A family of examples with quantum constraints

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Abstract. In [1, Part B] the authors describe a method for constructing directly (i.e. without using explicitly any field operator nor any concrete representation of the C*–algebra) nets of local C*–algebras associated to massless models with arbitrary helicity and that satisfy Haag–Kastler's axioms. In order to specify the sesquilinear– and the symplectic form of the CAR– and CCR–algebras, respectively, a certain operator–valued function $\beta(\cdot)$ is introduced. This function shows to be very useful to prove the covariance and the causality of the net and it also codes the degenerate character of the massless models with respect to the massive ones.
It is the intention of the present note to point out that the massless bosonic examples with helicity bigger than 0 fit completely into the general theory that Grundling and Hurst [2] used to describe systems with gauge degeneracy.

1 Introduction

In [1], [3], [4] and [5] the authors discuss several methods of constructing causal and covariant nets of local C*–algebras associated naturally to the parameters $\{m \geq 0,\, s,\, \pm\}$ that classify physically relevant unitary irreducible representations of $\widetilde{\mathfrak{P}_+^\uparrow}$, the universal covering group of the proper orthochronous component of the Poincaré group. The construction of the net is done directly, i.e. avoiding the explicit use of any field operator and of any concrete representation of the algebra. The reference spaces h_{F} and h_{B} of the CAR– and CCR–algebras, respectively, are defined through appropriate combinations of function spaces canonically given by the representation theory of $\widetilde{\mathfrak{P}_+^\uparrow}$. In order to specify the scalar product on h_{F} or the symplectic form on h_{B} a certain operator–valued function $\beta(\cdot)$ is introduced. This definition of the scalar product and of the symplectic form shows to be very useful in order to prove the covariance and the causality of the net.

The structural difference between the massive and the massless models can be related with the behaviour of $\beta(\cdot)$ in these examples. Indeed, for p an element of the positive mass shell $\beta(p)$ becomes a strictly positive operator on the (finite–dimensional) internal space of the models, while for p element of the positive light cone $\beta(p)$ turns out to be semi–definite for the models with helicity bigger than 0. This implies the degenerate character of the corresponding sesquilinear– and symplectic forms. For this reason in the massless models certain factor spaces are needed ([1, Section B.2]) in order to construct the net of local algebras following the spirit of [1, Part A]. It is the intention of the present note to show that, before introducing the factor spaces, the bosonic massless models with helicity bigger than 0 fit completely into the general frame developed by Grundling and Hurst in [2] to treat systems with gauge degeneracy. We will make explicit the relation of the models with the general theory. Summing up, if one tries to apply in the bosonic case the arguments that are needed to construct the net of local C*–algebras associated to massive models to the massless case, then quantum constraints appear naturally. One has to impose constraints (concretely apply the so–called T–procedure) in order to end up with the algebra of physical observables that contains the net of local C*–subalgebras satisfying the axioms of algebraic QFT.

One of the advantages of the examples presented in this note is that the degeneracy appears directly at the abstract level as a characteristic feature of the massless models in constrast with the massive ones. Since no field equations are explicitly used in the construction, the specification of the constraint set can not use heuristic arguments as in the example presented in [2, Section 5]. (But, at the same time, this last fact may be criticized, because we loose contact with Dirac's original theory of constraints [6].) We hope, nevertheless, that the present examples will also serve to clarify the relation of the theory of quantum constraints with the algebraic approach to QFT à la Haag and Kastler ([7, 8]).

2 Quantum constraints

In the present section we will summarize the algebric structures and the results of the papers by Grundling and Hurst [2, 9, 10, 11, 12] and by Manuceau et al. [13, 14] that are relevant for our examples. We refer to these papers for the proofs of the theorems and for further developments of the subject. We will stay as close as possible to the notation of Grundling and Hurst in [2] and [11, Section 2].

2.1 T–Procedure

From a mathematical point of view the so–called T–procedure is a process that starts with a unital C*–algebra, $\mathcal{F}$, with a subgroup, $\mathcal{U}$, of the unitary group of $\mathcal{F}$ and with the subset of states of $\mathcal{F}$ that give the value 1 on the elements of $\mathcal{U}$ and ends up with a unital and simple C*–algebra, $\mathfrak{R}$, which is the quotient algebra of a C*–subalgebra, $\mathfrak{O}$, of $\mathcal{F}$ by a proper closed 2–sided ideal, $\mathfrak{D}$, of the latter algebra. In the following we will define these quantities precisely.

If the starting objects of the T–procedure are associated to a physical system, then one of the main assumptions of the theory is that all physical information is contained in the triple $\Big(\mathcal{F},\, \mathcal{U},\, \mathfrak{S}_{\rm D}\Big)$, where we define by

$$\mathfrak{S}_{\rm D} := \Big\{\omega \in \text{states of } \mathcal{F}\colon\ \omega(U) = 1 \quad \text{for all} \quad U \in \mathcal{U}\Big\}\,, \tag{1}$$

the so–called set of *Dirac states* of $\mathcal{F}$. In this context $\mathcal{F}$ is said to be the *field algebra* and $\mathcal{U}$ the *set of state conditions.*

Lemma 2.1 *$\omega \in \mathfrak{S}_{\rm D}$ iff $\omega(FU) = \omega(F) = \omega(UF)$ for all $F \in \mathcal{F}$, $U \in \mathcal{U}$.*

Further, denote by $\mathcal{A}(\mathcal{U})$ the C*–subalgebra of the field algebra $\mathcal{F}$ generated by the set $\Big\{U - \mathbb{I}\colon\ U \in \mathcal{U}\Big\}$, i.e.

$$\mathcal{A}(\mathcal{U}) := \mathrm{C}^*\Big\{U - \mathbb{I}\colon\ U \in \mathcal{U}\Big\}\,. \tag{2}$$

Lemma 2.2 *$\mathfrak{S}_{\rm D} \neq \emptyset$ iff $\mathbb{I} \notin \mathcal{A}(\mathcal{U})$.*

Denote by $[\cdot]$ the closed linear space generated by its argument and with this convention define the sets

$$\mathfrak{D} \ := \ \Big[\mathcal{F}\mathcal{A}(\mathcal{U})\Big] \cap \Big[\mathcal{A}(\mathcal{U})\mathcal{F}\Big]\,, \tag{3}$$

$$\mathfrak{O} \ := \ \Big\{F \in \mathcal{F}\colon\ FD - DF \in \mathfrak{D} \quad \text{for all} \quad D \in \mathfrak{D}\Big\}\,. \tag{4}$$

Roughly, the set $\mathfrak{O}$ is the C*–algebraic version of Dirac's notion of observable given in [6]. See [2] and references cited therein for further motivation.

Theorem 2.3 *With the preceding notation we have*

(i) *$\mathfrak{D}$ is the largest* C*–*algebra annihilated by all Dirac states, i.e. $\mathfrak{D}$ is the unique maximal* C*–*algebra in* $\cap\Big\{\ker\omega\colon\ \omega \in \mathfrak{S}_{\rm D}\Big\}$.

(ii) $\mathfrak{O} = \mathcal{M}_{\mathcal{F}}(\mathfrak{D}) := \Big\{F \in \mathcal{F}\colon\ FD, DF \in \mathfrak{D} \quad \text{for all} \quad D \in \mathfrak{D}\Big\}$, *where the last expression is called the relative multiplier algebra*[1] *of $\mathfrak{D}$ in $\mathcal{F}$.*

The set $\mathcal{S}$, defined as the largest set in $\mathcal{F}$ such that $\mathcal{A}(\mathcal{U})\mathcal{S} \subset \Big[\mathcal{F}\mathcal{A}(\mathcal{U})\Big]$, proves to be very useful in order to characterize some of the objects defined before. Indeed, it can be shown

Theorem 2.4 *$\mathfrak{D} = \mathrm{clo}_{\|\cdot\|}\Big\{\mathcal{S}^*\mathcal{A}(\mathcal{U})\mathcal{S}\Big\}$ and $\mathfrak{O} = \mathcal{S}^* \cap \mathcal{S} \supset \mathcal{A}(\mathcal{U})'$, where $\|\cdot\|$ is the* C*–*norm in $\mathcal{F}$ and the relative commutant, $\mathcal{A}(\mathcal{U})'$, is taken with respect to the field algebra $\mathcal{F}$.*

The *maximal* C*–*algebra of physical observables* is defined as

$$\mathfrak{R} := \mathfrak{O}\big/_{\mathfrak{D}}. \tag{5}$$

In general, this construction procedure does not guarantee that $\mathfrak{R}$ will be a simple algebra. (See the remarks after Theorem 2.8 in [9]). But in our examples we will end up with a simple algebra $\mathfrak{R}$, so that no further comment is needed at this point.

Finally, we reproduce some results concerning the Dirac states.

[1] The multiplier algebra $\mathcal{M}_{\mathcal{F}}(\mathfrak{D})$ is the largest set that contains $\mathfrak{D}$ as a closed 2–sided ideal. $\mathcal{M}_{\mathcal{F}}(\mathfrak{D})$ is itself a C*–algebra and $\mathfrak{D}$ is a proper subalgebra of $\mathcal{M}_{\mathcal{F}}(\mathfrak{D})$ iff $\mathbb{I} \notin \mathfrak{D}$, [15].

Theorem 2.5 *The following statements hold*

(i) *Let ω be a state on $\mathcal{F}$ and $\left(\mathfrak{H}_\omega,\, \pi_\omega,\, \Omega_\omega\right)$ the corresponding GNS–data. Then $\omega \in \mathfrak{S}_{\rm D}$ iff $\pi_\omega(\mathfrak{D})\Omega_\omega = 0$.*

(ii) *There exists a w^*–continuous isometric bijection between the Dirac states on $\mathfrak{O}$ and the states on $\mathfrak{R}$, denoted by $\mathfrak{S}$.*

(iii) *Put $\alpha_{\rm U} := {\rm Ad}\,(U)$ for $U \in \mathcal{U}$. Then, $\alpha_{\rm U}$ leaves the Dirac states invariant, i.e. $\omega \circ \alpha_{\rm U} = \omega$ for all $\omega \in \mathfrak{S}_{\rm D}$, $U \in \mathcal{U}$. Further, for $A \in \mathcal{F}$ we have that $A \in \mathfrak{O}$ iff $\alpha_{\rm U}(A) - A \in \mathfrak{D}$ for all $U \in \mathcal{U}$.*

One expects that after the T–procedure no physical information is lost. With other words one assumes that after the T–procedure all physical information is contained in the pair $\left(\mathfrak{R}, \mathfrak{S}\right)$, where $\mathfrak{S}$ denotes the set of states on $\mathfrak{R}$.

2.2 Transformations

Consider the following subgroup of the automorphism group of $\mathcal{F}$:

$$\Upsilon := \left\{\alpha \in {\rm aut}\,\mathcal{F}\colon\ \alpha(\mathfrak{D}) = \mathfrak{D}\right\}. \tag{6}$$

Now from Theorem 2.3 (ii) we have that $\mathfrak{O} = \mathcal{M}_{\mathcal{F}}(\mathfrak{D})$, so that every $\alpha \in \Upsilon$ also preserves $\mathfrak{O}$ and, therefore, it defines canonically an automorphism on $\mathfrak{R}$, denoted by α'. Define the group homomorphism $T\colon\ \Upsilon \longrightarrow {\rm aut}\ \mathfrak{R}$ by $T(\alpha) := \alpha'$. In the present context one expects that $\ker T$ consists of gauge transformations, i.e. transformations that leave physical quantities invariant.

Theorem 2.6 *With the preceding notation we have that*

$$\ker T = \left\{\alpha \in {\rm aut}\,\mathcal{F}\colon\ \omega\Big(\alpha(F_1)F_2\Big) = \omega(F_1F_2) \quad \text{for all} \quad F_1, F_2 \in \mathfrak{O},\ \omega \in \mathfrak{S}_{\rm D}\right\}. \tag{7}$$

Finally, it is easy to see from Theorem 2.5 (iii) that the family of automorphism $\alpha_{\rm U}$, $U \in \mathcal{U}$, belong to $\ker T$, i.e.

$$\mathcal{G} := \left\{\alpha_{\rm U} \in {\rm aut}\,\mathcal{F}\colon\ \alpha_{\rm U} := {\rm Ad}\,U \quad \text{for some} \quad U \in \mathcal{U}\right\} \subset \ker T. \tag{8}$$

2.3 Manuceau's C*–algebra of the CCR

The field algebras of the examples presented in Section 4 will be special cases of Manuceau's C*–algebra of the CCR. Further, the maximal C*–algebras of physical observables will also prove to be isomorphic to this sort of algebra. For this reason we will sketch how to construct the C*–algebra of the CCR ([13, 14]) and state some useful results related to it.

Let h be a linear space and σ a (possibly degenerate) symplectic form on it. Denote by $\Delta(h,\sigma)$ the linear space of complex–valued functions on h with finite support. It is generated by the elements δ_φ, $\varphi \in h$, defined by

$$\delta_\varphi(\psi) := \begin{cases} 1 & \text{if} \quad \psi = \varphi \\ 0 & \text{if} \quad \psi \neq \varphi. \end{cases} \tag{9}$$

If we introduce a product and an involution in $\Delta(h,\sigma)$ defining $\delta_\varphi \cdot \delta_\psi := e^{-\frac{i}{2}\sigma(\varphi,\psi)}\,\delta_{\varphi+\psi}$ and $\left(\delta_\varphi\right)^* := \delta_{-\varphi}$, $\varphi, \psi \in h$, respectively, then $\Delta(h,\sigma)$ becomes a *–algebra with unit δ_0. Further, from the norm on $\Delta(h,\sigma)$, given by $\left\|\sum_{i=1}^{m}\alpha_i\delta_{\varphi_i}\right\|_1 := \sum_{i=1}^{m}|\alpha_i|$, $\alpha_i\in\mathbb{C}$, $i=1,\ldots,m$, we obtain the Banach algebra $\mathrm{clo}_{\|\cdot\|_1}\Delta(h,\sigma)$. Finally, we define

$$\overline{\Delta(h,\sigma)} := \mathrm{clo}_{\|\cdot\|}\left(\mathrm{clo}_{\|\cdot\|_1}\Delta(h,\sigma)\right), \tag{10}$$

where $\|\cdot\|$ is the minimal regular norm [13, Equation 3.2] of $\mathrm{clo}_{\|\cdot\|_1}\Delta(h,\sigma)$. With other words, Manuceau's algebra of the CCR, $\overline{\Delta(h,\sigma)}$, is the enveloping C*–algebra of $\mathrm{clo}_{\|\cdot\|_1}\Delta(h,\sigma)$.

Theorem 2.7 *$\overline{\Delta(h,\sigma)}$ is simple iff σ is nondegenerate.*

Next, consider a submanifold[2], $\mathcal{C}$, of h and the following definitions

$$\mathfrak{p} \;:=\; \left\{\varphi\in h\colon\ \sigma(\varphi,\psi)=0 \quad \text{for all} \quad \psi\in\mathcal{C}\right\} \tag{11}$$

$$\mathfrak{p}_0 \;:=\; \left\{\varphi\in \mathfrak{p}\colon\ \sigma(\varphi,\psi)=0 \quad \text{for all} \quad \psi\in\mathfrak{p}\right\}. \tag{12}$$

We conclude this section with a

Theorem 2.8 *Define by $h^{(d)} := \left\{\varphi\in h\colon\ \sigma(\varphi,\psi)=0 \quad \text{for all} \quad \psi\in h\right\}$ the degeneracy subspace of σ in h. Put $h' := h/h^{(d)}$ and denote by σ' the natural definition of σ on the factor space h'. Then we have the following algebraic isomorphism:*

$$\overline{\Delta(h',\sigma')} \cong \overline{\Delta(h,\sigma)}\Big/\overline{\mathrm{C}^*\left(\delta_{h^{(d)}}-\mathbb{I}\right)\overline{\Delta(h,\sigma)}}, \tag{13}$$

where, $\mathrm{C}^*\left(\delta_{h^{(d)}}-\mathbb{I}\right)$, denotes the C*–subalgebra of $\overline{\Delta(h,\sigma)}$ generated by $\left\{\delta_\varphi-\mathbb{I}\colon\ \varphi\in h^{(d)}\right\}$.

For further details see [11, Theorem 5.2, Theorem 5.3, Corollary 5.4 and Corollary 5.5].

3 Characterization of the family of bosonic examples

In this section we collect the main definitions that characterize the bosonic massless models and state without proof some useful results. See [1, Part B] for details and further references.

We start, motivated by the expression of the scalar product for the massive models [1, Eq. (8)], defining the following sesquilinear form: for φ,ψ a pair of $\mathcal{H}$–valued functions,

$$\left\langle\varphi,\,\psi\right\rangle_\beta := \int_{\mathcal{C}^+}\left(\varphi(p),\,\beta(p)\ \psi(p)\right)_{\mathcal{H}}\mu_0(\mathrm{d}p), \tag{14}$$

where μ_0 is the invariant measure on the mantle of the positive light cone, $\mathcal{C}^+$, and

$$\beta(p) \;:=\; D_{P^+} \;=\; \overset{n}{\otimes} P^+, \tag{15}$$

$$P^+ \;=\; \frac{1}{2}\left(p_0\sigma_0-\sum_{i=1}^{3}p_i\sigma_i\right). \tag{16}$$

We denote by σ_μ, $\mu=0,1,2,3$, the unit and the Pauli matrices, i.e.

$$\sigma_0 := \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1 := \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2 := \begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad\text{and}\quad \sigma_3 := \begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{17}$$

[2]In the examples that will be presented in the following two sections, the set of state conditions will be characterized by certain submanifolds of h.

Further, consider the set

$$\mathfrak{H}_n := \Big\{\varphi\colon\ \mathcal{C}^+ \longrightarrow \mathcal{H}, \quad \text{such that} \quad \Big\langle \varphi, \varphi \Big\rangle_\beta < \infty \Big\} \tag{18}$$

and define on it the following representations of $\widetilde{\mathfrak{P}_+^\uparrow}$:

$$\Big(V_1(g)\,\varphi\Big)(p) \;:=\; e^{-ipa}\, D_A\, \varphi\Big(\Lambda_A^{-1} p\Big), \qquad \text{for } \varphi \in \mathfrak{H}_n, \tag{19}$$

$$\Big(V_2(g)\,\varphi\Big)(p) \;:=\; e^{ipa}\, D_A\, \varphi\Big(\Lambda_A^{-1} p\Big), \qquad \text{for } \varphi \in \mathfrak{H}_n, \tag{20}$$

where $g = (A, a) \in \widetilde{\mathfrak{P}_+^\uparrow} = \mathrm{SL}(2,\mathbb{C})\circledS\mathbb{R}^4$, $A \mapsto D_A := \overset{n}{\otimes} A$ is the finite–dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ over $\mathcal{H} := \mathrm{Sym}\left(\overset{n}{\otimes}\mathbb{C}^2\right)$ of[3] the type $(n,\,0)$ with n even and Λ_A is the corresponding Lorentz transformation of the matrices $\pm A \in \mathrm{SL}(2,\mathbb{C})$.

The representations V_1 and V_2 leave the sesquilinear form, $\Big\langle\cdot,\cdot\Big\rangle_\beta$, invariant. From Equations (15) and (16) it is easy to see that the matrix $\beta(p)$ is singular for $p \in \mathcal{C}^+$, so that $\Big\langle\cdot,\cdot\Big\rangle_\beta$ is only positiv semi–definite. This fact serves to select the following two submanifolds of $\mathfrak{H}_n$:

Definition 3.1 With respect to the sesquilinear form defined above we can naturally define the following subspaces of $\mathfrak{H}_n$:

$$\mathfrak{H}_n^{(0)} \;:=\; \mathrm{span}\Big\{\varphi \in \mathfrak{H}_n\colon\ \Big\langle \varphi, \varphi \Big\rangle_\beta = 0\Big\} \tag{21}$$

$$\mathfrak{H}_n^{(>)} \;:=\; \mathrm{span}\left\{\varphi \in \mathfrak{H}_n\colon\ \varphi(p) = \overset{n}{\otimes}\begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix}\chi(p), \quad \text{for some scalar function } \chi\right\}. \tag{22}$$

Lemma 3.2 Using the preceding definitions we have that for $n > 0$

(i) $\mathfrak{H}_n = \mathfrak{H}_n^{(>)} \oplus \mathfrak{H}_n^{(0)}$.

(ii) The representations V_1 and V_2 leave the space $\mathfrak{H}_n^{(0)}$ invariant. On the contrary, the subspace $\mathfrak{H}_n^{(>)}$ is not invariant under the mentioned representations.

(iii) For any non zero $\varphi \in \mathfrak{H}_n^{(>)}$ we have $\|\varphi\|_\beta = \|V_{1,2}(g)\varphi\|_\beta > 0$ for all $g \in \widetilde{\mathfrak{P}_+^\uparrow}$.

Finally, we will establish the reference space, h_n, and the symplectic form, σ, on it that characterize uniquely the C*–algebra, $\mathrm{CCR}(h_n, \sigma)$ $\Big($[16, Section 8.2], [17]$\Big)$, associated with the bosonic models with helicity $\frac{n}{2}$. Put $h_n := \mathfrak{H}_n \oplus \mathfrak{H}_n$, n even, and interpret it as a real space. Consider further the following real–bilinear form: for $\varphi = \varphi_1 \oplus \varphi_2$, $\psi = \psi_1 \oplus \psi_2 \in h_n$ let

$$\Big\langle \varphi, \psi \Big\rangle := \Big\langle \varphi_1, \psi_1 \Big\rangle_\beta - \Big\langle \varphi_2, \psi_2 \Big\rangle_\beta \tag{23}$$

and define the symplectic form by

$$\sigma(\cdot,\cdot) := \mathrm{Im}\Big\langle \cdot, \cdot \Big\rangle. \tag{24}$$

Finally, the representation $V := V_1 \oplus V_2$ leaves the symplectic form invariant.

[3] "Sym" denotes the symmetrization operator.

4 Identification of the degenerate structures of the examples

In this section we will show that the bosonic models characterized in Section 3 specify a family[4] of examples with quantum constraints. They fall into the class of examples analyzed in detail in [2, Section 5], [9, Section IV], [11, Section 5] and [10, Section 6.A]. We will also identify for these models the sets $\mathcal{F}$, $\mathcal{U}$, $\mathcal{C}$, $\mathfrak{p}$, $\mathfrak{p}_0$, $\mathcal{S}$, $\mathfrak{O}$, $\mathfrak{D}$ and $\mathfrak{R}$ of Section 2.

4.1 T–Procedure

Consider the data (h_n, σ) given in Section 3. Using the results of Subsection 2.3 we define the field algebra as $\mathcal{F} := \overline{\Delta(h_n, \sigma)}$. Consider the degeneracy subspace of h_n

$$h_n^{(d)} := \Big\{ \varphi \in h_n\colon\ \sigma(\varphi, \psi) = 0 \quad \text{for all} \quad \psi \in h_n \Big\} . \tag{25}$$

Lemma 4.1 *The degeneracy subspace of the symplectic space (h_n, σ) is completely characterized by the "zero norm spaces" $\mathfrak{H}_n^{(0)}$. Concretely $h_n^{(d)} = \mathfrak{H}_n^{(0)} \oplus \mathfrak{H}_n^{(0)}$.*
Proof: Note that for $\varphi, \psi \in h_n$ ($\varphi := \varphi_a \oplus \varphi_b$, $\psi := \psi_a \oplus \psi_b$) we have

$$\sigma(\varphi, \psi) = \frac{1}{2i} \Big(\big\langle \varphi_a, \psi_a \big\rangle_\beta - \big\langle \psi_a, \varphi_a \big\rangle_\beta + \big\langle \psi_b, \varphi_b \big\rangle_\beta - \big\langle \varphi_b, \psi_b \big\rangle_\beta \Big) .$$

Since ψ_a and ψ_b are independent it suffices to show that for $0 \neq \varphi_a \notin \mathfrak{H}_n^{(0)}$ there exists a $\psi_a \in \mathfrak{H}_n$, such that $\frac{1}{2i} \Big(\big\langle \varphi_a, \psi_a \big\rangle_\beta - \big\langle \psi_a, \varphi_a \big\rangle_\beta \Big) > 0$. First consider the case $n = 1$ and denote $\varphi_a := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$; $\psi_a := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. We can choose without loss of generality $\varphi_a \in \mathfrak{H}_n^{(>)}$, i.e. from Definition 3.1, $\varphi_a(p) = \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \chi(p)$ for some scalar function χ and where $p \in \mathcal{C}^+$. Now, using the eigenvalue equation

$$\beta(p) \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \chi(p) = p_0 \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \chi(p), \ p \in \mathcal{C}^+, \tag{26}$$

we can explicitly compute

$$\begin{aligned}
&\frac{1}{2i} \Big(\big\langle \varphi_a, \psi_a \big\rangle_\beta - \big\langle \psi_a, \varphi_a \big\rangle_\beta \Big) \\
&= \frac{1}{2i} \Bigg\{ \int_{\mathcal{C}^+} p_0 \Big((-p_1 - ip_2)\overline{\chi(p)}\psi_1(p) + (p_0 + p_3)\overline{\chi(p)}\psi_2(p) \Big) \ \mu_0(\mathrm{d}p) \\
&\qquad - \int_{\mathcal{C}^+} p_0 \Big((-p_1 + ip_2)\chi(p)\overline{\psi_1(p)} + (p_0 + p_3)\chi(p)\overline{\psi_2(p)} \Big) \ \mu_0(\mathrm{d}p) \Bigg\} \\
&= \int_{\mathcal{C}^+} p_0^2(p_0 + p_3)|\chi(p)|^2 \ \mu_0(\mathrm{d}p) > 0,
\end{aligned}$$

where for the last equation we have chosen $\begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix} := ip_0 \begin{pmatrix} 0 \\ \chi(p) \end{pmatrix} \in \mathfrak{H}_1$.

[4] The elements of the family are given by the different integer values of the helicity, $\frac{n}{2}$.

For higher spins (in particluar for n even) similar arguments follow. For some scalar function λ, take $\mathfrak{H}_n^{(>)} \ni \varphi_a = \overset{n}{\otimes} \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \lambda(p)$ and denote $\psi_a = \overset{n}{\otimes} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathfrak{H}_n$. Motivated by the case $n = 1$ we put, for some determination of $(i)^{\frac{1}{n}}$ and for $p \in \mathcal{C}^+$, $\overset{n}{\otimes} \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix} := \overset{n}{\otimes} \begin{pmatrix} 0 \\ (i)^{\frac{1}{n}}\ p_0\ \lambda(p) \end{pmatrix}$, and get

$$\frac{1}{2i}\left(\Big\langle \varphi_a, \psi_a \Big\rangle_\beta - \Big\langle \psi_a, \varphi_a \Big\rangle_\beta\right) = \int\limits_{\mathcal{C}^+} (p_0)^{2n}(p_0+p_3)^n |\lambda(p)|^{2n}\, \mu_0(\mathrm{d}p) > 0.$$

This concludes the proof. ∎

We will characterize the set of state conditions through the subspaces $\mathfrak{H}_n^{(0)}$ that appear in the massless models, i.e. from the preceding lemma we have

$$\mathcal{C} \ := \ \mathfrak{H}_n^{(0)} \oplus \mathfrak{H}_n^{(0)} \ = \ h_n^{(d)} \quad \text{and} \tag{27}$$

$$\mathcal{U} \ := \ \Big\{ \delta_\varphi \colon \varphi \in \mathcal{C} \Big\} \ \subset \ \mathcal{F}. \tag{28}$$

This choice will guarantee that the algebra of physical observables $\mathfrak{R}$ constructed using the T–procedure will be simple (cf. Remark 4.3). The nontriviality condition, i.e. $\mathbb{I} \notin \mathcal{A}(\mathcal{U})$ (cf. Lemma 2.2), follows from Equation (27) and from [10, Lemma 6.1] (see also [12, Section 3]).

In the next theorem we will establish rest of the relations between our examples and the quantities defined in Section 2.

Theorem 4.2 *Using the notation of the Section* 2 *and* 3 *the following equations hold:*

(i) $\mathfrak{p} = h_n$

(ii) $\mathfrak{p}_0 = h_n^{(d)}$

(iii) $\mathcal{S} = \mathcal{S}^* = \mathcal{F} = \mathcal{A}(\mathcal{U})'$

(iv) $\mathfrak{O} = \mathcal{F}$

(v) $\mathfrak{D} = \overline{\mathcal{A}(\mathcal{U})\mathcal{A}(\mathcal{U})'}$

(vi) $\mathfrak{R} = \mathcal{A}(\mathcal{U})' \big/ \overline{\mathcal{A}(\mathcal{U})\mathcal{A}(\mathcal{U})'}$,

where we put $\overline{\mathcal{A}(\mathcal{U})\mathcal{A}(\mathcal{U})'} := \mathrm{clo}_{\|\cdot\|} \Big\{ \mathcal{A}(\mathcal{U})\mathcal{A}(\mathcal{U})' \Big\}$.

Proof: For the cases (i) and (ii) only the nontrivial inclusions $\mathfrak{p} \supset h_n$ and $\mathfrak{p}_0 \subset h_n^{(d)}$ have to be shown. But these inclusions are a direct consequence of the definition of the set $\mathcal{C}$ in Eq. (27), of the degeneracy of the operator–valued function $\beta(\cdot)$ and of Lemma 3.2.

To prove (iii) note that from (i) we have $\mathcal{F} = \mathcal{A}(\mathcal{U})'$, so that from the definition of $\mathcal{S}$ (cf. Subsecton 2.1) the largest set in $\mathcal{F}$ such that, $\mathcal{A}(\mathcal{U})\mathcal{S} \subset \Big[\mathcal{F}\mathcal{A}(\mathcal{U})\Big]$, is precisely $\mathcal{F}$. (Analogously for the set $\mathcal{S}^*$.)

The last three statements are a consequence of (iii) and of Theorem 2.4. ∎

Remark 4.3 From Theorem 2.8 we have that $\mathfrak{R} \cong \mathrm{CCR}\Big(h_n', \sigma'\Big)$, where $h_n' := h_n \big/ h_n^{(d)}$ and σ' is the natural definition of σ on the factor space h_n'. $\mathfrak{R}$ is simple, because σ' is nondegenerate on h_n' (cf. Lemma 2.7). But, $\mathrm{CCR}\Big(h_n', \sigma'\Big)$ coincides with the C*–algebra defined in [1, Section B.3.3.1] from which we could define an isotone, causal and covariant net of local C*–subalgebras. Therefore, we can carry over the net structure encoded in $\mathrm{CCR}\Big(h_n', \sigma'\Big)$ to the maximal algebra of physical observables, $\mathfrak{R}$. The existence of the net of local C*–subalgebras of $\mathfrak{R}$ confirms also the choice of the name 'algebra of physical observables' for $\mathfrak{R}$.

Remark 4.4 In the present remark we denote $\varphi = \varphi_1 \oplus \varphi_2$, $\psi = \psi_1 \oplus \psi_2 \in h_n$. It is now easy to lift the state of $\mathrm{CCR}\Big(h_n^{'}, \sigma^{'}\Big)$ defined in [1, Section B.4.2] and that satisfies the spectrality condition to a regular Dirac state on $\mathcal{F}$. Indeed, define on h_n the nontrivial complexification J by $J(\varphi_1 \oplus \varphi_2) := (i\,\varphi_1) \oplus (-i\,\varphi_2)$. It satisfies $J^2 = -\mathbb{1}$ and $\sigma(\varphi, J\psi) = -\sigma(J\varphi, \psi)$ as usual, but in contrast with [16, Definition 8.2.9] the real form, s, given by

$$s(\varphi,\varphi) := \sigma(\varphi, J\varphi) = \Big\langle \varphi_1, \varphi_1 \Big\rangle_\beta + \Big\langle \varphi_2, \varphi_2 \Big\rangle_\beta, \tag{29}$$

is only positive semi–definite as a consequence of Lemma 3.2. The sesquilinear form, k, induced by J,

$$k(\varphi,\psi) := s(\varphi,\psi) + i\sigma(\varphi,\psi) = \Big\langle \varphi_1, \psi_1 \Big\rangle_\beta + \Big\langle \psi_2, \varphi_2 \Big\rangle_\beta\,, \tag{30}$$

is, consequently, also positive semi–definite. The inequality $|\sigma(\varphi,\psi)| \leq \|\varphi\|_s\, \|\psi\|_s$ holds for all $\varphi, \psi \in h_n$.

The generating functional specifying the Dirac state is given by $\phi(\varphi) := e^{-\frac{1}{4}\|\varphi\|_s^2}$, $\varphi \in h_n$. It satisfies $\phi(\varphi) = 1$ for all $\varphi \in \mathcal{C} = h_n^{(d)}$ and from the properties of s we have that $\phi(\varphi) \leq 1$ for all $\varphi \in h_n$. Therefore we can extend it to a regular Dirac state on $\mathcal{F}$. This fact does not contradict [12, Theorem 3.1], since s is nonnegative. With other words, what Grundling and Hurst interpret as 'the set of nonphysical objects' is empty in our case, i.e. $h_n \backslash \mathfrak{p} = \emptyset$.

Finally, since $k(V\varphi, V\psi) = k(\varphi,\psi)$ for all $\varphi, \psi \in h_n$ (cf. Section 3), we have following [9, Definition 3.6 and Proposition 4.6] that the indefinite inner product space[5] $\Big(h_n,\, k(\cdot,\cdot)\Big)$ defines a strict Strocchi–Wightman structure.

4.2 Transformations

As a consequence of the definitions and results of Section 3 for any $g \in \widetilde{\mathfrak{P}_+^\uparrow}$ the map $V(g)$ generates a symplectic transformation on h_n. The corresponding Bogoljubov automorphism α_g satisfies the condition $\alpha_g(\mathcal{D}) = \mathcal{D}$ for all $g \in \widetilde{\mathfrak{P}_+^\uparrow}$, because for $\psi \in h_n^{(d)}$ and $\varphi \in h_n$ we get

$$\alpha_g\left[\Big(\delta_\psi - \mathbb{1}\Big)\,\delta_\varphi\right] = \Big(\delta_{V(g)\psi} - \mathbb{1}\Big)\,\delta_{V(g)\varphi} \tag{31}$$

and using Lemma 3.2 (ii) we have that $V(g)\psi \in h_n^{(d)}$. α_g is not a gauge transformation, because the associated automorphism $\alpha_g^{'}$ on $\mathfrak{R}$ is different from the identity. Indeed, $\alpha_g^{'}$ was precisely the automorphism that guaranteed the covariance of the net of local algebras in [1, Lemma B.3.6, Lemma B.3.7 and Theorem B.3.8]. Finally, since $\mathcal{F} = \mathcal{A}(\mathcal{U})^{'}$, it is easy to see that $\mathcal{G} = \Big\{id\Big\}$ (cf. Subsection 2.2).

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[5] This IIP–space is in a certain sense trivial, since it contains no vectors with negative k–norm. [18]

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