



Working Paper 10-28
Statistics and Econometrics Series 14
May 2010

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

BOOTSTRAP PREDICTION INTERVALS FOR VAR AND ES IN THE CONTEXT OF GARCH MODELS

María Rosa Nieto and Esther Ruiz

Abstract

In this paper, we propose a new bootstrap procedure to obtain prediction intervals of future Value at Risk (VaR) and Expected Shortfall (ES) in the context of univariate GARCH models. These intervals incorporate the parameter uncertainty associated with the estimation of the conditional variance of returns. Furthermore, they do not depend on any particular assumption on the error distribution. Alternative bootstrap intervals previously proposed in the literature incorporate the first but not the second source of uncertainty when computing the VaR and ES. We also consider an iterated smoothed bootstrap with better properties than traditional ones when computing prediction intervals for quantiles. However, this latter procedure depends on parameters that have to be arbitrarily chosen and is very complicated computationally. We analyze the finite sample performance of the proposed procedure and show that the coverage of our proposed procedure is closer to the nominal than that of the alternatives. All the results are illustrated by obtaining one-step-ahead prediction intervals of the VaR and ES of several real time series of financial returns.

Keywords: Expected Shortfall, Feasible Historical Simulation, Hill estimator, Parameter uncertainty, Quantile intervals, Value at Risk.

Universidad Carlos III de Madrid, Department of Statistics, Campus Getafe, Madrid, Spain. e-mail address: mrnieto@eco.uc3m.es (María Rosa Nieto) and ortega@est-econ.uc3m.es (Esther Ruiz). Financial support from Project ECO2009-08100 by the Spanish Government is gratefully acknowledged.

1 Introduction

Since it was proposed by the Basel Committee on Banking Supervision, the Value at Risk (VaR) is extensively used by financial institutions to measure the risk of their portfolios. The one-step ahead VaR is defined as the minimal potential loss that a portfolio can suffer in the $100\alpha\%$ worst cases, with $\alpha \in (0, 1)$, on some fixed time horizon. In particular, the VaR is given by

$$VaR_t^{(\alpha)} = -\sup \left[r \mid \Pr_{t-1} [R_t \leq r] \leq \alpha \right] \quad (1)$$

where R_t is the portfolio return at time t and the $t - 1$ under the probability means that it is taken conditional on the information available at time $t - 1$. In practice, the Basel Accord fixed $\alpha = 1\%$, and therefore, this is the value of α considered from now on, dropping the superscript α from the VaR_t .

It is well known that the VaR has the theoretical limitation of not being subadditive, i.e., a portfolio which is made of sub-portfolios may have more risk than the sum of the risks of the sub-portfolios. Furthermore, the VaR does not measure the risk in the tails; see Basak and Shapiro (2001) and Yamai and Yoshida (2005) for discussions on the tail risk of the VaR . Consequently, Artzner et al. (1997) propose to measure risk by the Conditional Value at Risk ($CVaR$) which measures the *expected loss* in the $100\alpha\%$ worst cases which is given by

$$CVaR_t = -E_{t-1} \{R_t \mid R_t \leq -VaR_t\}. \quad (2)$$

Given that the $CVaR$ is only coherent when the distribution of returns is continuous, Acerbi and Tasche (2002) proposed an alternative coherent measure of risk known as Expected Shortfall (ES) which is given by

$$ES_t = CVaR_t + (\lambda - 1)(CVaR_t - VaR_t) \quad (3)$$

where $\lambda \equiv \frac{\Pr_{t-1} [R_t \leq -VaR_t]}{\alpha} \geq 1$. The $CVaR$ and ES are equal when the distribution of returns is continuous as it is often assumed in practice.

There is a large literature devoted to point forecast, theoretical properties and backtesting of VaR and ES ; see Nieto and Ruiz (2009) for a recent survey in the context of univariate time series of returns. However, as it is the case in any forecast, there is also an interest in obtaining prediction intervals that measure the uncertainty associated with the corresponding point forecast. There are quite a few papers in the literature considering prediction intervals for the VaR and the ES . For example, Chan et al. (2007) propose to construct confidence intervals for the VaR by the tilting method of Hall and Yao (2003) and Peng and Qi (2003). Chen and Tang (2005) propose a nonparametric estimation of the VaR and its associated standard error. Chou et al. (2008) and Gilli and Killezi (2006) construct confidence intervals for the VaR and the ES respectively, using Extreme Value Theory (EVT). Finally, Lan et al. (2008) use the statistical theory of empirical likelihood to construct confidence intervals for the ES . However, these prediction intervals do not incorporate the uncertainty due to parameter estimation. Bams et al. (2005) shows that incorporating the parameter uncertainty within the prediction intervals for VaR and ES is important. Consequently, they use the asymptotic covariance matrix of the Maximum Likelihood (ML) estimator to quantify the uncertainty of the VaR by sampling from the asymptotic parameter distribution. However, this distribution can be an inadequate approximation of the finite sample distribution when the sample size is small.

Alternatively, it is possible to incorporate the parameter uncertainty by using bootstrap procedures which work well in prediction; see, for example, the survey by Ruiz and Pascual (2002). In that sense, Christoffersen and Gonçaves (2005) propose using bootstrap procedures to obtain prediction intervals for several parametric and non-parametric estimates of the VaR and ES . In the case of the parametric estimates, they consider a univariate $GARCH(1, 1)$ model for the conditional variances and implement the bootstrap procedure of Pascual et al. (2006). Then, in order to compute the corresponding quantile needed for the prediction of the VaR and ES , they consider several alternative assumptions about the distribution of the standardized returns. First, they consider Normal and Student- ν distributions. Second, they assume an Extreme Value distribution and compute the corresponding quantile by using the Hill estimator. Third, they approximate the distribution using the Cornish-Fisher and Gram-Charlier approximations. Finally, they implement Feasible Historical Simulation (FHS). When considering nonparametric estimates of the VaR and ES , Christoffersen and Gonçaves (2005) focus on the iid bootstrap procedure to obtain prediction intervals for the VaR and ES computed using Historical Simulation (HS). This bootstrap procedure is completely non-parametric avoiding any distributional assumption on the data. However, by implicitly assuming that returns are iid, this method fails to capture the dependence in returns when it exist. Therefore, the bootstrap procedures proposed by Christoffersen and Gonçaves (2005) either assume that returns are conditionally heteroscedastic with a particular assumption on their conditional distribution or when no error distribution is assumed returns are treated as iid. Only when implementing Cornish-Fisher and Gram-Charlier expansions of FHS there are not assumption on the error distribution. Within this context, they conclude that its bootstrap procedure has adequate coverage when the FHS is implemented to estimate the VaR . On the other hand, the Hill estimator has the best coverage for the ES but still well under the nominal. It is important to note that from a conservative risk management perspective under-coverage is worst than over-coverage.

In this paper, we focus on the parametric specification of returns assuming that they are represented by $GARCH$ -type models and propose to extend the bootstrap procedure of Christoffersen and Gonçaves (2005) by incorporating a second bootstrap step in the estimation of the quantile of the conditional distribution of the standardized returns. Furthermore, following Ho and Lee (2005), we also consider bootstrap prediction intervals for the quantile that overcome the limitations of the traditional prediction intervals. We show that, although our bootstrap procedure is very simple from a computational point of view, incorporating this second bootstrap step, improves the performance of the prediction intervals of the VaR and ES which have coverage much closer to the nominal. Furthermore, we show that the iterative smoothed bootstrap of Ho and Lee (2005) may have better coverages depending on the particular value of a smoothing parameter that has to be arbitrarily chosen.

The rest of the paper is organized as follows. In Section 2 we propose a new bootstrap procedure to obtain prediction intervals for the VaR and ES in the context of univariate $GARCH(1, 1)$ models. Section 3 reports the results of several Monte Carlo experiments carried out to analyze the finite sample performance of the proposed intervals and to compare them with alternative bootstrap intervals previously proposed in the literature. Section 4 illustrates the proposed procedures by implementing them to obtain prediction intervals of future VaR and ES of several real time series of financial returns. Finally, Section 5 concludes the paper.

2 Bootstrap prediction intervals for VaR and ES

In this section, we focus on the $GARCH(1,1)$ model for its simplicity and wide implementation to describe the dynamic evolution of the conditional variances of financial returns. However, the procedures described can be easily implemented for alternative specifications of the conditional variance as far as it is observable one-step ahead.

Consider that the return series of interest, R_t , is given by the following uncorrelated $GARCH(1,1)$ process,

$$R_t = \epsilon_t \sigma_t \quad (4)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \beta \sigma_{t-1}^2, \quad (5)$$

for $t = 2, \dots, T$, where σ_t is the conditional standard deviation of returns and $\sigma_1^2 = \frac{\alpha_0}{(1 - \alpha_1 - \beta)}$ is the marginal variance. The parameters α_0 , α_1 and β are assumed to satisfy the usual positivity and stationarity restrictions. The disturbances ϵ_t are assumed to be *iid* with zero mean and variance 1. If returns are given by (4), the one-step ahead VaR and ES are given by

$$VaR_t = \sigma_t q \quad (6)$$

and

$$ES_t = \sigma_t E_{t-1}[\epsilon_t | \epsilon_t \leq q], \quad (7)$$

respectively, where q is the 1% quantile of the distribution of ϵ_t . Expression (6), shows that in the parametric framework considered in this paper, we can express the VaR_t as the product of the conditional standard deviation, σ_t , and a constant, q , which depends on the distribution of the standardized returns, ϵ_t . Furthermore, the ES also depends on σ_t , q and on the expectation of the returns under q . By assuming a particular distribution of returns as, for example, the Normal or a Student- ν distribution, q and the expectation involved in (7) have known values; see Christoffersen and Gonçaves (2005) for the corresponding expressions. However, in the general case, when the distribution of ϵ_t is unknown, q and the expectation in (7) have to be estimated.

In any case, even if q is known, one needs to estimate the parameters of the conditional variance. Due to its well known asymptotic properties, in this paper, we consider the Quasi Maximum Likelihood (QML) estimator, denoted by $\{\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}\}$. Then, in practice, the VaR_t and ES_t are estimated as follows

$$\widehat{VaR}_t = \hat{\sigma}_t \hat{q} \quad (8)$$

and

$$\widehat{ES}_t = \hat{\sigma}_t \widehat{E}_{t-1}[\epsilon_t | \epsilon_t \leq \hat{q}], \quad (9)$$

where $\hat{\sigma}_t^2 = \hat{\alpha}_0 + \hat{\alpha}_1 R_{t-1}^2 + \hat{\beta} \hat{\sigma}_{t-1}^2$, for $t = 2, \dots, T$ and $\hat{\sigma}_1^2 = \frac{\hat{\alpha}_0}{(1 - \hat{\alpha}_1 - \hat{\beta})}$. Therefore, if the

$GARCH$ model is correctly specified, there are two sources of uncertainty associated with predicting VaR_{T+1} given $\{R_1, \dots, R_T\}$. One is the uncertainty in computing q and the other concerns the prediction of the volatility, σ_{T+1} . Furthermore, when computing the ES one also need to estimate the expectation beyond this quantile which also depends on the error distribution. In this section, we describe the Christoffersen and Gonçaves (2005) bootstrap procedure and propose another one with better properties in small samples.

2.1 Bootstrap based prediction intervals for VaR and ES

Consider the *GARCH* (1,1) model in equations (4) and (5) whose parameters have been estimated by *QML*. Then, one can obtain the standardized residuals, $\hat{\epsilon}_t = \frac{R_t}{\hat{\sigma}_t}$ where $\hat{\sigma}_t$ is defined as in equations (8) and (9). Pascual et al. (2006) propose to obtain a bootstrap replicate of the original returns, R_t^* , from the following recursions:

$$\sigma_t^{*2} = \hat{\alpha}_0 + \hat{\alpha}_1 R_{t-1}^{*2} + \hat{\beta} \sigma_{t-1}^{*2} \quad (10)$$

$$R_t^* = \epsilon_t^* \sigma_t^* \quad (11)$$

for $t = 2, \dots, T$, where $\sigma_1^{*2} = \frac{\hat{\alpha}_0}{(1 - \hat{\alpha}_1 - \hat{\beta})}$ and ϵ_t^* are random draws with replacement from the standardized residuals $\hat{\epsilon}_t$. Then, the parameters $(\hat{\alpha}_0^*, \hat{\alpha}_1^*, \hat{\beta}^*)$ are estimated from $\{R_1^*, \dots, R_T^*\}$ and used for the construction of one-step-ahead forecast of the volatility as follows

$$\hat{s}_{T+1}^{*2} = \hat{\alpha}_0^* + \hat{\alpha}_1^* R_T^2 + \hat{\beta}^* \hat{s}_T^* \quad (12)$$

where $\hat{s}_T^{*2} = \frac{\hat{\alpha}_0^*}{1 - \hat{\alpha}_1^* - \hat{\beta}^*} + \hat{\alpha}_1^* \sum_{j=0}^{T-2} \hat{\beta}^{*j} \left(R_{T-j-1}^2 - \frac{\hat{\alpha}_0^*}{1 - \hat{\alpha}_1^* - \hat{\beta}^*} \right)$, so that the forecast \hat{s}_{T+1}^{*2} is based on the original series of returns $\{R_1, \dots, R_T\}$ and on the bootstrap parameters. The algorithm above is repeated B times, obtaining B bootstrap replicates, denoted by $\hat{s}_{T+1}^{*2(i)}$, $i = 1, \dots, B$.

Christoffersen and Gonçaves (2005) propose to compute prediction intervals for the VaR_{T+1} and ES_{T+1} by obtaining bootstrap replicates of the VaR and ES by the following expressions

$$\widehat{VaR}_{T+1}^* = \hat{s}_{T+1}^* \hat{q}^* \quad (13)$$

and

$$\widehat{ES}_{T+1}^* = \hat{s}_{T+1}^* \hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*], \quad (14)$$

where \hat{s}_{T+1}^* is given by (12).

They consider several alternative estimators of \hat{q}^* and $\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*]$. First, they assume that ϵ_t has a Normal distribution; see also Hartz and Paolella (2006) who propose a modification to avoid biases in the estimates of the VaR . In this case, \hat{q}^* and $\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*]$ are given by

$$\hat{q}^* = \Phi_{0.01}^{-1} \quad (15)$$

$$\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*] = -\frac{\phi(\Phi_{0.01}^{-1})}{0.01} \quad (16)$$

where $\Phi_{0.01}$ is the notation of a Normal distribution. They also assume that ϵ_t has a standardized Student- ν distribution, where ν are the degrees of freedom. In this case, the expression for the quantile and the expectation are given by

$$\hat{q}^* = \sqrt{\frac{\nu - 2}{\nu}} t_{0.01}^{-1} \quad (17)$$

$$\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*] = \frac{\nu - 2}{\alpha(1 - \nu)} \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\sqrt{\pi(\nu - 2)} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{\hat{q}^{*2}}{\nu - 2}\right)^{\frac{1 - \nu}{2}} \quad (18)$$

where $t_{0.01}^{-1}$ is the 0.01 quantile of the Student- ν distribution. Then, these values which, do not depend on the standardized bootstrap residuals are substituted in (13) and (14) respectively to compute the VaR and ES . However, in the EVT framework, the tail of the conditional distribution of $\hat{\epsilon}_t^*$ is assumed to be well approximated by the distribution proposed by Gnedenko (1943). Then, the following Hill estimator is implemented to estimate the quantile

$$\hat{q}^* = -\hat{\epsilon}_{(k+1)}^* \left(\frac{0.01}{k/T} \right)^{-\hat{\xi}^{(k)}} \quad (19)$$

where $k \ll T$ is the number of observations in the tail, the threshold is defined at the $(k+1)$ th order statistic of the bootstrap residuals, $\hat{\epsilon}_{(k+1)}^*$, $\hat{\xi}^{(k)} = \frac{1}{k} \sum_{j=1}^k \log(\hat{\epsilon}_{(j)}^*) - \log(\hat{\epsilon}_{(k+1)}^*)$ and the standardized bootstrap residuals are computed as follows

$$\hat{\epsilon}_t^* = \frac{R_t^*}{\hat{\sigma}_t^*} \quad (20)$$

where $\hat{\sigma}_t^{*2} = \hat{\alpha}_0^* + \hat{\alpha}_1^* R_{t-1}^{*2} + \hat{\beta}_1^* \hat{\sigma}_{t-1}^{*2}$. It is important to notice the difference between \hat{s}_{T+1}^{*2} and $\hat{\sigma}_t^{*2}$. The latter is calculated using the bootstrap replicates and the former using the original data.

On the other hand, it can be shown that in the EVT framework the expectation in the tail is given by

$$\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*] = \frac{\hat{q}^*}{1 - \hat{\xi}^{(k)}}. \quad (21)$$

Implementing the Hill estimator requires to choose the cut-off point k which defines the subsample of extremes from which the tail index parameter ξ is estimated. Different procedures have been proposed in the literature in order to choose the optimal value of k , but there is not a formal method. Consequently, we choose the value of k such that $\hat{\xi}^{(k)}$ is stable. Christoffersen and Gonçalves (2005) compute the bias and root mean squared error of the \widehat{VaR}_{T+1}^* and \widehat{ES}_{T+1}^* and choose k as the value for which they are stable.

Finally, Christoffersen and Gonçalves (2005) propose two estimators of \hat{q}^* and $\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*]$ that do not rely on any particular error distribution. In particular, the FHS consists on estimating \hat{q}^* by

$$\hat{q}^* = \hat{\epsilon}_{(\omega)}^* \quad (22)$$

where $\hat{\epsilon}_{(\omega)}^*$ is the ω th-order statistic of the standardized bootstrap residuals $\{\hat{\epsilon}_1^*, \dots, \hat{\epsilon}_T^*\}$ and $\omega = [T \times 0.01] = \max\{m \mid m \leq T \times 0.01, m \in \mathbb{N}\}$. The corresponding expectation is estimated by

$$\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*] = \frac{\sum_{i=1}^{\omega} \hat{\epsilon}_{(i)}^*}{\omega}. \quad (23)$$

Alternatively, they propose using the Gram-Charlier and Cornish-Fisher ($GCCF$) expansions to approximate the conditional density of the standardized bootstrap residuals. The Cornish-Fisher expansion is used for the estimation of \hat{q}^* as follows

$$\hat{q}^* = \Phi_{0.01}^{-1} + \frac{\hat{\gamma}_1}{6} \left[(\Phi_{0.01}^{-1})^2 - 1 \right] + \frac{\hat{\gamma}_2}{24} \left[(\Phi_{0.01}^{-1})^3 - 3\Phi_{0.01}^{-1} \right] - \frac{\hat{\gamma}_1^2}{36} \left[2(\Phi_{0.01}^{-1})^3 - 5\Phi_{0.01}^{-1} \right] \quad (24)$$

where $\hat{\gamma}_1 = \frac{1}{T} \sum_{t=1}^T \hat{\epsilon}_t^{*3}$ and $\hat{\gamma}_2 = \frac{1}{T} \sum_{t=1}^T \hat{\epsilon}_t^{*4} - 3$. Giamouridis (2006) provides the following correction for the expression of the expectation needed to compute the ES

$$\hat{E}_T^* [\hat{\epsilon}_{T+1}^* | \hat{\epsilon}_{T+1}^* \leq \hat{q}^*] = \frac{\phi(\hat{q}^*)}{0.01} \left(1 + \frac{\hat{\gamma}_1}{6} (\hat{q}^*)^3 + \frac{\hat{\gamma}_2}{24} \left[(\hat{q}^*)^4 - 2(\hat{q}^*)^2 - 1 \right] \right) \quad (25)$$

where \hat{q}^* is given as in (24).

After computing B bootstrap replicates and their corresponding estimates of the VaR_{T+1} and ES_{T+1} , a set of B bootstrap estimates are obtained for both measures $\left\{ \widehat{VaR}_{T+1}^{*(1)}, \dots, \widehat{VaR}_{T+1}^{*(B)} \right\}$ and $\left\{ \widehat{ES}_{T+1}^{*(1)}, \dots, \widehat{ES}_{T+1}^{*(B)} \right\}$. The empirical distributions of $\widehat{VaR}_{T+1}^{*(i)}$ and $\widehat{ES}_{T+1}^{*(i)}$ are denoted by $Q_V^*(r) = \frac{\#\left\{ \widehat{VaR}_{T+1}^{*(i)} \leq r \right\}}{B}$ and $Q_E^*(r) = \frac{\#\left\{ \widehat{ES}_{T+1}^{*(i)} \leq r \right\}}{B}$ respectively, where $\#\{\cdot\}$ is the cardinality of $\{\cdot\}$. Then, the bootstrap prediction intervals for the VaR_{T+1} and ES_{T+1} are given by

$$\left[q_{\frac{\gamma}{2}}(Q_V^*(r)), q_{1-\frac{\gamma}{2}}(Q_V^*(r)) \right] \quad (26)$$

$$\left[q_{\frac{\gamma}{2}}(Q_E^*(r)), q_{1-\frac{\gamma}{2}}(Q_E^*(r)) \right]. \quad (27)$$

respectively, where $q_\gamma(\cdot)$ is the γ th empirical quantile of the corresponding empirical distribution.

They show that the bootstrap intervals constructed for the FHS estimator of VaR_{T+1} have coverages close to the nominal. However, the coverages of the bootstrap FHS estimator of ES_{T+1} are well under the nominal. On the other hand, the bootstrap intervals of the Hill estimator of ES_{T+1} are closer but still well under the nominal coverage. Therefore, the prediction intervals cannot be trusted for the ES risk measures.

Recently, Ho and Lee (2005) show that the traditional bootstrap procedures are not adequate when constructing prediction intervals for quantiles. Consequently, they propose the iterated smoothed bootstrap which corrects the errors in estimating quantiles by calibrating the nominal coverage level iteratively while smoothing the bootstrap amounts to drawing bootstrap samples from a kernel-smoothed empirical distribution instead of sampling with replacement from the row data. Ho and Lee (2005) assume that the original data is iid. Therefore, their procedure can be implemented, in our case, to the standardized returns $\hat{\epsilon}_t$, obtaining then, a confidence interval for q which allow us to construct a confidence interval for the one-step ahead VaR by multiplying it by $\hat{\sigma}_{T+1}$. Next, we describe the iterated smoothed bootstrap.

If F is the distribution of standardized returns, $F^{-1}(\alpha)$ is the α th quantile of $\hat{\epsilon}_t$. Let F_T be the empirical distribution function of $\hat{\epsilon}_t$ and $\hat{F}_{T,\eta}(t) = T^{-1} \sum_{i=1}^T K((t - \hat{\epsilon}_i)/\eta)$ its kernel-smoothed version defined for a kernel function K and a bandwidth $\eta > 0$. The smoothed bootstrap percentile method is the following: let $\hat{\epsilon}_t^\dagger = \left\{ \hat{\epsilon}_1^\dagger, \dots, \hat{\epsilon}_T^\dagger \right\}$ be a random sample from $\hat{F}_{T,\eta}$ which in practice is generated by $\hat{\epsilon}_i^\dagger = Y_i^* + \eta W_i^*$, where Y_i^* and W_i^* are independent random draws from F_T and K respectively. Let $F_{T,\eta}^*$ be the empirical distribution of $\hat{\epsilon}_t^\dagger$ and define

$$G_T(t) = P \left[T^{1/2} (F_T^{-1}(\alpha) - F^{-1}(\alpha)) \leq t \right], t \in \mathbb{R}. \quad (28)$$

The smoothed version of (28) is given by

$$\hat{G}_{T,\eta}(t) = P \left[T^{1/2} (F_{T,\eta}^{*-1}(\alpha) - \hat{F}_{T,\eta}^{-1}(\alpha)) \leq t \mid \hat{\epsilon}_t \right], t \in \mathbb{R}. \quad (29)$$

A noniterated smoothed bootstrap confidence interval for $F^{-1}(\alpha)$ is given by

$$I_{1,\gamma} = \left[F_T^{-1}(\alpha) - T^{-1/2} \hat{G}_{T,\eta}^{-1}(1-\gamma), F_T^{-1}(\alpha) - T^{-1/2} \hat{G}_{T,\eta}^{-1}(\gamma) \right] \quad (30)$$

consequently, the noniterated smoothed version for the VaR_{T+1} would be

$$\widehat{\sigma}_{T+1} I_{1,\gamma}. \quad (31)$$

The next step is to iterate the smoothed bootstrap. Let $\widehat{\epsilon}_t^{\dagger*} = \{\widehat{\epsilon}_1^{\dagger*}, \dots, \widehat{\epsilon}_T^{\dagger*}\}$ be a generic outer-level random sample from $\widehat{F}_{T,\beta}$, for a bandwidth $\beta > 0$ and $F_{T,\beta}^*$ be its empirical distribution function. Define the smoothed empirical distribution of $\widehat{\epsilon}_t^{\dagger*}$ as $\widehat{H}_{T,\eta}(t) = T^{-1} \sum_{i=1}^T K\left(\frac{t - \widehat{\epsilon}_i^{\dagger*}}{\eta}\right)$. Denote by $\widehat{\epsilon}_t^{**} = \{\widehat{\epsilon}_1^{**}, \dots, \widehat{\epsilon}_T^{**}\}$ a generic inner-level sample drawn from $\widehat{H}_{T,\eta}$ and by $H_{T,\eta}^*$ its empirical distribution. Define

$$\widehat{G}_{T,\eta}^*(t) = P\left[T^{1/2}\left(H_{T,\eta}^{*-1}(\alpha) - \widehat{H}_{T,\eta}^{-1}(\alpha)\right) \leq t \mid \widehat{\epsilon}_t, \widehat{\epsilon}_t^{\dagger*}\right], t \in \mathbb{R}. \quad (32)$$

Then, using $\widehat{J}_{n,\beta,\eta}$, the conditional distribution of $\widehat{G}_{T,\eta}^*\left(T^{1/2}\left(F_{T,\beta}^{*-1}(\alpha) - \widehat{F}_{T,\beta}^{-1}(\alpha)\right)\right)$ given $\widehat{\epsilon}_t$, the iterated smoothed bootstrap confidence interval for $F^{-1}(\alpha)$ is obtained by the following expression

$$I_{2,\gamma} = \left[F_T^{-1}(\alpha) - T^{-1/2}\widehat{G}_{T,\eta}^{-1}(\widehat{J}_{n,\beta,\eta}(1-\gamma)), F_T^{-1}(\alpha) - T^{-1/2}\widehat{G}_{T,\eta}^{-1}(\widehat{J}_{n,\beta,\eta}(\gamma))\right]. \quad (33)$$

Consequently, the iterated smoothed version for the VaR_{T+1} would be

$$\widehat{\sigma}_{T+1} I_{2,\gamma}. \quad (34)$$

There are two issues related to this method, one is the kernel for the smoothed distribution. Ho and Lee (2005) propose the triangular function given by $k(x) = 1 - |x|$, for $|x| \leq 1$. The second issue is the selection of an optimal bandwidth. There are in the literature many practical strategies for choosing it, but Ho and Lee (2005) propose a bootstrap procedure for the determination of the optimal bandwidth in practice.

2.2 A new bootstrap procedure

The new bootstrap procedure proposed in this paper extends the procedures proposed by Christoffersen and GonÇalves (2005) by estimating directly the quantile and the expectation in equations (6) and (7) with a second bootstrap step without making any particular assumption on the distribution of standardized returns. For each bootstrap replicate of the series of returns, we obtain \widehat{s}_{T+1}^{*2} as in equation (12). Then, we obtain $\epsilon_{T+1}^{*(i,n)}$ random draws from the empirical distribution of the standardized residuals $\widehat{\epsilon}_t$, for $n = 1, \dots, N$. Therefore, for each bootstrap replicate of the original series of returns, $i = 1, \dots, B$, we obtain a set of N disturbances $\{\epsilon_{T+1}^{*(i,1)}, \dots, \epsilon_{T+1}^{*(i,N)}\}$. The quantile $\widehat{q}^{(i)}$ can then be calculated as the 1% quantile of the empirical

distribution of $\epsilon_{T+1}^{*(i,n)}$ given by $F_B^*(\epsilon) = \frac{\#\{\epsilon_{T+1}^{*(i,n)} \leq \epsilon\}}{N}$. Therefore,

$$\widehat{q}^{*(i)} = q^{(i)}(F_B^*(\epsilon)) \quad (35)$$

$$\widehat{E}_T^* \left[\widehat{\epsilon}_{T+1}^* \mid \widehat{\epsilon}_{T+1}^* \leq \widehat{q}^{*(i)} \right] = \frac{\sum A}{M}$$

where $A = \{\epsilon_{T+1}^{*(i,n)} \leq q^{(i)}(F_B^*(\epsilon))\}$ and $M = \#A$. The algorithm is repeated B times, obtaining B measures of risk given by $\left\{\widehat{VaR}_{T+1}^{*(1)}, \dots, \widehat{VaR}_{T+1}^{*(B)}\right\}$ and $\left\{\widehat{ES}_{T+1}^{*(1)}, \dots, \widehat{ES}_{T+1}^{*(B)}\right\}$. Finally,

defining by $Q_{VB}^*(r)$ and $Q_{EB}^*(r)$ the bootstrap empirical distribution function of $\widehat{VaR}_{T+1}^{*(i)}$ and $\widehat{ES}_{T+1}^{*(i)}$ respectively, given by $Q_{VB}^*(r) = \frac{\#\left\{\widehat{VaR}_{T+1}^{*(i)} \leq r\right\}}{B}$ and $Q_{EB}^*(r) = \frac{\#\left\{\widehat{ES}_{T+1}^{*(i)} \leq r\right\}}{B}$, the $100(1 - \gamma)\%$ prediction interval for the VaR can be obtained by the percentile method of Efron (1981, 1982) given by¹

$$\left[q_{\frac{\gamma}{2}}(Q_{VB}^*(r)), q_{1-\frac{\gamma}{2}}(Q_{VB}^*(r)) \right] \quad (36)$$

and for the ES

$$\left[q_{\frac{\gamma}{2}}(Q_{EB}^*(r)), q_{1-\frac{\gamma}{2}}(Q_{EB}^*(r)) \right]. \quad (37)$$

Note that the FHS procedure proposed by Christoffersen and GonÇalves (2005) is based on computing the empirical quantile from $\widehat{\epsilon}_t^* = \frac{R_t^*}{\widehat{\sigma}_t^*} = \epsilon_t^* \frac{\sigma_t^*}{\widehat{\sigma}_t^*}$, while we propose to use again the original bootstrap draws from the empirical distribution of $\widehat{\epsilon}_t$ in order to estimate the quantile. Consequently, the difference between the procedure proposed in this paper and that of Christoffersen and GonÇalves (2005) depends on the ratio $\frac{\sigma_t^*}{\widehat{\sigma}_t^*}$. In very large sample sizes, if the estimator of the conditional variances is consistent, then both procedures should give similar prediction intervals. However, in small sample sizes or when using non-consistent estimators of the conditional variances, both procedures may give different results. We will show that although this difference can be relatively small when computing the quantile involved in the calculation of the VaR , it can be important when computing the expectation involved in the ES .

Finally, note that, the quantile of $\epsilon_{T+1}^{*(i,n)}$ can also be estimated by using the Hill estimator and the Gram-Charlier and Cornish-Fisher expansions as we will show in the next section.

3 Monte Carlo experiments

In this section, we carry out Monte Carlo experiments to analyze the finite sample performance of the proposed bootstrap procedure for constructing prediction intervals for the VaR and ES . We compare our procedure with those proposed by Christoffersen and GonÇalves (2005) and Ho and Lee (2005).

We generate replicates from a $GARCH(1, 1)$ model with parameters $\alpha_0 = 0.002$, $\alpha_1 = 0.05$ and $\beta = 0.9$ and a $EGARCH(1, 1)$ model with $\alpha_0 = -0.17$, $\alpha_1 = 0.15$, $\gamma = -0.13$ and $\beta = 0.9^2$. We consider four sample sizes, $T = 250, 500, 1000$ and 3000 and three alternative distributions of the standardized observations, ϵ_t , namely Normal, Student-8 and a Skewed-Student distribution, with 10 degrees of freedom and a coefficient of asymmetry equal to -0.11 ;

¹In order to improve the coverage of the confidence intervals they propose the bias corrected method that applied to the VaR case is given by $\left[Q_{VB}^{*-1}\left(\Phi\left[2z_0 + z_{\frac{\gamma}{2}}\right]\right), Q_{VB}^*\left(\Phi\left[2z_0 + z_{1-\frac{\gamma}{2}}\right]\right) \right]$, where z_γ is the γ th quantile of a Normal distribution and $z_0 = \Phi^{-1}\left[Q_{VB}^*\left(\widehat{VaR}_{T+1}\right)\right]$. Similar formulation is given by the ES case. Using this procedure we obtain the same conclusions than with the percentile method. Alternatively, Efron (1987) proposes a generalization of the bias corrected method named, the accelerated bias corrected method. Berkowitz and Kilian (2000) mention that the implementation of the latter method to time series has not been investigated and it is not straightforward. Therefore, we do not consider this alternative.

²The Exponential GARCH ($EGARCH$) model of Nelson (1991) is given by

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1 [|\epsilon_{t-1}| - E(|\epsilon_{t-1}|)] + \beta_1 \ln \sigma_{t-1}^2 + \gamma \epsilon_{t-1}$$

see Hansen (1994) for a complete description of the Skewed distribution. Although $T = 250$ may seem a rather small value for real life applications, it is important to note that the Basel Commission requires to compute the VaR with at least one year of data which corresponds approximately to 250 daily observations. For each replicate, we estimate the $GARCH(1,1)$ and the $EGARCH(1,1)$ parameters by ML by treating the degrees of freedom and asymmetry parameters as unknown parameters. We compute prediction intervals for the 1% VaR_{T+1} and ES_{T+1} based on 1000 Monte Carlo simulations, $B = 1000$ bootstrap replicates for nominal levels of 90% and 95%, by implementing the procedures proposed by Christoffersen and Gonçaves (2005) by assuming i) the true distribution ($C - G - D$), ii) the FHS ($C - G - FHS$), iii) EVT ($C - G - H$) and iv) Gram-Charlier and Cornish-Fisher expansions ($C - G - GCCF$). We also construct the prediction intervals by the bootstrap procedure proposed in this paper by i) FHS ($NR - FHS$), ii) EVT ($NR - H$) and iii) by the Gram-Charlier and Cornish-Fisher approximations ($NR - GCCF$). Finally, the prediction intervals are computed by the procedure proposed by Ho and Lee (2005).

Table 1 reports the average coverages of the 90% intervals³ for the VaR_{T+1} and ES_{T+1} when the series are generated by the $GARCH(1,1)$ model, shows that regardless of the particular distribution, the coverages of the VaR computed by assuming the true error distribution are well under the nominal. The same can be said when using the Hill estimator regardless of whether the bootstrap replicates are obtained as proposed by Christoffersen and Gonçaves (2005) or as proposed in this paper. The coverages are closer to the nominal when using either FHS or the $GCCF$ expansions and slightly better with the former. Furthermore note that the coverages are clearly closer to the nominal 90% when using the bootstrap procedure proposed in this paper instead of that proposed by Christoffersen and Gonçaves (2005). Therefore, with respect to the one-step ahead VaR , the bootstrap procedure proposed in this paper implemented to FHS gives coverages closer to the nominal for all assumed distributions and sample sizes. Additionally, Table 1 also reports the average coverages for the ES . In this case, we can observe that the intervals constructed assuming the true distribution are well under nominal coverage. Furthermore, when comparing the coverages obtained when implementing FHS , Hill estimator or the $GCCF$ expansions, it seems that the latter is, in general more appropriate. The exceptions are when the sample size is large and the error distribution has heavy tails. In any case, the results are in general better when implementing the second bootstrap step proposed in this paper.

Table 2 reports the average coverages of the 90% intervals for the VaR_{T+1} and ES_{T+1} when the series are generated by the $EGARCH$ model. The first clear difference between the results obtained for the $GARCH$ and the $EGARCH$ models is that in the former, the coverages are smaller than the nominal while in the latter, there is an overcoverage. In any case, although the differences between alternative procedures are smaller than those observed in Table 1 for the $GARCH$ model, we still observe that the coverages are closer to the nominal when using the two-step bootstrap proposed in this paper. In the context of the $EGARCH$ model, FHS seems to work better for both VaR and ES prediction intervals.

Finally, we also implement the proposal by Ho and Lee (2005). For the smoothed procedure we use $B = 1000$ bootstrap replicates, and for the smoothed iterated, $B = 1000$ replicates for

³Results for 95% intervals are similar and not reported to save space.

the outer level and $C = 500$ replicates for the inner level. Last column of Table 1 reports the coverage for a nominal level of 90% using the $GARCH(1,1)$ model. We can observe, that the coverages constructed with the smoothed bootstrap are better than those obtained with the percentile method, except when the data is generated using the Skewed-Student distribution and the sample size is 250. Similar results can be concluded when comparing the coverages obtained for the $EGARCH$ model. Last column of Table 2 shows that for all de distributions and sample sizes the coverages are improved.

One thing that we should notice is the importance of the selection of the bandwidth in those procedures. The idea is to fix a grid of proposed bandwidths, then generate smoothed and smoothed iterated intervals for each bandwidth and selecting the bandwidth which gives the best coverage. This procedure is very time consuming because of the use of many replicates bootstrap in each case, and even worse for the smoothed bootstrap because for each bootstrap replicate B , a bootstrap replicate C is generated. Therefore, we need a lot of time for selecting the bandwidth and then also for constructing the interval.

4 Empirical Application

In this section, we implement the methods described above to obtain prediction densities for VaR and ES of three series of daily returns, the $S\&P500$ index, the $IBEX35$ and the $Euro/Dollar$ observed from 01/09/2003 to 19/01/2010. The source of the data was the EcoWin database. Figure 1, that plots the series of returns, shows the effect of the crisis on an increased volatility at the end of the sample period. Figure 1 also plots for each of the three returns' series, the correlogram of absolute returns and the cross-correlations between returns, y_t , and future square returns, y_{t+h} together with their 95% confidence bands computed as suggested by Diebold (1988) to account for the presence of conditional heteroscedasticity. It is clear that the sample correlations of absolute returns are positive and highly persistent, being significantly different from zero even for very long lags. Therefore, returns could be conditionally heteroscedastic, possibly with long-memory. In the case of the $Euro/Dollar$ the sample correlations of absolute returns are also positive but the presence of conditional heteroscedasticity is less strong. However, the cross-correlations do not give any clear evidence of the presence of leverage effect.

Table 3 shows the 90% prediction intervals for the one-step ahead VaR and the ES using the same procedures implemented in the Monte Carlo simulations using the $GARCH(1,1)$ and the $EGARCH(1,1)$ models with Normal errors for the $S\&P500$, the $IBEX35$ and the $Euro/Dollar$ returns. We observe that the intervals for the VaR are clearly narrower than those of the ES and the intervals using the $GARCH(1,1)$ model are also narrower than those using the $EGARCH(1,1)$. In the Monte Carlo experiments we observe the same behavior because the coverages using the $EGARCH(1,1)$ were over the nominal. We can also notice that the differences among interval widths is smaller in the case of the $Euro/Dollar$. This could be expected because of the lack of heteroscedaticity. In the case of the $S\&P500$, the $IBEX35$ the differences among procedures are remarkable.

The intervals for the VaR calculated using the $GCCF$ approximations are clearly wider than the others when using the bootstrap procedure proposed in this paper and also with that proposed by Christoffersen and GonÇalves (2005). On the other hand, the intervals constructed

with the Normal distribution are the narrowest. Moreover, regardless the model, when implementing *FHS*, Hill estimator or the *GCCF* expansions the widths are larger for the *ES*. This is consistent with the results obtained in the Monte Carlo experiment where the best coverages were obtained when using these procedures.

We can also notice that the *VaR* prediction interval widths obtained with the procedure proposed in this paper using *FHS* are not always larger than the rest of the procedures. However, as we conclude in the last section, it provides the best coverages.

With respect to the intervals calculated using the procedure of Ho and Lee (2005) we can observe that the upper limits are greater than those obtained with the rest of the procedures when using the *GARCH*(1,1) model for the *S&P500*, the *IBEX35*. On the other hand, for the *IBEX35* the upper limit is near to the others. However, when using the *EGARCH*(1,1) model, the upper limit is, in general, lower than the others, except when the intervals are calculated using the *GCCF* expansions. As long as the upper limit of the prediction intervals using the procedure of Ho and Lee (2005) are greater than the rest, the coverages will be closer to the nominal.

5 Conclusions

In this paper, we extend the bootstrap procedure proposed by Christoffersen and Gonçaves (2005) for taking into account the uncertainty associated to parameter estimation when constructing prediction intervals for two of the most famous measures of risk, the *VaR* and the *ES*. We propose a second bootstrap step which avoid an extra source of uncertainty by bootstrapping directly from the original standardized residuals when computing the quantile of the error distribution.

Furthermore, we incorporate in our procedure the known fact that the financial series of returns have leverage effect. Thus, we calculate the volatility assuming the *EGARCH* model of Nelson (1991).

Several Monte Carlo experiments are carried out to analyze the finite sample properties of the proposed procedure and compare them with those of several alternatives. We show that in most of the cases, our procedure produces coverage closer to the nominal. The proposed procedure has coverages very close to the nominal when computing the *VaR* using the corresponding quantile of the bootstrap empirical distribution. However, when computing the *ES* the results can be better when using the Hill or *GCCF* approximations.

It will also be interesting which is the behavior of the proposed procedure in the presence of misspecification.

Regarding to the calculation of the confidence intervals, there are some alternative proposals to the typically used percentile method. We apply the bias corrected method of Efron (1987) and the smoothed iterated bootstrap method of Ho and Lee (2005). We observe that, in this case, the results obtained with the bias corrected method are very similar that those obtained with the percentile method. On the other hand, although with the smoothed iterated method we can improve the coverage of our procedure, there is an strong dependence with the optimal selection of the bandwidth, and additionally, it is very time consuming because for each outer

level bootstrap it is needed an inner level bootstrap, making the computational time grow.

For an illustration, we implement the proposed procedure to real series of returns, constructing the point estimates of the measures of risk and also the prediction intervals.

An interesting topic for further research is the extension of the analysis carried out in this paper to incorporate Stochastic Volatility (*SV*) models instead of *GARCH* models for the estimation of the volatility involved in the procedure.

Table 1. GARCH(1,1) 90% Prediction intervals coverage rates obtained by Monte Carlo simulations

Distribution		90% Prediction Intervals for the one-step-ahead 1%VaR									
		T	C-G-D	C-G-FHS	C-G-H	C-G-GCCF	NR-FHS	NR-H	NR-GCCF	Smoothed	
Normal	250	76.80%	81.70%	73.00%	80.10%	88.30%	72.90%	83.80%	91.40%		
	500	72.30%	83.30%	73.60%	79.20%	87.40%	73.30%	83.40%	91.60%		
	1000	71.40%	80.60%	74.40%	79.70%	84.20%	75.90%	83.50%	90.90%		
	3000	70.30%	80.60%	72.20%	79.10%	84.70%	72.10%	83.10%	90.60%		
Student-v	250	77.10%	82.80%	74.50%	76.40%	88.70%	72.30%	82.20%	88.70%		
	500	74.30%	86.17%	76.83%	80.02%	89.01%	76.60%	84.16%	89.10%		
	1000	73.00%	84.70%	76.60%	81.10%	87.80%	77.50%	87.20%	91.50%		
	3000	78.30%	86.30%	83.20%	77.70%	89.30%	84.20%	84.70%	89.40%		
Skewed Student-v	250	76.40%	84.20%	73.50%	77.40%	89.50%	71.40%	84.70%	91.90%		
	500	76.20%	87.30%	79.40%	82.10%	89.60%	78.90%	87.70%	89.90%		
	1000	75.50%	87.10%	78.10%	82.80%	89.50%	79.90%	87.80%	90.60%		
	3000	76.60%	86.40%	81.70%	81.20%	89.70%	83.30%	87.70%	90.20%		

Distribution		90% Prediction Intervals for the one-step-ahead 1%ES									
		T	C-G-D	C-G-FHS	C-G-H	C-G-GCCF	NR-FHS	NR-H	NR-GCCF		
Normal	250	76.80%	70.90%	80.90%	87.80%	74.40%	84.90%	86.20%			
	500	72.30%	73.90%	82.40%	88.10%	76.00%	86.50%	88.70%			
	1000	71.40%	76.80%	81.40%	84.20%	79.30%	88.80%	90.00%			
	3000	70.30%	78.60%	79.10%	80.60%	80.60%	86.90%	88.00%			
Student-v	250	77.10%	63.30%	77.80%	93.40%	67.30%	80.10%	97.90%			
	500	74.30%	70.09%	82.27%	87.12%	71.28%	72.60%	96.45%			
	1000	73.00%	77.20%	84.50%	85.50%	78.30%	87.30%	96.60%			
	3000	78.30%	84.60%	86.60%	88.40%	85.70%	91.40%	94.60%			
Skewed Student-v	250	76.40%	60.50%	78.00%	89.90%	65.60%	80.90%	94.50%			
	500	76.20%	75.20%	84.50%	86.00%	75.70%	85.50%	92.60%			
	1000	75.50%	81.30%	87.60%	86.20%	82.90%	90.40%	95.10%			
	3000	76.60%	81.70%	85.30%	77.90%	83.90%	89.10%	91.50%			

Table 2. EGARCH(1,1) 90% Prediction intervals coverage rates obtained by Monte Carlo simulations

Distribution		90% Prediction Intervals for the one-step-ahead 1%VaR									
		T	C-G-D	C-G-FHS	C-G-H	C-G-GCCF	NR-FHS	NR-H	NR-GCCF	Smoothed	
Normal	250	93.10%	93.30%	94.40%	93.40%	93.40%	93.60%	94.70%	88.90%		
	500	93.90%	93.90%	93.80%	94.70%	93.80%	93.70%	94.30%	87.90%		
	1000	94.30%	94.50%	95.00%	94.70%	94.20%	94.70%	94.50%	87.80%		
	3000	92.50%	93.10%	93.20%	93.10%	93.00%	93.20%	92.50%	89.60%		
Student-v	250	92.70%	94.00%	93.70%	94.90%	92.70%	92.70%	93.70%	90.50%		
	500	93.40%	94.80%	93.40%	94.50%	93.80%	93.30%	93.90%	91.20%		
	1000	90.70%	91.10%	90.90%	91.00%	90.80%	90.80%	90.60%	88.80%		
	3000	91.60%	91.70%	91.70%	91.50%	91.60%	91.50%	91.50%	89.70%		
Skewed Student-v	250	92.60%	94.20%	92.60%	94.10%	92.60%	92.50%	93.30%	90.70%		
	500	93.40%	93.50%	94.00%	94.10%	92.70%	93.20%	93.50%	90.40%		
	1000	92.10%	92.40%	91.40%	92.60%	91.90%	91.70%	91.80%	89.20%		
	3000	88.60%	88.60%	88.20%	88.70%	88.80%	88.30%	88.80%	89.80%		

Distribution		90% Prediction Intervals for the one-step-ahead 1%ES									
		T	C-G-D	C-G-FHS	C-G-H	C-G-GCCF	NR-FHS	NR-H	NR-GCCF		
Normal	250	93.10%	94.00%	94.20%	91.60%	93.40%	95.50%	91.60%			
	500	93.90%	94.10%	94.40%	94.00%	94.20%	94.90%	93.50%			
	1000	94.30%	93.70%	94.00%	94.30%	94.00%	94.10%	94.10%			
	3000	92.50%	92.90%	92.50%	93.00%	92.50%	92.40%	92.40%			
Student-v	250	92.70%	93.00%	93.50%	95.70%	91.70%	93.10%	94.90%			
	500	93.40%	92.70%	94.20%	94.60%	92.80%	93.40%	94.20%			
	1000	93.50%	93.00%	93.80%	92.90%	92.80%	93.00%	93.70%			
	3000	91.60%	92.00%	91.50%	91.50%	91.80%	91.80%	91.50%			
Skewed Student-v	250	92.60%	91.80%	93.10%	95.40%	90.50%	93.20%	94.10%			
	500	93.40%	94.00%	93.50%	94.00%	93.30%	94.00%	94.40%			
	1000	92.10%	92.10%	92.50%	92.70%	92.00%	92.50%	92.50%			
	3000	88.60%	88.20%	88.90%	88.70%	88.30%	89.10%	89.10%			

Figure 1. S&P500, IBEX35 and Euro/Dollar returns observed from 1st September 2003 up to 19th January 2010, correlogram of absolute returns and cross-correlogram of returns and squared returns together with their corresponding 95% confidence bands.

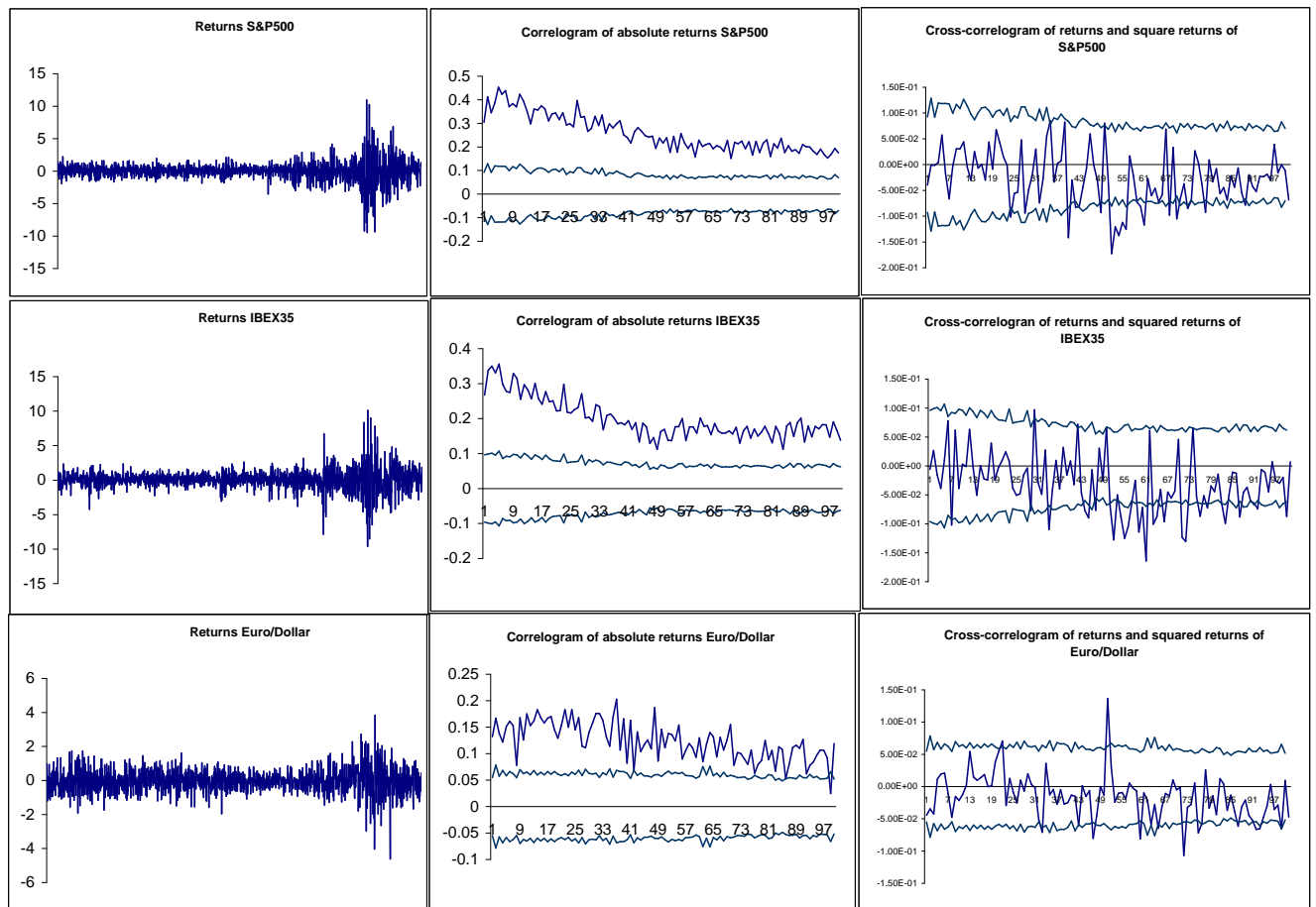


Table 3. 90% prediction intervals for 1% VaR and 1%ES for S&P500, IBEX35 and Euro/Dollar returns

		S&P500			Euro/Dollar			IBEX35		
Model		Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width
GARCH-N	C-G-N	-0.0196	-0.0178	0.0018	-0.0146	-0.0130	0.0016	-0.0224	-0.0205	0.0020
	C-G-FHS	-0.0220	-0.0194	0.0026	-0.0164	-0.0134	0.0030	-0.0278	-0.0222	0.0056
	C-G-H	-0.0222	-0.0192	0.0030	-0.0155	-0.0132	0.0023	-0.0267	-0.0224	0.0043
	C-G-GCCF	-0.0278	-0.0193	0.0085	-0.0167	-0.0137	0.0030	-0.0322	-0.0233	0.0089
	NR-FHS	-0.0223	-0.0191	0.0032	-0.0166	-0.0131	0.0035	-0.0283	-0.0217	0.0066
	NR-H	-0.0223	-0.0188	0.0035	-0.0155	-0.0129	0.0026	-0.0270	-0.0221	0.0049
	NR-GCCF	-0.0286	-0.0190	0.0097	-0.0170	-0.0135	0.0035	-0.0336	-0.0229	0.0107
H-L		-0.0183	0.0183		-0.0135	0.0135		-0.0204	0.0204	
EGARCH-N	C-G-N	-0.0455	-0.0118	0.0337	-0.0240	-0.0085	0.0155	-0.0453	-0.0146	0.0307
	C-G-FHS	-0.0519	-0.0132	0.0388	-0.0255	-0.0089	0.0165	-0.0547	-0.0171	0.0376
	C-G-H	-0.0507	-0.0129	0.0378	-0.0248	-0.0086	0.0162	-0.0515	-0.0164	0.0352
	C-G-GCCF	-0.0575	-0.0138	0.0437	-0.0262	-0.0091	0.0171	-0.0538	-0.0169	0.0369
	NR-FHS	-0.0522	-0.0134	0.0388	-0.0260	-0.0091	0.0170	-0.0552	-0.0178	0.0375
	NR-H	-0.0509	-0.0129	0.0380	-0.0248	-0.0087	0.0161	-0.0520	-0.0164	0.0356
	NR-GCCF	-0.0617	-0.0139	0.0478	-0.0264	-0.0093	0.0171	-0.0551	-0.0171	0.0380
H-L		-0.0136	0.0136		-0.0084	0.0084		-0.0168	0.0168	

		S&P500			Euro/Dollar			IBEX35		
Model		Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width	Lower Limit	Upper Limit	Width
GARCH-N	C-G-N	-0.0224	-0.0204	0.0020	-0.0167	-0.0149	0.0018	-0.0257	-0.0234	0.0023
	C-G-FHS	-0.0293	-0.0219	0.0073	-0.0198	-0.0158	0.0040	-0.0372	-0.0280	0.0092
	C-G-H	-0.0282	-0.0221	0.0062	-0.0205	-0.0159	0.0046	-0.0380	-0.0281	0.0099
	C-G-GCCF	-0.0285	-0.0210	0.0075	-0.0209	-0.0166	0.0043	-0.0330	-0.0231	0.0099
	NR-FHS	-0.0292	-0.0216	0.0075	-0.0197	-0.0157	0.0040	-0.0374	-0.0275	0.0099
	NR-H	-0.0277	-0.0209	0.0069	-0.0205	-0.0152	0.0053	-0.0379	-0.0267	0.0112
	NR-GCCF	-0.0286	-0.0155	0.0131	-0.0212	-0.0160	0.0052	-0.0328	-0.0164	0.0165
EGARCH-N	C-G-N	-0.0521	-0.0135	0.0386	-0.0275	-0.0097	0.0178	-0.0519	-0.0168	0.0352
	C-G-FHS	-0.0639	-0.0160	0.0479	-0.0306	-0.0105	0.0201	-0.0651	-0.0207	0.0444
	C-G-H	-0.0645	-0.0160	0.0485	-0.0312	-0.0106	0.0206	-0.0678	-0.0214	0.0463
	C-G-GCCF	-0.0623	-0.0139	0.0484	-0.0326	-0.0109	0.0217	-0.0664	-0.0211	0.0453
	NR-FHS	-0.0657	-0.0160	0.0497	-0.0308	-0.0106	0.0202	-0.0646	-0.0203	0.0443
	NR-H	-0.0642	-0.0156	0.0486	-0.0310	-0.0104	0.0206	-0.0672	-0.0210	0.0462
	NR-GCCF	-0.0573	-0.0106	0.0466	-0.0325	-0.0113	0.0212	-0.0658	-0.0210	0.0448

References

- Acerbi, C. and D. Tasche (2002). Expected shortfall: a natural coherent alternative to value at risk. *Economic Notes* 31, 379–388.
- Artzner, P., F. Delbaen, J. Eber, and D. Heath (1997). Thinking coherently. *Risk* 10, 68–71.
- Bams, D., T. Lehnert, and C. Wolff (2005). An evaluation framework for alternative var models. *Journal of International Money and Finance* 24, 944–958.
- Basak, S. and A. Shapiro (2001). Value-at-risk based risk management: Optimal policies and asset prices. *Review of Financial Studies* 14, 371–405.
- Berkowitz, J. and L. Kilian (2000). Recent developments in bootstrapping time series. *Econometric Review* 19, 1–48.
- Chan, N., S. Deng, L. Peng, and Z. Xia (2007). Interval estimation for the conditional value at risk based on GARCH models with heavy tailed innovations. *Journal of Econometrics* 137, 556–576.
- Chen, S. and C. Tang (2005). Nonparametric inference of value-at-risk for dependent financial returns. *Journal of Financial Econometrics* 3, 227–255.
- Chou, J., H. Yu, and Z. Chen (2008). Interval estimation of value at risk for Taiwan weighted stock index based on extreme value theory. *Journal of the Chinese Institute of Industrial Engineers* 25, 31–42.
- Christoffersen, P. and S. GonÇalves (2005). Estimation risk in financial risk management. *Journal of Risk* 7, 1–28.
- Diebold, F. (1988). *Empirical Modeling of Exchange Rate Dynamics*. New York: Springer-Verlag.
- Efron, B. (1981). Nonparametric standard errors and confidence intervals. *The Canadian Journal of Statistics*, 139–172.
- Efron, B. (1982). The jackknife, the bootstrap, and other resampling plans. *Regional Conference Series in Applied Mathematics*, No.38. Philadelphia: SIAM.
- Efron, B. (1987). Better bootstrap confidence intervals. *Journal of the American Statistical Association*, 171–185.
- Giamouridis, D. (2006). Estimation risk in financial risk management: A correction. *Journal of Risk* 8, 121–125.
- Gilli, M. and E. Killezi (2006). An application of extreme value theory for measuring financial risk. *Computational Economics* 27, 207–228.
- Gnedenko, B. (1943). Sur la distribution limite du terme maximum d’une série aléatoire. *Annals of Mathematics* 44, 423–453.
- Hall, P. and Q. Yao (2003). Data tilting for time series. *Journal of the Royal Statistical Society, Series B* 65, 425–442.

- Hansen, B. (1994). Autoregressive conditional density estimation. *International Economic Review* 35, 705–730.
- Hartz, C., M. S. and M. Paoletta (2006). Accurate value-at-risk forecasting based on the normal-GARCH model. *Computational Statistics & Data Analysis* 51, 2295–2312.
- Ho, Y. and S. Lee (2005). Iterated smoothed bootstrap confidence intervals for population quantiles. *The Annals of statistics*, 437–462.
- Lan, H., B. Nelson, and J. Staum (2008). A confidence interval procedure for expected shortfall risk measurement via two-level simulation. *Working Paper, Department of Industrial Engineering and Management Sciences, Northwestern University*.
- Nelson, D. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* 59, 347–370.
- Nieto, M. and E. Ruiz (2009). Measuring financial risk: Comparison of alternative procedures to estimate VaR and ES. *Working Paper 08-73, Universidad Carlos III de Madrid*.
- Pascual, L., J. Romo, and E. Ruiz (2006). Bootstrap prediction for returns and volatilities in GARCH models. *Computational Statistics and Data Analysis* 50, 2293–2312.
- Peng, L. and Y. Qi (2003). Confidence intervals for high quantiles of a heavy tailed distribution. *Technical Report*, Georgia Institute of Technology, United States.
- Ruiz, E. and L. Pascual (2002). Bootstrapping financial time series. *Journal of Economic Surveys* 16, 271–300.
- Yamai, Y. and T. Yoshihara (2005). Value-at-risk versus expected shortfall: A practical perspective. *Journal of Banking & Finance* 29, 997–1015.