Recurrences and continued fractions for Kummer functions

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1 Confluent hypergeometric functions

Let us consider Kummer differential equation:

$$x\frac{d^2y(x)}{dx^2} + (c-x)\frac{dy(x)}{dx} - ay(x) = 0 \tag{1}$$

If $a \neq 0, -1, -2, \ldots$ two independent solutions are:

- Confluent hypergeometric function of the first kind.

$$M(a;c;x) = {}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\,\frac{x^n}{n!} \tag{2}$$

- Confluent hypergeometric function of the second kind.

$$U(a;c;x) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)}\,M(a;c;x) + \frac{\Gamma(c-1)}{\Gamma(a)}\,x^{1-c}\,M(a+1-c;2-c;x) \tag{3}$$

Particular cases of this functions include several elementary functions, Laguerre and Hermite polynomials, incomplete gamma functions, error functions and parabolic cylinder functions.

2 Three term recurrence relations. Minimal and dominant solutions

Let us consider the functions $M(a+kn;c+mn;x)$ and $U(a+kn;c+mn;x)$ where $a,c,x>0$ are fixed, $k,m=0,\pm 1$ and n is an integer parameter that we allow to be large. When suitably normalized both functions satisfy the same three term recurrence relation (TTRR):

$$y_{n+1} + b_n y_n + a_n y_{n-1} = 0 \tag{4}$$

When using TTRR for numerical computation the notions of minimal and dominant solutions are crucial for stability. We recall here the basic definitions due to Gautschi and refer to [2] for more details.

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Given a TTRR we say that a solution f_n is ***minimal*** if given another solution g_n of the recurrence, not proportional to f_n, then:

$$\lim_{n\to\infty}\frac{f_n}{g_n}=0$$

We say that g_n is ***dominant***. We note that the minimal solution is uniquely defined up to a constant (not depending on n), whereas any dominant solution plus an arbitrary multiple of a minimal solution remains to be dominant.

Any computed solution y_n of a TTRR can be written as:

$$y_n = Af_n + Bg_n$$

In general we cannot guarantee that $B = 0$, so y_n will always contain a dominant factor that will rule the behavior of the solutions when n is large. Hence a minimal solution cannot be computed using the TTRR in the forward direction.

We list here minimal and dominant solutions for several cases of confluent hypergeometric functions. This information has been obtained using Perron theorem (see [2] and [7]) and asymptotic information on the solutions (see [8]):

(k,m)	Minimal	Dominant
$(0,1)$	$f_n = \dfrac{1}{\Gamma(c+n)}\, M(a;c+n;x)$	$g_n = \dfrac{1}{\Gamma(c+n-a)}\, U(a;c+n;x)$
$(0,-1)$	$f_n = \dfrac{\Gamma(a+1-c+n)}{\Gamma(2-c+n)}\, x^n\, M(a+1-c+n;2-c+n;x)$	$g_n = \dfrac{1}{\Gamma(c-n)}\, M(a;c-n;x)$
$(1,1)$	$f_n = \dfrac{1}{\Gamma(c+n)}\, M(a+n;c+n;x)$	$g_n = (-1)^n\, U(a+n;c+n;x)$
$(-1,-1)$	$f_n = (-1)^n\dfrac{\Gamma(1+n-a)}{\Gamma(2-c+n)}\, x^n M(a+1-c,2-c+n,x)$	$g_n = \dfrac{1}{\Gamma(c-n)}\, U(a-n;c-n;x)$
$(1,0)$	$f_n = U(a+n;c;x)$	$g_n = \dfrac{1}{\Gamma(a+n+1-c)}\, M(a+n;c;x)$
$(-1,0)$	–	–

3 Continued fractions

Continued fractions are useful expressions that can be used for approximating confluent hypergeometric functions. We refer the reader to classical references like [5] and [6] for more details and classification of continued fractions. Of particular interest are the C-fractions available for $M(a+kn;c+mn;x)$ functions in the cases $(k,m)=(1,1)$ and $(k,m)=(1,0)$, obtained through QD and related algorithms.

Here we will concentrate on the continued fractions derived from the corresponding TTRR, namely:

$$\frac{f_n}{f_{n-1}} = \frac{-a_n}{b_n+}\,\frac{-a_{n+1}}{b_{n+1}+}\,\frac{-a_{n+2}}{b_{n+2}+}\cdots \tag{5}$$

The question about convergence of this continued fraction is answered by Pincherle theorem:

Theorem (Pincherle). The continued fraction (5) converges if and only if the recurrence (4) has a minimal solution f_n. In case of convergence, for each $n = 1, 2, 3, \dots$

$$\frac{f_n}{f_{n-1}} = \frac{-a_n}{b_n +} \frac{-a_{n+1}}{b_{n+1} +} \frac{-a_{n+2}}{b_{n+2} +} \dots$$

provided $f_n \neq 0$, $n = 0, 1, 2, \dots$

4 Numerical strategies

Given a TTRR with a minimal solution we can proceed as follows:

- If we want to compute a dominant solution we can use the TTRR in the forward direction from two initial values.
- If we want to compute a minimal solution we can use the associated CF. A normalization identity is necessary to obtain a single value of the function instead of a ratio of consecutive values. See [9] for more details.

Is this information enough in order to compute safely using recurrences?

5 Pseudostability and pseudoconvergence

We note that minimal and dominant solutions are defined in the limit $n \to \infty$. It can be shown that in many cases the behavior for moderate values of n can be opposite to what is given by asymptotic information.

We will speak of **pseudostability** of the recurrence when there is a transitory region for moderate values of n where a dominant solution is not correctly computed and a minimal solution is accurately calculated within machine precision. Analogously we will speak of **pseudoconvergence** of the associated continued fraction when the successive approximants converge temporarily to a ratio of dominant solutions. Roughly speaking, the roles of dominant and minimal solutions are reversed.

Both phenomena occur within a finite (although possibly large) range of n, and when n is large enough the expected behavior is recovered. Some striking examples in the case of confluent hypergeometric functions are the following:

- Recurrence $(k, m) = (1, 1)$ when $x > 0$ is large. This case was already noted by Gautschi in [3].
- Recurrence $(k, m) = (1, 0)$ when $c > 0$ is large.
- Recurrence $(k, m) = (0, -1)$ when $x > 0$ is large.

A complete study of three term recurrence relations for confluent hypergeometric functions, including the problemas of pseudostability, is not yet available in the literature. This information is absolutely necessary for instance in order to design an algorithm of computation for these functions.

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