$$H(t, x, \widehat{\phi}, \Gamma(t, x, \widehat{\phi})) \geq H(t, x, u, \Gamma(t, x, \widehat{\phi})). \tag{26}$$

Then $\widehat{\phi}$ is an optimal Markov control for problem (1)–(2).

Proof. Let u be any admissible Markov control and ξ^u the associated process with initial condition (t, x). We will omit the dependence of ξ^u on the initial condition in order to facilitate the exposition. Let $u(s)$ be $u(s, \xi^u(s))$. Applying Itô's rule to $J(s, \xi^u(s); u)$, $s \geq t$. We have

$$dJ(s, \xi^u(s); u) = \mathcal{A}^u J(s, \xi^u(s); u)\, ds + J_x(s, \xi^u(s); u)^\top \sigma(s, \xi^u(s))\, dw(s). \tag{27}$$

On the other hand, as shown in Yong and Zhou (Ref. 5), we can write the objective functional as

$$J(s, \xi^u(s); u) = \mathrm{E}\left\{ \int_s^T L(r, \xi^u(r), u(r))\, dr + S(T, \xi^u(T)) \,|\, \mathcal{F}_s^t \right\} \quad \forall s \in [t, T],\ \text{P–a.s.}, \tag{28}$$

where $\{\mathcal{F}_s^t\}_{s \geq t}$ is the filtration of the σ–fields generated by Brownian motion in the interval $[t, s]$. The process

$$m(s) = \mathrm{E}\left\{ \int_t^T L(r, \xi^u(r), u(r))\, dr + S(T, \xi^u(T)) \,|\, \mathcal{F}_s^t \right\}, \quad s \in [s, T]$$

is a square–integrable $\{\mathcal{F}_s^t\}_{s \in [t,T]}$–martingale, hence by the martingale representation theorem, we have

$$m(s) = m(t) + \int_t^s M(r)\, dw(r),$$

with $M \in (L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$. Let us observe that $m(t) = J(t, x; u)$, therefore

$$m(s) = J(t, x; u) + \int_t^s M(r)\, dw(r). \tag{29}$$

By (28) and (29)

$$\begin{aligned} J(s,\xi^u(s);u) &= m(s) - \mathrm{E}\left\{\int_t^s L(r,\xi^u(r),u(r))\,dr\right\} \\ &= J(t,x;u) - \mathrm{E}\left\{\int_t^s L(r,\xi^u(r),u(r))\,dr\right\} + \int_t^s M(r)\,dw(r). \end{aligned}$$

It then follows

$$dJ(s,\xi^u(s);u) = -\mathrm{E}\{L(s,\xi^u(s),u(s))\}\,ds + M(s)\,dw(s). \tag{30}$$

We get from (27) and (30)

$$\begin{aligned} \mathrm{E}\Big\{ & J_s(s,\xi^u(s);u) + L(s,\xi^u(s),u(s)) + J_y^\top(s,\xi^u(s);u) f(s,\xi^u(s),u(s)) \\ & + \frac{1}{2}\,\mathrm{Tr}\{(\sigma\sigma^\top)(s,\xi^u(s)) J_{yy}(s,\xi^u(s);u)\}\Big\} = 0, \end{aligned}$$

This equality holds for all admissible $u \in \mathcal{U}$, for all $s \in [t,T]$. In particular, it holds for $\widehat{\phi}$, hence

$$\begin{aligned} 0 &= \mathrm{E}\Big\{ J_s(s,\xi^u(s);\widehat{\phi}) + H(s,\xi^u(s),\widehat{\phi}, J_y(s,\xi^u(s);\widehat{\phi})) \\ &\quad + \frac{1}{2}\,\mathrm{Tr}\{(\sigma\sigma^\top)(s,\xi^u(s)) J_{yy}(s,\xi^u(s);\widehat{\phi})\}\Big\} \\ &\geq \mathrm{E}\Big\{ J_s(s,\xi^u(s);\widehat{\phi}) + H(s,\xi^u(s),u, J_y(s,\xi^u(s);\widehat{\phi})) \\ &\quad + \frac{1}{2}\,\mathrm{Tr}\{(\sigma\sigma^\top)(s,\xi^u(s)) J_{yy}(t,\xi^u(s);\widehat{\phi})\}\Big\}, \end{aligned}$$

because $J_y \equiv \Gamma$, an identity which is proven in Proposition 4.1, and because of (26). Expanding the Hamiltonian function we find that the latter inequality is equivalent to $\mathrm{E}\big\{L(s,\xi^u(s),u(s))) + \mathcal{A}^u J(s,\xi^u(s);\widehat{u})\big\} \leq 0$. Integrating and exchanging expectation and integration we have

$$\mathrm{E}\left\{\int_t^T (L(s,\xi^u(s),u(s)) + \mathcal{A}^u J(s,\xi^u(s);\widehat{\phi}))\,ds\right\} \leq 0. \tag{31}$$

Given that by the assumptions made

$$\int_t^T J_x(s,\xi^u(s);\widehat{\phi})^\top \sigma(s,\xi^u(s))\,dw(s)$$

is a martingale, the application of Dynkin's formula to (30) leads to

$$\mathrm{E}\{S(T,\xi^u(T))\} - J(t,x;\widehat{\phi}) = \mathrm{E}\left\{\int_t^T \mathcal{A}^u J(s,\xi^u(s);\widehat{\phi})\,ds\right\}. \tag{32}$$

Substituting (32) into (31) we obtain

$$\mathrm{E}\left\{\int_t^T L(s,\xi^u(s),u(s))\,ds\right\} \leq J(t,x;\widehat{\phi}) - \mathrm{E}\{S(T,\xi^u(T))\}, \tag{33}$$

that is, $J(t,x;u) \leq J(t,x;\widehat{\phi})$. □

Remark 4.2 Condition (26) automatically holds when $\widehat{\phi}$ is interior to the control set U and the Hamiltonian function is concave with respect to u, for every t,x,p. To see this, note that $H_u(t,x,\widehat{\phi},\Gamma(t,x,\widehat{\phi})) = 0$ is trivially fulfilled by the definition of Γ, hence $\widehat{\phi}$ is a critical point of the concave function $u \mapsto H(\cdot,\cdot,u,\cdot)$, so $\widehat{\phi}$ is a global maximum of H. On the other hand, it is worth noting that the full strength of (26) is not really needed in the proof. It only suffices that for every initial condition (t,x) and for every admissible Markov control u the following holds

$$\mathrm{E}\left\{H(s,\xi^u(s),\Gamma(s,\xi^u(s),\widehat{\phi}),\widehat{\phi})\right\} \geq \mathrm{E}\left\{H(s,\xi^u(s),\Gamma(s,\xi^u(s),\widehat{\phi}),u)\right\}, \tag{34}$$

where ξ^u is the state variable process associated to u.

Remark 4.3 (Infinite Horizon). Proposition 4.1 can be extended to the infinite–horizon case, $T = \infty$, when the following transversality condition holds

$$\lim_{T\to\infty} \mathrm{E}\big\{\gamma^\top(T,\xi(T))\,\xi_x(T)\big\} = 0. \tag{35}$$

By Proposition 4.1 (35) is the same as $\lim_{T\to\infty} \mathrm{E}\big\{\partial_x V(T,\xi(T))\big\} = 0$ for every initial condition (t,x), that is, the long run behavior of the expected value function along the optimal trajectory is independent of the initial condition x, for every x. With respect to Theorem 4.1, two assumptions about the limit of $J(t,x;\widehat{\phi})$ as $t \to \infty$ must be added to the hypotheses, in order to assure the optimality of $\widehat{\phi}$. One of them is (35) which assures the equality between p^i and $J_{x^i}(t,x;\widehat{\phi})$, for $i = 1,\dots,n$. The other one is obtained by substituting $\mathrm{E}\{S(T,\xi^u(T))\}$ by $J(T,\xi^u(T);\widehat{\phi})$ in (33), given that, in the infinite horizon problem, there is no residual function S. Taking limits when T tends to infinite in expression (33), if the conditions

$$\limsup_{T\to\infty} J(T,\xi^u(T);\widehat{\phi}) = \limsup_{T\to\infty} V(T,\xi^u(T)) \geq 0 \tag{36}$$

and

$$\lim_{T\to\infty} \mathrm{E}\left\{\int_t^T L(s,\xi^u(s),u(s))\,ds\right\} = \mathrm{E}\left\{\int_t^\infty L(s,\xi^u(s),u(s))\,ds\right\} < \infty$$

hold for all admissible control u, then $J(t,x;u) \leq J(t,x;\widehat{\phi})$. The latter equality simply means that the cost functional of the infinite–horizon problem makes sense for the class of admissible controls.

Remark 4.4 In the deterministic case, $\sigma \equiv 0$, the system of partial differential equations (9) is of course of first order, and quasilinear. The system for this case was first derived in Bourdache–Siguerdidjane and Fliess (Ref. 14). Clearly, the results remain valid now for $\mathcal{C}^1$ solutions and **(A2)** is not needed. In Rincón–Zapatero et al. (Ref. 15) and Rincón–Zapatero (Ref. 16) an extension to differential games is provided.

Example 4.1 This is an *ad hoc* example, showing the advantages of our approach in some models. Suppose $n = m = 1$, $U = \mathbb{R}$, $T = \infty$ and that the functions $L(t,x,u) = e^{-\rho t}\ell_1(u)\ell_2(x)$, $f(x,u) = f_1(u)f_2(x)$ and σ are of class $\mathcal{C}^2$. Assume that f_1' and f_2 are different from zero and that the Hamiltonian is concave with respect to u. Furthermore, suppose:

(i) there exists a unique constant $\widehat{u}$ such that $\ell_1(\widehat{u})f_1'(\widehat{u}) - \ell_1'(\widehat{u})f_1(\widehat{u}) = 0$;

(ii) the function $k = \ell_2/f_2$ satisfies the linear second order differential equation

$$-\rho k(x) + \frac{1}{2}(\sigma^2(x)k'(x))' = 0.$$

We have a lot of information about the problem and the question is whether this is enough to obtain a solution to the HJB equation, which is given by

$$-\rho V(x) + \max_{u\in\mathbb{R}} \Big\{ \ell_1(u)\ell_2(x) + V'(x)f_1(u)f_2(x) \Big\} + \frac{\sigma^2(x)}{2}V''(x) = 0,$$

since we are considering stationary Markov controls. At first sight is not apparent what the solution is; it is even difficult to get an idea of the explicit form of this non linear equation, given that the maximization cannot be done explicitly. Let us turn our attention to the PDE (25) for the optimal control which, in contradistinction, is always *explicit*. We have

$$\Gamma(x,u) = -\frac{\ell_1'(u)}{f_1'(u)}k(x), \qquad \mathcal{H}(x,u) = \frac{\ell_2(x)}{f_1'(u)}(\ell_1(u)f_1'(u) - \ell_1'(u)f_1(u))$$

and then $\mathcal{H}(x,\widehat{u}) = 0$ for all $x \in \mathbb{R}$ by (i). If we look at equation (25), we see that the *constant control* $\widehat{u}$ is a solution if

$$\frac{\ell_1'(\widehat{u})}{f_1'(\widehat{u})}\Big(\rho k(x) - \frac{1}{2}(\sigma^2(x)k'(x))'\Big) = 0$$

holds and this is asserted in (ii). It can be easily shown that the solution of the HJB is

$$V(x) = -\frac{1}{2\rho}\frac{\ell_1'(\widehat{u})}{f_1'(\widehat{u})}\sigma^2(x)k'(x). \tag{37}$$

Further assumptions on the coefficient functions would imply that the constant control $\widehat{u}$ is the solution of the stochastic control problem and that V is the value function. Once this solution is known, it is obvious that $u = \widehat{u}$ is the maximizing argument in the HJB equation, but without this knowledge, it is difficult to guess a tentative form for the solution. It could be argued that the example is somewhat academic but it is worth noting that it is the PDE (25) that has allowed their construction because it directly characterizes the optimal control.

As an specific example, consider $\ell_1(u) = (au+b)^{1-1/a}$, $\ell_2(x) = x^\alpha$, $f_1(u) = \mu - u$, $f_2(x) = x$, and $\sigma(x) = \sigma x$, with $0 < \alpha < 1$, $a > 1$, $\mu \geq 0$ and $\sigma > 0$. The problem is

$$\max_u \mathrm{E} \int_0^\infty e^{-\rho t}\xi^\alpha(au+b)^{1-1/a}\,dt$$

subject to $d\xi = \big(\mu - u\big)\xi\,dt + \sigma\xi\,d\omega$, $\xi(0) = x > 0$. This formulation models the exploitation of a renewable resource by a single agent that derives utility both from the consumption rate u and the stock level, ξ. The dynamics shows a technology that makes costly to obtain the resource as it becomes scarce. We are supposing that the recruitment function is linear.

If the constants ρ, σ and α are linked by $\rho = (1/2)(1-\alpha)\alpha\sigma^2$, then (ii) holds. Imposing (i), we find that the optimal policy is the constant $\widehat{u} = (1-a)\mu - b$, which is non–negative for suitable values of the parameters involved. From (37), the value function is $V(x) = (1/\alpha)(a\widehat{u}+b)^{1-1/a}x^\alpha$.

5 Application to a consumption–savings problem

In this section we consider the classic economic problem of maximizing the utility derived from consumption along a fixed period of time, in a stochastic environment. Our aim is to give necessary and sufficient conditions for existence and uniqueness of the optimal consumption process, by means of the theory developed in the previous sections. Given the similarity of the model with those of the one–sector optimal economic growth, Merton (Ref. 19), and the optimal management of a natural resource, Clark (Ref. 20), the results apply also to these models with minor modifications.

More precisely, we study a simplification of the classic consumption–investment model of Merton (Ref. 6). We suppose that the agent's wealth is totally invested in a single risky asset, without possibility for diversification in a portfolio of different financial assets.

5.1 Statement of the problem

In a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ let $w(t)$ be a d–dimensional Brownian motion, adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. The wealth $\xi(t)$ is consumed at rate $c(t)$ and the remaining is invested in an asset with stochastic rate of return $S^1(t)$ given by

$$dS^1(t) = S^1(t)\left(\mu dt + \sigma dw(t)\right), \quad S^1(0) = s_0,$$

with $\mu \geq 0$ and where σ is a strictly positive d–dimensional row vector. Thus, given initial wealth x, the wealth process obeys the SDE

$$d\xi(s) = \xi(s)\frac{dS^1(s)}{S^1(s)} - c(s)ds = \left(\mu\xi(s) - c(s)\right)ds + \xi(s)\sigma dw(s), \qquad \xi(t) = x. \tag{38}$$

The class of admissible controls is given as in Definition 2.1, but incorporating the obvious condition $c \geq 0$. Given an initial state x and time t, the agent's objective is to choose a consumption process $c(t) = \phi(t, \xi(t))$ maximizing the expected total utility of consumption, discounted at rate $\rho \geq 0$ and over a fixed time horizon $[0, T]$,

$$J(t, x; c) = E_{tx}\left\{\int_t^T e^{-\rho(s-t)}U(c(s))\,ds + e^{-\rho(T-t)}S(\xi(T))\right\}. \tag{39}$$

We suppose that both the instantaneous utility U and the function S giving the utility of wealth at the final time T are of class $\mathcal{C}^3$ on $(0, \infty)$; furthermore, U is monotone increasing and strictly concave.

5.2 Solving the problem in the general case

Given the assumptions, the Hamiltonian, $H(t, x, c, p) = U(c) + (\mu x - c)p$, is strictly concave with respect to c. By (8), $\Gamma(s, x, c) = e^{-\rho(s-t)}U'(c)$ and the partial differential equation (25) characterizing the optimal consumption c is

$$-\rho U'(c) + U''(c)c_t + \partial_x\big(U(c) + (\mu x - c)\,U'(c) + \frac{1}{2}\sigma\sigma^\top x^2 U''(c)c_x\big) = 0, \tag{40}$$

with final condition $c(T,x) \equiv \varphi(x) := (U')^{-1}(S'(x))$, for $x > 0$; obviously, $\varphi(x) = x$ if $S = U$. Since positive consumption is not feasible if the wealth level is zero, we impose the boundary condition $c(t,0) = 0$ for all $t \leq T$. This assures, moreover, that if the wealth hits zero at some instant of time, then it remains zero. In order to obtain smooth solutions, a natural hypothesis is $\varphi(0) = 0$. If $\varphi(0) > 0$, then the final condition associated to the partial differential equation would be discontinuous, because $c(T,0) = 0$.

Taking derivatives in (40) and defining $\tau = T - t$ the equation is

$$c_\tau - \frac{\sigma\sigma^\top}{2}x^2 c_{xx} - (\rho - \mu)\mathcal{E}_1(c) - (\mu x - c + \sigma\sigma^\top x)c_x - \frac{\sigma\sigma^\top}{2}x^2\mathcal{E}_2(c)c_x^2 = 0, \tag{41}$$

where $\mathcal{E}_1 := -U'/U''$ and $\mathcal{E}_2 := U'''/U''$. Note that this PDE is degenerate.

In the following we will make another transformation in order to apply some recent results about global existence and uniqueness of solutions of non–linear parabolic equations. Specifically, Theorem 3.1 of Constantin and Escher (Ref. 21) apply to initial boundary value problems of the following type

$$\begin{cases} u_t - \partial_x(a(t,x,u)u_x) = F(t,x,u,u_x), & t > 0,\ x \in D, \\ u(t,x) = 0, & t > 0,\ x \in \partial D, \\ u(0,x) = \varphi(x), & x \in \overline{D}, \end{cases} \tag{42}$$

where D is a bounded open domain in $\mathbb{R}$. Note that we consider only the one–dimensional case. The functions a and F are supposed to be of class $\mathcal{C}^2$ and $\mathcal{C}^1$, respectively, with $a(t,x,u) > \eta > 0$ for all $(t,x,u) \in \mathbb{R}_+ \times \overline{D} \times \mathbb{R}$. We will suppose that φ is of class $\mathcal{C}^2$, though in the formulation of Constantin and Escher (Ref. 21) less regularity is required. It suffices that φ belongs to the Sobolev space $W_0^{r,u}(D)$, the closure of the test functions in $W^{r,u}(D)$, with $0 \leq r < 1 + 1/u$. Then, if F satisfies

$$|F(t,x,v,z)| \leq A(v)z^2 + B(v)|z| + C(v), \quad (t,x,v,z) \in \mathbb{R}_+ \times \overline{D} \times \mathbb{R} \times \mathbb{R},$$

for continuous and non–negative functions A, B and a continuous and positive function C such that

$$\int_0^\infty \frac{dv}{C(v)} = \int_{-\infty}^0 \frac{dv}{C(v)} = \infty, \tag{43}$$

then there exists a unique solution of class $\mathcal{C}^{1,2}$ of (42) globally defined on $[0,\infty) \times D$. It is interesting to remark that the bounding condition appearing in F, of quadratic type with respect to u_x, is of special usefulness in the semilinear equations that we are studying, because the type of non–linearity appearing in these equations is precisely quadratic with respect to u_x, as it was previously stated.

Now we will accommodate our problem to this result. Define $\widetilde{c} = c - \varphi$. The equation that $\widetilde{c}$ satisfies is:

$$\widetilde{c}_\tau - \frac{\sigma\sigma^\top}{2}\partial_x(x^2\widetilde{c}_x) = F(x, \widetilde{c}, \widetilde{c}_x), \tag{44}$$

where

$$\begin{aligned} F(x, v, z) = &\frac{\sigma\sigma^\top}{2}\partial_x(x^2\varphi'(x)) + (\rho - \mu)\mathcal{E}_1(v + \varphi(x)) + (\mu x - v - \varphi(x))(z + \varphi'(x)) \\ &+ \frac{\sigma\sigma^\top}{2}x^2(z + \varphi'(x))^2\mathcal{E}_2(v + \varphi(x)). \end{aligned}$$

We consider for all $k \in \mathbb{N}$, $k > 1$, the open sets $D_k = (1/k, k)$ and the family of problems (44) on each set $[0, \infty) \times D_k$. In this way, each of the sets D_k are open and bounded, the condition $\widetilde{c} \equiv 0$ on $\{0\} \times \partial D_k$ holds and $\sigma\sigma^\top x^2 \geq \eta \equiv \sigma\sigma^\top/k^2 > 0$ is satisfied. On the other hand, for all $k > 1$ the following bound holds

$$|F(x, v, z)| \leq A_k(v)z^2 + B_k(v)z + C_k(v),$$

where

$$A_k(v) = \frac{\sigma\sigma^\top}{2}\max_{x\in\overline{D}_k} x^2|\mathcal{E}_2(v + \varphi(x))|,$$

$$B_k(v) = \max_{x\in\overline{D}_k}\left\{|\mu x - v - \varphi(x)| + |\sigma\sigma^\top\mathcal{E}_2(v + \varphi(x))x^2\varphi'(x)|\right\}$$

and

$$\begin{aligned} C_k(v) = \max_{x\in\overline{D}_k}&\left\{\varphi'(x)|ax - v - \varphi(x)| + \frac{\sigma\sigma^\top}{2}x^2|\mathcal{E}_2(v + \varphi(x))|(\varphi'(x))^2 + |\rho - \mu|\mathcal{E}_1(v + \varphi(x))\right. \\ &\left.+ \frac{\sigma\sigma^\top}{2}\frac{\partial}{\partial x}(x^2\varphi'(x))\right\}, \end{aligned}$$

are continuous functions. Evidently, $C_k(u) \leq \widetilde{C}_k(u)$, where

$$\widetilde{C}_k(v) = a_0 + a_1 v + a_2|\mathcal{E}_2(v + a_3)| + |\rho - \mu|\mathcal{E}_1(v + a_4) \tag{45}$$

for some positive constants $a_0, \ldots, a_4$ depending on k.

Clearly, the hypotheses of Theorem 3.1 in Constantin and Escher (Ref. 21) are satisfied in every set $[0, \infty) \times \overline{D}_k$, with the possible exception of (43). Thus we proceed to impose the condition

$$\int_0^\infty \frac{dv}{\widetilde{C}_k(v)} = \infty. \tag{46}$$

Notice that the other side condition, $\int_{-\infty}^{0} \big(1/\widetilde{C}_k(v)\big)\, dv = \infty$, need not be checked here because we are restricted to non–negative values of consumption. If (46) holds, then there exists a unique global solution of class $\mathcal{C}^{1,2}$ in $[0,\infty) \times \overline{D}_k$ for each k, that we denote by $\widetilde{c}_k$. By the uniqueness and the regularity of the solution, a global solution $\widetilde{c}$ can be defined as $\widetilde{c}(t,x) = \widetilde{c}_k(t,x)$ if $(t,x) \in \mathbb{R}_+ \times D_k$. It is evident that $\widetilde{c}$ is the unique solution of class $\mathcal{C}^{1,2}$ of (44) such that $\widetilde{c} \equiv 0$ in $\{0\} \times [0,\infty)$ and, therefore, $c = \widetilde{c} + \varphi$ is the the unique solution of (41) satisfying $c(0,x) = \varphi(x)$. Whence we have proved the following result.

Proposition 5.1 Let us suppose that φ is of class $\mathcal{C}^2$, $\varphi(0) = 0$ and (46) is satisfied for every set of positive constants a_i, for $i = 0, \ldots, 4$. Then there exists a unique solution of class $\mathcal{C}^{1,2}$ of (41) satisfying the appropriate final and boundary conditions.

A solution of (41), whose existence and uniqueness is assured in the conditions stated in the above proposition, is not necessarily an optimal consumption process unless we prove:

- it is admissible, that is, there exists a pathwise unique strong solution of (38);
- assumption **(B)** holds.

In order to verify the admissibility of the solution, we introduce the following hypothesis, which guarantees the linear growth of the optimal consumption with respect to the level of wealth. This and the local Lipschitzianity of consumption, guarantees the existence of a pathwise strong unique solution of the SDE governing the wealth evolution.

(C1) There exist constants $a > 0$, $A > 0$ such that for all c, x, $\mathcal{E}_1(c) \leq a(1+c)$ and $\varphi(x) \leq A(1+x)$.

To obtain the linear growth of c from assumption **(C1)** we will use the concept and properties of viscosity solution, which was introduced in Lions (Ref. 7) for equations of second order. We refer the reader to the Appendix of this paper for the definitions and the statement and proof of two lemmas that will be used in this section, and to Fleming and Soner (Ref. 3), which constitutes a good presentation of the theory and their applications.

Proposition 5.2 Under the hypotheses of Proposition 5.1, suppose $\lim_{c \to 0^+} U'(c) < \infty$ and that **(C1)** holds. Then there exists a unique optimal consumption function, which is given by the solution of (41) satisfying the appropriate final and boundary conditions.

Proof. Let us show that the solution of class $\mathcal{C}^{1,2}$, whose existence is assured by Proposition 5.1, satisfies all hypotheses of Theorem 4.1. It is evident that the Hamiltonian is strictly concave with respect to the control variable. From this, any admissible control satisfies (26), as was stated in Remark 4.2. With respect to the admissibility of the solution, the equality $c(t, 0) = 0$ guarantees that wealth never takes negative values. On the other hand, the solution is locally Lipschitz because it is of class $\mathcal{C}^{1,2}$. Given that Lemma 7.1 implies that the consumption growth is at most linear, we have assured the existence of a unique global solution of the SDE (38) associated to c. Finally, all assumptions stated in Remark 4.1, which assure **(B)**, are obviously fulfilled in our model. In particular, the second inequality displayed in (i) does not hold in general, but it does for the optimal $\widehat{\phi}$, because it grows at most linearly. □

Example 5.1 The problem with the exponential utility function $U(c) = 1 - e^{-\delta c}$, with $\delta > 0$, and S such that φ satisfies **(C1)** is framed within the requirements of Proposition 5.2. Let us show this, point by point:

1. U is strictly concave and smooth.

2. For this utility function, $\mathcal{E}_1(c) = 1/\delta$ is independent of c, thus **(C1)** holds.

3. The function $\widetilde{C}_k$ is given by

$$\widetilde{C}_k(v) = a_0 + a_1 v + a_2 \delta + \frac{|\rho - \mu|}{\delta},$$

   for positive constants a_0, a_1, a_2. Of course, $\int_0^\infty \left(1/\widetilde{C}_k(v)\right) dv = \infty$.

The situation is more delicate when the marginal utility at zero is infinite, $\lim_{c \to 0^+} U'(c) = \infty$, because checking hypothesis **(B)** is then difficult if the wealth may become null with positive probability. Another difficulty is that Γ is not well defined when the consumption is zero. Therefore, we will impose conditions in order that the wealth process associated with the optimal solution is positive with probability one. In this case, the maximization condition (34) obviously holds.

We introduce the following hypothesis:

C2. There exist constants $a > 0$, $A > 0$ such that for positive and small values of c and x, $\mathcal{E}_1(c) \le ac$ and $\varphi(x) \le Ax$.

Now, given that when the wealth values ξ are near to 0, it is bounded below by a geometric Brownian motion, thus the probability for ξ becoming 0 is null. In fact, for values of ξ near zero, the following inequality holds by virtue of Lemma 7.2

$$\begin{aligned}\xi(t) &= \xi(0) + \int_0^t (\mu\xi(s) - c(s,\xi(s)))\,ds + \int_0^t \sigma\xi(s)\,dw(s) \\ &\geq \xi(0) + \int_0^t (\mu\xi(s) - B\xi(s))\,ds + \int_0^t \sigma\xi(s)\,dw(s) > 0\end{aligned}$$

with positive probability if $\xi(0) > 0$. In consequence, the same arguments as those used in the proof of Proposition 5.2 now show that **(B)** holds.

We have proved the following result.

Proposition 5.3 Under the hypotheses of Proposition 5.1, suppose that $\lim_{c\to 0^+} U'(c) = \infty$ and that **(C1)** and **(C2)** hold. Then there exists a unique optimal consumption function, continuous on $[0,T]\times[0,\infty)$ and of class $\mathcal{C}^{1,2}$ on $[0,T]\times(0,\infty)$, which is given by the solution of (41) satisfying the appropriate final and boundary conditions.

Example 5.2 Consider $U(c) = c^\delta/\delta$, with $\delta < 1$ and let S be such that φ satisfies **(C1)**. Let us show that all hypotheses in the above proposition hold.

1. U is strictly concave.

2. For this utility function, $\mathcal{E}_1(c) = -c/(\delta - 1)$ satisfies **(C1)** and **(C2)**.

3. The function $\widetilde{C}_k$ is given by

$$\widetilde{C}_k(v) = \frac{(1+a_2)v^2 + a_1 p + a_0}{v + a_3}$$

   for adequate non negative arbitrary constants a_i, $i = 0, \ldots, 3$. The corresponding improper integral of $1/\widetilde{C}_k(v)$ is divergent.

6 Conclusions

This paper provides an alternative method for the analysis of stochastic optimal control problems to the classical ones based on dynamic programming, duality, and the maximum principle. The novelty of the approach we propose in this paper does not consists in the tools we use in the construction of the theoretical framework—which heavily depend on dynamic programming concepts and the maximum principle—but in the optimality conditions, necessary and sufficient, that are obtained. These are entirely new. We do not pretend to convey the reader the idea that our approach is superior to the existing ones—we have remarked the limitations of the method in the Introduction—but to provide a different perspective, based in a system of PDEs which directly characterize the optimal controls, without resorting to the value function. A useful feature of the system of PDEs introduced in the paper is that the gradient of the optimal control enters in a quadratic way. This allows us to use recent results in the theory of parabolic PDEs to obtain existence and uniqueness of the optimal policy in the classical consumption–savings model with rather general utility functions.

7 Appendix. Auxiliary results

Consider a non–linear partial differential equation of the form

$$u_t + F(t, x, u, u_x, u_{xx}) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}, \tag{47}$$

with initial condition

$$u(0, x) = \varphi(x), \tag{48}$$

where $F : [0, T) \times \mathbb{R}^4 \longmapsto \mathbb{R}$ is continuous and elliptic, that is,

$$F(t, x, u, u_x, a + b) \le F(t, x, u, u_x, a), \qquad \text{if } b \ge 0,$$

and where $\varphi : \mathbb{R} \longmapsto \mathbb{R}$ is continuous also.

Definition 7.1 A continuous function $u : [0, T] \longmapsto \mathbb{R}$ is a viscosity subsolution (*resp.* supersolution) of (47)–(48) if $u(0, x) \le \varphi(x)$ (*resp.* $u(0, x) \ge \varphi(x)$) and for all smooth function ω, such that $u - \omega$ attains a local maximum (*resp.* minimum) at (τ_0, x_0),

$$\omega_t(\tau_0, x_0) + F(\tau_0, x_0, u(\tau_0, x_0), \omega_x(\tau_0, x_0), \omega_{xx}(\tau_0, x_0)) \le 0 \quad (\textit{resp.} \ \ge 0).$$

If u is simultaneously a viscosity subsolution and a viscosity supersolution of (47)–(48), then we will say that it is a viscosity solution of (47)–(48).

For equations where F is elliptic any solution of class $\mathcal{C}^{1,2}$ of the initial value problem is a viscosity solution. On the other hand, there holds an important comparison property for viscosity sub– and supersolutions, under certain additional hypotheses on F and on the initial data. Specifically, any subsolution is always less than or equal to any supersolution, as shown in Fleming and Soner (Ref. 3).

Lemma 7.1 Suppose that **(C1)** and the hypotheses of Proposition 5.1 are satisfied. Then there exists a constant $B > 0$ such that for all $t \in [0, T]$, $c(t, x) \le B(1 + x)$ holds for all x.

Proof. The function $c_+(x) = B(1 + x)$ is a viscosity supersolution of the equation (44) with initial data $c(0, x) = \varphi(x)$, if B is chosen as

$$B \ge \max\{A, \alpha + \sigma\sigma^\top + \max\{0, (\rho - \alpha)a\}\},$$

such that $(\rho-\alpha)a(1+B)-B^2 \leq 0$. In order to prove this fact, let ω be of class $\mathcal{C}^{1,2}$ such that $c-\omega$ has a minimum in (τ_0, x_0) and $c(\tau_0, x_0) = \omega(\tau_0, x_0)$. In this case it is satisfied

$$\begin{aligned}
\omega_x(\tau_0, x_0) &= c_+'(x_0) = B, \\
\omega_\tau(\tau_0, x_0) &= 0, \\
\omega_{xx}(\tau_0, x_0) &\leq 0.
\end{aligned}$$

It is thus easy to see that

$$\omega_\tau(\tau_0, x_0) + F(\tau_0, x_0, \omega(\tau_0, x_0), \omega_x(\tau_0, x_0), \omega_{xx}(\tau_0, x_0)) \geq 0$$

by the choice of B, where the meaning of F is obvious from (41). Moreover, $c_+(x) \geq \varphi(x)$ by the selection of B. Given that c is of class $\mathcal{C}^{1,2}$, it is a viscosity solution and, in particular, it is a subsolution and therefore $c \leq c_+$ as claimed. □

Lemma 7.2 Suppose that **(C2)** and the hypotheses of Proposition 5.1 are satisfied. Then there exists $B > 0$ such that the solution of (41) satisfies $c(\tau, x) \leq Bx$ for x positive and small enough.

Proof. As in the proof of Lemma 7.1, it is easy to see that in a certain interval $(0, \varepsilon)$ the function $c_+(x) = Bx$ is a viscosity supersolution of (41), taking

$$B = \max\{A, \alpha + \sigma\sigma^\top + \max\{0, (\rho-\alpha)a\}\}.$$

Given that the equation satisfies the comparison principle and that c is a viscosity solution because it is of class $\mathcal{C}^{1,2}$ (and, therefore, a subsolution), $c(\tau, x) \leq c_+(x) = Bx$ in $(0, \varepsilon)$. □

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