



Statistical analysis of the mechanical properties of composite materials

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Abstract

The Weibull statistic is currently used in designing mechanical components made of composite materials. This work presents useful formulae to describe the behaviour of the Weibull modulus estimator, which in turn may be described by means of a three parameter Weibull distribution. Expressions for the parameters of this latter distribution, dependent on the sample size, are also given in the paper, so, the percentage points, published until now in tabular form, may be directly calculated. Empirical expressions are derived for determining the *A basis* and the *B basis* material properties as a function of the sample size.

Keywords: C. Statistical methods; B. Mechanical properties; B. Strength; Composite materials

1. Introduction

Unlike most traditional structural materials, whose mechanical behaviour is assumed to be homogeneous and isotropic, mechanical properties of composite materials exhibit intrinsic statistical dependence. In particular, their strength properties are usually scattered due to their inhomogeneity and anisotropic characteristics and to the brittleness of the matrices and fibres. So, careful statistical analysis is indispensable for the understanding of the mechanical characterisation of these materials.

The Weibull statistic [1] has been widely used in the recent years to describe the statistical behaviour of the strength properties of many materials, such as advanced ceramics [2], metallic matrix composites [3], ceramic matrix composites [4], and polymeric matrix composites [5]. The Weibull distribution also describes the fatigue behaviour of materials and the scatter of the fracture toughness of steels in the ductile–brittle transition region, where failure occurs by cleavage [6,7].

The two-parameter Weibull distribution function is given by:

$$F(\sigma) = 1 - \exp \left[- \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad (1)$$

where F is the probability of rupture of the material under uniaxial tensile stress σ , m is the shape parameter or

Weibull modulus, and σ_0 is the scale parameter of the distribution. Weibull modulus, m , is related to the scatter of the data: the higher the m the lower the dispersion of fracture stress. It becomes the most important parameter of the distribution. The scale parameter is closely related to the mean fracture stress.

Since the evaluation of the parameters of Weibull distribution is made from a finite number of tests, the estimators of their true values have a statistical character, and thus the uncertainty of the estimation must be known. In this work a three-parameter Weibull distribution is also proposed for the estimator of the Weibull modulus. Formulae for the three parameters defining such a distribution, dependent only on the sample size, are given below.

To design structural and mechanical components, the determination of the σ -values, corresponding to a predefined failure probability, is of great interest to the engineer. These values coincide with their percentiles of the distribution. In particular, the values corresponding to the percentiles of 90 and 99%, estimated with a confidence level of 95%, are known [8] as the *A basis* and *B basis* material property, respectively. These values are obtained from the estimations of m , σ_0 , and in terms of two parameters, P_A and P_B , which depend only on the sample size and on the estimation method used, and are usually obtained from tables [8]. In this paper, to simplify the calculus of these parameters, fitted equations are developed for two estimation methods: maximum-likelihood and weighted regression.

2. Theoretical background

Several procedures are available for the determination of the Weibull distribution parameters. From the maximum likelihood method, the estimators of the Weibull parameters, \hat{m} and $\hat{\sigma}_0$, should satisfy the following equations:

$$\sum_{i=1}^{i=n} \ln \sigma_i - n \left(\frac{\sum_{i=1}^{i=n} \ln \sigma_i (\sigma_i)^{\hat{m}}}{\sum_{i=1}^{i=n} (\sigma_i)^{\hat{m}}} \right) + \frac{n}{\hat{m}} = 0 \quad (2)$$

and

$$\hat{\sigma}_0 = \left(\frac{1}{n} \times \sum_{i=1}^{i=n} (\sigma_i)^{\hat{m}} \right)^{1/\hat{m}} \quad (3)$$

Although Eq. (2) is non-linear, it has a unique positive solution [9], and may be solved by the Newton–Raphson iteration technique or by any other method.

Eq. (1) becomes a straight line if a double logarithmic transformation is made, i.e.

$$\ln \left[\ln \left(\frac{1}{1 - F(\sigma)} \right) \right] = m \ln \sigma - m \ln \sigma_0 \quad (4)$$

The F values are assigned on the basis of the i th position of a value among the n ordered σ -values forming the sample. Thus the general linear regression method may be used to obtain \hat{m} and $\hat{\sigma}_0$.

Several estimators of F can be considered

$$F_i = \frac{i - 0.5}{n} \quad (5a)$$

$$F_i = \frac{i - 0.3}{(n + 0.4)} \quad (5b)$$

$$F_i = \frac{i}{(n + 1)} \quad (5c)$$

Different authors [10,11] have pointed out the convenience of using a weight function, W_i , in performing the linear regression. Bergman [10] proposed the weight factor given by

$$W_i = [(1 - F_i) \times \ln(1 - F_i)]^2 \quad (6)$$

and Faucher and Tyson [11] suggested the following:

$$W_i = 3.3F_i - 27.5[1 - (1 - F_i)^{0.025}] \quad (7)$$

Using the general linear regression method, from the n pairs of values $(\sigma - F)$, the estimators of the Weibull parameters can be calculated as

$$\hat{m} = \frac{(\sum W_i)(\sum W_i z_i y_i) - (\sum W_i z_i)(\sum W_i y_i)}{(\sum W_i)(\sum W_i z_i^2) - (\sum W_i z_i)^2} \quad (8)$$

and

$$\hat{\sigma}_0 = \exp \left(-\frac{a}{\hat{m}} \right) \quad (9)$$

where

$$z_i = \ln \sigma_i \quad (10)$$

$$y_i = \ln \left[\ln \left(\frac{1}{1 - F_i} \right) \right] \quad (11)$$

$$a = \frac{\sum W_i y_i - \hat{m} \sum W_i z_i}{\sum W_i} \quad (12)$$

Finally, applying the moments method, in which the sample moments are equated to those of the distribution, the estimators of m and σ_0 are the solutions of

$$\frac{\bar{\sigma}}{S} = \frac{\Gamma \left(1 + \frac{1}{\hat{m}} \right)}{\left[\Gamma \left(1 + \frac{2}{\hat{m}} \right) - \Gamma^2 \left(1 + \frac{1}{\hat{m}} \right) \right]^{1/2}} \quad (13)$$

and

$$\hat{\sigma}_0 = \frac{\hat{\sigma}}{\Gamma \left(1 + \frac{1}{\hat{m}} \right)} \quad (14)$$

where $\bar{\sigma}$ and S are, respectively, the mean and the standard deviation of the experimental data, and Γ is the Gamma function. Eq. (13) may be solved using the Newton–Raphson procedure.

Thoman et al. [12] showed that if the maximum likelihood method were used, the variables \hat{m}/m and $\hat{m} \ln(\hat{\sigma}_0/\sigma_0)$ are distributed independently of the true values of m and σ_0 , and they have the same distribution as \hat{m}_{11} and $\hat{m}_{11} \ln(\hat{\sigma}_0)_{11}$, respectively, which correspond to $m = 1$ and $\sigma_0 = 1$. The percentage points of these distributions were calculated by Thoman et al. [12] from Monte Carlo simulations.

For a two-parameter Weibull distribution, the A-basis, σ_A , and B-basis material properties, σ_B , might be obtained [8] as

$$\sigma_A = \hat{\sigma}_0 \left(\ln \left(\frac{1}{P_A} \right) \right)^{1/\hat{m}} \quad (15)$$

$$\sigma_B = \hat{\sigma}_0 \left(\ln \left(\frac{1}{P_B} \right) \right)^{1/\hat{m}} \quad (16)$$

Usually the values of the parameters P_A and P_B are calculated by means of the maximum-likelihood method and, so far, their values have been presented in a tabular form [8]. Herein the values of P_A and P_B may be computed as

$$P_A = (1 - 0.01)^{\exp(-C_{0.01})} \quad (17a)$$

$$P_B = (1 - 0.1)^{\exp(-C_{0.1})} \quad (17b)$$

where the constants $C_{0.01}$ and $C_{0.1}$ are, respectively, the 95

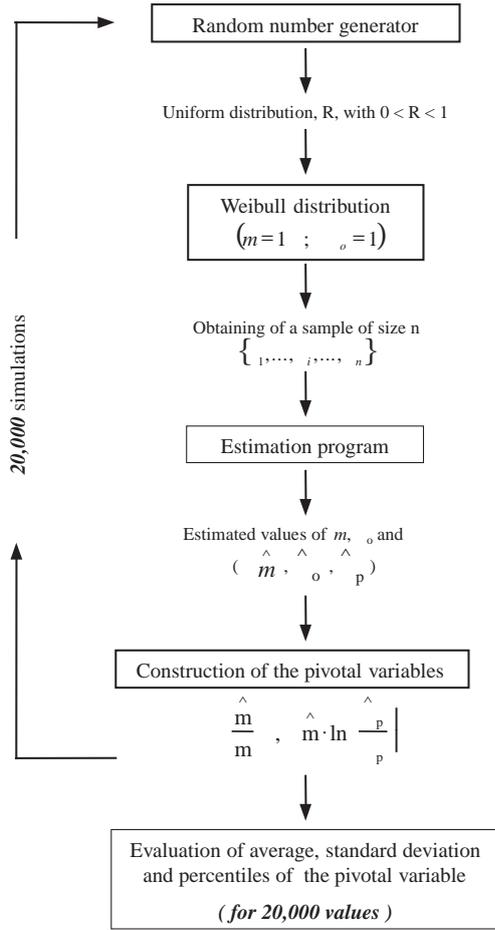


Fig. 1. Flow chart of the simulation procedure.

percentage points of the pivotal variables $\hat{m} \ln(\hat{\sigma}_{0.01}/\sigma_{0.01})$ and $\hat{m} \ln(\hat{\sigma}_{0.1}/\sigma_{0.1})$, i.e. should verify

$$\Pr \left[\hat{m} \ln \left(\frac{\hat{\sigma}_{0.01}}{\sigma_{0.01}} \right) \leq C_{0.01} \right] = 0.95 \quad (18a)$$

$$\Pr \left[\hat{m} \ln \left(\frac{\hat{\sigma}_{0.1}}{\sigma_{0.1}} \right) \leq C_{0.1} \right] = 0.95 \quad (18b)$$

Fernández-Sáez et al. [13] showed that the variable $\hat{m} \ln(\hat{\sigma}_p/\sigma_p)$ is distributed independently of the true percentile value and, Barbero et al. [14] have proved that, in order to estimate the percentiles of the two-parameter Weibull distribution, the best method is the weighted regression one, using Eq. (5c) to estimate F_i , and Eq. (7) to calculate W_i , if sample size is above 7, or Eq. (6) otherwise.

Table 1
Average and standard deviation of variable \hat{m}/m , according to Eq. (20)

	Parameter			
	A	B	C	D
Average	0.9807	1.7001	2.5873	1.0408
Standard deviation	-0.1357	0.5297	0.7303	0.3087

3. Numerical simulation

To obtain the statistical distribution of the pivotal variables \hat{m}/m and $\hat{m} \ln(\hat{\sigma}_p/\sigma_p)$, a simulation procedure, based on the Monte Carlo method, has been used. In this procedure (see Fig. 1 for better understanding) a set of n values was generated as

$$\sigma_i = \sigma_0 \ln \left(\frac{1}{R} \right)^{1/m} \quad (19)$$

In this work we select $m = 1$ and $\sigma_0 = 1$. Note that, as stated above, the analysis is independent of the true values [12].

R is a random variable with uniform distribution in the $[0,1]$ interval. From each sample so obtained, $\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n\}$, estimations of the Weibull parameters, \hat{m} and $\hat{\sigma}_0$, and the p -percentile $\hat{\sigma}_p$, were obtained. From these estimations, the variables \hat{m}/m and $\hat{m} \ln(\hat{\sigma}_p/\sigma_p)$ could also be built. Repeated application of this procedure provides a statistical distribution for the two latter variables. In this work we have computed 20,000 values for \hat{m}/m and $\hat{m} \ln(\hat{\sigma}_p/\sigma_p)$ for each sample size, which, in turn was progressively increased from 5 up to 120.

The values of $C_{0.01}$ and $C_{0.1}$, defined by Eqs. (18a) and (18b), were also computed, and from these, the values of P_A and P_B were obtained using Eqs. (17a) and (17b), respectively.

For the estimation of \hat{m}/m , the maximum-likelihood method (method 1) was utilised, while the authors have used three estimation methods for the calculation of P_A and P_B : maximum-likelihood (method 1) and weighted regression, with two different weighted functions: Eq. (6) (method 2), and Eq. (7) (method 3).

4. Parameter fitting

From the 20,000 values of the pivotal variable \hat{m}/m , its average value, $(\hat{m}/m)_{ave}$ was calculated. In order to fit this value to the sample size, the following four-parameter function is proposed:

$$\left(\frac{\hat{m}}{m} \right)_{ave} = A + B \left(\frac{1}{\ln(Dn)} \right)^C \quad (20)$$

where A , B , C , and D are the fit parameters given in Table 1, and n is the sample size. As shown in Fig. 2, the fit seems to be very good, with a maximum error of 0.23 %. If parameter D were taken as unity, the increment of the maximum error would become negligible and thus three parameters would suffice. The average value of the pivotal variable approaches unity (the estimator approach to the true value) when sample size n increases, which agrees with the Khalili et al. [15] results.

The standard deviation of the variable \hat{m}/m was computed as a function of the sample size. Its value decreases to zero as the sample size increases, as Khalili et al. [15] observed.

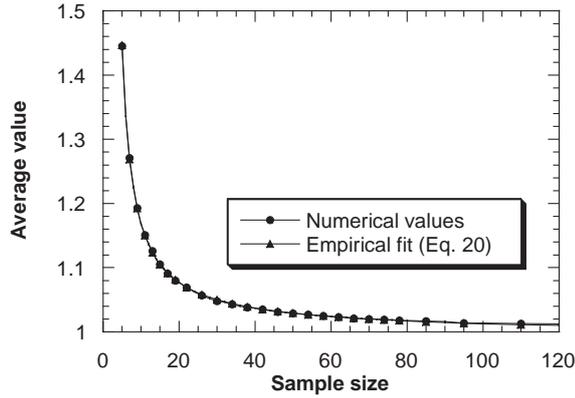


Fig. 2. Variation of the average of the variable \hat{m}/m as a function of the sample size, n . Empirical fit according to Eq. (20).

To fit the standard deviation we used the same type of function as before for the average value. Fig. 3 shows the dependence of the standard deviation on the sample size. The fit parameters are given in Table 1 and the maximum error was 2.07%.

Percentage points of the pivotal variable were also calculated numerically. These results, published by Thoman et al. [12] in tabular form, showed that the variable \hat{m}/m is normal asymptotically with an average value of 1 and standard deviation $\sqrt{0.608/n}$, which means that the percentiles of the variable could be calculated from those of a normal distribution. One improvement to this approach would be to consider that the variable \hat{m}/m is normally distributed with the average and the standard deviation given by Eq. (20) using the parameter obtained from Table 1.

In this paper a three-parameter Weibull distribution is proposed for the variable \hat{m}/m , i.e.

$$F = 1 - \exp \left[- \left(\frac{\hat{m}/m - P_1}{P_2} \right)^{P_3} \right] \quad (21)$$

where P_1 , P_2 , and P_3 are, respectively, the position, scale and form parameters, that may be fitted as a function of the

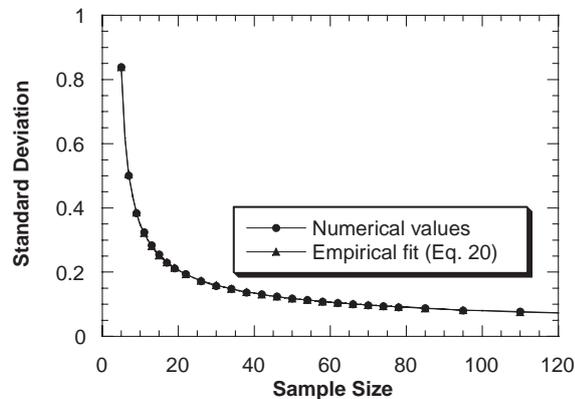


Fig. 3. Variation of the standard deviation of the variable \hat{m}/m as a function of sample size, n . Empirical fit according to Eq. (20).

Table 2

Three-parameter Weibull distribution for the variable \hat{m}/m , according to Eqs. (21) and (22)

	Parameter		
	m_1	m_2	m_3
P_1	0.5268	0.0591	1.0572
P_2	-0.2109	1.5299	-0.8318
P_3	0.2297	0.7140	0.8027

sample size, n , by

$$P_i = m_{i1} + m_{i2}(\ln n)^{m_{i3}}; \quad i = 1, 2, 3 \quad (22)$$

the empirical parameters m_{i1} , m_{i2} , and m_{i3} being those shown in Table 2.

This last approach seems to be better than those cited above for the calculation of the percentiles, as shown in Fig. 4, which gives the variation of the error with the sample size for the 0.985 percentile of the variable \hat{m}/m using different methods: the three-parameter Weibull distribution; normal distribution of average 1 and standard deviation $\sqrt{0.608/n}$; and finally normal distribution with the average and standard deviation values calculated from Eq. (20).

Also, the values of P_A and P_B , obtained using Eqs. (17a) and (17b) from the numerical simulation results have been fitted by means of the following equation:

$$P_A \text{ (or } P_B) = M_1 + M_2 \exp(n^{-M_3}) \quad (23)$$

The parameters M_1 , M_2 , and M_3 are shown in Tables 3 and 4. Fig. 5 shows a comparison of the numerical values of P_B with the corresponding ones obtained from Eq. (23). The fits for both P_A and P_B are very good, with errors below 0.3% in all the cases. Once P_A and P_B values are known, the A-basis and B-basis material properties may be obtained from Eqs. (15) and (16).

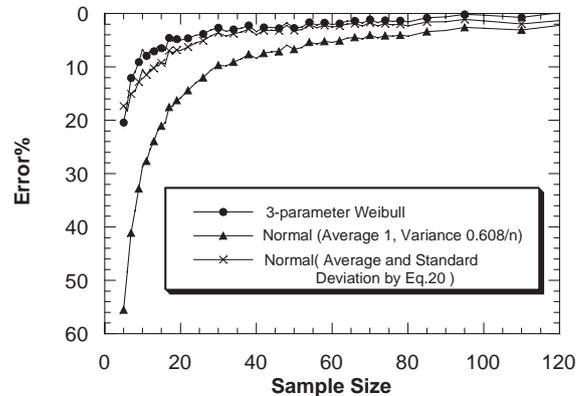


Fig. 4. Error of estimation of the 0.985 percentile of the variable \hat{m}/m for different methods. Curve 1: three-parameter Weibull distribution. Curve 2: normal of average 1 and variance $0.608/n$. Curve 3: normal of average and variance calculated using Eq. (20).

Table 3
Estimation of P_A as a function of sample size, according to Eq. (23)

Method	m_1	m_2	m_3	R	Maximum error (%)
1	0.2758	0.2685	0.0030	0.9991	0.22
2	0.4777	0.1928	0.0028	0.9981	0.01
3	0.9766	0.0103	0.1250	0.9992	0.03

Table 4
Estimation of P_B as a function of sample size, according to Eq. (23)

Method	m_1	m_2	m_3	R	Maximum error (%)
1	-1.3102	0.8651	0.0106	0.9980	0.30
2	0.8132	0.0840	0.2227	0.9977	0.27
3	0.8070	0.1003	0.3529	0.9914	0.17

5. Practical examples

To illustrate the above methodology two examples are included in this paper, although it is worth noting that they can be applied in their integrity to any kind of composite material. In the first example, the estimation of the confidence intervals of the Weibull modulus for the static flexural strength of a CMC is considered, whereas in the second, the A-basis and B-basis values of the dynamic flexural strength of a CFRP are computed. In these examples, the loading rate and the sample size are different.

5.1. Example 1: estimation of confidence intervals of Weibull modulus

In the first example, the confidence intervals of the Weibull modulus (90% and 99% confidence levels), for the static flexural strength of a ceramic matrix composite, are estimated.

The composite material was an alumina (CONDEA APA-0.5, 99.97% of purity) matrix reinforced by 30% of zirconia particles (Tosoh 3YB), processed by uniaxial pressing under 87 MPa and pressureless sintering at 1650°C for 2 h [16].

Three-point bending tests of 15 prismatic specimens of

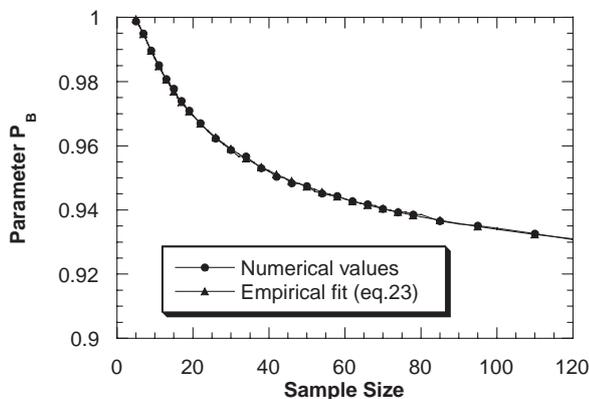


Fig. 5. Variation of the parameter P_B as a function of sample size, n . Empirical fit according to Eq. (23).

Table 5
Example 1: flexural strength (MPa) of a 30% zirconia alumina composite

Sample	Strength (MPa)	Sample	Strength (MPa)
1	203	9	310
2	225	10	325
3	239	11	345
4	265	12	360
5	270	13	365
6	275	14	400
7	289	15	425
8	295		

$10 \times 10 \times 55 \text{ mm}^3$ were carried out using a universal testing machine (INSTRON 8016) with a 5 kN load cell. The tests results (flexural strength) are shown in Table 5. From these values an estimation of the Weibull modulus and scale parameters were obtained by the maximum likelihood method [14], which gave

$$\hat{m} = 5.398; \quad \hat{\sigma}_0 = 331.5 \text{ MPa}$$

Fig. 6 gives the experimental results and the fitted values using the Weibull distribution.

From the parameters A , B , C and D of Table 1, the values of the average and standard deviation of the pivotal variable are calculated by Eq. (20) with $n = 15$ were

$$\text{Average of } \left(\frac{\hat{m}}{m} \right) = 1.1050$$

$$\text{Standard deviation of } \left(\frac{\hat{m}}{m} \right) = 0.2521$$

To find the percentage points of the estimator of the Weibull modulus, its distribution function, Eq. (21), may be used. The values of P_i in this equation were computed from Eq. (22) and the parameters m_1 , m_2 and m_3 from Table 2. For $n = 15$ the following values were obtained:

$$P_1 = 0.6962 \quad P_2 = 0.4571 \quad P_3 = 1.8182$$

The percentiles of variable \hat{m}/m to be used to calculate the confidence levels of the 99 and 90% are shown in Table 6. From such values the interval of the variable m , corresponding to the 99% confidence level, is

$$2.934 < m < 7.487$$

Table 6
Example 1: percentiles of variable \hat{m}/m , calculated according Eqs. (21) and (22)

Percentile	Value
<i>Confidence level: 99%</i>	
0.005	0.72103
0.995	1.8398
<i>Confidence level: 90%</i>	
0.050	0.78544
0.950	1.5320

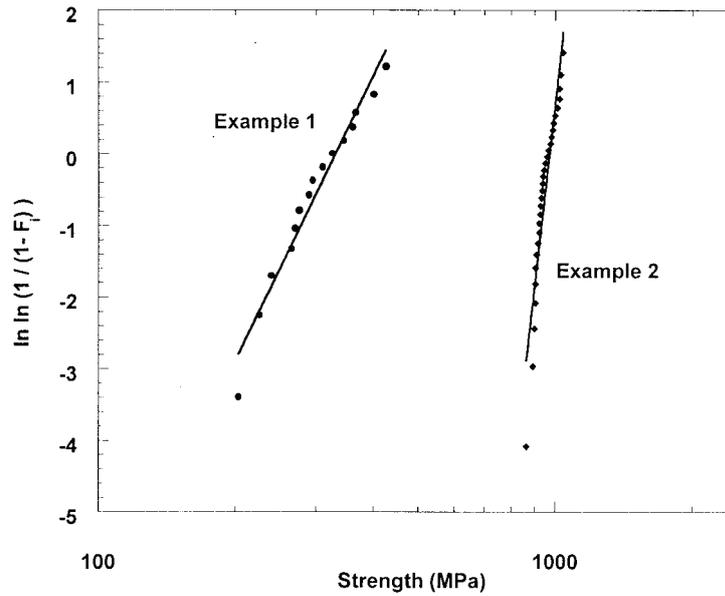


Fig. 6. Flexural strength distribution of an alumina 30% zirconia composite (Example 1) and 0/90 carbon/epoxy laminate (Example 2).

and for the 90% confidence level

$$3.523 < m < 6.873$$

5.2. Example 2: estimation of the A basis and B basis material properties

In this example, the A-basis and B-basis material properties (dynamic flexural strength) of a polymeric matrix composite were calculated.

The composite material is a 2.4 mm thick carbon/epoxy, Hexcel AS4/350-61, [0/90]_{2S} laminate, manufactured by SACESA.

Thirty dynamic three-point bending tests were performed using a CEAST Fractovis drop weight tester. The specimens were $2.4 \times 20 \times 105 \text{ mm}^3$ beams. The drop weight was

Table 7
Example 2: flexural strength (MPa) of [0/90] carbon/epoxy composite

Sample	Strength (MPa)	Sample	Strength (MPa)
1	864	16	941
2	891	17	945
3	900	18	950
4	904	19	959
5	904	20	965
6	904	21	975
7	910	22	980
8	915	23	986
9	922	24	990
10	922	25	998
11	926	26	1010
12	928	27	1022
13	931	28	1022
14	936	29	1027
15	938	30	1040

3.608 kg and its velocity 4.08 m/s. The flexural strength results are given in Table 7. For this example the estimations of the Weibull modulus and scale parameters were

$$\hat{m} = 21.67; \quad \hat{\sigma}_0 = 972.29 \text{ MPa}$$

Fig. 6 gives the experimental results and the fitted values using the Weibull distribution.

From Eq. (23) and Tables 3 and 4, the parameters P_A and P_B are estimated for a sample size of 30. The values obtained for the three methods cited in this article are shown in Table 8, which also gives the A-basis and B-basis material properties.

6. Summary

This work presents useful formulae to analyse the variability of the mechanical properties of composite materials. The study is focused on the two-parameter Weibull distribution, currently used to describe statistically the strength properties of many kinds of materials. In order to obtain the percentage points of the estimator of the Weibull modulus, published until now in tabular form, a three-parameter Weibull distribution is proposed. Empirical expressions for these three parameters, dependent only on the sample size, are also given. The A-basis and B-basis material

Table 8
Example 2: estimation of A-basis and B-basis

Method	P_A	P_B	σ_A (MPa)	σ_B (MPa)
1	0.9995	0.9728	694	834
2	0.9976	0.9562	745	853
3	0.9974	0.9519	749	857

properties based on Weibull statistics can be derived from the estimations of the Weibull parameters and the values P_A and P_B , which depend only on the sample size and the type of estimation method. Expressions for P_A and P_B , estimated by the maximum likelihood, and two selected weighted-regression methods, are given.

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