GENERALIZED WEIGHTED SOBOLEV SPACES AND APPLICATIONS TO SOBOLEV ORTHOGONAL POLYNOMIALS II

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1. Introduction and main results.

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics. In the classical books [Ku], [KS], we can find the point of view of Partial Differential Equations. See also [T] and [HKM]. (The main topic of [HKM] is non-linear Partial Differential Equations and its applications to quasiconformal and quasiregular maps.) We are interested in the relationship between this topic and Approximation Theory in general, and Sobolev Orthogonal Polynomials in particular.

The specific problems we want to solve are the following:

1) Given a Sobolev scalar product with general measures in $\mathbf{R}$, find hypotheses on the measures, as general as possible, so that we can define a Sobolev space whose elements are functions.

2) If a Sobolev scalar product with general measures in $\mathbf{R}$ is well defined for polynomials, what is the completion, $P^{k,2}$, of the space of polynomials with respect to the norm associated to that scalar product? This problem has been studied in some very particular cases (see e.g. [ELW1], [EL], [ELW2]), but at this moment no general theory has been built.

3) What are the most general conditions under which the multiplication operator, $Mf(x) = x\,f(x)$, is bounded in the space $P^{k,2}$? We know by a theorem in [LPP] that the zeroes of the Sobolev orthogonal polynomials are contained in the disk $\{z : |z| \le \|M\|\}$. The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]). In this paper we answer the question stated also in [LP] about general conditions for M to be bounded (see Section 5). A more detailed study of this operator can be found in [R2].

This last question is very close to the definition of Sobolev spaces associated to these norms, the study of their completeness and the density of C^∞ functions. The problem of the definition of Sobolev spaces has been solved in [RARP] with the concept of p-admissible measures (see definitions 8 and 9 below); in that paper we also prove their completeness under very general conditions, not only for $p = 2$, but for $1 \le p \le \infty$.

One of the main problems in the theory of weighted Sobolev spaces is the study of the density of smooth functions. In particular, when all the measures are finite, have compact support and $C_c^\infty(\mathbf{R})$ is dense in a Sobolev space that is complete, then the closure of the polynomials is the whole Sobolev space. This is deduced from Bernstein's proof of Weierstrass' theorem, where the polynomials he builds approximate uniformly up to the k-th derivative any function in $C^k([a,b])$ (see e.g. [D, p.113]).

For the case $L^2(\mathbf{R},\mu)$, a classical result by M. Riesz gives conditions to obtain the density of polynomials when μ is not of compact support (see [R], [F, Chapter II.4]). In [R3] we can find results on density of polynomials in weighted Sobolev spaces in $\mathbf{R}$.

Here we prove density theorems for $C^\infty(\mathbf{R})$, with $1 \le p < \infty$. See also [R3] for other results with $1 \le p < \infty$, and [R1] for the case $p = \infty$. When the measures have compact support, this implies density theorems for $C_c^\infty(\mathbf{R})$ (multiplying by a suitable $C_c^\infty(\mathbf{R})$ function).

These generalized Sobolev spaces can be extended to the context of curves instead of $\mathbf{R}$ (see [APRR]).

We should remark that there exists another generalization of Sobolev spaces in the context of metric spaces (see [H], [M]). In these papers the treatment of this topic is from a different point of view.

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Now, let us state our main results here. We refer to the definitions in the next section. Let us present first the theorem about completeness, which is proved in [RARP].

Theorem A. ([RARP, Theorem 5.1]) *Let us consider* $1 \le p \le \infty$, *an open set* $\Omega \subseteq \mathbf{R}$ *and a* p*-admissible vectorial measure* $\mu = (\mu_0, \dots, \mu_k)$ *in* $\overline{\Omega}$ *with* $(\overline{\Omega}, \mu) \in \mathcal{C}$. *Then the Sobolev space* $W^{k,p}(\overline{\Omega}, \mu)$ *is complete.*

Remark. The condition $(\overline{\Omega}, \mu) \in \mathcal{C}$ is not very restrictive (see Definition 16 and its remarks in Section 3).

We cannot obtain such a general result for the density of $C^\infty(\mathbf{R})$; we need some additional hypotheses on the measures. The measures we are dealing with have been divided in five types and we have results which allow "gluing" these measures (see theorems 4.2, 4.3 and 4.4 below).

These are our main results on density.

Theorem 4.1. *Let us consider* $1 \le p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a* p*-admissible vectorial measure in* $[a, b]$. *If* μ *is a measure of type* $1, 2, 3, 4$ *or* 5, *then* $C_c^\infty(\mathbf{R})$ *is dense in the Sobolev space* $W^{k,p}([a, b], \mu)$.

Theorems A and 4.1 have the following important consequence.

Corollary 4.1. *Let us consider* $1 \le p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a* p*-admissible vectorial measure in* $[a, b]$. *If* μ *is a measure of type* $1, 2, 3, 4$ *or* 5, *then* $W^{k,p}([a, b], \mu)$ *is the closure of the polynomials in the norm of* $W^{k,p}([a, b], \mu)$.

Observe that we cannot expect $C_c^\infty(\mathbf{R})$ to be dense in $W^{k,\infty}([a, b], \mu)$ since $C([a, b])$ is not dense in $L^\infty([a, b])$. In [R1] one of the authors studies what is the closure of smooth functions in $W^{k,\infty}$.

In the following theorems we present density results for measures which can be obtained by "gluing" simpler ones (for example, gluing measures of types 1 to 5).

Theorem 4.2. *Let us consider* $1 \le p < \infty$ *and* $-\infty \le a < b < c < d \le \infty$. *Let* $\mu = (\mu_0, \dots, \mu_k)$ *be a* p*-admissible vectorial measure in* $[a, d]$, *and assume that there exists an interval* $I \subseteq [b, c]$ *with* $(I, \mu) \in \mathcal{C}_0$ *and* $\mu_j(I) < \infty$ *for* $0 \le j \le k$. *Then* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a, d], \mu)$ *if and only if* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a, c], \mu)$ *and* $W^{k,p}([b, d], \mu)$.

Remark. If $a, d \in \mathbf{R}$, we can write in Theorem 4.2 $C_c^\infty(\mathbf{R})$ instead of $C^\infty(\mathbf{R})$.

Theorem 4.3. *Let us consider* $1 \le p < \infty$ *and* $\{a_n\}$, $\{b_n\}$ *strictly increasing sequences of real numbers* (n *belonging to a finite set, to* $\mathbf{Z}$, $\mathbf{Z}^+$ *or* $\mathbf{Z}^-$) *with* $a_{n+1} < b_n$ *for every* n. *Let us consider* $(\alpha, \beta) := \cup_n (a_n, b_n)$ *with* $-\infty \le \alpha < \beta \le \infty$ *and a* p*-admissible vectorial measure,* μ, *in* $[\alpha, \beta]$. *Assume that for each* n *there exists an interval* $I_n \subseteq [a_{n+1}, b_n]$ *with* $(I_n, \mu) \in \mathcal{C}_0$ *and* $\mu_j(I_n) < \infty$ *for* $0 \le j \le k$. *Then* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([\alpha, \beta], \mu)$ *if and only if* $C^\infty(\mathbf{R})$ *is dense in every* $W^{k,p}([a_n, b_n], \mu)$.

Corollary 4.2. *Let us consider* $1 \le p < \infty$ *and* $\{a_n\}$, $\{b_n\}$ *strictly increasing sequences of real numbers* (n *belonging to a finite set, to* $\mathbf{Z}$, $\mathbf{Z}^+$ *or* $\mathbf{Z}^-$) *with* $a_{n+1} < b_n$ *for every* n. *Let us consider* $(\alpha, \beta) := \cup_n (a_n, b_n)$ *with* $-\infty \le \alpha < \beta \le \infty$ *and a* p*-admissible vectorial measure,* μ, *in* $[\alpha, \beta]$. *If, for each* n, $\mu|_{[a_n, b_n]}$ *is of type* $1, 2, 3, 4$ *or* 5, *then* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([\alpha, \beta], \mu)$.

The main result about the multiplication operator is the following.

Theorem 5.1. *For* $1 \le p < \infty$, *if* μ *is a* p*-admissible vectorial measure in* $[a, b]$ *of type* $1, 2, 3$ *or* 4, *and the multiplication operator is well defined in* $W^{k,p}([a, b], \mu)$, *then it is bounded. The result is also true for measures of type* 5 *verifying the additional condition* $w_k \le c\, w_{k-1}$ *in* $[a_0 - \delta, a_0 + \delta] \cap [a, b]$.

In the paper, the results are numbered according to the section where they are proved. Now we present the notation we use.

Notation. In the paper $k \ge 1$ denotes a fixed natural number; obviously $W^{0,p}(\overline{\Omega}, \mu) = L^p(\overline{\Omega}, \mu)$. All the measures we consider are Borel and positive. Also, all the weights are non-negative Borel measurable functions. If the measure does not appear explicitly, we mean that we are using Lebesgue measure. We allow measures μ_j which are not necessarily σ-finite but always assume that $d\mu_j = d(\mu_j)_s + w_j\, dx$, where $(\mu_j)_s$ is singular with respect to Lebesgue measure and w_j is a Lebesgue measurable function (which can be infinite in a set of positive Lebesgue measure). We denote by $\operatorname{supp} \nu$ the support of the measure ν. If A is a Borel set, $|A|, \chi_A$, $\overline{A}$, $\operatorname{int}(A)$ and $\#A$ denote, respectively, the Lebesgue measure, the characteristic function, the closure, the interior and the cardinality of A. By $f^{(j)}$ we mean the j-th distributional derivative of f. When we work in the space $W^{k,p}(\overline{\Omega},\mu)$ we denote by $W^{k-r,p}(\overline{\Omega},\mu)$ the space $W^{k-r,p}(\overline{\Omega},(\mu_r,\dots,\mu_k))$. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property. P_n denotes the set of polynomials with degree less than or equal to n, and a, b arbitrary real numbers with $a<b$; they are finite unless the contrary is specified. Finally, the constants in the formulae can vary from line to line and even in the same line.

The outline of the paper is as follows. Section 2 presents most of the definitions we need to state our results. We collect and prove some useful technical results in Section 3. In Section 4 we prove the results on completeness. Finally, we prove the results for Sobolev orthogonal polynomials in Section 5.

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2. Definitions.

There are two standard ways to define classical Sobolev spaces $W^{k,p}(\Omega)$ (with $1\le p<\infty$) in an open subset Ω of an Euclidean space:

(1) the completion of smooth functions $C^\infty(\Omega)$ with the norm

$$\|f\|_{k,p}:=\sum_{|\alpha|\le k}\|D^\alpha f\|_p\,,$$

where $\|g\|_p$ denotes the $L^p(\Omega)$ norm of g with respect to Lebesgue measure, and

(2) the functions f belonging to $L^p(\Omega)$ such that their weak derivatives up to order k belong also to $L^p(\Omega)$.

It is well-known that these two definitions are equivalent for $1\le p<\infty$ (see e.g. [A, p.52], [Ma, p.12]). However (1) and (2) coincide with the completion of $C^\infty(\mathbf{R}^n)$ only for smooth domains (see e.g. [A, p.54], [Ma, p.14]).

It is possible to define some particular weighted Sobolev spaces, where the weights considered are powers of $d(x)=dist(x,K)$ with $K\subseteq\partial\Omega$, and even $h(d(x))$ with h a monotone function, following the text [Ku]. If we want to define more general weighted Sobolev spaces we can use the approach in [KO]. Before we state the definition in [KO], let us observe that the distributional derivative of a Sobolev function is also a function belonging to $L^1_{loc}(\Omega)$. In order to get the inclusion

$$L^p(\Omega,u)\subseteq L^1_{loc}(\Omega)\,,\qquad \text{for } 1<p<\infty\,,$$

a sufficient condition, by Hölder inequality, is that the weight u satisfies $u^{-1/(p-1)}\in L^1_{loc}(\Omega)$ (see [KO, Theorem 1.5] or Lemma A below). With this fact in mind we can understand the definition in [KO]:

Given a weight u in Ω let us denote by $M_p(u)$, for $1<p<\infty$, the closed set

$$M_p(u):=\Big\{x\in\Omega:\ \int_{\Omega\cap U(x)}u^{-1/(p-1)}(y)\,dy=\infty\quad\text{for every neighbourhood } U(x) \text{ of } x\Big\}.$$

Given $w=(w_\alpha)_{|\alpha|\le k}$ a vectorial weight in Ω we can define the exceptional set $B:=\cup_{|\alpha|\le k}M_p(w_\alpha)$ and the Sobolev space $W^{k,p}(\Omega,w)$ with weight w, as the set of all functions $f\in L^p(\Omega\setminus B,w_0)$ such that their weak derivatives $D^\alpha f$ are elements of $L^p(\Omega\setminus B,w_\alpha)$ for all α with $|\alpha|\le k$.

With this definition, the weighted Sobolev space $W^{k,p}(\Omega,w)$ is a Banach space (see [KO, Section 3]).

In general, this is not true without removing the set B (see some examples in [KO]). However, note that if some w_α is identically zero, then $M_p(w_\alpha)=\Omega$ and $\Omega\setminus B=\emptyset$.

But now, we want to define a more general class of Sobolev spaces appearing in the context of orthogonal polynomials. Since we are interested in orthogonal polynomials on the real line we only need to consider the case $\Omega\subseteq\mathbf{R}$. In this field it is usual to work with Sobolev spaces for which the measures $w_j(x)\,dx$ are changed by general measures $d\mu_j(x)$ and some of them may have $\mu_j(\delta\Omega)>0$; so we consider in our definition Sobolev spaces in $\overline{\Omega}$, where Ω is an open set. Therefore, in general, these spaces do not match the definition in [KO].

Let us start with some preliminary definitions.

Definition 1. *We say that two functions u, v are comparable on the set A if there are positive constants c_1, c_2 such that $c_1 v(x) \le u(x) \le c_2 v(x)$ for almost every $x \in A$. Since measures and norms are functions on measurable sets and vectors, respectively, we can talk about comparable measures and comparable norms. We say that two vectorial weights or vectorial measures are comparable if each component is comparable.*

In what follows, the symbol $a \asymp b$ means that a and b are comparable for a and b functions, measures or norms.

Obviously, the spaces $L^p(A, \mu)$ and $L^p(A, \nu)$ are the same and have comparable norms if μ and ν are comparable on A. Therefore, in order to obtain results on completeness or density we can change a measure μ to any comparable measure ν.

Next, we shall define a class of weights which plays an important role in our results.

Definition 2. *We say that a weight w belongs to $B_p([a, b])$ if and only if*

$$\begin{aligned} &w^{-1} \in L^{1/(p-1)}([a, b]) , &&\text{for } 1 \le p < \infty , \\ &w^{-1} \in L^1([a, b]) , &&\text{for } p = \infty . \end{aligned}$$

Also, if J is any interval we say that $w \in B_p(J)$ if $w \in B_p(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint intervals $\cup_{i\in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

Observe that if $v \ge w$ in J and $w \in B_p(J)$, then $v \in B_p(J)$.

This class contains the classical Muckenhoupt A_p weights appearing in Harmonic Analysis (see [GR]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbf{R}^n$, and $A_p(\mathbf{R}^n)$ $(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces in [KO] and [K] respectively.

Definition 3. *We denote by $AC([a, b])$ the set of functions absolutely continuous in $[a, b]$, i.e. the functions $f \in C([a, b])$ such that $f(x) - f(a) = \int_a^x f'(t)\, dt$ for all $x \in [a, b]$. If J is any interval, $AC_{loc}(J)$denotes the set of functions absolutely continuous in every compact subinterval of J.*

Definition 4. *Let us consider $1 \le p \le \infty$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$. For $0 \le j \le k$ we define the open set*

$$\Omega_j := \{x \in \mathbf{R} \,:\, \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_p(V)\} .$$

Observe that we always have $w_j \in B_p(\Omega_j)$ for any $0 \le j \le k$. In fact, Ω_j is the greatest open set U with $w_j \in B_p(U)$. Obviously, Ω_j depends on p and μ, although p and μ do not appear explicitly in the symbol Ω_j. Lemma A below gives that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $0 \le j \le k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$, and therefore $f^{(j-1)} \in AC_{loc}(\Omega_j)$ if $1 \le j \le k$.

Hypothesis. From now on we assume that w_j is identically 0 on the complement of Ω_j.

We need this hypothesis in order to obtain complete Sobolev spaces (see [KO] and [RARP]).

Remark. This hypothesis is satisfied, for example, if we can modify w_j in a set of zero Lebesgue measure in such a way that there exists a sequence $\alpha_n \searrow 0$ with $w_j^{-1}\{(\alpha_n, \infty]\}$ open for every n. If w_j is lower semicontinuous, then this condition is satisfied.

Let us consider $1 \le p \le \infty$, $w = (w_0, \dots, w_k)$ a vectorial weight in an open set $\Omega \subseteq \mathbf{R}$ and $y \in \overline{\Omega}$. To obtain a greater regularity of the functions in a Sobolev space we construct a modification of the weight w in a neighbourhood of y, using Muckenhoupt weighted version of Hardy inequality (see [Ma, p.44] or Section 3 below). This modified weight is equivalent in some sense to the original one (see Theorem B).

Definition 5. *A vectorial weight $\overline{w} = (\overline{w}_0, \dots, \overline{w}_k)$ is a right completion of w with respect to y, if $\overline{w}_k := w_k$ and there is an $\varepsilon > 0$ such that $\overline{w}_j := w_j$ in the complement of $[y, y + \varepsilon]$ and*

$$\overline{w}_j(x) := w_j(x) + \tilde{w}_j(x) , \qquad \text{for } x \in [y, y + \varepsilon] \text{ and } 0 \le j < k ,$$

where $\tilde{w}_j$ is any weight satisfying:
i) $\tilde{w}_j \in L^1([y, y+\varepsilon])$ *if* $1 \le p < \infty$,
ii) $\tilde{w}_j \in L^\infty([y, y+\varepsilon])$ *if* $p = \infty$,
iii) $\Lambda_p(\tilde{w}_j, \overline{w}_{j+1}) < \infty$, *with*

$$\Lambda_p(u, v) := \sup_{y<r<y+\varepsilon} \left(\int_y^r u \right) \|v^{-1}\|_{L^{1/(p-1)}([r,y+\varepsilon])} , \qquad \textit{for} \;\; 1 \le p < \infty ,$$
$$\Lambda_\infty(u, v) := \operatorname*{ess\,sup}_{y<r<y+\varepsilon} u(r) \int_r^{y+\varepsilon} v^{-1} .$$

Example. It can be shown that the following construction is always a completion: we choose $\tilde{w}_j := 0$ if $\overline{w}_{j+1} \notin B_p((y, y+\varepsilon])$; if $\overline{w}_{j+1} \in B_p([y, y+\varepsilon])$ we set $\tilde{w}_j(x) := 1$ in $[y, y+\varepsilon]$; and if $\overline{w}_{j+1} \in B_p((y, y+\varepsilon]) \setminus B_p([y, y+\varepsilon])$ we take $\tilde{w}_j(x) := 1$ for $x \in [y+\varepsilon/2, y+\varepsilon]$, and

$$\tilde{w}_j(x) := \frac{d}{dx} \Big\{ \Big(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1/(p-1)} \Big)^{-p+1} \Big\} = \frac{(p-1)\, \overline{w}_{j+1}(x)^{-1/(p-1)}}{\Big(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1/(p-1)} \Big)^p} , \qquad \text{if } 1 < p < \infty ,$$
$$\tilde{w}_j(x) := \|\overline{w}_{j+1}^{-1}\|^{-1}_{L^\infty([x,y+\varepsilon])} + \frac{d}{dx}\Big(\|\overline{w}_{j+1}^{-1}\|^{-1}_{L^\infty([x,y+\varepsilon])} \Big) , \qquad \text{if } p = 1 ,$$
$$\tilde{w}_j(x) := \min \Big\{ 1, \Big(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1} \Big)^{-1} \Big\} , \qquad \text{if } p = \infty ,$$

for $x \in (y, y+\varepsilon/2)$.

Remarks.

1. We can define a left completion of w with respect to y in a similar way.

2. If for every $0 < \eta \le \eta_0 \le \varepsilon$ we have $\overline{w}_{j+1} \notin B_p((y, y+\eta])$, then there exists some $\delta > 0$ such that every $\tilde{w}_j$ must be 0 almost everywhere in $(y, y+\delta)$ (where ε is the constant corresponding to $\overline{w}$). Moreover, the constant δ depends on η_0 and $\overline{w}_{j+1}$, but not on $\tilde{w}_j$.

3. If $\overline{w}_{j+1} \in B_p([y, y+\varepsilon])$, then $\Lambda_p(\tilde{w}_j, \overline{w}_{j+1}) < \infty$ for any weight $\tilde{w}_j \in L^1([y, y+\varepsilon])$ if $1 \le p < \infty$ and for any bounded weight $\tilde{w}_j$ if $p = \infty$. In particular, $\Lambda_p(1, \overline{w}_{j+1}) < \infty$.

4. If w, v are two weights such that $w_j \ge c\, v_j$ for $j = 0, \dots, k$ and $\overline{v}$ is a right completion of v, then there is a right completion $\overline{w}$ of w, with $\overline{w}_j \ge c\, \overline{v}_j$ for $j = 0, \dots, k$ (it is enough to take $\tilde{w}_j = \tilde{v}_j$). Also, if w, v are comparable weights, $\overline{v}$ is a right completion of v if and only if it is comparable to a right completion $\overline{w}$ of w.

5. The hypotheses i) and ii) are not restrictive at all; if we are interested in the regularity of Sobolev functions we must choose weights without "big" singularities.

6. We always have $\overline{w}_k = w_k$ and $\overline{w}_j \ge w_j$ for $0 \le j < k$.

7. If $\overline{w}$ is a right completion of w with constant $\varepsilon > 0$, the weight $\overline{w}^* = (\overline{w}_0^*, \dots, \overline{w}_k^*)$ defined by

$$\overline{w}_j^*(x) = \begin{cases} \overline{w}_j(x) , & x \in [y, y+\delta] , \\ w_j(x) , & x \notin [y, y+\delta] , \end{cases}$$

for some $0 < \delta < \varepsilon$, is a right completion of w with constant δ.

Definition 6. *For $1 \le p \le \infty$ and w a vectorial weight in an open set $\Omega \subseteq \mathbf{R}$, we say that a point $y \in \overline{\Omega}$ is right j-regular (respectively, left j-regular), if there exist $\varepsilon > 0$, a right completion $\overline{w}$ (respectively, left completion) and $j < i \le k$ such that $\overline{w}_i \in B_p([y, y+\varepsilon])$ (respectively, $B_p([y-\varepsilon, y])$). Also, we say that a point $y \in \overline{\Omega}$ is j-regular, if it is right and left j-regular.*

Remarks.

1. A point $y \in \overline{\Omega}$ is right j-regular (respectively, left j-regular), if at least one of the following properties is verified:

(a) There exist $\varepsilon > 0$ and $j < i \le k$ such that $w_i \in B_p([y, y+\varepsilon])$ (respectively, $B_p([y-\varepsilon, y])$). Here we have chosen $\tilde{w}_j = 0$ and $\overline{w} = w$.

(b) There exist $\varepsilon > 0$, $j < i \le k$, $\alpha > 0$, $\delta < \delta_p$, with $\delta_p := (i-j)p-1$ if $1 \le p < \infty$ and $\delta_\infty := i-j-1$, such that

$$w_i(x) \ge \alpha\,|x-y|^{\delta}\,, \qquad \text{for almost every } x \in [y, y+\varepsilon]$$

(respectively, $[y-\varepsilon, y]$). See Lemma 3.4 in [RARP].

2. If y is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \le i \le j$.

3. We can take $i = j+1$ in this definition since by the third remark after Definition 5 we can choose $\overline{w}_l = w_l + 1 \in B_p([y, y+\varepsilon])$ for $j < l < i$, if $j+1 < i$.

4. If we define

$$k_0 := \max\{0 \le j \le k \,:\, \exists\, \eta > 0 \text{ with } w_j \in B_p((y, y+\eta])\}\,,$$

the completion $\overline{w}$ in Definition 6 can be chosen as $\overline{w}_j = w_j$ for $k_0 \le j \le k$ and $\overline{w}_{k_0} = w_{k_0} \in B_p((y, y+\varepsilon])$. This is an immediate consequence of remarks 2 and 7 to Definition 5.

When we use this definition we think of a point $\{b\}$ as the union of two half-points: $\{b^+\}$ and $\{b^-\}$. With this convention, each one of the following sets

$$(a,b) \cup (b,c) \cup \{b^+\} = (a,b) \cup [b^+, c) \ne (a,c)\,,$$
$$(a,b) \cup (b,c) \cup \{b^-\} = (a,b^-] \cup (b,c) \ne (a,c)\,,$$

has two connected components, and the set $(a,b) \cup (b,c) \cup \{b^-\} \cup \{b^+\} = (a,b) \cup (b,c) \cup \{b\} = (a,c)$ is connected.

We only use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity in an interval, if $f \in C([a,b)) \cap C([b,c])$ then we do not have $f \in C([a,c])$. Of course, we have $f \in C([a,c])$ if and only if $f \in C([a,b^-]) \cap C([b^+,c])$, where, by definition, $C([b^+,c]) = C([b,c])$ and $C([a,b^-]) = C([a,b])$. This idea can be formalized with a suitable topological space.

Let us introduce some more notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $y \in \Omega^{(j)}$ if and only if y is j-regular, we say that $y^+ \in \Omega^{(j)}$ if and only if y is right j-regular, and we say that $y^- \in \Omega^{(j)}$ if and only if y is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Observe that $\Omega^{(j)}$ depends on p (see Definition 6).

Remark. If $0 \le j < k$ and I is an interval, $I \subseteq \Omega^{(j)}$, then the set $I \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$ is discrete. If $y^+ \in I \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$, there exist $\varepsilon > 0$, a right completion $\overline{w}$ and $j < i \le k$ with $\overline{w}_i \in B_p([y, y+\varepsilon])$. Then there exist $\delta > 0$ and $i \le l \le k$ with $w_l \in B_p((y, y+\delta])$ and consequently $(y, y+\delta) \subseteq \Omega_{j+1} \cup \cdots \cup \Omega_k$ (see the second remark to Definition 5). Obviously the same is true for y^-.

Definition 7. *We say that a function h belongs to the class $AC_{loc}(\Omega^{(j)})$ if $h \in AC_{loc}(I)$ for every connected component I of $\Omega^{(j)}$.*

Definition 8. *We say that the vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ is p-admissible if $(\mu_j)_s(\mathbf{R} \setminus \Omega^{(j)}) = 0$, for $1 \le j \le k$, and $(\mu_k)_s \equiv 0$. We say that a p-admissible vectorial measure, μ, is strongly p-admissible if* $\operatorname{supp}(\mu_j)_s \subseteq \Omega^{(j)}$, *for $1 \le j \le k-1$.*

We use the letter p in p-admissible in order to emphasize the dependence on p (recall that $\Omega^{(j)}$ depends on p).

Remarks.

1. Observe that there is not any restriction on $\operatorname{supp}(\mu_0)_s$.

2. Every absolutely continuous measure is p-admissible, and even strongly p-admissible.

3. We want to remark that this definition of p-admissibility does not coincide with the one in [HKM].

Definition 9. (Sobolev space in the closure of an open set.) *Let us consider an open set $\Omega \subseteq \mathbf{R}$ and a p-admissible vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $\overline{\Omega}$. We define the Sobolev space $W^{k,p}(\overline{\Omega}, \mu)$ as the space of equivalence classes of*

$$V^{k,p}(\overline{\Omega}, \mu) := \Big\{ f : \overline{\Omega} \to \mathbf{C} \ / \ f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } j = 0, 1, \dots, k-1 \text{ and } \|f^{(j)}\|_{L^p(\overline{\Omega}, \mu_j)} < \infty \text{ for } j = 0, 1, \dots, k \Big\},$$

with respect to the seminorms

$$\|f\|_{W^{k,p}(\overline{\Omega},\mu)} := \Big(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(\overline{\Omega},\mu_j)} \Big)^{1/p}, \qquad \text{for } 1 \le p < \infty,$$
$$\|f\|_{W^{k,\infty}(\overline{\Omega},\mu)} := \max_{0 \le j \le k} \|f^{(j)}\|_{L^\infty(\overline{\Omega},\mu_j)}.$$

Remarks.

1. This definition is natural since when the $(\mu_j)_s$-measure of the set where $|f^{(j)}|$ is not continuous is positive, the integral $\int |f^{(j)}|^p d(\mu_j)_s$ does not make sense.

2. If we consider Sobolev spaces with real valued functions, every result in this paper also holds.

An example of Sobolev space as we have just defined is the following: $W^{2,2}([0,6], \mu)$, where

$$\|f\|^2_{W^{2,2}([0,6],\mu)} = \int_4^6 |f|^2 + |f(6)|^2 + \int_0^1 |f'|^2 \sqrt{x} + \int_3^5 |f'|^2 \sqrt{x-3} + |f'(1)|^2 + \int_1^3 |f''|^2 (3-x).$$

In this example, $w_0 \in B_2([4,6])$, $w_1 \in B_2([0,1] \cup [3,5])$, $w_2 \in B_2([1,3))$, and consequently $\Omega_0 = (4,6)$, $\Omega_1 = (0,1) \cup (3,5)$ and $\Omega_2 = (1,3)$; therefore, $\Omega^{(1)} = [1,3)$ and $\Omega^{(0)} = [0,5]$. Observe that 3 is right 0-regular since $w_1 \in B_2([3,5])$, and that 3 is left 0-regular since we can take $\tilde{w}_1 = 1$ in $[1,3]$. If we add δ_a to μ_1, we obtain a p-admissible measure (and the Sobolev space is well defined) if and only if $a \in [1,3)$. We can add δ_a to μ_0 for any $a \in \mathbf{R}$, and we can not add δ_a to μ_2 for any $a \in \mathbf{R}$. Obviously, in this definition $f'(1)$ stands for $f'(1^+)$, since $f' \in AC_{loc}([1,3))$.

In the results on density we consider the following five types of measures.

Definition 10. *Consider $1 \le p < \infty$. We say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $[a,b]$ is of type* 1 *if it is p-admissible, finite and $w_k \in B_p([a,b])$.*

Observe that the finiteness of μ_j is not an important restriction if every polynomial must be integrable (and the function x^j in particular). Observe also that a function $f \in W^{k,p}([a,b], \mu)$ is very regular if μ is of type 1 (Lemma A below says that $f^{(k-1)} \in AC([a,b])$), since we are working with dimension one.

Definition 11. *Consider $1 \le p < \infty$. We say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $[a,b]$ is of type* 2 *if it is strongly p-admissible, finite and there exist real numbers $a \le a_1 < a_2 < a_3 < a_4 \le b$ such that*

(1) *$w_k \in B_p([a_1, a_4])$,*
(2) *if $a < a_1$, then w_j is comparable to a non-decreasing weight in $[a, a_2]$, for $0 \le j \le k$,*
(3) *if $a_4 < b$, then w_j is comparable to a non-increasing weight in $[a_3, b]$, for $0 \le j \le k$.*

Observe that the measures of type 1 are also of type 2.

Definition 12. *Consider $1 \le p < \infty$. We say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $[a,b]$ is of type* 3 *if it is strongly p-admissible, finite and there exist real numbers $a \le a_1 < a_2 < a_3 < a_4 \le b$ and integers $k_1, k_2 \ge 0$ such that*

(1) *$w_k \in B_p([a_1, a_4])$,*
(2) *if $a < a_1$, then w_j is comparable to a non-decreasing weight in $[a, a_2]$, for $k_1 \le j \le k$,*
(3) *if $a_4 < b$, then w_j is comparable to a non-increasing weight in $[a_3, b]$, for $k_2 \le j \le k$,*
(4) *a is right $(k_1 - 1)$-regular if $k_1 > 0$ and b is left $(k_2 - 1)$-regular if $k_2 > 0$.*

Observe that the measures of type 2 are also of type 3.

Definition 13. *Consider $1 \le p < \infty$. We say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $[a, b]$ is of type 4 if it is p-admissible, finite and there exist $a_0 \in [a, b]$ and positive constants c, δ such that*

(1) $w_k(x) \le c\,|x - a_0|^p w_{k-1}(x)$ *for almost every x in* $[a_0 - \delta, a_0 + \delta] \cap [a, b]$,

(2) $w_k \in B_p([a, b] \setminus \{a_0\})$,

(3) *if* $k > 1$, a_0 *is* $(k - 2)$*-regular.*

Remark. If $a_0 = a$ (respectively, $a_0 = b$), condition (3) means that a is right $(k-2)$-regular (respectively, b is left $(k-2)$-regular).

The last type of measures that we consider is a variant of measures of type 4.

Definition 14. *Consider $1 \le p < \infty$. We say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $[a, b]$ is of type 5 if it is p-admissible, finite and there exist $a_0 \in [a, b]$ and positive constants c, δ such that*

(1) *if* $p > 1$, $w_k(x) \le c\,|x - a_0|^{p-1}$ *for almost every x in* $[a_0 - \delta, a_0 + \delta] \cap [a, b]$*; if* $p = 1$, w_k *can be modified in a set of zero Lebesgue measure in such a way that* $\lim_{x \to a_0} w_k(x) = 0$,

(2) $w_k \in B_p([a, b] \setminus \{a_0\})$,

(3) *if* $k > 1$, a_0 *is* $(k - 2)$*-regular.*

Observe that by condition (1) we know that the weight w_k does not belong to $B_p([a_0 - \delta, a_0 + \delta] \cap [a, b])$.

3. Technical results.

One of the classical results we are using in this paper is the known Muckenhoupt inequality, that we state as follows.

Muckenhoupt inequality. *Let us consider $1 \le p < \infty$ and μ_0, μ_1 measures in $(a, b]$ with $w_1 := d\mu_1/dx$. Then there exists a positive constant c such that*

$$\Big\| \int_x^b g(t)\, dt \Big\|_{L^p((a,b],\mu_0)} \le c\, \|g\|_{L^p((a,b],\,\mu_1)}$$

for any measurable function g in $(a, b]$, if and only if

$$\Lambda_p(\mu_0, \mu_1) := \sup_{a<r<b} \mu_0((a, r])\big\|w_1^{-1}\big\|_{L^{1/(p-1)}([r,b])} < \infty\,.$$

In our proofs we use some technical results which appear in [RARP]. For completeness we include the statements here. Some of them use Muckenhoupt inequality.

Lemma A. ([RARP, Lemma 3.1]) *Let us consider $1 \le p \le \infty$ and $w \in B_p((a, b))$. For any compact interval $I \subseteq (a, b)$, there is a positive constant c_1, which only depends on p, w and I, such that*

$$\|g\|_{L^1(I)} \le c_1 \|g\|_{L^p(I,w)} \le c_1 \|g\|_{L^p([a,b],w)}\,, \qquad \textit{for any } g \in L^p([a, b], w).$$

If furthermore $w \in B_p([a, b])$, there is a positive constant c_2, which only depends on p and w such that

$$\|g\|_{L^1([a,b])} \le c_2 \|g\|_{L^p([a,b],w)}\,, \qquad \textit{for any } g \in L^p([a, b], w).$$

Consequently, if $w \in B_p([a, b])$ and $f' \in L^p([a, b], w)$, then $f \in AC([a, b])$.

The following result generalizes Muckenhoupt inequality.

Lemma B. ([RARP, Lemma 3.2]) *Let us consider* $1 \le p \le \infty$, $t > 0$ *and* μ_0, μ_1 *measures in* $(a, b]$ *with* $a+t \le b$, $w_0 := d\mu_0/dx$ *and* $w_1 := d\mu_1/dx$, *verifying:* (i) $w_1 \in B_p([a+t, b])$ *if* $a+t < b$; (ii) $\mu_0((a, b]) < \infty$ *if* $a+t < b$ *and* $1 \le p < \infty$, (iii) $w_0 \in L^\infty([a+t, b])$ *if* $a+t < b$ *and* $p = \infty$. *Let us assume that* $\Lambda'_p(\mu_0, \mu_1) < \infty$, *where*

$$\Lambda'_p(\mu_0, \mu_1) := \sup_{a<r<a+t} \mu_0((a, r]) \, \|w_1^{-1}\|_{L^{1/(p-1)}([r,b])} \,, \qquad \text{for } 1 \le p < \infty,$$

$$\Lambda'_\infty(\mu_0, \mu_1) := \begin{cases} \operatorname*{ess\,sup}_{a<r<a+t} w_0(r) \displaystyle\int_r^b w_1^{-1} \,, & \text{if } (\mu_0)_s((a, b]) = 0 \,, \\ \max\Big\{ \operatorname*{ess\,sup}_{a<r<a+t} w_0(r) \displaystyle\int_r^b w_1^{-1}, \int_\alpha^b w_1^{-1} \Big\} \,, & \text{if } (\mu_0)_s((a, b]) > 0 \,, \end{cases}$$

where $\alpha := \min(\operatorname{supp}(\mu_0)_s)$. *Then* $\Lambda_p(\mu_0, \mu_1) < \infty$ *and this implies that there exists a positive constant* c *such that*

$$\Big\| \int_x^b g(s)\, ds \Big\|_{L^p((a,b],\, \mu_0)} \le c \, \|g\|_{L^p((a,b],\, \mu_1)}$$

for any measurable function g *in* $(a, b]$, *where* $\Lambda_p(\mu_0, \mu_1)$ *is defined changing* $a + t$ *by* b *in the definition of* $\Lambda'_p(\mu_0, \mu_1)$.

Before we state some other results from [RARP] we need the following definitions.

Definition 15. *Let us define the subspace* $\mathcal{K}(\overline{\Omega}, \mu)$ *as*

$$\mathcal{K}(\overline{\Omega}, \mu) := \big\{ g : \Omega^{(0)} \longrightarrow \mathbf{C} / \ g \in V^{k,p}\big(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}}\big), \ \|g\|_{W^{k,p}\left(\overline{\Omega^{(0)}},\, \mu|_{\Omega^{(0)}}\right)} = 0 \big\} \,.$$

Definition 16. *Let us consider* $1 \le p \le \infty$, Ω *an open subset of* $\mathbf{R}$ *and a* p*-admissible vectorial measure* μ *in* $\overline{\Omega}$. *We say that* $(\overline{\Omega}, \mu)$ *belongs to the class* $\mathcal{C}_0$ *if there exist compact sets* M_n, *which are a finite union of compact intervals, such that*

i) M_n *intersects at most a finite number of connected components of* $\Omega_1 \cup \cdots \cup \Omega_k$,

ii) $\mathcal{K}(M_n, \mu) = \{0\}$,

iii) $M_n \subseteq M_{n+1}$,

iv) $\cup_n M_n = \Omega^{(0)}$.

We say that $(\overline{\Omega}, \mu)$ *belongs to the class* $\mathcal{C}$ *if there exists a measure* $\mu'_0 = \mu_0 + \sum_{m \in D} c_m \delta_{x_m}$ *with* $c_m > 0$, $\{x_m\} \subset \Omega^{(0)}$, $D \subseteq \mathbf{N}$ *and* $(\overline{\Omega}, \mu') \in \mathcal{C}_0$, *where* $\mu' = (\mu'_0, \mu_1, \dots, \mu_k)$ *is minimal in the following sense: there exists* $\{M_n\}$ *corresponding to* $(\overline{\Omega}, \mu') \in \mathcal{C}_0$ *such that if* $\mu''_0 = \mu'_0 - c_{m_0} \delta_{x_{m_0}}$ *with* $m_0 \in D$ *and* $\mu'' = (\mu''_0, \mu_1, \dots, \mu_k)$, *then* $\mathcal{K}(M_n, \mu'') \ne \{0\}$ *if* $x_{m_0} \in M_n$.

Remarks.

1. The condition on $(\overline{\Omega}, \mu)$ is very general. In fact, the Remark after the proof of Theorem B ([RARP, Theorem 4.3]) that we only state below, gives that if $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, then $(\overline{\Omega}, \mu) \in \mathcal{C}$. If, furthermore, $\mathcal{K}(\overline{\Omega}, \mu) = \{0\}$, we have $(\overline{\Omega}, \mu) \in \mathcal{C}_0$.

2. Since the restriction of a function of $\mathcal{K}(\overline{\Omega}, \mu)$ to M_n is in $\mathcal{K}(M_n, \mu)$ for every n, then $(\overline{\Omega}, \mu) \in \mathcal{C}_0$ implies $\mathcal{K}(\overline{\Omega}, \mu) = \{0\}$.

3. If $(\overline{\Omega}, \mu) \in \mathcal{C}_0$, then $(\overline{\Omega}, \mu) \in \mathcal{C}$, with $\mu' = \mu$.

4. As a consequence of Remark 1, if μ is a measure of type 1, 2, 3, 4 or 5 in $[a, b]$ then $([a, b], \mu) \in \mathcal{C}$; if, furthermore, $\mathcal{K}([a, b], \mu) = \{0\}$, then $([a, b], \mu) \in \mathcal{C}_0$.

5. By the proof of Theorem B ([RARP, Theorem 4.3]) we know that if for every connected component Λ of $\Omega_1 \cup \cdots \cup \Omega_k$ we have $\mathcal{K}(\overline{\Lambda}, \mu) = \{0\}$, then $(\overline{\Omega}, \mu) \in \mathcal{C}_0$. Condition $\#\operatorname{supp} \mu_0|_{\overline{\Lambda} \cap \Omega^{(0)}} \ge k$ implies $\mathcal{K}(\overline{\Lambda}, \mu) = \{0\}$.

Theorem B. ([RARP, Theorem 4.3]) *Let us consider* $1 \le p \le \infty$, *an open set* $\Omega \subseteq \mathbf{R}$ *and a* p*-admissible vectorial measure,* μ, *in* $\overline{\Omega}$. *Let* K_j *be a finite union of compact intervals contained in* $\Omega^{(j)}$, *for* $0 \le j < k$ *and* $\overline{w}$ *a right* (*or left*) *completion of* w. *Then:*

(a) *If* $(\overline{\Omega},\mu)\in\mathcal{C}_0$ *there exist positive constants* $c_1=c_1(K_0,\dots,K_{k-1})$ *and* $c_2=c_2(\overline{w},K_0,\dots,K_{k-1})$ *such that*

$$c_1\sum_{j=0}^{k-1}\|g^{(j)}\|_{L^\infty(K_j)}\le\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)},\qquad c_2\,\|g\|_{W^{k,p}(\overline{\Omega},\overline{w})}\le\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)},\qquad\forall g\in V^{k,p}(\overline{\Omega},\mu).$$

(b) *If* $\big(\overline{\Omega},\mu\big)\in\mathcal{C}$ *there exist positive constants* $c_3=c_3(K_0,\dots,K_{k-1})$ *and* $c_4=c_4(\overline{w},K_0,\dots,K_{k-1})$ *such that for every* $g\in V^{k,p}(\overline{\Omega},\mu)$*, there exists* $g_0\in V^{k,p}(\overline{\Omega},\mu)$*, independent of* $K_0,\dots,K_{k-1}$*,* c_3*,* c_4 *and* $\overline{w}$*, with*

$$\|g_0-g\|_{W^{k,p}(\overline{\Omega},\,\mu)}=0\,,$$

$$c_3\sum_{j=0}^{k-1}\|g_0^{(j)}\|_{L^\infty(K_j)}\le\|g_0\|_{W^{k,p}(\overline{\Omega},\,\mu)}=\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)},\qquad c_4\,\|g_0\|_{W^{k,p}(\overline{\Omega},\overline{w})}\le\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)}.$$

Furthermore, if g_0,f_0 *are these representatives of* g,f *respectively, we have for the same constants* c_3,c_4

$$c_3\sum_{j=0}^{k-1}\|g_0^{(j)}-f_0^{(j)}\|_{L^\infty(K_j)}\le\|g-f\|_{W^{k,p}(\overline{\Omega},\,\mu)},\qquad c_4\,\|g_0-f_0\|_{W^{k,p}(\overline{\Omega},\overline{w})}\le\|g-f\|_{W^{k,p}(\overline{\Omega},\,\mu)}.$$

We also have the following corollaries of Theorem B.

Corollary A. ([RARP, Corollary 4.3]) *Let us consider* $1\le p\le\infty$*, an open set* $\Omega\subseteq\mathbf{R}$ *and a* p*-admissible vectorial measure* μ *in* $\overline{\Omega}$*. Let* K_j *be a finite union of compact intervals contained in* $\Omega^{(j)}$*, for* $0\le j<k$*. Then:*

(a) *If* $(\overline{\Omega},\mu)\in\mathcal{C}_0$ *there exists a positive constant* $c_1=c_1(K_0,\dots,K_{k-1})$ *such that*

$$c_1\sum_{j=0}^{k-1}\|g^{(j+1)}\|_{L^1(K_j)}\le\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)},\qquad\forall g\in V^{k,p}(\overline{\Omega},\mu).$$

(b) *If* $\big(\overline{\Omega},\mu\big)\in\mathcal{C}$ *there exists a positive constant* $c_2=c_2(K_0,\dots,K_{k-1})$ *such that for every* $g\in V^{k,p}(\overline{\Omega},\mu)$*, there exists* $g_0\in V^{k,p}(\overline{\Omega},\mu)$ *(the same function as in Theorem B), with*

$$\|g_0-g\|_{W^{k,p}(\overline{\Omega},\,\mu)}=0\,,\qquad c_2\sum_{j=0}^{k-1}\|g_0^{(j+1)}\|_{L^1(K_j)}\le\|g_0\|_{W^{k,p}(\overline{\Omega},\,\mu)}=\|g\|_{W^{k,p}(\overline{\Omega},\,\mu)}.$$

Furthermore, if g_0,f_0 *are these representatives of* g,f *respectively, we have for the same constant* c_2

$$c_2\sum_{j=0}^{k-1}\|g_0^{(j+1)}-f_0^{(j+1)}\|_{L^1(K_j)}\le\|g-f\|_{W^{k,p}(\overline{\Omega},\,\mu)}.$$

Corollary B. ([RARP, Corollary 4.4]) *Let us suppose that* $1\le p\le\infty$ *and that* $\mu=(\mu_0,\dots,\mu_k)$ *is a* p*-admissible vectorial measure in* $[a,b]$ *with* $w_k\in B_p([a,b])$*. Then:*

(a) *There exists a positive constant* c_1 *such that*

$$c_1\sum_{j=0}^{k-1}\|g^{(j)}\|_{L^\infty([a,b])}\le\|g\|_{W^{k,p}([a,b],\,\mu)},\qquad\forall\,g\in V^{k,p}([a,b],\mu),$$

if and only if $\mathcal{K}([a,b],\mu)=\{0\}$*.*

(b) *There exists a positive constant* c_2 *such that for every* $g\in V^{k,p}([a,b],\mu)$*, there exists* g_0 *with*

$$\|g_0-g\|_{W^{k,p}([a,b],\,\mu)}=0\,,\qquad c_2\sum_{j=0}^{k-1}\|g_0^{(j)}\|_{L^\infty([a,b])}\le\|g_0\|_{W^{k,p}([a,b],\,\mu)}=\|g\|_{W^{k,p}([a,b],\,\mu)}.$$

But we need some new technical results, which are the following.

Lemma 3.1. *Let us consider $1 \le p \le \infty$ and $w_1 \in B_p((a,b])$. If w_1 is comparable to a non-decreasing function in (a,b), then $\Lambda_p(w_1,w_1) < \infty$.*

Remark. For each $1 \le p \le \infty$ there exists a weight $w_1 \in L^\infty([a,b])$ with $\Lambda_p(w_1,w_1) = \infty$.

Proof. Without loss of generality we can assume that w_1 is a non-decreasing function in (a,b). In the case $1 \le p < \infty$ we have that

$$\Big(\int_a^r w_1\Big) \|w_1^{-1}\|_{L^{1/(p-1)}([r,b])} \le (r-a)w_1(r)(b-r)^{p-1}w_1(r)^{-1} \le (b-a)^p \, .$$

The proof is similar in the case $p = \infty$.

We usually multiply Sobolev functions in the results that we prove here. The following result allows us to control the norm of the product.

Theorem 3.1. *Let us consider $1 \le p < \infty$, an open set $\Omega \subseteq \mathbf{R}$ and $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure in $\overline{\Omega}$, with $(\overline{\Omega}, \mu) \in \mathcal{C}_0$. Assume that K is a finite union of compact intervals $J_1, \dots, J_n$ and that for every J_m there is an integer $0 \le k_m \le k$ verifying $J_m \subseteq \Omega^{(k_m-1)}$, if $k_m > 0$, and $\mu_j(J_m) = 0$ for $k_m < j \le k$, if $k_m < k$. If $\mu_j(K) < \infty$ for $0 < j \le k$, then there exists a positive constant c_0 such that*

$$c_0 \, \|fg\|_{W^{k,p}(\overline{\Omega},\,\mu)} \le \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \Big(\sup_{x\in\overline{\Omega}} |g(x)| + \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)} \Big)$$

for every $f, g \in V^{k,p}(\overline{\Omega}, \mu)$ with g constant in each connected component of $\overline{\Omega} \setminus K$.

Remarks.

1. The case $p = \infty$ is also true if we change the hypothesis μ finite by $w_1, \dots, w_k \in L^\infty(K)$. (We only use this hypothesis in order to obtain the inequality (3.3) below.)

2. The theorem is not true without the hypothesis $\mu_j(J_m) = 0$ for $k_m < j \le k$, if $k_m < k$, as shows the following example: Let us consider $w_0 = w_1 = 1$ and $w_2 = \infty$ in $[a,b]$. Then $x \in V^{2,p}([a,b], w)$ but $x \cdot x = x^2 \notin V^{2,p}([a,b], w)$.

Proof. Let us fix $1 \le m \le n$. If $k_m > 0$, then f and g belong to $C^{k_m-1}(J_m)$. Applying Theorem B(a) (with $K_1 = \cdots = K_{k_m-1} = J_m$) we obtain

$$c_m \sum_{j=0}^{k_m-1} \|h^{(j)}\|_{L^\infty(J_m)} \le \|h\|_{W^{k,p}(\overline{\Omega},\,\mu)} \, , \tag{3.1}$$

for every $h \in V^{k,p}(\overline{\Omega}, \mu)$. Let us consider now f, g as in the hypotheses of the theorem; we have

$$\|f^{(j)}g\|_{L^p(\overline{\Omega},\,\mu_j)} \le \|f^{(j)}\|_{L^p(\overline{\Omega},\,\mu_j)} \sup_{x\in\overline{\Omega}} |g(x)| \le \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \sup_{x\in\overline{\Omega}} |g(x)| \, , \tag{3.2}$$

for $0 \le j \le k$. Using (3.1) we also have for $0 \le i < j \le k_m$, except for $j = k_m$ and $i = 0$,

$$\begin{aligned} \|f^{(i)}g^{(j-i)}\|_{L^p(J_m,\,\mu_j)} &\le c \, \|f^{(i)}g^{(j-i)}\|_{L^\infty(J_m)} \\ &\le c \, \|f^{(i)}\|_{L^\infty(J_m)} \|g^{(j-i)}\|_{L^\infty(J_m)} \le c \, \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)} \, , \end{aligned} \tag{3.3}$$

since $\mu_j(J_m) < \infty$. If $j = k_m$ and $i = 0$ we have, using (3.1) again,

$$\|fg^{(k_m)}\|_{L^p(J_m,\,\mu_{k_m})} \le \|f\|_{L^\infty(J_m)} \|g^{(k_m)}\|_{L^p(J_m,\,\mu_{k_m})} \le c \, \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)} \, . \tag{3.4}$$

We obtain as a consequence of (3.3) and (3.4)

$$\|f^{(i)}g^{(j-i)}\|_{L^p(J_m,\,\mu_j)} \le c \, \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)} \tag{3.5}$$

for $0 \le i < j \le k_m$. Applying Leibniz rule, (3.2) and (3.5) we obtain

$$
\begin{aligned}
\|(fg)^{(j)}\|_{L^p(J_m,\,\mu_j)} &\le \|f^{(j)}g\|_{L^p(J_m,\,\mu_j)} + c\sum_{i=0}^{j-1} \|f^{(i)}g^{(j-i)}\|_{L^p(J_m,\,\mu_j)} \\
&\le c\,\|f\|_{W^{k,p}(\overline{\Omega},\,\mu)}\Big(\sup_{x\in\overline{\Omega}}|g(x)| + \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)}\Big)
\end{aligned}
\tag{3.6}
$$

for $0 \le j \le k_m$ and $k_m > 0$ (the case $j = 0$ is (3.2)). If $k_m = 0$, we obtain

$$
\|fg\|_{L^p(J_m,\,\mu_0)} \le \|f\|_{L^p(J_m,\,\mu_0)} \sup_{x\in\overline{\Omega}}|g(x)| \le \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \sup_{x\in\overline{\Omega}}|g(x)|\,,
$$

and we have proved that (3.6) is also true in this case. As a consequence of (3.6) we have

$$
\|fg\|_{W^{k,p}(J_m,\,\mu)} \le c\,\|f\|_{W^{k,p}(\overline{\Omega},\,\mu)}\Big(\sup_{x\in\overline{\Omega}}|g(x)| + \|g\|_{W^{k,p}(\overline{\Omega},\,\mu)}\Big)\,,
$$

since $\mu_j(J_m) = 0$ for $k_m < j \le k$, if $k_m < k$. Besides

$$
\|(fg)^{(j)}\|_{L^p(\overline{\Omega}\setminus K,\,\mu_j)} = \|f^{(j)}g\|_{L^p(\overline{\Omega}\setminus K,\,\mu_j)} \le \|f\|_{W^{k,p}(\overline{\Omega},\,\mu)} \sup_{x\in\overline{\Omega}}|g(x)|\,,
$$

for $0 \le j \le k$, since g is constant in each connected component of $\overline{\Omega} \setminus K$.

The theorem follows now from the two last inequalities, since there is only a finite number of J_m.

Corollary 3.1. *Let us consider $1 \le p < \infty$, $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure in $[a, b]$ with $w_k \in B_p([a, b])$, $\mu_1, \dots, \mu_k$ finite and $\mathcal{K}([a, b], \mu) = \{0\}$. Then there exists a positive constant c_0 such that*

$$
c_0\,\|fg\|_{W^{k,p}([a,b],\,\mu)} \le \|f\|_{W^{k,p}([a,b],\,\mu)}\|g\|_{W^{k,p}([a,b],\,\mu)} \qquad \textit{for every } f, g \in V^{k,p}([a, b], \mu)\,.
$$

Proof. It is enough to apply Theorem 3.1 and Corollary B(a), with $m = 1$, $K = J_1 = [a, b]$ and $k_1 = k$.

Lemma 3.2. *Let us consider $1 \le p < \infty$, $a_1 < a_0 < a_2$, $\beta_1, \beta_2 \in \mathbf{C}$, and $\delta, \varepsilon > 0$. Then, there exists a function $f \in W^{1,\infty}((a_1, a_2))$ such that*

1) *$f \in C^\infty([a_1, a_2] \setminus \{a_3, a_0, a_4\})$ with $a_3 < a_0 < a_4$ and a_3, a_4 as close to a_0 as we wish.*
2) *$f((a_1 + a_0)/2) = \beta_1$, $f((a_0 + a_2)/2) = \beta_2$, $f(a_0) = 0$.*
3) *f verifies*

$$
\|f\|_{L^\infty([a_0,a_2])} \le 2\,|\beta_2|\,, \qquad \|f\|_{L^\infty([a_1,a_0])} \le 2\,|\beta_1|\,.
$$

4) *$\|f'\|_{L^p([a_1,a_2],u)} < \varepsilon$, where $u(x) := |x - a_0|^{p-1}$ if $p > 1$, and u is any integrable function in $[a_1, a_2]$ with $\lim_{x\to a_0} u(x) = 0$ if $p = 1$.*

Proof. Without loss of generality we can assume that $a_0 = 0$. For $0 < \alpha \le 1$, $a_4 \in (0, a_2/4)$ and $a_3 = -a_4 \in (a_1/4, 0)$, let us consider the function

$$
g(x) := \begin{cases}
\beta_2\Big(\frac{2x}{a_2}\Big)^\alpha, & \text{if } x \in [a_4, a_2]\,, \\
\beta_2\Big(\frac{2a_4}{a_2}\Big)^\alpha \frac{x}{a_4}\,, & \text{if } x \in [0, a_4]\,, \\
-\beta_1\Big(\frac{-2a_4}{a_1}\Big)^\alpha \frac{x}{a_4}\,, & \text{if } x \in [-a_4, 0]\,, \\
\beta_1\Big(\frac{2x}{a_1}\Big)^\alpha, & \text{if } x \in [a_1, -a_4]\,.
\end{cases}
$$

It is obvious that $g \in W^{1,\infty}((a_1, a_2))$ and satisfies 1), 2) and 3). We want to show that $\|g'\|_{L^p([a_1,a_2],|x|^{p-1})} < \varepsilon$ if $p > 1$. We have that

$$
\|g'\|^p_{L^p([a_1,a_2],|x|^{p-1})} \le \int_0^{a_2} |\beta_2|^p (2/a_2)^{\alpha p}\alpha^p x^{p(\alpha-1)}x^{p-1}dx + \int_{a_1}^0 |\beta_1|^p|2/a_1|^{\alpha p}\alpha^p|x|^{p(\alpha-1)}|x|^{p-1}dx
$$

$$+\int_0^{a_4} |\beta_2|^p (2\,a_4/a_2)^{\alpha p} a_4^{-p} x^{p-1} dx + \int_{-a_4}^0 |\beta_1|^p |2\,a_4/a_1|^{\alpha p} a_4^{-p} |x|^{p-1} dx$$

$$= (|\beta_2|^p + |\beta_1|^p) 2^{\alpha p} \alpha^{p-1}/p + |\beta_2|^p (2\,a_4/a_2)^{\alpha p}/p + |\beta_1|^p |2\,a_4/a_1|^{\alpha p}/p\,.$$

Let us fix $\alpha > 0$ small enough so that $(|\beta_2|^p + |\beta_1|^p) 2^{\alpha p} \alpha^{p-1}/p < \varepsilon/2$. Let us also fix now $a_4 > 0$ small enough so that $|\beta_2|^p (2\,a_4/a_2)^{\alpha p}/p + |\beta_1|^p |2\,a_4/a_1|^{\alpha p}/p < \varepsilon/2$. This finishes the proof of Lemma 3.2 with $f = g$ for this choice of α and a_4 if $p > 1$.

If $p = 1$, we can take the function

$$h(x) := \begin{cases} \beta_2\,, & \text{if } x \in [a_4, a_2]\,, \\ \beta_2 \frac{x}{a_4}\,, & \text{if } x \in [0, a_4]\,, \\ -\beta_1 \frac{x}{a_4}\,, & \text{if } x \in [-a_4, 0]\,, \\ \beta_1\,, & \text{if } x \in [a_1, -a_4]\,. \end{cases}$$

It is obvious that $h \in W^{1,\infty}((a_1, a_2))$ and satisfies 1), 2) and 3). Also we have

$$\|h'\|_{L^1([a_1,a_2],u)} = |\beta_2| \frac{1}{a_4} \int_0^{a_4} u(x)\,dx + |\beta_1| \frac{1}{a_4} \int_{-a_4}^0 u(x)\,dx\,,$$

and this expression tends to $(|\beta_2| + |\beta_1|) \lim_{x\to 0} u(x) = 0$ as a_4 tends to zero.

This finishes the proof of Lemma 3.2 with $f = h$ for an appropriate choice of a_4 if $p = 1$.

4. Proof of theorems on density.

Observe that if ν is a finite measure in a Borel set $D \subseteq \mathbf{R}^n$ then $C_c(D)$ is dense in $L^p(D, \nu)$ for $1 \le p < \infty$. Consequently $C_c^\infty(\mathbf{R}^n)$ is dense in $L^p(D, \nu)$ for $1 \le p < \infty$.

Proof of Theorem 4.1. It is enough to prove the density of $C^\infty(\mathbf{R})$, since multiplying any function of this class by a function in $C_c^\infty(\mathbf{R})$ with value 1 in $[a, b]$ we obtain a function in $C_c^\infty(\mathbf{R})$. We have in fact the density of $C_c^\infty([a-\varepsilon, b+\varepsilon])$ for any fixed $\varepsilon > 0$.

If a function f belongs to $V^{k,p}([a, b], \mu)$, since $(a, b)^{(0)} \ne \emptyset$, there exists a compact interval $I \subseteq (a, b)^{(0)}$ such that f is a continuous function on I and it belongs to $V^{k,p}([a, b], \tilde\mu)$ with $\tilde\mu = (\tilde\mu_0, \mu_1, \dots, \mu_k)$ and $d\tilde\mu_0 = d\mu_0 + \chi_I dx$. It is obvious that it is more complicated to approximate f in $W^{k,p}([a, b], \tilde\mu)$ than in $W^{k,p}([a, b], \mu)$. Therefore, without loss of generality we can assume that $\mathcal{K}([a, b], \mu) = \{0\}$ and even $([a, b], \mu) \in \mathcal{C}_0$ in order to study the density of the set $C^\infty(\mathbf{R})$, since $(a, b)^{(0)} \setminus ((a, b)_1 \cup \dots \cup (a, b)_k)$ has at most three points (see Remark 1 to Definition 16).

We divide this proof into five parts; each one of them is devoted to a different type of measure.

Measures of type 1. Let $f \in V^{k,p}([a, b], \mu)$. Let $g \in C_c^\infty((a, b)) \subseteq C^\infty(\mathbf{R})$ be a function which approximates $f^{(k)}$ in the $L^p([a, b], w_k)$ norm. Consider the function

$$h(x) := \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{k-1}}{(k-1)!}\,dt\,.$$

Obviously we have that $f^{(j)}(x) - h^{(j)}(x) = \displaystyle\int_a^x (f^{(k)}(t) - g(t)) \frac{(x-t)^{k-j-1}}{(k-j-1)!}\,dt$, for $j = 0, \dots, k-1$.

This gives the inequalities

$$|f^{(j)}(x) - h^{(j)}(x)| \le \int_a^x \big|f^{(k)}(t) - g(t)\big| \frac{|x-t|^{k-j-1}}{(k-j-1)!}\,dt \le c_1 \,\|f^{(k)} - g\|_{L^1([a,b])} \le c_2 \,\|f^{(k)} - g\|_{L^p([a,b],w_k)}\,,$$

for $j = 0, \dots, k-1$, since $w_k \in B_p([a, b])$. Consequently, the finiteness of μ_j implies

$$\|f^{(j)} - h^{(j)}\|_{L^p([a,b],\mu_j)} \le c_3 \,\|f^{(k)} - g\|_{L^p([a,b],w_k)}\,, \qquad \text{for } j = 0, \dots, k-1,$$

and then

$$\|f - h\|_{W^{k,p}([a,b],\mu)} \le c_4 \,\|f^{(k)} - g\|_{L^p([a,b],w_k)}\,, \qquad \text{with } h \in C^\infty(\mathbf{R})\,.$$

Measures of type 2. Without loss of generality we can assume that w_j is a non-decreasing weight in $[a, a_2]$ (if $a < a_1$) and a non-increasing weight in $[a_3, b]$ (if $a_4 < b$) for $0 \le j \le k$.

Let $f \in V^{k,p}([a,b],\mu)$. Let us consider $\{\psi_1,\psi_2,\psi_3\} \subseteq C_c^\infty(\mathbf{R})$ a partition of unity satisfying: $\psi_1+\psi_2+\psi_3=1$ in $[a,b]$, $\psi_1|_{[a,a_1]} \equiv 1$, $\psi_2|_{[a_4,b]} \equiv 1$, $\psi_3|_{[a_2,a_3]} \equiv 1$, $\psi_1|_{[a_2-\delta,\infty)} \equiv 0$, $\psi_2|_{(-\infty,a_3+\delta]} \equiv 0$, $\psi_3|_{(a_1+\delta,a_4-\delta)^c} \equiv 0$, for some $\delta > 0$. Consider also the functions $f_i = f\psi_i$ for $i = 1, 2, 3$. If $a = a_1$ and $a_4 < b$ (or $a_4 = b$ and $a < a_1$), we consider a partition of unity with only two functions. If $a = a_1$ and $a_4 = b$, μ is a measure of type 1 in $[a,b]$. Then we only consider the case $a < a_1$ and $a_4 < b$, since the other cases are easier.

Observe that by Theorem 3.1 we know that each f_i belongs to $V^{k,p}([a,b],\mu)$, since $w_k \in B_p([a_1,a_4])$ and $\operatorname{supp}\psi_i' \subseteq [a_1,a_4]$.

It is enough to show that each f_i can be approximated in $W^{k,p}([a,b],\mu)$ by functions belonging to $C^\infty(\mathbf{R})$, since $f = f_1+f_2+f_3$ in $[a,b]$.

(1) Approximation of f_1.

Let us see first that we can approximate f_1 in $W^{k,p}([a,b],w)$ by functions in $C^\infty(\mathbf{R})$. In order to do this approximation (with the weight w, the absolutely continuous part of μ) we follow the ideas of [Ku, Theorem 7.2], though there the weights are only of the form $w_j(x) = \operatorname{dist}(x,\partial\Omega)^\alpha$ with the same $\alpha > 0$ for every j.

For fixed $0 \le j \le k$, consider the functions $g(x) := f_1^{(j)}(x)$ and $g_\lambda(x) := g(x+\lambda)$ for $0<\lambda<\delta$. It is clear that g_λ also belongs to $L^p([a,b],w_j)$, since $w_j|_{[a,a_2]}$ is non-decreasing for $j=0,\dots,k$ and $\operatorname{supp} g \subseteq [a, a_2-\delta]$.

Next, we show that g_λ tends to g in $L^p([a,b],w_j)$ as $\lambda \to 0^+$. We need to estimate the integral

$$J(\lambda) := \int_a^b |g(x)-g(x+\lambda)|^p w_j(x)\,dx\,.$$

Recall that $g(x)=0$ for $x \ge a_2-\delta$. Then, we have that $J(\lambda) \le 2^{p-1}\big[J_1(\lambda)+J_2(\lambda)\big]$, where

$$J_1(\lambda) := \int_a^{a_2-\delta} \big|g(x)w_j(x)^{1/p} - g(x+\lambda)w_j(x+\lambda)^{1/p}\big|^p\,dx,$$

$$J_2(\lambda) := \int_a^{a_2-\delta} |g(x+\lambda)|^p \big|w_j(x+\lambda)^{1/p} - w_j(x)^{1/p}\big|^p\,dx.$$

It is clear that $J_1(\lambda) \to 0$ as $\lambda \to 0^+$ since $g(x)w_j(x)^{1/p} \in L^p([a,a_2])$ (see [SW, p.10]). On the other hand,

$$\big|w_j(x+\lambda)^{1/p} - w_j(x)^{1/p}\big|^p = w_j(x+\lambda)\Big|\Big(\frac{w_j(x)}{w_j(x+\lambda)}\Big)^{1/p} - 1\Big|^p, \qquad \text{for } x \in [a, a_2-\delta],$$

if we consider that $0/0=1$. Then, we can write, since $0<\lambda<\delta$,

$$\begin{aligned} J_2(\lambda) &= \int_a^{a_2-\delta} |g(x+\lambda)|^p w_j(x+\lambda)\Big|\Big(\frac{w_j(x)}{w_j(x+\lambda)}\Big)^{1/p} - 1\Big|^p\,dx = \\ &= \int_a^{a_2} |g(x)|^p w_j(x)\Big|\Big(\frac{w_j(x-\lambda)}{w_j(x)}\Big)^{1/p} - 1\Big|^p \chi_{(a+\lambda,a_2+\lambda-\delta)}(x)\,dx\,. \end{aligned}$$

If we define $w_j := 0$ in $(-\infty,a)$, w_j is a non-decreasing function in $(-\infty,a_2]$ and we have $0 \le \frac{w_j(x-\lambda)}{w_j(x)} \le 1$ for $x \le a_2$. The following bound

$$|g(x)|^p w_j(x)\Big|\Big(\frac{w_j(x-\lambda)}{w_j(x)}\Big)^{1/p} - 1\Big|^p \chi_{(a+\lambda,a_2+\lambda-\delta)}(x) \le |g(x)|^p w_j(x), \qquad \text{for } x \in [a,a_2],$$

and the dominated convergence Theorem give $J_2(\lambda) \to 0$ as $\lambda \to 0^+$, since $\lim_{\lambda\to 0^+} w_j(x-\lambda) = w_j(x)$ for almost every $x \in [a,a_2]$ (w_j is monotone there). Hence $J(\lambda) \to 0$ as $\lambda \to 0^+$.

Then, it is enough to approximate $f_1(x+\lambda)$ in $W^{k,p}([a,b],w)$ for $\lambda > 0$ small enough.

Let $\{\phi_t\}_{t>0}$ be an usual approximation of the identity: $\phi_t(x) = \phi(x/t)/t$ for all $x \in \mathbf{R}$, $t>0$, and for some $\phi \in C_c^\infty((-1,1))$ verifying $\phi \ge 0$ and $\int \phi = 1$. Set u_t the convolution $u_t := \phi_t * (f_1)_\lambda$ with $0<t<\lambda<\delta/2$, when we take $f_1 \equiv 0$ on $(-\infty, a) \cup (a_2 - \delta, \infty)$. Then $u_t \in C_c^\infty(\mathbf{R})$ and define $v_t := u_t^{(j)} = \phi_t * g_\lambda$ for some fixed $0 \le j \le k$. We only need $v_t \to g_\lambda$ in $L^p([a,b], w_j)$ as $t \to 0^+$. But

$$
\begin{aligned}
\|v_t - g_\lambda\|_{L^p([a,b],w_j)} &= \Big(\int_a^b \Big| \int_{-t}^t \phi_t(y) g_\lambda(x-y)\, dy - \int_{-t}^t \phi_t(y)\, dy\, g_\lambda(x) \Big|^p w_j(x)\, dx \Big)^{1/p} \\
&\le \int_{-t}^t \phi_t(y) \Big(\int_a^b |g_\lambda(x-y) - g_\lambda(x)|^p w_j(x)\, dx \Big)^{1/p} dy \\
&\le \sup_{|y| \le t} \Big(\int_a^b |g(x+\lambda-y) - g(x+\lambda)|^p w_j(x)\, dx \Big)^{1/p} \\
&\le \sup_{|y| \le t} \Big\{ \Big(\int_a^b |g(x) - g(x+\lambda-y)|^p w_j(x)\, dx \Big)^{1/p} \\
&\quad + \Big(\int_a^b |g(x) - g(x+\lambda)|^p w_j(x)\, dx \Big)^{1/p} \Big\} \\
&= \sup_{|y| \le t} \big\{ J(\lambda - y)^{1/p} + J(\lambda)^{1/p} \big\} \le 2 \sup_{0 < s < 2\lambda} J(s)^{1/p},
\end{aligned}
$$

and this last term tends to zero since $J(\lambda) \to 0$ as $\lambda \to 0^+$.

Therefore, given $\varepsilon > 0$, there is a function $f_{1,\varepsilon} \in C_c^\infty(\mathbf{R})$ such that $\|f_1 - f_{1,\varepsilon}\|_{W^{k,p}([a,b],w)} < \varepsilon$.

Let us show now that $f_{1,\varepsilon}^{(j)}$ also approximates $f_1^{(j)}$ in the norm $L^p([a,b], \mu_j)$ for $0 \le j \le k$. This is trivial if $(\mu_j)_s = 0$. If $(\mu_j)_s \ne 0$, let us consider

$$
\alpha_j := \inf(\text{ supp } (\mu_j)_s)\,, \qquad \beta_j := \sup(\text{ supp } (\mu_j)_s)\,.
$$

Then, $\operatorname{supp}(\mu_j)_s$ is a compact set contained in $[\alpha_j, \beta_j]$. By the definition of strongly p-admissible measure, α_j is right j-regular and β_j is left j-regular. We also have that every $x \in (\alpha_j, \beta_j)$ is j-regular, by properties of measures of type 2. Theorem B(a) gives that there exists a positive constant c such that

$$
c\, \|h^{(j)}\|_{L^\infty([\alpha_j,\beta_j])} \le \|h\|_{W^{k,p}([a,b],w)}\,, \qquad \forall h \in V^{k,p}([a,b], \mu)\,,
$$

since $([a,b], \mu) \in \mathcal{C}_0$. So, $f_{1,\varepsilon}^{(j)}$ also approximates uniformly $f_1^{(j)}$ in $[\alpha_j, \beta_j]$, and therefore in the norm of $L^p([a,b], \mu_j)$, since μ_j is finite.

(2) Approximation of f_2.
We obtain the result applying a symmetric argument to (1).

(3) Approximation of f_3.
This is an immediate consequence of the first part of this theorem, since the support of f_3 is contained in $[a_1, a_4]$, $w_k \in B_p([a_1, a_4])$ and then $\mu^* = (\mu_0, \dots, \mu_{k-1}, \mu_k^*)$ is a measure of type 1 in $[a,b]$ with $d\mu_k^* = d\mu_k + \chi_{[a,a_1] \cup [a_4,b]}\, dx$.

Measures of type 3. Without loss of generality we can assume that w_j is a non-decreasing weight in $[a, a_2]$ for $k_1 \le j \le k$ (if $a < a_1$) and a non-increasing weight in $[a_3, b]$ for $k_2 \le j \le k$ (if $a_4 < b$). Consider $f \in V^{k,p}([a,b], \mu)$ and $f_i = f \psi_i$ for $i = 1, 2, 3$, as in the proof of this theorem for measures of type 2. If $a = a_1$ and $a_4 < b$ (or $a_4 = b$ and $a < a_1$), we consider a partition of unity with only two functions. If $a = a_1$ and $a_4 = b$, μ is a measure of type 1 in $[a,b]$. Then we only consider the case $a < a_1$ and $a_4 < b$ since the other cases are easier. It is enough to show that each f_i can be approximated by functions in $C^\infty(\mathbf{R})$.

(1) Approximation of f_1.
If $k_1 = 0$, we can approximate f_1 as in the case of measures of type 2. Assume now $k_1 > 0$. The proof in the case of measures of type 2 gives that it is possible to approximate $f_1^{(k_1)}$ by functions in $C^\infty(\mathbf{R})$ in the norm of $W^{k-k_1,p}([a,b], \mu)$, since $f_1^{(k_1)} \in W^{k-k_1,p}([a,b], \mu)$. Recall that we write $W^{k-k_1,p}([a,b], \mu)$ instead of

$W^{k-k_1,p}([a,b],(\mu_{k_1},\dots,\mu_k))$. Without loss of generality we can assume that $w_j \ge 1$ in $[a_2,b]$ for $k_1 \le j \le k$, since $f_1 = 0$ in this interval. Then $\mathcal{K}([a,b],(\mu_{k_1},\dots,\mu_k)) = \{0\}$ and even $([a,b],(\mu_{k_1},\dots,\mu_k)) \in \mathcal{C}_0$.

If $g \in C^\infty(\mathbf{R})$ approximates $f_1^{(k_1)}$ in $W^{k-k_1,p}([a,b],\mu)$, consider the function

$$h(x) := f_1(a) + f_1'(a)(x-a) + \cdots + f_1^{(k_1-1)}(a)\frac{(x-a)^{k_1-1}}{(k_1-1)!} + \int_a^x g(t)\frac{(x-t)^{k_1-1}}{(k_1-1)!}\,dt\,.$$

Then we have $f_1^{(j)}(x) - h^{(j)}(x) = \displaystyle\int_a^x (f^{(k_1)}(t) - g(t))\frac{(x-t)^{k_1-j-1}}{(k_1-j-1)!}\,dt$, for $j = 0,\dots,k_1-1$.

Now, by Corollary A(a), we have for $j = 0,\dots,k_1-1$, since $([a,b],(\mu_{k_1},\dots,\mu_k)) \in \mathcal{C}_0$ and $[a,b] \in \Omega^{(k_1-1)}$,

$$\|f_1^{(j)} - h^{(j)}\|_{L^\infty([a,b])} \le c\,\|f_1^{(k_1)} - g\|_{L^1([a,b])} \le c\,\|f^{(k_1)} - g\|_{W^{k-k_1,p}([a,b],\mu)}\,.$$

(2) Approximation of f_2.

We use the same proof with the appropriate symmetry.

(3) Approximation of f_3.

We proceed as in the proof of the case of measures of type 2.

Measures of type 4. Let us assume that $a_0 \in (a,b)$. If $a_0 = a$ or $a_0 = b$ our work is simpler. Without loss of generality we can assume that $a_0 = 0$. Let us consider a function $f \in V^{k,p}([a,b],\mu)$. In order to approximate f by functions in $C^\infty(\mathbf{R})$, without loss of generality we can assume also that

$$\int_{\{0\}} |f^{(k-1)}|^p d\mu_{k-1} = 0\,.$$

This is obvious if $\mu_{k-1}(\{0\}) = 0$. If $\mu_{k-1}(\{0^-\}) > 0$ and/or $\mu_{k-1}(\{0^+\}) > 0$, we can change $f(x)$ by

$$f(x) - f^{(k-1)}(0^-)\,\frac{x^{k-1}}{(k-1)!} \qquad \text{or} \qquad f(x) - f^{(k-1)}(0^+)\,\frac{x^{k-1}}{(k-1)!}\,,$$

since $x^{k-1} \in C^\infty(\mathbf{R})$. Recall that we always write $(\alpha+\beta)\,\delta_0$ instead of $\alpha\,\delta_{0^-} + \beta\,\delta_{0^+}$.

For each big enough natural number n we can choose points $x_n \in (0,1/n]$ and $y_n \in [-1/n,0)$ such that

$$\begin{aligned} &|f^{(k-1)}(x_n)|^p \int_0^{1/n} d\mu_{k-1}(x) \le \int_0^{1/n} |f^{(k-1)}(x)|^p\,d\mu_{k-1}(x)\,,\\ &|f^{(k-1)}(y_n)|^p \int_{-1/n}^0 d\mu_{k-1}(x) \le \int_{-1/n}^0 |f^{(k-1)}(x)|^p\,d\mu_{k-1}(x)\,. \end{aligned} \tag{4.1}$$

Now, we can define the following sequence approximating $f^{(k-1)}$:

$$h_n(x) := \begin{cases} f^{(k-1)}(x)\,, & x \notin (y_n,x_n)\,,\\ f^{(k-1)}(x_n)x/x_n\,, & x \in [0,x_n]\,,\\ f^{(k-1)}(y_n)x/y_n\,, & x \in [y_n,0]\,. \end{cases}$$

By the first inequality in (4.1) we have that

$$\int_0^{x_n} |h_n(x)|^p d\mu_{k-1}(x) = |f^{(k-1)}(x_n)|^p \int_0^{x_n} \Big(\frac{x}{x_n}\Big)^p d\mu_{k-1}(x) \le \|f^{(k-1)}\|^p_{L^p([0,1/n],\,\mu_{k-1})}$$

and

$$\begin{aligned} \Big(\int_0^b |f^{(k-1)}(x) - h_n(x)|^p d\mu_{k-1}(x)\Big)^{1/p} &= \Big(\int_0^{x_n} |f^{(k-1)}(x) - h_n(x)|^p d\mu_{k-1}(x)\Big)^{1/p}\\ &\le \|f^{(k-1)}\|_{L^p([0,x_n],\,\mu_{k-1})} + \|f^{(k-1)}\|_{L^p([0,1/n],\,\mu_{k-1})}\\ &\le 2\,\|f^{(k-1)}\|_{L^p([0,1/n],\,\mu_{k-1})}\,. \end{aligned}$$

Similarly,

$$\Big(\int_a^0 |f^{(k-1)}(x)-h_n(x)|^p d\mu_{k-1}(x)\Big)^{1/p} \le 2\,\|f^{(k-1)}\|_{L^p([-1/n,0],\,\mu_{k-1})}\,,$$

and then

$$\|f^{(k-1)}-h_n\|_{L^p([a,b],\,\mu_{k-1})} \le 2\,\|f^{(k-1)}\|_{L^p([-1/n,1/n],\,\mu_{k-1})}\,.$$

As n tends to infinity, this last norm converges to $\|f^{(k-1)}\|_{L^p(\{0\},\,\mu_{k-1})}=0$.

Furthermore, property (1) and (4.1) give for big enough n

$$\begin{aligned}\Big(\int_0^b |f^{(k)}(x)-h_n'(x)|^p w_k(x)\,dx\Big)^{1/p} &= \Big(\int_0^{x_n} |f^{(k)}(x)-f^{(k-1)}(x_n)/x_n|^p w_k(x)\,dx\Big)^{1/p}\\ &\le \Big(\int_0^{x_n} |f^{(k)}(x)|^p w_k(x)\,dx\Big)^{1/p} + \Big(\int_0^{x_n} |f^{(k-1)}(x_n)|^p c\, w_{k-1}(x)(x/x_n)^p\,dx\Big)^{1/p}\\ &\le \Big(\int_0^{1/n} |f^{(k)}(x)|^p w_k(x)\,dx\Big)^{1/p} + c\Big(\int_0^{1/n} |f^{(k-1)}(x)|^p d\mu_{k-1}(x)\Big)^{1/p}.\end{aligned}$$

We can estimate $\|f^{(k)}-h_n'\|_{L^p([a,0],w_k)}$ in a similar way. Therefore, we have proved that it is possible to approximate the function $f^{(k-1)}$ in the Sobolev norm $W^{1,p}([a,b],\mu)$ by functions which are in $W^{1,\infty}$ of a neighbourhood of 0.

Consider now the weight

$$w_{k,n}(x)=\begin{cases} w_k(x)\,, & x\notin (y_n,x_n)\,,\\ 1+w_k(x)\,, & x\in (y_n,x_n)\,,\end{cases}$$

and the vectorial measure $\mu^n=(\mu_{k-1},\mu_{k,n})$ with $d\mu_{k,n}(x)=w_{k,n}(x)dx$.

Since $w_{k,n}\in B_p([a,b])$ by property (2) of measures of type 4, the result for measures of type 1 gives that the function h_n can be approximated by functions of $C^\infty(\mathbf{R})$ in the norm $W^{1,p}([a,b],\mu^n)$, and consequently in the norm $W^{1,p}([a,b],\mu)$, since $w_k(x)\le w_{k,n}(x)$. This finishes the proof if $k=1$.

If $k>1$, conditions (2) and (3) prove that $f^{(k-2)}$ belongs to $AC([a,b])$. An integration argument as in the proof of the case of measures of type 1 or 3 (using Corollary A(a)) gives that f can be approximated by functions of $C^\infty(\mathbf{R})$ in the norm $W^{k,p}([a,b],\mu)$.

Measures of type 5. Let us assume that $a_0\in(a,b)$. If $a_0=a$ or $a_0=b$ our work is simpler. Without loss of generality we can assume that $a_0=0$. Let us consider a function $f\in V^{k,p}([a,b],\mu)$. As in the proof for measures of type 4 we can assume that $\int_{\{0\}}|f^{(k-1)}|^p\,d\mu_{k-1}=0$.

For each big enough natural number n we can choose points $x_n\in(0,1/n]$ and $y_n\in[-1/n,0)$ verifying (4.1). If we take

$$a_1=2\,y_n\,,\qquad a_2=2\,x_n\,,\qquad \beta_1=f^{(k-1)}(y_n)\,,\qquad \beta_2=f^{(k-1)}(x_n)\,,$$

and $\varepsilon>0$, then Lemma 3.2 allows us to choose a function f_n, for each big n, verifying

$$(4.2)\qquad \|f_n\|_{L^\infty([0,x_n])}\le 2\,|f^{(k-1)}(x_n)|\,,\qquad \|f_n\|_{L^\infty([y_n,0])}\le 2\,|f^{(k-1)}(y_n)|\,,\qquad \|f_n'\|_{L^p([y_n,x_n],w_k)}<\varepsilon\,,$$

such that the function

$$h_n(x):=\begin{cases} f^{(k-1)}(x)\,, & x\notin (y_n,x_n)\,,\\ f_n(x)\,, & x\in [y_n,x_n]\,,\end{cases}$$

belongs to $W^{1,p}([a,b],\mu)\cap W^{1,\infty}((y_n,x_n))$.

Using (4.2), we also have, as in the proof of the case of measures of type 4, that

$$\|f^{(k-1)}-h_n\|_{L^p([a,b],\,\mu_{k-1})}\le c_1\,\|f^{(k-1)}\|_{L^p([-1/n,1/n],\,\mu_{k-1})}<\varepsilon\,,$$

if n is big, since $\|f^{(k-1)}\|_{L^p(\{0\},\,\mu_{k-1})}=0$. Furthermore

$$\|f^{(k)}-h_n'\|_{L^p([a,b],\,\mu_k)}=\|f^{(k)}-f_n'\|_{L^p([y_n,x_n],w_k)}\le \|f^{(k)}\|_{L^p([y_n,x_n],w_k)}+\|f_n'\|_{L^p([y_n,x_n],w_k)}\,.$$

Therefore, we have proved that it is possible to approximate the function $f^{(k-1)}$ in the Sobolev norm $W^{1,p}([a,b],\mu)$ by functions which are in $W^{1,\infty}$ in a neighbourhood of 0.

Finally, the proof finishes as in the case of measures of type 4.

This finishes the proof of Theorem 4.1.

We have examples which prove that the techniques in the proof of Theorem 4.1 for measures of type 2 do not work if we remove hypothesis (2) or (3) of this kind of measures.

Proof of Theorem 4.2. We only prove the non-trivial implication. Let us consider $J=[\alpha,\beta]\subseteq I$ and an integer $0\le k_1\le k$, such that $J\subseteq (b,c)^{(k_1-1)}$ if $k_1>0$, and $\mu_j(J)=0$ for $k_1<j\le k$ if $k_1<k$.

Let us consider $f\in V^{k,p}([a,d],\mu)$ and $\varphi_1,\varphi_2\in C^\infty(\mathbf{R})$ such that φ_1 approximates f in $W^{k,p}([a,c],\mu)$ and φ_2 approximates f in $W^{k,p}([b,d],\mu)$.

Set $\theta\in C^\infty(\mathbf{R})$ a fixed function with $0\le\theta\le 1$, $\theta=0$ in $(-\infty,\alpha]$ and $\theta=1$ in $[\beta,\infty)$. It is enough to see that $\theta\varphi_2+(1-\theta)\varphi_1$ approximates f in $W^{k,p}([a,d],\mu)$, or equivalently, in $W^{k,p}(J,\mu)$. Theorem 3.1 gives

$$\begin{aligned}\|f-\theta\varphi_2-(1-\theta)\varphi_1\|_{W^{k,p}(J,\mu)} &\le \|f-\theta\varphi_2-(1-\theta)\varphi_1\|_{W^{k,p}(I,\mu)}\\ &\le \|\theta(f-\varphi_2)\|_{W^{k,p}(I,\mu)}+\|(1-\theta)(f-\varphi_1)\|_{W^{k,p}(I,\mu)}\\ &\le c\left(\|f-\varphi_2\|_{W^{k,p}(I,\mu)}+\|f-\varphi_1\|_{W^{k,p}(I,\mu)}\right),\end{aligned}$$

since $(I,\mu)\in\mathcal{C}_0$.

Remark. The same conclusion of Theorem 4.2 is achieved if we change the hypothesis $(I,\mu)\in\mathcal{C}_0$ by $([a,c],\mu)\in\mathcal{C}_0$ and $([b,d],\mu)\in\mathcal{C}_0$.

Proof of Theorem 4.3. We only prove the non-trivial implication. Let us consider $\varphi_n\in C^\infty(\mathbf{R})$ which approximates f in $W^{k,p}([a_n,b_n],\mu)$. By the proof of Theorem 4.2 we know that there exist $\theta_n\in C^\infty(\mathbf{R})$ and positive constants c_n such that

$$\|f-\theta_n\varphi_{n+1}-(1-\theta_n)\varphi_n\|_{W^{k,p}([a_{n+1},b_n],\mu)}\le c_n\left(\|f-\varphi_n\|_{W^{k,p}([a_{n+1},b_n],\mu)}+\|f-\varphi_{n+1}\|_{W^{k,p}([a_{n+1},b_n],\mu)}\right).$$

Now, given $\varepsilon>0$, it is enough to approximate f in $[a_n,b_n]$ with error less than $2^{-|n|}\varepsilon\,\min\{1,c_n^{-1},c_{n-1}^{-1}\}$.

Proof of Corollary 4.2. Let us fix $f\in V^{k,p}([\alpha,\beta],\mu)$ and choose compact intervals $I_n\subseteq(a_{n+1},b_n)^{(0)}$ and constants $c_n>0$ such that $\|f\|_{L^p([\alpha,\beta],\mu_0^*)}<\infty$, where $d\mu_0^*:=d\mu_0+\sum_n c_n\chi_{I_n}dx$ (observe that $f\in C(I_n)$). If $\mu^*=(\mu_0^*,\mu_1,\dots,\mu_k)$, it is enough to see that we can approximate f by functions of $C^\infty(\mathbf{R})$ in $W^{k,p}([\alpha,\beta],\mu^*)$. This is a consequence of theorems 4.1 and 4.3, since $(I_n,\mu^*)\in\mathcal{C}_0$ and each $\mu^*|_{[a_n,b_n]}$ is of type 1, 2, 3, 4 or 5 (see Remark 4 to Definition 16).

We have another "gluing" theorem. In this result it is not necessary that the intervals overlap when the contact is at "different levels".

Theorem 4.4. *Let us consider $1\le p<\infty$, $a<b<c$, μ a p-admissible vectorial measure in $[a,c]$ and $0\le k_0\le k$, with the following properties:*

i) $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,b],\mu)$,

ii) $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}([b,c],\mu)$,

iii) $\mu_{k_0}([a,b])<\infty$,

iv) if $k_0>0$, $\mu_j([a,c])<\infty$, for $j=0,\dots,k_0-1$,

v) if $k_0<k$, $w_{k_0+1}=\dots=w_k=0$ in $[a,b]$,

vi) if $k_0>0$, $w_{k_0}\in B_p([a,b])$,

vii) if $k_0>0$, $[b,c]=(b,c)^{(k_0-1)}$,

viii) if $k_0>0$, $([b,c],(\mu_{k_0},\dots,\mu_k))\in\mathcal{C}_0$.

Then $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,c],\mu)$.

Remarks.

1. These conditions are not very restrictive. Conditions $i)$ to $iv)$ are natural. Condition $v)$ includes the case where at the contact point b the last nonzero derivative is of different order at both sides. With respect to conditions $vi)$ and $vii)$, they give the necessary regularity at the contact point. Condition $viii)$ allows us to use Corollary A(a).

2. If $-\infty \le a < b < c < d \le \infty$ and μ is identically zero in (b,c), it is clear that $C^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,d],\mu)$ if and only if $C^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,b],\mu)$ and $W^{k,p}([c,d],\mu)$.

Proof. Let us consider $f \in V^{k,p}([a,c],\mu)$. By property $iii)$ we can approximate $f^{(k_0)}$ in $L^p([a,b],w_{k_0})$ by a function $\psi_1 \in C_c^\infty(\mathbf{R})$ with supp $\psi_1 \subseteq [a, b-\nu]$ for some $0<\nu<b-a$. By property $ii)$ we can approximate $f^{(k_0)}$ in $W^{k-k_0,p}([b,c],\mu)$ by a function $\psi_2 \in C_c^\infty(\mathbf{R})$ with supp $\psi_2 \subseteq [b-\nu/2,\infty)$. Also, we can assume by $v)$ that $\|\psi_2\|_{W^{k-k_0,p}([a,b],\mu)} = \|\psi_2\|_{L^p([a,b],w_{k_0})}$ is small.

The function $h=\psi_1+\psi_2$ belongs to $C_c^\infty(\mathbf{R})$ (in particular, $h \equiv 0$ in $[b-\nu, b-\nu/2]$) and approximates $f^{(k_0)}$ in $W^{k-k_0,p}([a,c],\mu)$. This finishes the proof if $k_0=0$.

Whenever $k_0>0$, properties $vi)$ and $vii)$ give $[a,c]=(a,c)^{(k_0-1)}$ and then $f^{(k_0-1)} \in AC([a,c])$. Let us define for $x \in \mathbf{R}$ the function

$$u(x) := f(b) + \cdots + f^{(k_0-1)}(b)\frac{(x-b)^{k_0-1}}{(k_0-1)!} + \int_b^x h(t)\frac{(x-t)^{k_0-1}}{(k_0-1)!}\,dt\,.$$

We have $u \in C^\infty(\mathbf{R})$ and $f(x)-u(x) = \displaystyle\int_b^x (f^{(k_0)}(t)-h(t))\frac{(x-t)^{k_0-1}}{(k_0-1)!}\,dt$. Therefore we obtain by $iv)$

$$\sum_{j=0}^{k_0-1} \|f^{(j)}-u^{(j)}\|_{L^p([a,c],\,\mu_j)} \le c_1 \sum_{j=0}^{k_0-1} \|f^{(j)}-u^{(j)}\|_{L^\infty([a,c])} \le c_2\,\|f^{(k_0)}-h\|_{L^1([a,c])}\,.$$

By Lemma A and $vi)$ we know that

$$\|f^{(k_0)}-h\|_{L^1([a,b])} \le c_3\|f^{(k_0)}-h\|_{L^p([a,b],w_{k_0})}.$$

On the other hand, since $([b,c],(\mu_{k_0},\dots,\mu_k)) \in \mathcal{C}_0$, the argument in the proof of Corollary A(a) and $vii)$ give

$$\|f^{(k_0)}-h\|_{L^1([b,c])} \le c_4\,\|f^{(k_0)}-h\|_{W^{k-k_0,p}([b,c],\,\mu)}.$$

These inequalities finish the proof.

5. Applications to Sobolev orthogonal polynomials.

We denote by $P^{k,p}([a,b],\mu)$ the completion of polynomials with the norm of $W^{k,p}([a,b],\mu)$. By a theorem in [LPP] we know that the zeroes of the Sobolev orthogonal polynomials in $W^{k,2}([a,b],\mu)$ are contained in the disk $\{z : |z| \le \|M\|\}$, where the multiplication operator $(Mf)(x)=xf(x)$ is considered in the space $P^{k,2}([a,b],\mu)$. Consequently, the set of the zeroes of the Sobolev orthogonal polynomials is bounded if the multiplication operator is bounded.

In [LP] also appears the following result: If μ is a finite measure in $[a,b]$ sequentially dominated, then M is a bounded operator in $P^{k,2}([a,b],\mu)$, where the vectorial measure μ is sequentially dominated if $\#\text{supp}\,\mu_0=\infty$ and $d\mu_j = f_{j-1}\,d\mu_{j-1}$ with f_{j-1} bounded for $0<j\le k$. In that paper the authors ask for other conditions on M to be bounded.

We have the following results.

Theorem 5.1. *For $1 \le p < \infty$, if μ is a p-admissible vectorial measure in $[a,b]$ of type $1,2,3$ or 4, and the multiplication operator is well defined in $W^{k,p}([a,b],\mu)$, then it is bounded. The result is also true for measures of type 5 verifying the additional condition $w_k \le c\,w_{k-1}$ in $[a_0-\delta,a_0+\delta]\cap[a,b]$.*

Remark. In this situation, Corollary 4.1 gives $P^{k,p}([a,b],\mu)=W^{k,p}([a,b],\mu)$.

Obviously, the multiplication operator M is well defined in $W^{k,p}(\overline{\Omega},\mu)$ if and only if it is well defined in $V^{k,p}(\overline{\Omega},\mu)$ (i.e. $xf\in V^{k,p}(\overline{\Omega},\mu)$ for every $f\in V^{k,p}(\overline{\Omega},\mu)$) and $\|xf\|_{W^{k,p}(\overline{\Omega},\mu)}=0$ for every $f\in V^{k,p}(\overline{\Omega},\mu)$ with $\|f\|_{W^{k,p}(\overline{\Omega},\mu)}=0$. This second condition can be written as $M(\mathcal{K}(\overline{\Omega},\mu))\subseteq\mathcal{K}(\overline{\Omega},\mu)$.

Theorem 5.2. *Let us consider $1\le p<\infty$, an open set $\Omega\subseteq\mathbf{R}$ and a p-admissible vectorial measure μ in $\overline{\Omega}$. Assume that the multiplication operator M is well defined in $V^{k,p}(\overline{\Omega},\mu)$. Then M is well defined in $W^{k,p}(\overline{\Omega},\mu)$ if and only if $\mathcal{K}(\overline{\Omega},\mu)=\{0\}$.*

Proof. Suppose first that $\mathcal{K}(\overline{\Omega},\mu)=\{0\}$. Then, if $f\in V^{k,p}(\overline{\Omega},\mu)$ with $\|f\|_{W^{k,p}(\overline{\Omega},\mu)}=0$ we have $\|f\|_{W^{k,p}\left(\overline{\Omega^{(0)}},\,\mu\right)}=0$. Consequently $f|_{\Omega^{(0)}}\equiv 0$ and so $\|xf\|_{W^{k,p}\left(\overline{\Omega^{(0)}},\,\mu\right)}=0$. On the other hand, we also have $\|f\|_{L^p(\overline{\Omega},\mu_0)}=0$, and so $f(x)=0$ for μ_0-almost every x. Then $xf(x)=0$ for μ_0-almost every x and $\|xf\|_{L^p(\overline{\Omega},\mu_0)}=0$. We deduce from these two arguments that

$$\|xf\|^p_{W^{k,p}(\overline{\Omega},\,\mu)}\le\|xf\|^p_{L^p(\overline{\Omega},\,\mu_0)}+\|xf\|^p_{W^{k,p}\left(\overline{\Omega^{(0)}},\,\mu\right)}=0\,,$$

and therefore the multiplication operator is well defined in $W^{k,p}(\overline{\Omega},\mu)$.

On the converse, let us suppose that there is $f\in V^{k,p}(\overline{\Omega},\mu)$ such that

$$\|f\|_{W^{k,p}\left(\overline{\Omega^{(0)}},\,\mu|_{\Omega^{(0)}}\right)}=0\,,$$

but f is not identically zero in $\Omega^{(0)}$. We know that there exists an interval $I_0\subseteq\Omega^{(0)}$ such that $f|_{I_0}\neq 0$, and therefore there is another interval $I\subseteq I_0$ such that $I\subseteq\Omega_i$ for some $1\le i\le k$ and $f|_I\neq 0$. If g belongs to $\mathcal{K}(\overline{\Omega},\mu)$, we have that $g^{(i)}(x)=0$ for almost every $x\in\Omega_i$, and therefore that $g^{(i-1)}$ is constant in each connected component of Ω_i. Then $g|_I\in P_{i-1}$. Let us choose now $h\in\mathcal{K}(\overline{\Omega},\mu)$ such that $\deg h|_I\ge\deg g|_I$ for all $g\in\mathcal{K}(\overline{\Omega},\mu)$ (we have $\deg h|_I\ge 0$ since the function f is not identically zero in I). Then, $\deg xh|_I>\deg h|_I$; therefore $xh\notin\mathcal{K}(\overline{\Omega},\mu)$ and M is not well defined.

Proof of Theorem 5.1. We shall divide this proof into five parts; each one of them will be devoted to a different type of measure. Remember that in our hypotheses we always have $\mathcal{K}([a,b],\mu)=\{0\}$ by Theorem 5.2. Therefore $([a,b],\mu)\in\mathcal{C}_0$, since $(a,b)^{(0)}\setminus((a,b)_1\cup\cdots\cup(a,b)_k)$ has at most two points.

Measures of type 1. By Corollary 3.1 we have directly

$$\|xf\|_{W^{k,p}([a,b],\,\mu)}\le c\,\|x\|_{W^{k,p}([a,b],\,\mu)}\|f\|_{W^{k,p}([a,b],\,\mu)}\,,$$

for all $f\in V^{k,p}([a,b],\mu)$, since $\mathcal{K}([a,b],\mu)=\{0\}$.

Measures of type 2. We can write each function $f\in V^{k,p}([a,b],\mu)$ as the sum $f=f_1+f_2+f_3$ following the proof of Theorem 4.1 for measures of type 2. Then we have

$$\|xf\|_{W^{k,p}([a,b],\,\mu)}\le\|xf_1\|_{W^{k,p}([a,b],\,\mu)}+\|xf_2\|_{W^{k,p}([a,b],\,\mu)}+\|xf_3\|_{W^{k,p}([a,b],\,\mu)}\,.$$

Theorem 3.1 gives for $i=1,2,3$

$$\|f_i\|_{W^{k,p}([a,b],\,\mu)}\le c\,\|f\|_{W^{k,p}([a,b],\,\mu)}$$

with a positive constant c independent of f (although c depends on the partition of unity chosen). Then it is clear that we only need to prove

$$\|xf_1\|_{W^{k,p}([a,b],\,\mu)}\le c\,\|f_1\|_{W^{k,p}([a,b],\,\mu)}$$

for a positive constant c independent of f, since the argument for xf_2 is symmetric and what we have just proved for measures of type 1 is the corresponding inequality for xf_3.

Observe that as $(xf_1)^{(j)} = xf_1^{(j)} + jf_1^{(j-1)}$ and $\|xf_1^{(j)}\|_{L^p([a,b],\,\mu_j)} \le \|x\|_{L^\infty([a,b])}\|f_1^{(j)}\|_{L^p([a,b],\,\mu_j)}$, it is enough to prove for $0 < j \le k$ that

$$\|f_1^{(j-1)}\|_{L^p([a,b],\,\mu_j)} \le c\,\|f_1\|_{W^{k,p}([a,b],\,\mu)}\,. \tag{5.1}$$

To prove this, we apply lemmas B (with $a + t = b$) and 3.1 and obtain

$$\|f_1^{(j-1)}\|_{L^p([a,b],w_j)} = \|f_1^{(j-1)}\|_{L^p([a,a_2],w_j)} \le c\,\|f_1^{(j)}\|_{L^p([a,a_2],w_j)}\,, \tag{5.2}$$

since $f_1^{(j-1)}(x) = \int_{a_2}^x f_1^{(j)}$, supp $f_1 \subseteq [a, a_2]$ and w_j is comparable to a non-decreasing function in $[a, a_2]$.

The strong p-admissibility of μ and properties of measures of type 2 give the following inclusion

$$[A_j, B_j] := [\min(\operatorname{supp}(\mu_j)_s), \max(\operatorname{supp}(\mu_j)_s)] \subseteq (a, b)^{(j)}$$

if $(\mu_j)_s$ is not identically zero. Then the finiteness of μ and Theorem B(a) imply that

$$\|f_1^{(j-1)}\|_{L^p([a,b],(\mu_j)_s)} \le c\,\|f_1^{(j-1)}\|_{L^\infty([A_j,B_j])} \le c\,\|f_1\|_{W^{k,p}([a,b],\,\mu)}\,, \tag{5.3}$$

since $([a, b], \mu) \in \mathcal{C}_0$. Obviously, this is also true if $(\mu_j)_s$ is identically zero.

Now, inequalities (5.2) and (5.3) give (5.1).

Measures of type 3. We split each function $f \in V^{k,p}([a, b], \mu)$ as in the previous case and also like in the proof of the previous case, we only need to prove (5.1) for $0 < j \le k$. If $k_1 \le j \le k$, we obtain (5.1) as in the proof of the theorem for measures of type 2. If $k_1 = 0$ we have finished. If $k_1 > 0$, since a is right $(k_1 - 1)$-regular, we have $[a, a_2] \subseteq (a, b)^{(k_1-1)}$ by (2), and then the finiteness of μ and Theorem B(a) give

$$\sum_{j=0}^{k_1-1} \|f_1^{(j)}\|_{L^p([a,b],\,\mu_{j+1})} = \sum_{j=0}^{k_1-1} \|f_1^{(j)}\|_{L^p([a,a_2],\,\mu_{j+1})} \le c_1 \sum_{j=0}^{k_1-1} \|f_1^{(j)}\|_{L^\infty([a,a_2])} \le c_2\,\|f_1\|_{W^{k,p}([a,b],\,\mu)}\,.$$

Measures of type 4. Let us denote by I the interval $[a_0 - \delta, a_0 + \delta] \cap [a, b]$ and by J the closure of $[a, b] \setminus [a_0 - \delta, a_0 + \delta]$. As in the proof of this theorem for measures of type 2, it is enough to show for $0 < j \le k$ and $f \in V^{k,p}([a, b], \mu)$

$$\|f^{(j-1)}\|_{L^p([a,b],\,\mu_j)} \le c\,\|f\|_{W^{k,p}([a,b],\,\mu)}\,. \tag{5.4}$$

On one hand, by property (2) of measures of type 4, we know that $J \subseteq (a, b)^{(k-1)}$. The finiteness of μ and Theorem B(a) give

$$\|f^{(j-1)}\|_{L^p(J,\,\mu_j)} \le c\,\|f^{(j-1)}\|_{L^\infty(J)} \le c\,\|f\|_{W^{k,p}([a,b],\,\mu)}\,, \tag{5.5}$$

for $0 < j \le k$, since $([a, b], \mu) \in \mathcal{C}_0$.

On the other hand, applying property (1) of measures of type 4, we obtain

$$\|f^{(k-1)}\|^p_{L^p(I,\,\mu_k)} \le c\int_I |f^{(k-1)}(x)|^p|x - a_0|^p\, d\mu_{k-1}(x) \le c\,\|f^{(k-1)}\|^p_{L^p([a,b],\,\mu_{k-1})}\,. \tag{5.6}$$

If $k > 1$, properties (2) and (3) of measures of type 4 imply $I \subseteq (a, b)^{(k-2)}$. Then the finiteness of μ and Theorem B(a) give for $0 < j \le k - 1$

$$\|f^{(j-1)}\|_{L^p(I,\,\mu_j)} \le c\,\|f^{(j-1)}\|_{L^\infty(I)} \le c\,\|f\|_{W^{k,p}([a,b],\,\mu)}\,. \tag{5.7}$$

Then (5.5), (5.6) and (5.7) give (5.4).

Measures of type 5. The proof is similar to the case of measures of type 4, except for (5.6). Instead, it is enough to observe

$$\|f^{(k-1)}\|^p_{L^p(I,\,\mu_k)} = \int_I |f^{(k-1)}(x)|^p w_k(x)\, dx \le c\int_I |f^{(k-1)}(x)|^p w_{k-1}(x)\, dx \le c\,\|f^{(k-1)}\|^p_{L^p([a,b],\,\mu_{k-1})}\,.$$

This finishes the proof of Theorem 5.1.

Finally, the following obvious result allows us to study the bounds of M in each connected component of $\Omega^{(0)}$.

Theorem 5.3. *Let us consider $1 \le p < \infty$, an open set $\Omega \subseteq \mathbf{R}$ and a p-admissible vectorial measure μ in $\overline{\Omega}$. Let us consider also the connected components $\{I_i\}_i$ of $\Omega^{(0)}$. If M is well defined in $W^{k,p}(\overline{\Omega},\mu)$, then the multiplication operator is bounded in $W^{k,p}(\overline{\Omega},\mu)$ if and only if $\operatorname{supp}\mu_0 \setminus \Omega^{(0)}$ is bounded and there exists a positive constant c such that*

$$\|xf\|_{W^{k,p}(\overline{I_i},\,\mu|_{I_i})} \le c\,\|f\|_{W^{k,p}(\overline{I_i},\,\mu|_{I_i})}\,,$$

for every i and every f belonging to $W^{k,p}(\overline{\Omega},\mu)$.

In [R2] we find further results on the multiplication operator.

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