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Circular Bernstein polynomial distributions

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This paper introduces a new non-parametric approach to the modeling of circular data, based on the use of Bernstein polynomial densities which generalizes the standard Bernstein polynomial model to account for the specific characteristics of circular data. It is shown that the trigonometric moments of the proposed circular Bernstein polynomial distribution can all be derived in closed form. We comment on how to fit the Bernstein polynomial density approximation to a sample of data and illustrate our approach with a real data example.

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1. Introduction

Problems where the data are angular directions occur in many different scientific fields such as biology (direction of movement of migrating animals), meteorology (wind directions) and geology (directions of joints and faults). Also, phenomena that are periodic in time such as times of hospital admittance for births or the times when crimes are committed may also be converted to angular data via a simple transformation modulo some period. Data of this type are commonly known as circular data and are usually represented as points on the circumference of an unit circle or as angles, θ, where $0 \leq \theta < 2\pi$ radians, which represent the positive angle of rotation from some arbitrarily chosen origin, $\theta = 0$.

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A number of parametric models for circular data have been developed using a variety of techniques, see e.g. Mardia and Jupp (1999) for a full review. However, most parametric models developed for circular data are unimodal and symmetric but in many cases,both multimodal and asymmetric data may be encountered. In such cases, semi-parametric or non-parametric approaches might be preferred. Semi-parametric approaches based on trigonometric sums and mixtures of von Mises or circular normal distributions have been introduced in Fernández-Durán (2004) and Mooney et al. (2003) respectively, but with the exception of Fisher (1989), where a kernel based approach is considered, there has been little work on non-parametric modeling of circular data.

In this paper, we introduce an alternative non-parametric approach based on the use of Bernstein polynomials. It is well known that the Bernstein polynomial is a useful tool for interpolating functions defined on a closed interval. Bernstein polynomials have been proposed as density estimators for variables with finite support in a number of articles, see e.g. (Vitale, 1975; Petrone, 1999a,b; Petrone and Wassermann, 2002; Babu et al., 2002; Kakizawa, 2004). However, a problem with generalizing standard Bernstein polynomial density approaches to circular data is that these can lead to fitted densities with discontinuities which is generally unreasonable for continuous circular data. Here, we show that this problem is easily solved.

The article is organized as follows. In Section 2 we define the Bernstein polynomial density approximation and show how this can be extended to circular variables. In Section 3 we demonstrate how to calculate the circular moments of a Bernstein polynomial density and in Section 4 we comment on how the model can be fitted to a sample of data. Finally, we illustrate our results with a real data set in Section 5.

2. The circular Bernstein polynomial distribution

Let X be a random variable with support $[0, 1]$ and continuous distribution function $F_X(\cdot)$. Then the Bernstein polynomial distribution function of order k

is defined to be

$$B_k(x) = \sum_{j=0}^{k} F_X\left(\frac{j}{k}\right)\left(\begin{array}{c} k \\ j \end{array}\right) x^j(1-x)^{k-j} \quad \text{for } 0 \le x \le 1 \text{ and } k \in \mathbb{N} \tag{1}$$

It is well known that $B_k(x)$ converges uniformly to $F_X(x)$ as k goes to infinity, see e.g. Vitale, (1975). The associated Bernstein density function is given by

$$b_k(x) = \sum_{j=1}^{k}\left(F_X\left(\frac{j}{k}\right) - F_X\left(\frac{j-1}{k}\right)\right)\beta(x \mid j, k-j+1), \tag{2}$$

where $\beta(\cdot \mid a, b)$ is a beta density function:

$$\beta(x \mid a, b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \tag{3}$$

and $B(a,b) = (a+b-1)!/(a-1)!(b-1)!$, for $a, b \in \mathbb{N}$, is the beta function.

Clearly it is straightforward to extend the use of Bernstein polynomials to densities defined on any closed interval, e.g. $[0, 2\pi]$. However, in order to define a distribution on the circle, it is first necessary to formally define the density function of a circular random variable.

The density function, $f_\Theta(\theta)$ of a continuous, circular random variable, Θ, is a non-negative, continuous function such that

$$f_\Theta(\theta + 2\pi) = f_\Theta(\theta) \quad \text{for } \theta \in \mathbb{R}$$

and $\int_0^{2\pi} f_\Theta(\theta)\, d\theta = 1$. Then, in order to define a cumulative distribution function of a circular random variable, it is necessary to establish an origin, $0 \le \nu < 1$. Given this origin, the cumulative distribution function is

$$F_\Theta^\nu(\theta) = \int_\nu^{\nu+\theta} f_\Theta(u)\, du \quad \text{for } 0 \le \theta < 2\pi.$$

However, if we wish to consider the Bernstein polynomial density approximation of order k, with respect to the origin ν, that is

$$f_k^\nu(\theta) = \frac{1}{2\pi}\sum_{j=1}^{k}\left(F_\Theta^\nu\left(\frac{2\pi j}{k}\right) - F_\Theta^\nu\left(\frac{2\pi(j-1)}{k}\right)\right)\beta\left(\frac{\theta}{2\pi}\middle| j, k-j+1\right) \tag{4}$$

for this to be a strictly continuous, circular density then it is necessary that,

$$F_\Theta^\nu\left(\frac{2\pi}{k}\right) = 1 - F_\Theta^\nu\left(\frac{2\pi(k-1)}{k}\right), \tag{5}$$

where $F_\Theta^\nu(\theta)$. The following theorem guarantees the existence of at least one origin satisfying (5).

Theorem 1. Let f be a density function for a continuous, circular random variable. Then there exists at least one point $\nu \in [0, 2\pi)$ such that for any $k \in \mathbb{N}$,

$$\int_\nu^{\nu+\frac{2\pi}{k}} f(\theta)\, d\theta = \int_{\nu-\frac{2\pi}{k}}^{\nu} f(\theta)\, d\theta.$$

Proof. Define $G(\nu) = \int_\nu^{\nu+\frac{2\pi}{k}} f(\theta)\, d\theta - \int_{\nu-\frac{2\pi}{k}}^{\nu} f(\theta)\, d\theta$. If there exist two points, $0 \leq \nu_1 \neq \nu_2 < 2\pi$ such that $G(\nu_1) \leq 0$ and $G(\nu_2) \geq 0$, then by Bolzano's theorem, there exists at least one point, $0 \leq \nu_0 < 2\pi$ such that $G(\nu_0) = 0$. Otherwise, suppose that $G(\nu)$ is always positive. Then, we have

$$\int_{\nu-\frac{2\pi}{k}}^{\nu} f(\theta)\, d\theta < \int_\nu^{\nu+\frac{2\pi}{k}} f(\theta)\, d\theta < \int_{\nu+2\pi-\frac{2\pi}{k}}^{\nu+2\pi} f(\theta)\, d\theta$$

which is impossible, as, due to the periodicity of f, we have that

$$\int_{\nu-\frac{2\pi}{k}}^{\nu} f(\theta)\, d\theta = \int_{\nu+2\pi-\frac{2\pi}{k}}^{\nu+2\pi} f(\theta)\, d\theta.$$

Similarly, G cannot always be negative and so the theorem is proved. □

3. Trigonometric moments of the circular Bernstein polynomial distribution

For a circular random variable, Θ, the p'th trigonometric moment is defined to be

$$\begin{aligned}
\boldsymbol{\mu}'_p &= E\left[e^{ip\Theta}\right] = E[\cos p\Theta] + iE[\sin p\Theta] \quad \text{where } i = \sqrt{-1} \\
&\stackrel{\text{def}}{=} \rho_p\left(\cos \mu'_p + i \sin \mu'_p\right) \quad \text{where} \\
\rho_p &= \sqrt{E[\cos p\Theta]^2 + E[\sin p\Theta]^2} \quad \text{and} \qquad (6) \\
\mu'_p &= \begin{cases} \tan^{-1} \frac{E[\sin p\Theta]}{E[\cos p\Theta]} & \text{if } \; E[\sin p\Theta] > 0 \text{ and } E[\cos p\Theta] > 0 \\ \tan^{-1} \frac{E[\sin p\Theta]}{E[\cos p\Theta]} + \pi & \text{if } \; E[\cos p\Theta] < 0 \\ \tan^{-1} \frac{E[\sin p\Theta]}{E[\cos p\Theta]} + 2\pi & \text{if } \; E[\sin p\Theta] < 0 \text{ and } E[\cos p\Theta] > 0 \end{cases} \qquad (7)
\end{aligned}$$

for $p = 1, 2, \ldots$. In particular, when $p = 1$, we write μ for μ_1' and ρ for ρ_1. Then, μ is the mean direction and ρ is the mean resultant length, see e.g. Mardia and Jupp (1999) for more details.

The trigonometric moments of the circular Bernstein polynomial distribution can be derived using the following theorem.

Theorem 2. The p'th trigonometric moments of a circular Bernstein polynomial distribution are given by

$$\boldsymbol{\mu}_p' = \sum_{j=1}^{k} w_j \left(E[\cos 2\pi p \mathcal{B}_j] + iE[\sin 2\pi p \mathcal{B}_j] \right)$$

where

$$\omega_j = F_\Theta^\nu \left(\frac{j}{k} \right) - F_\Theta^\nu \left(\frac{j-1}{k} \right)$$

and where $\mathcal{B}_j$ is a beta random variable with density function $\beta(\cdot \mid j, k-j+1)$ as defined in (3) such that

$$\begin{aligned}
E[\cos(2\pi p \mathcal{B}_j)] &= \tfrac{1}{B(j,k-j+1)} \textstyle\sum_{r=0}^{k-j} (-1)^r \begin{pmatrix} k-j \\ r \end{pmatrix} I_p(j+r-1) && (8) \\
&= \tfrac{1}{B(j,k-j+1)} \textstyle\sum_{r=0}^{j-1} (-1)^r \begin{pmatrix} j-1 \\ r \end{pmatrix} I_p(k-j+r) && (9) \\
E[\sin(2\pi p \mathcal{B}_j)] &= \tfrac{1}{B(j,k-j+1)} \textstyle\sum_{r=0}^{k-j} (-1)^r \begin{pmatrix} k-j \\ r \end{pmatrix} J_p(j+r-1) && (10) \\
&= \tfrac{1}{B(j,k-j+1)} \textstyle\sum_{r=0}^{j-1} (-1)^{r+1} \begin{pmatrix} j-1 \\ r \end{pmatrix} J_p(k-j+r) && (11)
\end{aligned}$$

where

$$I_p(j) = \int_0^1 \cos(2\pi x) x^j \, dx \qquad J_p(j) = \int_0^1 \sin(2\pi x) x^j \, dx \quad \text{for } j = 0, 1, 2, \ldots$$

and $I_p(0) = J_p(0) = I_p(1) = 0$, $J_p(1) = -\frac{1}{2\pi p}$ and for $C = 2, 3, 4, \ldots$,

$$\begin{aligned}
I_p(C) &= \textstyle\sum_{c=1}^{\lfloor \frac{C}{2} \rfloor} (-1)^{c-1} \frac{C!}{(C-2c+1)!} \frac{1}{(2\pi p)^{2c}} && (12) \\
J_p(C) &= \textstyle\sum_{c=1}^{\lfloor \frac{C+1}{2} \rfloor} (-1)^{c} \frac{C!}{(C-2c+2)!} \frac{1}{(2\pi p)^{2c-1}}. && (13)
\end{aligned}$$

PROOF. First note that

$$\begin{aligned}
E[\cos(2\pi p\mathcal{B}_j)] &= \int_0^1 \cos(2\pi px)\frac{1}{B(j,k-j+1)}x^{j-1}(1-x)^{k-j}\,dx \\
&= \frac{1}{B(j,k-j+1)}\sum_{r=0}^{k-j}(-1)^r \left(\begin{array}{c} k-j \\ r \end{array}\right)\int_0^1 \cos(2\pi px)x^{j-1+r}\,dx \\
&= \int_0^1 \cos(2\pi py)\frac{1}{B(j,k-j+1)}(1-y)^{j-1}y^{k-j}\,dy \\
&= \frac{1}{B(j,k-j+1)}\sum_{r=0}^{j-1}(-1)^r \left(\begin{array}{c} j-1 \\ r \end{array}\right)\int_0^1 \cos(2\pi py)y^{k-j+r}\,dy
\end{aligned}$$

which gives the expressions for (8) and (9). In a similar way, the expressions (10) and (11) can be derived, recalling that $\sin(2\pi-\theta) = -\sin(\theta)$. Now we can demonstrate formulas (12) and (13) by induction.

Now observe that

$$\begin{aligned}
I_p(0) &= \int_0^1 \cos(2\pi px)\,dx = 0 \\
J_p(0) &= \int_0^1 \sin(2\pi px)\,dx = 0 \\
I_p(1) &= \int_0^1 x\cos(2\pi px)\,dx = \frac{1}{2\pi p}\left[x\sin(2\pi px)\right]_0^1 - \frac{1}{2\pi p}\int_0^1 \sin(2\pi px)\,dx = 0 \\
J_p(1) &= \int_0^1 x\sin(2\pi px)\,dx = -\frac{1}{2\pi p}\left[x\cos(2\pi px)\right]_0^1 + \frac{1}{2\pi p}\int_0^1 \cos(2\pi px)\,dx = -\frac{1}{2\pi p}
\end{aligned}$$

Now consider $I_p(C)$. For $C \geq 2$,

$$\begin{aligned}
I_p(C) &= \int_0^1 x^C\cos(2\pi px)\,dx \\
&= \frac{1}{2\pi p}\left[x^C\sin(2\pi px)\right]_0^1 - \frac{C}{2\pi p}\int_0^1 x^{C-1}\sin(2\pi px)\,dx \\
&= -\frac{C}{2\pi p}\int_0^1 x^{C-1}\sin(2\pi px)\,dx \\
&= \frac{C}{(2\pi p)^2}\left[x^{C-1}\cos(2\pi px)\right]_0^1 - \frac{C(C-1)}{(2\pi p)^2}\int_0^1 x^{C-2}\cos(2\pi px)\,dx \\
&= \frac{C}{(2\pi p)^2} - \frac{C(C-1)}{(2\pi p)^2}I_p(C-2) \qquad (14)
\end{aligned}$$

and therefore $I_p(2) = \frac{2}{(2\pi p)^2}$ and $I_p(3) = \frac{3}{(2\pi p)^2}$ which satisfy (12). Assume now that the formula is valid for $c = 2, \ldots, C$. Then

$$\begin{aligned}
I_p(C+2) &= \frac{C+2}{(2\pi p)^2} - \frac{(C+2)(C+1)}{(2\pi p)^2} I_p(C) \quad \text{from (14)} \\
&= \frac{C+2}{(2\pi p)^2} - \sum_{c=1}^{\lfloor \frac{C}{2} \rfloor} (-1)^{c-1} \frac{C!}{(C-2c+1)!} \frac{1}{(2\pi p)^{2c}} \quad \text{from the induction assumption} \\
&= \frac{C+2}{(2\pi p)^2} + \sum_{c=1}^{\lfloor \frac{C}{2} \rfloor} (-1)^{c+1-1} \frac{(C+2)!}{(C+2-2(c+1)+1)!} \frac{1}{(2\pi p)^{2(c+1)}} \\
&= \frac{C+2}{(2\pi p)^2} + \sum_{c=2}^{\lfloor \frac{C+2}{2} \rfloor} (-1)^{c-1} \frac{(C+2)!}{(C+2-2c+1)!} \frac{1}{(2\pi p)^{2c}} \\
&= \sum_{c=1}^{\lfloor \frac{C+2}{2} \rfloor} (-1)^{c-1} \frac{(C+2)!}{(C+2-2c+1)!} \frac{1}{(2\pi p)^{2c}}
\end{aligned}$$

which demonstrates (12).

Equally, we have the recurrence relation

$$J_p(C) = -\frac{1}{2\pi p} - \frac{C(C-1)}{(2\pi p)^2} J_p(C-2) \tag{15}$$

which implies that $J_p(2) = -\frac{1}{2\pi p}$ and $J_p(3) = -\frac{1}{2\pi p} + \frac{3!}{(2\pi p)^3}$ which satisfy (13). Assuming the formula is valid for $c = 2, \ldots, C$ then

$$\begin{aligned}
J_p(C+2) &= -\frac{1}{2\pi p} - \frac{(C+2)(C+1)}{(2\pi p)^2} J_p(C) \quad \text{from (15)} \\
&= -\frac{1}{2\pi p} - \frac{(C+2)(C+1)}{(2\pi p)^2} \sum_{c=1}^{\lfloor \frac{C+1}{2} \rfloor} (-1)^{c} \frac{C!}{(C-2c+2)!} \frac{1}{(2\pi p)^{2c-1}} \\
&\quad \text{from the induction assumption} \\
&= -\frac{1}{2\pi p} - \sum_{c=1}^{\lfloor \frac{C+1}{2} \rfloor} \frac{(C+2)!}{(C-2c+2)!} \frac{1}{(2\pi p)^{2c+1}} \\
&= \sum_{c=1}^{\lfloor \frac{C+2+1}{2} \rfloor} (-1)^{c} \frac{(C+2)!}{(C+2-2c+2)!} \frac{1}{(2\pi p)^{2c-1}}
\end{aligned}$$

which demonstrates (13) and proves the theorem. □

4. Estimation for the Circular Bernstein Polynomial

Given a sample of n data generated from a linear variable, X, with support $[0, 1]$, then from (2), the natural Bernstein polynomial estimator of order k for the density of X is given by,

$$\hat{b}_k(x) = \sum_{j=1}^{k} \left(\hat{F}_X \left(\frac{j}{k} \right) - \hat{F}_X \left(\frac{j-1}{k} \right) \right) \beta(x \mid j, k-j+1),$$

where $\hat{F}_X(\cdot)$ is the empirical distribution function, see e.g. Vitale (1975). Observe that this estimator can be seen as a smoothed histogram because a histogram is simply a function

$$h(x) = \sum_{j=1}^{k} \left(\hat{F}_X \left(\frac{j}{k} \right) - \hat{F}_X \left(\frac{j-1}{k} \right) \right) I_{\left[\frac{j-1}{k}, \frac{j}{k} \right]}(x),$$

where $I_A(x)$ is the indicator function in the set A.

Suppose that we have a sample of n circular data. Then the Bernstein polynomial density approximation can be fitted in various steps. Firstly, for a given order, k, we need to select an origin. Analogous to (5), it may be that there exist multiple origins, ν, such that the empirical distribution function, $\hat{F}^{\nu}(\cdot)$, satisfies the condition

$$\hat{F}^{\nu} \left(\frac{2\pi}{k} \right) = 1 - \hat{F}^{\nu} \left(\frac{2\pi(k-1)}{k} \right) \tag{16}$$

when the fitted Bernstein polynomial density is from (4)

$$\hat{f}_k^{\hat{\nu}}(\theta) = \frac{1}{2\pi} \sum_{j=1}^{k} \left(\hat{F}^{\hat{\nu}} \left(\frac{2\pi j}{k} \right) - \hat{F}^{\hat{\nu}} \left(\frac{2\pi(j-1)}{k} \right) \right) \beta \left(\frac{\theta}{2\pi} \middle| j, k-j+1 \right)$$

and $\hat{\nu}$ is chosen to maximize the log-likelihood estimate

$$\sum_{r=1}^{n} \log \sum_{j=1}^{k} \left(\hat{F}^{\nu} \left(\frac{2\pi j}{k} \right) - \hat{F}^{\nu} \left(\frac{2\pi(j-1)}{k} \right) \right) \beta \left(\frac{\theta_r}{2\pi} \middle| j, k-j+1 \right).$$

However, as the observed data are discrete, the existence of an origin which satisfies (16) is not guaranteed and if no such origin exists, an origin $\hat{\nu}$ is chosen such that the distance,

$$\left| \hat{F}^{\nu} \left(\frac{2\pi}{k} \right) + \hat{F}^{\nu} \left(\frac{2\pi(k-1)}{k} \right) - 1 \right|$$

is minimized. Then the Bernstein polynomial density estimate is given by

$$\hat{f}_k^{\hat{\nu}}(\theta) \quad = \quad \frac{1}{2\pi} \left[\frac{1}{2} \left\{ \hat{F}^{\hat{\nu}} \left(\frac{2\pi}{k} \right) + 1 - \hat{F}^{\hat{\nu}} \left(\frac{2\pi(k-1)}{k} \right) \right\} \beta \left(\frac{\theta}{2\pi} \middle| 1, k \right) + \right.$$

$$\sum_{j=2}^{k-1}\left\{\hat{F}^{\hat{\nu}}\left(\frac{2\pi j}{k}\right)-\hat{F}^{\hat{\nu}}\left(\frac{2\pi(j-1)}{k}\right)\right\}\beta\left(\frac{\theta}{2\pi}\middle| j,k-j+1\right)+$$
$$\left.\frac{1}{2}\left\{\hat{F}^{\hat{\nu}}\left(\frac{2\pi}{k}\right)+1-\hat{F}^{\hat{\nu}}\left(\frac{2\pi(k-1)}{k}\right)\right\}\beta\left(\frac{\theta}{2\pi}\middle| k,1\right)\right]$$

so that the weights of the first and last beta densities are equal and the fitted density is circular. This procedure is this analogous to that of minimizing enclosure in operational research problems, see e.g. Lutterkort et al. (2001).

The problem of choosing the order, k, of the Bernstein polynomial can be viewed as similar to the problem of choosing the number of bars for the histogram. Small values of k will lead to a very poor approximation and high values will lead to overfitting. One possibility is to choose $k = \lceil\sqrt{n}\rceil$. The goodness of fit of the fitted Bernstein polynomial density can then be tested using standard tests such as those of Watson (1961) or Kuiper (1960) and if the fitted model is rejected at a given significance level (e.g. 10%), k can be increased until the fitted model is accepted.

5. Example

Here we consider data which correspond to the twenty four hour clock times of 1297 crimes perpetrated in Chicago on May 11$^{\text{th}}$, 2007, obtained from `www.chicagocrime.org`. A Bernstein polynomial density approximation of order $k = 37 = \lceil\sqrt{1297}\rceil$ was fitted to these data. Figure 5 shows a histogram of the data and the fitted density. The data are plotted so that the origin is set to the fitted origin time, $\hat{\nu} = 06:20$ hours. The fitted and empirical mean direction and 17:39 hours and 17:42 hours and the fitted and empirical mean circular resultant lengths are 0.1998 and 0.1880 respectively. Thus, there is good agreement between the fitted and empirical moments.

Both the Kuiper and Watson tests were used to test the goodness of fit of the Bernstein polynomial density and in both cases, the model was not rejected at a 10% level.

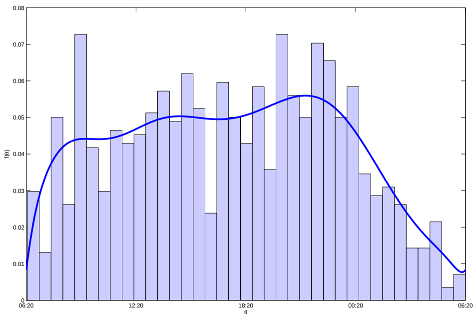

Figure 1: Histogram of the Chicago crime data with fitted Bernstein polynomial density approximation

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