New inequalities from classical Sturm theorems

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Abstract

Inequalities satisfied by the zeros of the solutions of second order hypergeometric equations are derived through a systematic use of Liouville transformations together with the application of classical Sturm theorems. This systematic study allows us to improve previously known inequalities and to extend their range of validity as well as to discover inequalities which appear to be new. Among other properties obtained, Szegö's bounds on the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$ for $|\alpha|<1/2$, $|\beta|<1/2$ are completed with results for the rest of parameter values, Grosjean's inequality (Grosjean, C.C., J. Approx. Theory 50 (1987) 84–88) on the zeros of Legendre polynomials is shown to be valid for Jacobi polynomials with $|\beta|<1$, bounds on ratios of consecutive zeros of Gauss and confluent hypergeometric functions are derived as well as an inequality involving the geometric mean of zeros of Bessel functions.

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1 Introduction

Sturm theorems for second order ODEs, in its various forms, are well known results from which a large variety of properties have been obtained (see for instance [12,7,5,10]). In particular, bounds on the distances of zeros and convexity properties of the zeros of hypergeometric functions can be derived.

A common characteristic of these results is that they are usually based on adequate changes of the dependent and the independent variables which allow for a simple analysis of the resulting differential equation. For example, given a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, the function

$$u(\theta)=\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos\theta) \tag{1}$$

satisfies a differential equation in normal form

$$d^2u/d\theta^2+A(\theta)u(\theta)=0 \tag{2}$$

where the coefficient $A(\theta)$ satisfies [12]

$$A(\theta)>(n+\frac{1}{2}(\alpha+\beta+1))^2\equiv A_M \text{ when } |\alpha|<1/2 \text{ and } |\beta|<1/2\,. \tag{3}$$

From this, Sturm's comparison theorem provides the following bound for the distance between two consecutive zeros of $u(\theta)$ [12]:

$$\theta_{k+1}-\theta_k>\frac{\pi}{\sqrt{A_M}}=\frac{\pi}{n+(\alpha+\beta+1)/2} \text{ when } |\alpha|<1/2\,,\ |\beta|<1/2 \tag{4}$$

A similar analysis can be done, for instance, for Laguerre polynomials by considering the function $v(x)=\exp(-x^2)x^{\alpha+1/2}L_n^{(\alpha)}(x^2)$. This gives a lower bound for the differences of square roots of consecutive zeros of Laguerre polynomials and a bound for distances on zeros of Hermite polynomials $H_n(x)$ [12] (because $H_n(\sqrt{x})$, $x>0$ satisfies the differential equation for Laguerre polynomials with $\alpha=-1/2$). Also the functions $\sqrt{x}C_\nu(x)$, being $C_\nu(x)$ a cylinder function (Bessel function), satisfy differential equations in normal form for which the Sturm comparison Theorem can be applied with ease [13].

A question remains regarding this type of analysis: why these changes of the dependent and independent variables and not other ones?. Which are the changes amenable to a simple application of the Sturm theorems?. In this paper, we perform a systematic study of Liouville transformations of the hypergeometric equations (Gauss and confluent) which allow for a simple analysis, in a sense to be made explicit later, of the monotony properties of the coefficient of the resulting differential equation (in normal form); this is followed by a detailed analysis of this coefficient. The above mentioned results for Jacobi, Laguerre and Hermite polynomials and for Bessel functions will be particular cases of the more general results provided by this systematic study.

Our analysis will also reveal convexity properties of the zeros and of simple functions of the zeros. For instance, we will see how Grosjean's convexity property [5] (see also [7]) for the zeros of Legendre polynomials

$$(1-x_k)^2 < (1-x_{k-1})(1-x_{k+1}) \tag{5}$$

also holds for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $|\beta| \leq 1$ (Legendre polynomials being a particular case) and to the zeros in $(0,1)$ of any other solution of the corresponding differential equation.

In addition, inequalities which appear to be new can be obtained, like, for instance, bounds on ratios of consecutive zeros.

Our results apply for any non-trivial solution of the corresponding differential equation. We will restrict to real intervals where the coefficients of the differential equation are analytic and to cases where we may have solutions with at least two zeros in this interval. These were the oscillatory cases studied in [4].

2 Methodology

We will consider the Sturm comparison and convexity properties in the following form:

Theorem 1 (Sturm) *Let $y'' + A(x)y = 0$ with $A(x)$ continuous in (a,b). Let $y(x)$ be a non-trivial solution of the differential equation in (a,b). Let $x_k < x_{k+1} < ...$ denote consecutive zeros of y in (a,b) arranged in increasing order. Then*

(1) If $\exists A_M > 0$ such that $A(x) < A_M$ in (a,b) then $\Delta x \equiv x_{k+1} - x_k > \dfrac{\pi}{\sqrt{A_M}}$

(2) If $\exists A_m > 0$ such that $A(x) > A_m$ in (a,b) then $\Delta x \equiv x_{k+1} - x_k < \dfrac{\pi}{\sqrt{A_m}}$

(3) If $A(x)$ is strictly increasing in (a,b) then $\Delta^2 x \equiv x_{k+2} - 2x_{k+1} + x_k < 0$

(4) If $A(x)$ is strictly decreasing in (a,b) then $\Delta^2 x \equiv x_{k+2} - 2x_{k+1} + x_k > 0$

Remark 2 *Of course, the first result still holds if there is one point in (a,b) for which $A(x) = A_M$ but $A(x) < A_M$ in the rest of the interval. For instance,* *we will find this case when $A(x)$ reaches a relative maximum in (a,b) and it is an absolute maximum in (a,b). The same can be said for the second result of this theorem. Clearly, this also holds under much broader conditions.*

The third and fourth results of Theorem 1 are usually known as convexity theorem [7], which admits the following relaxation of hypothesis:

Theorem 3 (Sturm convexity theorem) *Let $y'' + A(x)y = 0$ with $A(x)$ continuous in (a,b) and such that it may change sign in (a,b) at one point $(x = c)$ at most. Let $A(x)$ be positive in an interval $I \subseteq (a,b)$ and, if $A(x)$ changes sign, let $A(x) < 0$ in the rest of the interval (except at $x = c$).*

(1) If $A(x)$ is strictly increasing in I then $\Delta^2 x \equiv x_{k+1} - 2x_k + x_{k-1} < 0$.
(2) If $A(x)$ is strictly decreasing in I then $\Delta^2 x \equiv x_{k+1} - 2x_k + x_{k-1} > 0$.

These are well known results. We provide a brief sketch of the proofs in Appendix A.

We will apply these theorems for confluent and Gauss hypergeometric functions, which are solutions of differential equations

$$y'' + B(x)y' + A(x)y = 0. \tag{6}$$

with one (confluent functions at $x = 0$) or two finite singular regular points (Gauss hypergeometric function at $x = 0$ and $x = 1$).

Our goal will be to obtain bounds on the spacing between zeros and convexity properties of the zeros, or of simple functions of these zeros, which are valid for all the zeros inside a given maximal interval of continuity of $B(x)$ and $A(x)$. In particular, we will focus on the intervals $(0, +\infty)$ for confluent functions and $(0, 1)$ for Gauss hypergeometric functions; as we later discuss, properties in the rest of the maximal intervals can be obtained using Kummer's relations [2].

The differential equations satisfied by the hypergeometric functions are not in normal formal, but they can be transformed to normal form by a change of function or by a change of variables or by both. Given a solution $y(x)$ of a differential equation in standard form (Eq. (6)), the function $\tilde{y}(x)$ defined as

$$\tilde{y}(x) = \exp\left(\frac{1}{2}\int^x B(x)\right) y(x) \tag{7}$$

satisfies the equation

$$\tilde{y}'' + \tilde{A}(x)\tilde{y} = 0 \text{ with } \tilde{A}(x) = A - B'/2 - B^2/4 \tag{8}$$

which is in the form of Theorem 1. In addition to changes of the dependent variable, we can also consider changes of the independent variable $z(x)$, followed by a transformation to normal form. It is straightforward to check that given a function $y(x)$ which is a solution of Eq. (6) then the function $Y(z)$, with $Y(z(x))$ given by

$$Y(z(x)) = \sqrt{z'(x)} \exp\left(\frac{1}{2}\int^x B(x)\right) y(x), \tag{9}$$

satisfies the equation in normal form

$$\ddot{Y}(z) + \Omega(z)Y(z) = 0 \tag{10}$$

where the dots mean differentiation with respect to z and [8]

$$\Omega(z) = \dot{x}^2 \tilde{A}(x(z)) + \frac{1}{2}\{x, z\} \tag{11}$$

where $\{x, z\}$ is the Schwarzian derivative of $x(z)$ with respect to z and $\tilde{A}(x)$ is given by Eq. (8). This transformation of the differential equation is called a Liouville transformation, of crucial importance in the asymptotic analysis of second order ODEs [8]. In the variable x, this can also be written as:

$$\begin{aligned} \Omega(x) \equiv \Omega(z(x)) &= \frac{1}{z'(x)^2}(\tilde{A}(x) - \tfrac{1}{2}\{z, x\})) \\ &= \frac{1}{z'(x)^2}\left(A(x) - \frac{B'(x)}{2} - \frac{B(x)^2}{4} + \frac{3d'(x)^2}{4d(x)^2} - \frac{d''(x)}{2d(x)}\right) \end{aligned} \tag{12}$$

where $\{z, x\}$ is the Schwarzian derivative of $z(x)$ with respect to x.

The transformed function $Y(x) \equiv Y(z(x))$, Eq. (9), has the same zeros as $y(x)$ in (a, b) provided that $B(x)$ is continuous in (a, b), and because $Y(z)$ satisfies (10) the equation is in the form of the Sturm theorems.

We will use the freedom to chose $d(x)$ conveniently so that the problem becomes tractable in the sense that the monotony properties of $\Omega(z)$ are easily obtained. For this purpose, it is preferable to study the monotony properties of $\Omega(x)$ instead of those of $\Omega(z)$; both functions have the same monotony properties provided we consider changes of variable such that $z'(x) > 0$ (because $\Omega'(x) = \dot{\Omega}(z)z'(x)$). In addition we introduce a further simplification of the problem by restricting to those changes of variable for which the equation $\Omega'(x) = 0$ is equivalent to a simple quadratic equation. Within these restrictions, we will perform a detailed study of the monotony of the $\Omega(x)$ coefficients for the available changes of variable.

We will now consider separately the case of the differential equations satisfied by the hypergeometric functions ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$, starting from ${}_p\mathrm{F}_1$ $p=2$ and decreasing p. With this, the study includes the whole family of hypergeometric functions satisfying second order ODEs for real parameters. The case of the differential equation satisfied by the ${}_2\mathrm{F}_0$:

$$x^2 y'' + [-1 + x(a+b+1)]y' + ab\, y = 0\,. \tag{13}$$

does not need to be considered separately, because given $y(\alpha, \beta, x)$ a set of solutions of the confluent hypergeometric equation (with one of the solutions ${}_0\mathrm{F}_1(\alpha;\beta;x)$), then $w(x) = |x|^{-a} y(a, 1+a-b, -1/x)$, for $x > 0$ or $x < 0$, are solutions of Eq. (13). In other words, the properties of the zeros of solutions of Eq. (13) can be related to properties of the zeros of confluent hypergeometric functions.

3 Gauss hypergeometric equation

We consider the hypergeometric equation, satisfied by the Gauss hypergeometric functions ${}_2\mathrm{F}_1(a, b; c; x)$

$$x(1-x)\, y'' + [c - (a+b+1)x]\, y' - ab\, y = 0 \tag{14}$$

with the restrictions on the parameters necessary for having oscillatory solutions in $(0,1)$ (see [4]), namely:

$$a < 0\,,\ b > 1\,,\ c - a > 1\,,\ c - b < 0 \tag{15}$$

or, by symmetry, the same relation with a interchanged with b.

Properties of zeros in the other two maximal intervals of continuity ($(-\infty, 0)$ and $(1, +\infty)$), for the corresponding ranges of parameters consistent with oscillation, can be obtained from the properties of the zeros in $(0,1)$ using linear transformations of the differential equations mapping these other two intervals into $(0,1)$ (see [2], Vol. I, Chap. II) . Indeed, if we denote by $\psi(\alpha, \beta; \gamma, x)$ the solutions of the hypergeometric equation $x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0$ in the interval $(0,1)$ one can write solutions in the other two intervals by using that both

$$y(a, b; c; x) = (1-x)^{-a} \psi(a, c-b; c; x/(x-1))\,,\, x < 0 \tag{16}$$

and

$$y(a,b;c;x) = x^{-a}\psi(a, a+1-c; a+b+1-c; 1-1/x)\,, x>1 \tag{17}$$

are solutions of the hypergeometric differential equation $x(1-x)y''+(c-(a+b+1)x)y'-aby=0$.

Instead of the parameters a, b and c, we will normally use the real parameters

$$n=-a\,,\ \alpha=c-1\,, \beta=a+b-c\,, \tag{18}$$

corresponding to the notation for Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} {}_2\mathrm{F}_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2). \tag{19}$$

The oscillatory conditions in the interval $(0,1)$ (Eq. (15)) can be written, in terms of the Jacobi parameters, as:

$$n>0\,,\ n+\alpha+\beta>0\,,\ n+\alpha>0\,,\ n+\beta>0\,. \tag{20}$$

Transforming the hypergeometric differential equation (14) with the transformations (7) and (8) we arrive to an equation in normal form with

$$4\tilde{A}(x) = \frac{L^2-\alpha^2-\beta^2+1}{x(1-x)} + \frac{1-\alpha^2}{x^2} + \frac{1-\beta^2}{(1-x)^2} \tag{21}$$

where

$$L=b-a=2n+\alpha+\beta+1\,. \tag{22}$$

Although in principle possible, the study of the monotony properties of $\tilde{A}(x)$ for all ranges of the parameters L, α and β, with the conditions (15) seems a difficult task. For this sake, we should solve a cubic equation depending on three parameters in order to obtain the points were $\tilde{A}'(x)=0$. Instead, we will consider the restriction before mentioned of considering changes of variable for which the coefficient $\Omega(x)$ for the equation in normal form admits a simple analysis of its monotony properties, meaning that $\Omega'(x)=0$ is equivalent to a quadratic equation in the interval $(0,1)$ for any values of the parameters. This will allow us to obtain global inequalities, satisfied for all the zeros inside each interval of continuity of $\tilde{A}(x)$; classical inequalities [12] as well as new inequalities or generalizations of previous inequalities [5] will be obtained in a systematic way.

For the Gauss hypergeometric equation, there are several different types of changes of variables which provide such simple coefficients $\Omega(x)$. Looking at Eq. (12) it is easy to see that the term $\tilde{A}(x)/z'(x)^2$ will be simple for all parameters if

$$1/z'(x)^2 \propto x(1-x), x^2, (1-x)^2, x^2(1-x), x(1-x)^2, x^2(1-x)^2. \qquad (23)$$

On the other hand, one can check that the Schwarzian derivative term gives a contribution of the same type, and that the resulting $\Omega(x)$ is such that $\Omega'(x) = 0$ is equivalent to a quadratic equation in $(0, 1)$.

These are not the only changes of variable which lead to a simple $\Omega(x)$. A more systematic analysis can be used to prove that the changes of variable $z(x)$ such that $z'(x) \equiv d(x) = x^{p-1}(1-x)^{q-1}$, with $p = 0$ or $q = 0$ or $p+q = 1$ are also valid (see Appendix B). We will only study in detail the changes of variable given by (Eq. (23)), for which inequalities in terms of elementary functions of the zeros can be obtained. A general study of all the possible admissible changes of variable and the study of the monotony properties of $\Omega(x)$ (depending on the selection of 4 independent parameters) is in principle possible but it seems a very laborious task which lies outside the scope of the present work.

In addition, it is interesting to note that the changes of variable corresponding to Eq. (23) are those associated to the different fixed point methods, stemming from first order DDEs, available for the computation of the zeros of Gauss hypergeometric functions [9,3,4]. Interlacing properties between the zeros of contiguous hypergeometric functions are easily available from a simple analysis of these DDEs, similarly as was done in [11]. We will not explore here this type of properties.

Because, as shown in Appendix B, interchanging the values of p and q is equivalent to interchanging α with β and x with $1-x$, it is enough to consider, for instance, that $q \geq p$. The analogous properties when interchanging p and q follow immediately. In this way, it is enough to take into account the cases $(p, q) = (1/2, 1/2), (0, 1), (0, 1/2), (0, 0)$ in order to complete the analysis of the changes of variable implied by Eq. (23).

3.1 The change $z(x) = \arccos(1-2x)$. Szegö's bounds for Jacobi polynomials and related results.

For $p = q = 1/2$, we can, for instance, take as change of variable $z(x) = \arccos(1-2x)$, which transforms the interval $(0,1)$ to $(0,\pi)$. The new variable $z(x)$ is the angle θ in Eq. (1). We will use the notation $\theta(x)$ for the change of variables instead of $z(x)$. With the corresponding Liouville transformation we get (Eq. (10)):

$$\omega(x) \equiv 4\Omega(x) = L^2 - \frac{\alpha^2 - 1/4}{x} - \frac{\beta^2 - 1/4}{1-x} \tag{24}$$

where

$$L = 2n + \alpha + \beta + 1\,. \tag{25}$$

This differential equation corresponding to Eq. (24) is the differential equation for the function defined in Eq. (1), studied by Szegö [12]. Not surprisingly, the study of the monotony of $\omega(x)$ leads to Szegö's bound when $|\alpha|, |\beta| \leq 1/2$ (in a slightly improved version: compare Eqs. (4) and (28)). It is straightforward to check that, when the oscillatory conditions (Eq. (20)) are satisfied, we have the following properties in the interval $(0,1)$:

(1) If $|\alpha| = |\beta| = 1/2$ then $\Omega'(x) = 0$,
(2) otherwise:
   (a) If $|\alpha| \leq 1/2$ and $|\beta| \leq 1/2$, $\Omega(x)$ has one and only absolute extrema in $[0,1]$ and it is a minimum.
   (b) If $|\alpha| \geq 1/2$ and $|\beta| \geq 1/2$, $\Omega(x)$ has one and only absolute extrema in $[0,1]$ and it is a maximum.
   (c) If $|\alpha| \geq 1/2$ and $|\beta| \leq 1/2$, $\Omega'(x) > 0$.
   (d) If $|\alpha| \leq 1/2$ and $|\beta| \geq 1/2$, $\Omega'(x) < 0$.

In the cases for which there is an extremum, it is reached at

$$x_e = \frac{\sqrt{|1/4 - \alpha^2|}}{\sqrt{|1/4 - \alpha^2|} + \sqrt{|1/4 - \beta^2|}} \tag{26}$$

and the values of $\omega(x) = 4\Omega(x)$ at these points are:

$$\omega(x_e) = L^2 \pm \left(\sqrt{|1/4 - \alpha^2|} + \sqrt{|1/4 - \beta^2|}\right)^2 > 0, \tag{27}$$

where the $+$ sign applies when the extremum is a maximum and the $-$ sign is for the minimum. Accordingly, the following relations are obtained in the variable $\theta(x)$:

Theorem 4 *Let x_k, $k=1,...,N$ be the zeros of any solution of the hypergeometric equation in $(0,1)$ and let $\theta_k = \arccos(1-2x_k)$, $k=1,...,N$, then*

$$\begin{aligned}
&\textit{If } |\alpha| = |\beta| = 1/2 \textit{ then } \Delta\theta = \tfrac{2\pi}{L}\textit{, otherwise:}\\
&\textit{1.- } \Delta\theta < \frac{2\pi}{\sqrt{L^2+\left(\sqrt{1/4-\alpha^2}+\sqrt{1/4-\beta^2}\right)^2}}\,,\ |\alpha|\le 1/2\,,\ |\beta|\le 1/2.\\
&\textit{2.- } \Delta\theta > \frac{2\pi}{\sqrt{L^2-\left(\sqrt{\alpha^2-1/4}+\sqrt{\beta^2-1/4}\right)^2}}\,,\ |\alpha|\ge 1/2\,,\ |\beta|\ge 1/2.\\
&\textit{3.- } \Delta^2\theta < 0\,,\ |\alpha|\ge 1/2\,,\ |\beta|\le 1/2.\\
&\textit{4.- } \Delta^2\theta > 0\,,\ |\alpha|\le 1/2\,,\ |\beta|\ge 1/2.
\end{aligned} \tag{28}$$

These results refine Szegö's the bounds on distances of the θ-zeros of Jacobi polynomials, for $|\alpha| < 1/2$ and $|\beta| < 1/2$, and complete them to general values of α and β.

Additional monotony results for the first two cases can be obtained when we only consider zeros which lie in the same side of the extremum x_e (either in the increasing or the decreasing side of $\omega(x)$). Indeed, given $\theta_e = \arccos(1-2x_e)$, if we denote by $\mathrm{sign}(\theta-\theta_e)$ the common sign of all the values of $\theta-\theta_e$ for θ any of the zeros involved in the expression of $\Delta^2\theta$, we have:

$$\begin{aligned}
&\text{1.- If } |\alpha|\le 1/2 \text{ and } |\beta|\le 1/2 \text{ (but not both equal to 1/2) then:}\\
&\qquad \mathrm{sign}(\theta-\theta_e)\Delta^2\theta < 0.\\
&\text{2.- If } |\alpha|\ge 1/2 \text{ and } |\beta|\ge 1/2 \text{ (but not both equal to 1/2) then:}\\
&\qquad \mathrm{sign}(\theta-\theta_e)\Delta^2\theta > 0.
\end{aligned} \tag{29}$$

In the particular cases for which $|\alpha| = |\beta|$, the possible extrema are reached at $x_e = 1/2$, that is $\theta_e = \pi/2$, and Szegö's monotony results are obtained ([12], pg. 126, Thm. 6.3.3) as a particular case. In [1], a similar property, true for $|\alpha| < 1/2$ and $|\beta| \ge |\alpha|$, is proved; this is related to result number 4 in Eq. (28) and to the first result in Eq. (29). In the sequel, we will not insist in showing these partial monotony results and we will only consider bounds and inequalities satisfied globally for x-zeros (or simple functions of these zeros) in the whole interval $(0,1)$.

3.2 The change $z(x) = \log(x)$. Generalization of Grosjean's inequality.

Taking $p = 0$, $q = 1$, we have the change $z(x) = \log(x)$. The corresponding $\Omega(x)$ function is given by:

$$w(x) = 4\Omega(x) = -L^2 + \frac{L^2 - \alpha^2 + \beta^2 - 1}{1 - x} + \frac{1 - \beta^2}{(1 - x)^2}, \tag{30}$$

where we see that the singularity in $x = 0$ has been absorbed by the new variable $z(x)$ and it has disappeared in $\Omega(x)$.

Then, assuming that the oscillation conditions (Eq. 20) are fulfilled, we have the following monotony properties in $(0, 1)$:

(1) If $|\beta| \leq 1$, $\Omega'(x) > 0$.
(2) If $|\beta| > 1$, $\Omega(x)$ has only one absolute maximum, which is located at

$$0 < x_e = \frac{L^2 - \alpha^2 - (\beta^2 - 1)}{L^2 - \alpha^2 + \beta^2 - 1} < 1, \tag{31}$$

where

$$\Omega(x_e) = \frac{1}{16} \frac{[(L + \alpha)^2 - (\beta^2 - 1)][(L - \alpha)^2 - (\beta^2 - 1)]}{\beta^2 - 1} > 1. \tag{32}$$

Consequently, we have that:

Theorem 5 *Let $z(x) = \log(x)$ then the zeros of hypergeometric functions in $(0, 1)$ satisfy:*

(1) If $|\beta| \leq 1$ then $\Delta^2 z < 0$. Therefore (undoing the change of variables) the zeros of the hypergeometric function satisfy the inequality

$$x_k^2 > x_{k-1} x_{k+1}. \tag{33}$$

(2) If $|\beta| > 1$ then $\Delta z > f(L, \alpha, \beta)$ where

$$f(L, \alpha, \beta) = 4\pi \sqrt{\frac{\beta^2 - 1}{[(L + \alpha)^2 - (\beta^2 - 1)][(L - \alpha)^2 - (\beta^2 - 1)]}} \tag{34}$$

or, in terms of the zeros of the hypergeometric function

$$\frac{x_{k+1}}{x_k} > \exp(f(L, \alpha, \beta)) \text{ if } |\beta| > 1. \tag{35}$$

In terms of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, and denoting its zeros by $\tilde{x_k}$, we have:

Corollary 6 *The zeros of Jacobi polynomials satisfy:*

$$\begin{aligned} &(1-\tilde{x}_k)^2 > (1-\tilde{x}_{k-1})(1-\tilde{x}_{k+1}) \ \textit{when}\ |\beta| < 1 \\ &\frac{1-\tilde{x}_k}{1-\tilde{x}_{k+1}} > \exp(f(L,\alpha,\beta)) \ \textit{if}\ |\beta| > 1. \end{aligned} \tag{36}$$

This result was proved by Grosjean [5] in the particular case of Legendre polynomials (see also [6]). Therefore, our result is a generalization of Grosjean's [5] inequality for Jacobi polynomials. Furthermore, these are true for the zeros in $(0,1)$ of any solution of the corresponding differential equation (for Jacobi polynomials with $\alpha, \beta > -1$ all the zeros lie in $(0,1)$).

Interchanging the values of p and q we have the change $z(x) = -\log(1-x)$, for which we get similar results as before but the α and β parameters are interchanged, as well as x is interchanged with $1-x$ in Eqs. (33) and (35). In terms of the zeros of Jacobi polynomials, we get:

Corollary 7 *The zeros of Jacobi polynomials satisfy:*

$$\begin{aligned} &(1+\tilde{x}_k)^2 > (1+\tilde{x}_{k-1})(1+\tilde{x}_{k+1}) \ \textit{when}\ |\alpha| \leq 1 \\ &\frac{1+\tilde{x}_{k+1}}{1+\tilde{x}_k} > \exp(f(L,\beta,\alpha)) \ \textit{if}\ |\alpha| > 1. \end{aligned} \tag{37}$$

3.3 The change $z(x) = -\tanh^{-1}(\sqrt{1-x})$.

For $p=0$ and $q=1/2$, we take the change of variables $z(x) = -\tanh^{-1}(\sqrt{1-x})$. After the corresponding Liouville transformation, the singularity at $x=0$ disappears in $\Omega(x)$, namely:

$$\Omega(x) = \beta^2 - \alpha^2 - \frac{1}{4} + \left(L^2 - 1/4\right) x - \frac{\beta^2 - 1/4}{1-x}. \tag{38}$$

Again, always assuming that the oscillation conditions are fulfilled, it is easy to check the following monotony properties in $[0,1]$:

(1) If $|\beta| \leq 1/2$, $\Omega'(x) > 0$.
(2) If $|\beta| \geq 1/2$, $\Omega(x)$ has only one absolute maximum, which is located at

$$0 < x_e = 1 - \sqrt{\frac{\beta^2 - 1/4}{L^2 - 1/4}} \leq 1, \tag{39}$$

where

$$\Omega(x_e) = \left(\sqrt{L^2 - 1/4} - \sqrt{\beta^2 - 1/4}\right)^2 - \alpha^2 > 0. \tag{40}$$

Consequently, we have that:

Theorem 8 *Let $z(x) = -\tanh^{-1}(\sqrt{1-x})$, the zeros of hypergeometric functions in $(0,1)$ verify the following inequalities:*

(1) If $|\beta| \leq 1/2$ then $\Delta^2 z < 0$, or, in terms of the zeros x_k of the hypergeometric function:

$$\frac{x_{k+1}x_{k-1}}{x_k^2} < \frac{h(x_{k+1})h(x_{k-1})}{h(x_k)^2} \tag{41}$$

with

$$h(x) \equiv (1 + \sqrt{1-x})^2. \tag{42}$$

(2) If $|\beta| \geq 1/2$ then $\Delta z > p(L, \alpha, \beta)$ where

$$p(L,\alpha,\beta) = \frac{\pi}{\sqrt{\left(\sqrt{L^2 - 1/4} - \sqrt{\beta^2 - 1/4}\right)^2 - \alpha^2}}. \tag{43}$$

This implies that

$$\frac{1 + \sqrt{1 - x_k}}{\sqrt{x_k}} \frac{\sqrt{x_{k+1}}}{1 + \sqrt{1 - x_{k+1}}} > \exp(p(L,\alpha,\beta)). \tag{44}$$

Similarly as before, if we interchange p and q by considering the change of variables $z(x) = \tanh^{-1}(\sqrt{x})$, we have similar relations with α interchanged with β and x with $1-x$. Namely:

Corollary 9 *The zeros of hypergeometric functions in $(0,1)$ satisfy:*

(1) If $|\alpha| \leq 1/2$ then

$$\frac{(1 - x_{k+1})(1 - x_{k-1})}{(1 - x_k)^2} < \frac{g(x_{k+1})g(x_{k-1})}{g(x_k)^2} \tag{45}$$

with

$$g(x) \equiv (1 + \sqrt{x})^2. \tag{46}$$

(2) If $|\alpha| \geq 1/2$ then $\Delta z > g(L, \beta, \alpha)$ for $z(x) = \tanh^{-1}(\sqrt{x})$, this means

that:

$$\frac{\sqrt{1-x_k}}{1+\sqrt{x_k}}\frac{1+\sqrt{x_{k+1}}}{\sqrt{1-x_{k+1}}} > \exp(p(L,\beta,\alpha)). \tag{47}$$

3.4 The change $z(x)=\log(x/(1-x))$.

This change, corresponding to $p=q=0$, treats in the same way the singularities at $x=0$ and $x=1$, similarly as the case $p=q=1/2$. This is why it is invariant with respect to the replacement $x\leftrightarrow 1-x$. Both singularities are eliminated in $\Omega(x)$, which becomes

$$4\Omega(x) = -(L^2-1)x^2+(L^2+\alpha^2-\beta^2-1)x-\alpha^2. \tag{48}$$

This is a parabola with one absolute maximum in

$$0<x_e<\frac{1}{2}\frac{L^2+\alpha^2-\beta^2-1}{L^2-1}<1, \tag{49}$$

where $\Omega(x)$ gets the value:

$$\Omega(x_e)=\frac{1}{16}\frac{(L^2-1-(\alpha-\beta)^2)(L^2-1-(\alpha+\beta)^2)}{L^2-1}. \tag{50}$$

This is true for any set of values of the parameters consistent with oscillation. As a consequence of this we have:

$$\Delta z > f(\beta,\alpha,L)=f(\alpha,\beta,L) \tag{51}$$

with f defined in Eq. (34).

In terms of the zeros of the hypergeometric function, we have the global bound:

Theorem 10 *The zeros of hypergeometric functions in* $(0,1)$ *verify:*

$$\frac{1-x_k}{x_k}\frac{x_{k+1}}{1-x_{k+1}} > \exp(f(\alpha,\beta,L)) \tag{52}$$

for all values of the parameters consistent with oscillation (Eq. (20)).

In terms of the zeros of hypergeometric functions for $x<0$ this result can be expressed in an even simpler form. Indeed, using Eq. (16) it is straightforward to check that:

Theorem 11 *given a solution of the hypergeometric equation (14) which oscillates in* $(-\infty, 0)$*, the successive zeros in this interval verify:*

$$\frac{x_{k+1}}{x_k} > \exp(f(c-1, a-b, c-b-a)) \tag{53}$$

for all the values of a*,* b *and* c *consistent with oscillation in* $(-\infty, 0)$ *(Remark 12).*

Remark 12 *For* $x < 0$ *the oscillating conditions are*

$$\begin{aligned} &a < 0\,,\, b < 0\,,\, c-a > 1\,,\, c-b > 1 \text{ or} \\ &a > 1\,,\, b > 1\,,\, c-a < 0\,,\, c-b < 0 \end{aligned} \tag{54}$$

When this conditions are not satisfied, there are no solutions with two zeros in $(-\infty, 0)$*, see [4].*

Going back to our original discussion in the interval $(0, 1)$, we notice that Theorem 10 resembles a combination of the bound for the $p = 0$ and $q = 1$ (Eq. (35) and the related bound for the $p = 1$ and $q = 0$, which reads:

$$\frac{1-x_k}{1-x_{k+1}} > \exp(f(L, \beta, \alpha)) \text{ for } |\alpha| > 1 \tag{55}$$

Combining both we have, when $|\alpha| > 1$ and $|\beta| > 1$ simultaneously,

$$\frac{1-x_k}{x_k}\frac{x_{k+1}}{1-x_{k+1}} > \exp(f(L, \alpha, \beta) + f(L, \beta, \alpha)) \tag{56}$$

which is weaker than Eq. (52), both because there is no restriction in the parameters in Eq. (52) and also in an asymptotic sense, because $f(L, \alpha, \beta)/f(\alpha, \beta, L) \to 0$ as $L \to \infty$

In terms of the zeros of Jacobi polynomials, Eq. (52) can also be written as:

Theorem 13 *The zeros (in* $(0, 1)$*) of Jacobi polynomials satisfy*

$$\frac{1-\tilde{x}_k}{1+\tilde{x}_k}\frac{1+\tilde{x}_{k+1}}{1-\tilde{x}_{k+1}} > \exp(f(\alpha, \beta, L)) \tag{57}$$

for all values of the parameters consistent with oscillation (Eq. (20)).

4 Kummer's confluent hypergeometric equation

The confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0 \tag{58}$$

is satisfied by the confluent hypergeometric series ${}_1\mathrm{F}_1(a; c; x)$. We concentrate on the positive zeros of this or any other function which is solution of (58). For the possible negative zeros of these functions the relations are similar because, if $y_1(x) \equiv y(a; c; x)$ is a solution of (58) then also $y_2(x) \equiv e^x y(c - a, c, -x)$ is a solution of the same equation.

Instead of the parameters a and c, we will normally use

$$n = -a\,,\ \alpha = c - 1 \tag{59}$$

corresponding to the notation of Laguerre polynomials:

$$L_n^{(\alpha)}(x) = \begin{pmatrix} n+\alpha \\ \alpha \end{pmatrix} {}_1\mathrm{F}_1(-n; \alpha + 1; x) \tag{60}$$

In terms of these parameters the oscillatory conditions [4] for the solutions of Eq. (58) in $(0, +\infty)$ are given by

$$n > 0\,,\ n + \alpha > 0 \tag{61}$$

Hermite polynomials are also related to the confluent hypergeometric equation because

$$H_n(x) = 2^n \mathrm{U}(-n/2; 1/2; x^2) \tag{62}$$

where $\mathrm{U}(a; c; x)$ is a solution of (58).

Let us now study the differential equations in normal form after convenient changes of variable. As before, we write this transformed equation as:

$$\ddot{Y}(z) + \Omega(z)Y(z) = 0 \tag{63}$$

and we study the monotony properties of $\Omega(x) \equiv \Omega(z(x))$.

Directly transforming the equation to normal form we obtain:

$$4\Omega(x) = -1 + \frac{2L}{x} + \frac{1-\alpha^2}{x^2} \tag{64}$$

where we now define

$$L = 2n + \alpha + 1\,. \tag{65}$$

This means that the trivial change $z(x) = x$ already provides information. Also, it is easy to convince oneself that other tractable changes of variable are $z(x) = \sqrt{x}$ and $z(x) = \log(x)$.

We can do the analysis for these and other cases by considering the effect of those changes of variable for which $d(x) = z'(x) = x^{m-1}$ (and therefore $z(x) = x^m/m\,, m \neq 0$ and $z(x) = \log(x)\,, m = 0$). With this change we have:

$$\Omega(x) = -\frac{1}{4}x^{-2m}(x^2 - 2Lx + \alpha^2 - m^2) \tag{66}$$

A careful analysis of this function for all values of the parameters reveals the following behaviour:

Lemma 14 *Let $\Omega(x)$ given by Eq. (66) and let*

$$x_e = \frac{m-1/2}{m-1}L - \frac{\sqrt{\Delta}}{m-1}\,,\ \Delta = (m-1/2)^2L^2 + m(1-m)(\alpha^2 - m^2) \tag{67}$$

then, except for some cases when $|\alpha| < |m|$ and $m \in (0, 1/2)$ simultaneously, one of the following situations necessarily takes place (independently for n and for values of the parameters consistent with oscillation) :

(1) Either $\Omega(x)$ has only one absolute extrema for $x \geq 0$ and it s a maximum, located at x_e, where $\Omega(x_e) > 0$.
(2) Or $\Omega(x)$ satisfies the conditions of Theorem 3 in $(0, 1)$, being $\Omega(x)$ strictly decreasing when it is positive.

The situations (1) and (2) take place for the following values:

(I) If $|\alpha| > |m|$ then the situation (1) takes place for all values of $|\alpha|$.
(II) If $|\alpha| = |m|$ then:
- *(a) If $m \leq 1/2$ then (1).*
- *(b) If $m \geq 1/2$ then (2).*

(III) If $|\alpha| < |m|$ then
- *(a) If $m < 0$ then (1).*

(b) If $m > 1/2$ then (2).

In the previous lemma, it is understood that the corresponding limit should be taken when a given expression loses meaning. For instance, when $m = 1$ and $|\alpha| > |m|$ we understand that $x_e = \lim_{m\to 1} \dfrac{m-1/2}{m-1} L - \dfrac{\sqrt{\Delta}}{m-1} = (\alpha^2-1)/L$. As a consequence of this Lemma and Theorem 1 (see Remark 2) and Theorem 3 we have

Theorem 15 *Let $x_k, x_{k+1}, ...,$ $x_k < x_{k+1} < ...,$ positive consecutive zeros of $xy'' + (\alpha + 1 - x)y' + ny = 0$, with $n > 0$ and $n + \alpha > 0$. Let*

$$\begin{aligned}
\delta_m x_k &= \Delta z(x_k) = z(x_{k+1}) - z(x_k) = \frac{x_{k+1}^m - x_k^m}{m}, \\
\delta_0 x_k &= \lim_{m\to 0} \frac{x_{k+1}^m - x_k^m}{m} = \log(x_{k+1}/x_k), \\
\delta_m^2 x_k &= \Delta^2 z(x_k) = (x_{k+2}^m - 2x_{k+1}^m - x_k^m)/m, \\
\delta_0^2 x_k &= \log(x_{k+2}) - 2\log(x_{k+1}) + \log(x_k)\,.
\end{aligned} \tag{68}$$

Then:

(1) If $|\alpha| \leq |m|$ and $m \geq 1/2$ (simultaneously) then $\delta_m^2 x_k > 0$
(2) If:
(a) $|\alpha| > |m|$ or
(b) $|\alpha| = |m|$ and $m \leq 1/2$ or
(c) $|\alpha| < |m|$ and $m < 0$
then

$$\delta_m x_k < \frac{\pi}{\sqrt{\Omega(x_e)}} = 2\pi x_e^m \sqrt{\frac{1-m}{Lx_e - \alpha^2 + m^2}} \tag{69}$$

where x_e and $\Omega(x_e)$ are given by Eqs. (67) and (66) respectively.
For $m = 1$ the right side of Eq. (69) should be understood as a limit:

$$\delta_1 x_k < \lim_{m\to 1} \frac{\pi}{\sqrt{\Omega(x_e)}} = \pi \sqrt{\frac{\alpha^2 - 1}{L^2 - (\alpha^2 - 1)}} \tag{70}$$

We illustrate Theorem 15 we three simple examples, the cases $m = 1$, $m = 1/2$ and $m = 0$. The cases $m = 1/2$ and $m = 0$ correspond to two linear difference differential equations of first order satisfied by confluent hypergeometric functions. As commented for the Gauss hypergeometric functions, interlacing properties between the zeros of contiguous functions can be obtained by using Sturm methods as described in [11].

4.1 $m = 1$

This corresponds to the trivial change of variable $z(x) = x$. In this case:

$$4\Omega(x) = -1 + \frac{2L}{x} + \frac{1-\alpha^2}{x^2} \tag{71}$$

which is strictly decreasing if $|\alpha| \leq 1$; the relative extremum for $|\alpha| > 1$ in $(0, +\infty)$ is reached at $x_e = \dfrac{\alpha^2 - 1}{L}$ where $\Omega(x_e) = \dfrac{L^2 - (\alpha^2 - 1)}{\alpha^2 - 1} > 0$.

We have that:

Theorem 16 *The zeros of confluent hypergeometric functions in* $(0, +\infty)$ *and, in particular, the zeros of Laguerre polynomials* $L_n^{(\alpha)}(x)$*, satisfy the following properties under oscillatory conditions (Eq.(61))*

(1) If $|\alpha| \leq 1$ *then* $\Delta^2 x > 0$*, in other words:*

$$x_k < (x_{k+1} + x_{k-1})/2. \tag{72}$$

(2) If $|\alpha| > 1$ *then*

$$x_{k+1} - x_k > \pi \frac{\sqrt{\alpha^2 - 1}}{\sqrt{L^2 - (\alpha^2 - 1)}}. \tag{73}$$

The zeros of Hermite polynomials $H_n(x)$ *(*$\alpha = -1/2$*),* $\tilde{x}_k$*, satisfy*

$$\tilde{x}_k^2 < (\tilde{x}_{k-1}^2 + \tilde{x}_{k+1}^2)/2. \tag{74}$$

4.2 $m = 1/2$

This corresponds to a change of variable $z(x) = 2\sqrt{x}$. We have:

$$\Omega(x) = -x + 2L - \frac{\alpha^2 - 1/4}{x}. \tag{75}$$

This function is monotonically decreasing for $|\alpha| \leq 1/2$. For $|\alpha| > 1/2$, it has only one local extremum for $x > 0$, which is a maximum and it is reached at $x_e = \sqrt{\alpha^2 - 1/4}$ where $\Omega(x_e) = 2(L - \sqrt{\alpha^2 - 1/4})$. For $|\alpha| = 1/2$ this value is also an upper bound for the function $\Omega(x)$, because its maximum value is reached at $x = 0$ in this case. Therefore:

Theorem 17 *The zeros of confluent hypergeometric functions in* $(0, +\infty)$ *and, in particular, the zeros of Laguerre polynomials* $L_n^{(\alpha)}(x)$*, satisfy the following properties under oscillatory conditions (Eq.(61))*

(1) If $|\alpha| \leq 1/2$ *then* $\Delta^2 \sqrt{x} > 0$*, that is:*

$$\sqrt{x_k} < \frac{\sqrt{x_{k+1}} + \sqrt{x_{k-1}}}{2}. \tag{76}$$

(2) If $|\alpha| \geq 1/2$ *then*

$$\Delta\sqrt{x} = \sqrt{x_{k+1}} - \sqrt{x_k} > \frac{\pi}{\sqrt{2(L - \sqrt{\alpha^2 - 1/4})}}. \tag{77}$$

The zeros of Hermite polynomials $H_n(x)$ *(*$L = n + 1/2$ *and* $\alpha = -1/2$*) satisfy simultaneously the properties:*

$$\begin{aligned} x_k &< \frac{x_{k+1} + x_{k-1}}{2}, \\ x_{k+1} - x_k &> \frac{\pi}{\sqrt{2n+1}}. \end{aligned} \tag{78}$$

The bound for Hermite polynomials is given by Szegö's formula (6.31.21), pg. 31.

4.3 $m = 0$

This corresponds to the change of variables $z(x) = \log(x)$. The singularities at $x = 0$ disappear from $\Omega(x)$, which becomes a parabola:

$$4\Omega(x) = -x^2 + 2Lx - \alpha^2 . \tag{79}$$

The maximum is reached at $x_e = L$, where $4\Omega(x_e) = L^2 - \alpha^2$. Therefore, the zeros of the confluent hypergeometric functions (like Laguerre polynomials) satisfy $\Delta \log(x) > \frac{2\pi}{\sqrt{L^2 - \alpha^2}}$ that is:

Theorem 18 *The zeros of confluent hypergeometric functions in* $(0, +\infty)$ *and, in particular, the zeros of Laguerre polynomials* $L_n^{(\alpha)}(x)$*, satisfy the following properties for any values of the parameters consistent with oscillation (Eq.(61))*

$$\frac{x_{k+1}}{x_k} > \exp\left(2\frac{\pi}{\sqrt{L^2 - \alpha^2}}\right) . \tag{80}$$

The zeros of Hermite polynomials satisfy:

$$\frac{\tilde{x}_{k+1}}{\tilde{x}_k} > \exp\left(\frac{\pi}{\sqrt{L^2-\alpha^2}}\right) . \tag{81}$$

5 The confluent equation for the ${}_0\mathbf{F}_1(;c;x)$ series: Bessel functions

The confluent hypergeometric equation

$$x^2y'' + (\nu+1)xy' + xy = 0 \tag{82}$$

has as one of its solutions the hypergeometric series ${}_0\mathrm{F}_1(;\nu+1;-x)$. The differential equation has oscillatory solutions only for $x>0$ and they have an infinite number of zeros. We used $-x$ as argument and $c=\nu+1$ as parameter in the series because the relation with Bessel functions is, in this way, simpler: is $\phi(\nu,x)$ is a solution of (82), the function

$$y(x) = x^{\nu/2}\phi(;\nu;x^2/4) \tag{83}$$

is a solution of Bessel equation

$$x^2y'' + xy' + (\nu^2 - x^2)y = 0 \tag{84}$$

for $x>0$.

In particular, the regular Bessel function $J_\nu(x)$ is related to the ${}_0\mathrm{F}_1(;\nu{+}1;-x)$ series.

We will express the results in this section in terms of the zeros of Bessel functions $c_{\nu,k}$ as well as for the zeros of the solutions of (82).

With the changes of variable $z(x)$ such that $z'(x) = d(x) = x^{m-1}$ we arrive to

$$\Omega(x) = \frac{4x + m^2 - \nu^2}{4x^{2m}} \tag{85}$$

and, depending on the values of m and ν, all the possibilities in Theorems 1 and 3 (or Remark 2) are possible. Namely:

Lemma 19 *Let $\Omega(x)$ given by Eq. (85) and let*

$$x_e = \frac{m(\nu^2-m^2)}{4(m-1/2)} \, , \; \Omega(x_e) = \frac{1}{2mx_e^{2m-1}} \tag{86}$$

then:

(1) If
  (a) $|\nu| > |m|$ *and* $m \leq 1/2$,
  (b) or $|\nu| = |m| < 1/2$,
  (c) or $|\nu| < |m|$ *and* $m < 0$,
  then the hypothesis of Theorem 3 (1) are satisfied.
(2) If
  (a) $|\nu| = |m| > 1/2$,
  (b) or $|\nu| < |m|$ *and* $m \geq 1/2$,
  then the hypothesis of Theorem 3 (2) are satisfied
(3) If $|\nu| > |m|$ *and* $m > 1/2$, $\Omega(x)$ *reaches only one absolute extremum for* $x > 0$ *and its is a maximum located at* $x = x_e$, *where* $\Omega(x_e) > 0$. *Theorem 1 (1) (with Remark 2) can be applied.*
(4) If $|\nu| < |m|$ *and* $m \in (0, 1/2)$, $\Omega(x)$ *reaches only one absolute extremum for* $x > 0$ *and its is a minimum located at* $x = x_e$, *where* $\Omega(x_e) > 0$. *Theorem 1 (2) (with Remark 2) can be applied.*

In addition, when $m = 1/2$, *we have that, for* $x > 0$

(1) If $|\nu| > 1/2$, *then* $\Omega'(x) > 0$ *and* $\Omega(x) < 1$.
(2) If $|\nu| = 1/2$, *then* $\Omega(x) = 1$.
(3) If $|\nu| < 1/2$, *then* $\Omega'(x) < 0$ *and* $\Omega(x) > 1$.

Then, using these results we arrive at:

Theorem 20 *Let* $x_k, x_{k+1}, \ldots,$ $x_k < x_{k+1} < \ldots$, *positive consecutive zeros of solutions of* $x^2y'' + (\nu+1)y' + xy = 0$. *Let* $\delta_m x_k$ *and* $\delta_m^2 x_k$ *as in Eq. (68), then*

(1) If
  (a) $|\nu| > |m|$ *and* $m \leq 1/2$,
  (b) or $|\nu| = |m|$ *and* $m < 1/2$,
  (c) or $|\nu| < |m|$ *and* $m < 0$,
  then $\delta_m^2 x_k < 0$.
(2) If
  (a) $|\nu| = |m|$ *and* $m > 1/2$,
  (b) or $|\nu| < |m|$ *and* $m \geq 1/2$,
  then $\delta_m^2 x_k > 0$.
(3) If $|\nu| > |m|$ *and* $m \geq 1/2$ *then* $\delta_m x_k > \pi/\sqrt{\Omega(x_e)}$.
(4) If $|\nu| = |m|$ *and* $m = 1/2$ *then* $\delta_m x_k = \pi$.
(5) If $|\nu| < |m|$ *and* $m \in (0, 1/2]$ *then* $\delta_m x_k < \pi/\sqrt{\Omega(x_e)}$.

where $x_e = \frac{m}{4}\frac{\nu^2 - m^2}{(m-1/2)}$, $m \neq 1/2$ *and*

$$\Omega(x_e) = \begin{cases} 1\,,\ m = 1/2 \\ \dfrac{1}{2m x_e^{2m-1}}\,,\ m \neq 1/2 \end{cases}$$

Relations for the zeros of Bessel functions can be obtained from Theorem 20 by replacing the x_k by $c_{\nu,k}^2/4$. For instance, when $m = 1/2$ we obtain the well known result:

Theorem 21 *The zeros of Bessel functions* $c_{\nu,k}$ *satisfy:*

(1) If $|\nu| > 1/2$ *then* $c_{\nu,k+1} - c_{\nu,k} > \pi$.
(2) If $|\nu| = 1/2$ *then* $c_{\nu,k+1} - c_{\nu,k} = \pi$.
(3) If $|\nu| < 1/2$ *then* $c_{\nu,k+1} - c_{\nu,k} < \pi$.

As a further example, we consider the case $m = 0$. When $m = 0$, $z(x) = \log(x)$ and $\delta_0^2 x_k = \log(x_{k+1}) - 2\log(x_k) + \log(x_{k-1}) < 0$ and then $x_k > \sqrt{x_{k-1}x_{k+1}}$ or, in terms of the zeros Bessel functions:

Theorem 22 *Let* $c_{\nu,k}$ *be consecutive zeros of a Bessel function of order* ν*, then*

$$c_{\nu,k} > \sqrt{c_{\nu,k-1}c_{\nu,k+1}}. \tag{87}$$

Using a variant of Sturm theorems, a related inequality was proved in [10], namely, that the extrema $c'_{\nu,k}$ between to consecutive zeros $c_{\nu,k}$ and $c_{\nu,k+1}$ satisfies $c'_{\nu,k} > \sqrt{c_{\nu,k}c_{\nu,k+1}}$.

6 Conclusions

We have developed a systematic study of transformations to normal form of second order hypergeometric equations by means of Liouville transformations. We selected transformations for which the problem of computing the extrema of the resulting coefficient reduces to solving a quadratic equation. Classical results on distances of zeros and convexity properties [12] are particular cases of the obtained properties. Other results, like the convexity property proved by Grosjean [5] for Legendre polynomials can be also obtained and generalized with our approach. In particular, Grosjean's inequality has been proved to be also valid for Jacobi polynomials. Other properties have been also derived, like bounds for ratios of consecutive zeros for Gauss and confluent hypergeometric functions and an inequality involving the geometric mean of the zeros of Bessel functions.

A Proof of Sturm theorems

The bounds on distances of consecutive zeros of Theorem 1 (and Remark 2) can be obtained with ease by using Sturm comparison theorem in the form given, for instance, in [13]. An even more direct proof can be found using the Ricatti equation associated to $y'' + A(x)y = 0$, similarly as done in [10]. We prove the second result in Theorem 1 (also with the comments in Remark 2) and the second result in Theorem 3 (which implies the forth result in Theorem 1). The rest of results can be proved in an analogous way.

Let $x_k < x_{k+1}$ be consecutive zeros of $y(x)$, which is a non-trivial twice differentiable solution of $y'' + A(x)y = 0$ in (a, b), being $A(x)$ continuous in (a, b). Because $y(x)$ is non-trivial necessarily $y'(x_k)y'(x_{k+1}) \neq 0$. Considering that, for instance $y(x)$ is positive in (x_k, x_{k+1}) then $y'(x_k) > 0$ and $y'(x_{k+1}) < 0$; therefore, the function

$$h(x) = -y'(x)/y(x) \tag{A.1}$$

satisfies that $\lim_{x\to x_k^+} h(x) = -\infty$ and $\lim_{x\to x_{k+1}^-} h(x) = +\infty$; besides $h(x)$ is differentiable in (x_k, x_{k+1}), where

$$h'(x) = A(x) + h(x)^2 \tag{A.2}$$

Assuming now that $A(x) > A_m > 0$ in (a, b) (with the exception of one point if Remark 2 is considered) then $h' > A_m + h^2$ in (x_k, x_{k+1}) and then $g(x) \equiv h(x)'/(A_m + h(x)^2) - 1 > 0$. Therefore

$$\lim_{\epsilon\to 0^+} \int_{x_k-\epsilon}^{x_{k+1}+\epsilon} g(x)dx > 0 \Rightarrow \frac{\pi}{\sqrt{A_m}} - (x_{k+1} - x_k) > 0$$

This proves (2) of Theorem 1 (of course, this result keeps being valid for the situations described in Remark 2).

For proving the second result of Theorem 3 we consider the hypothesis of that theorem with $A'(x) < 0$ when $A(x) > 0$ in (a, b); therefore, $A(x)$ may become negative at the right of the interval. Within these hypothesis, it is obvious that if $\exists c \in (a, b)$ such that $A(x) < 0\, \forall x \in (c, b)$ then, for any non-trivial solution $y(x)$ in (a, b), there is at most one zero in $[c, b)$: this is implied by the fact that $A(x) < 0$ in (c, b) and then $y(x)y''(x) > 0$ in (c, b). Let

$x_k < x_{k+1} < x_{k+2}$ be consecutive zeros such that $A(x_k) > 0$ and $A(x_{k+1}) > 0$. Because $A(x) > A(x_{k+1})$ in (x_k, x_{k+1}), we have, similarly as before, that

$$\frac{\pi}{\sqrt{A(x_{k+1})}} > x_{k+1} - x_k \tag{A.3}$$

and, no matter which is the sign of $A(x_{k+2})$, we have that $A(x) < A(x_{k+1})$ in (x_{k+1}, x_{k+2}) and therefore

$$\frac{\pi}{\sqrt{A(x_{k+1})}} < x_{k+2} - x_{k+1} \tag{A.4}$$

Eqs. (A.3) and (A.4) imply that $\Delta x_k = x_{k+2} - 2x_{k+1} + x_k > 0$, which proves the second result of Theorem 3.

B General changes of variable for the Gauss hypergeometric equation

Starting from the Gauss hypergeometric equation (14) written in standard form (6), and considering a Liouville transformation with change of variable $z(x)$ such that $z'(x) = x^{p-1}(1-x)^{q-1}$ we find (Eq. (12)) that

$$\begin{aligned}\Omega(x) = \frac{1}{4}x^{2(1-p)}(1-x)^{2(1-q)} \Bigg(& \frac{L^2 - \alpha^2 - \beta^2 + 1 - 2(p-1)(q-1)}{x(1-x)} \\ & + \frac{p^2-\alpha^2}{x^2} + \frac{q^2-\beta^2}{(1-x)^2} \Bigg)\end{aligned} \tag{B.1}$$

Let us notice that interchanging the values of p and q is equivalent to interchanging α with β and x with $1-x$.

We want to obtain the values of p and q for which, for any values of the parameters L, α and β, the equation $P(x) = 0$ for $x \in (0,1)$ is equivalent to solving a quadratic equation (or maybe a linear one). Taking the derivative, we find that it has the following structure:

$$\Omega'(x) = x^{-2p-1}(1-x)^{-2q-1}P(x) \tag{B.2}$$

where $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is a polynomial of degree 3 with coefficients depending on five parameters: L, α, β, p and q. Now, $\Omega'(x) = 0$ will be equivalent to a quadratic equation in $(0,1)$ when when $a_3 = 0$, when $P(0) = 0$ and then $P(x) = x(b_2x^2 + b_1x + b_0)$ or similarly when $P(1) = 0$. A lengthy but straightforward calculation gives:

$$
\begin{aligned}
a_3 &= \tfrac{1}{2}(1-p-q)\left[L^2 - (1-p-q)^2\right] \\
P(0) &= -\tfrac{1}{2}p(p^2 - \alpha^2) \\
P(1) &= \tfrac{1}{2}q(q^2 - \beta^2)
\end{aligned}
\tag{B.3}
$$

therefore, the equivalence with a quadratic equation is true if and only if one of these conditions is satisfied:

1. $p + q = 1$
2. $p = 0$ (B.4)
3. $q = 0$

which confirms that the changes implied by Eq. (23) are indeed valid. The general changes induced by these conditions are themselves related to hypergeometric functions. Of course, given any of these valid changes of variables, $z(x)$, $\tilde{z}(x) = K_1 z(x) + K_2$, being K_1 and K_2 constants are also valid changes and equivalent to $z(x)$ in the sense that they provide the same properties. As before mentioned, we always take $z(x)$ such that $z'(x) > 0\,\forall x$.

For $p > 0$ we can take as $z(x)$ the following incomplete beta function:

$$
\begin{aligned}
z(x) &= \textstyle\int_0^x t^{p-1}(1-t)^{q-1}dt = B_x(p,q) \\
&= \frac{x^p}{p}\,{}_2\mathrm{F}_1(1-q, p; p+1; x)
\end{aligned}
\tag{B.5}
$$

and for $q > 0$ we may consider

$$
z(x) = -B_{1-x}(q,p) = -\frac{(1-x)^q}{q}\,{}_2\mathrm{F}_1(1-p, q; q+1; 1-x) \tag{B.6}
$$

These changes of variable do not make sense when $p = 0$ or $q = 0$, but the differences, $z(x_{k+1}) - z(x_k)$ do make sense in the limit $p \to 0$ (or $q \to 0$). Of course, these cases can be also considered separately.

References

[1] S. Ahmed, A. Laforgia, M.E. Muldoon. *On the spacing of the zeros of some classical orthogonal polynomials.* J. London Math. Soc. (2), 25 (1982) 246–252.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. *Higher trascendental functions (Bateman Manuscript Project).* McGraw-Hill, 1953.

[3] A. Gil, J. Segura. *Computing the zeros and turning points of solutions of second order homogeneous linear ODEs*, SIAM J. Numer. Anal. 41(3) (2003) 827-855

[4] A. Gil, W. Koepf, J. Segura. *Numerical algorithms for the real zeros of hypergeometric functions* (submitted).

[5] C.C. Grosjean *A property of the zeros of the Legendre polynomials.* J. Approx. Theory 50 (1987) 84–88.

[6] A. Laforgia *Monotonicity properties for the zeros of orthogonal polynomials and Bessel functions.* Orthogonal polynomials and applications (Bar-le-Duc, 1984), 267–277, Lecture Notes in Math., 1171, Springer, Berlin, 1985.

[7] A. Laforgia, M.E. Muldoon *Some consequences of the Sturm comparison theorem.* Am. Math. Monthly 93 (1986) 89-94.

[8] F.W.J. Olver. *Asymptotics and Special Functions.* AK Peters. Wellesley, 1997.

[9] J. Segura. *The zeros of special functions from a fixed point method*, SIAM J. Numer. Anal. 40 (2002) 114-133.

[10] J. Segura. *On a conjecture regarding the extrema of Bessel functions and its generalization*, J. Math. Anal. Appl. 280 (2003) 54–62.

[11] j. Segura. *On the zeros and turning points of special functions*, J. Comput. Appl. Math. 153 (2003) 433-440.

[12] G. Szegö. *Orthogonal Polynomials.* American Mathematical Society. Providence, 1939.

[13] Nico M. Temme. *Special functions. An introduction to the classical functions of Mathematical Physics.* John Wiley and Sons, 1996.