Orthogonal forms: a simple tool for proving the irrationality of $\zeta(3)$*

J. Arvesú

Department of Mathematics, Universidad Carlos III de Madrid,
Avda. de la Universidad, 30, 28911, Leganés, Madrid, Spain

August 24, 2009

Abstract

A new proof of the irrationality of $\zeta(3)$ is given. The orthogonality relation among certain known forms [17] constitutes a novel ingredient used in the present approach. Here, the same sequences of integers obtained in [10] appear. Apéry's recurrence relation for the sequence of rational approximants to $\zeta(3)$ are constructively obtained. A simultaneous rational approximation problem is used for such purposes.

2000 Mathematics Subject Classification: Primary 42C05, 11B37, 11J72, 11M06; Secondary 30B70, 11A55, 11J70, 33C47

Keywords: Simultaneous rational approximation, multiple orthogonal polynomials, irrationality, recurrence relations.

1 Introduction

Since Apéry's proof on the irrationality of $\zeta(3)$ in 1978 [1], several authors have constructed different approaches to explain the irrationality of $\zeta(3)$,

*This research was partially supported by 'Ramón y Cajal' research program of Spain, the research project MTM2006-13000-C03-02 of the Ministerio de Educación y Ciencia of Spain, and Projects CCG07-UC3M/ESP-3339 and CCG08-UC3M/ESP-4516 from Comunidad Autónoma de Madrid.

mostly leading to the same sequence of rational approximants. This surprising fact, gives rise to the so-called Apéry's phenomenon. Exceptions to this issue are the researches dealing with the irrationality measure (see for instance [8, 16] and the references contained there). As a consequence, a quite reasonable amount of material can be gathered under the title 'irrationality of $\zeta(3)$', and therefore, any recent contribution on this topic must necessarily be summarized. Mainly, two different approaches have been developed to explain the irrationality of $\zeta(3)$, namely, complex contour integrals proposed by Nesterenko [10] -in a similar fashion that [7]- and triple integrals involving Legendre polynomials considered by Beukers [3]. Recently, in [11] was established certain equivalency between multiple and contour integrals, which is related with hypergeometric type series. An elementary explanation on how the solutions of the hypergeometric equation can be expressed by means of countour integrals can be found in [12].

Specially relevant for its consequences in the study of linear forms involving odd zeta values [21] is the next lemma:

Suppose that for $n = 1, 2, \dots,$ one has the sequences

$$\alpha_n = -2\sum_{k=0}^{n} b_k^{(n)}, \quad \text{where} \quad b_k^{(n)} = \binom{n+k}{k}^2 \binom{n}{k}^2,$$

$$\beta_n = 2\sum_{k=1}^{n} b_k^{(n)} \sum_{r=1}^{k} \left(r^{-3} - r^2\gamma_{k,n}\right), \quad \text{with} \quad \gamma_{k,n} = \sum_{j=1}^{n} \frac{1}{k+j} - \sum_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{k-j}.$$

Lemma 1.1 *(Nesterenko [10]) The following expression is valid:*

$$\alpha_n\zeta(3) - \beta_n = \frac{1}{2\pi i}\int_L \left(\frac{\pi(1-z)_n}{(z)_{n+1}\sin \pi z}\right)^2 dz, \tag{1}$$

where L is the vertical line $\Re z = C$, $0 < C < n+1$, oriented from top to bottom, and $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ denotes the Pochhammer symbol.

The above lemma is the cornerstone that the author uses in [10] for the irrationality proof of $\zeta(3)$. Indeed, the turning point method (also known as Laplace's method) gives an estimation to the above contour integral (1) yielding the irrationality of $\zeta(3)$. In addition, the author obtained a new continuous fraction expansion of $\zeta(3)$.

Going back to the late seventies, in [5] the author considers a simultaneous rational approximation problem in the neighborhood of zero for polylogarithms which allows to prove the irrationality of $\zeta(3)$. Similarly, other simultaneous rational approximation problem in the neighborhood of infinity involved the same functions (polylogarithms) is formulated in [17], and consequently proved the irrationality of $\zeta(3)$ (see also [18]).

The aim of the present contribution is twofold. First, give a constructive but a the same time elementary proof of the irrationality of $\zeta(3)$. This goal is achieved by a simple approach to deduce the Apéry's difference equation (6). In [10], p.629, the author notes the interest in obtaining an elementary proof of the irrationality of $\zeta(3)$. In an elementary proof one expects to find the use of simple methods -basic mathematical tools if possible- to estimate the sequence of rational approximants to $\zeta(3)$ (see formula (28)) and prove in a simple way that none of the remainder approximation terms vanishes. Lemma 3.1 gives a simple answer to the last question. For contrasting our approach two other proofs of the irrationality of $\zeta(3)$ are included in section 4. Second, present a general framework based on a suitable simultaneous rational approximation problem which comprises some previous independent proofs on the irrationality of $\zeta(3)$. Indeed, this approximation problem serves as a main stem in which the complex contour integral and multiple integral approaches find their common roots. Usually, simultaneous approximation has been introduced to obtain the Beukers's integrals (triple integrals over a unitary cube) for the evaluation of the irrationality of $\zeta(3)$ [17, 18], but not for the complex contour integrals. This paper covers this lack and intents to use elementary techniques -from the point of view of rational approximation- in its exposition. A compressed collection of references including some proofs dealing with the aforementioned two approaches are: [1, 3, 4, 5, 10, 17, 18], and [20]. See also [21] for a reasonable relation of references on the topic.

The contents of this paper are as follows. In Section 2, we familiarize the reader with some preliminary notions and well-known results on linear difference equations with varying coefficients, more specifically, the Perron's theorem; this theorem is essential to obtain a proof of the irrationality of $\zeta(3)$ as a consequence of Apéry's difference equation (6) obtained in lemma 3.3 of the next section. In Section 3, we carefully discuss the simultaneous rational approximation problem; generated by equations (8)-(10) that has associated the orthogonal forms (16). Since the orthogonal forms contain the logarithmic function we avoid referring to them as linear forms. Likely, the Plato's sense of Forms is more adequate. Here, the novelty in the given irrationality proof consists only in the simplicity of the ingredients for proving the positiveness of the remainder sequence of approximation to $\zeta(3)$ as well as for obtaining the well-known Apéry's difference equation in a simple and constructive way. From the simultaneous rational approximation problem two key sequences appear, namely, the Apéry's sequence (see $(q_n)_{\geq 0}$ in (6)) and the remainder sequence $(r_n)_{\geq 0}$ (see (30)). Indeed, it will be obtained that the verification of the Apéry's difference equation indistinctly by one of these sequences -no matter which of them- is a necessary and sufficient condition in order the other one also verifies it (see also remark 4.1). Subsection 3.2 contains an interesting result, namely, lemma 3.4 which establishes a usefull expression for the remainder sequence in terms of complex contour integrals (in similar fashion lemma 1.1 was proved in [10]). This result allows us to connect naturally in the last subsection of this paper the two aforementioned approaches: triple integrals and complex contour integrals (58). To address the question whether or not our two sequences of integers generate good rational approximants to $\zeta(3)$, we avoid to use turning point method for estimating the complex contour integral (42) as well as an upper bound for triple integrals, i.e., for the remainder of the equation (9) of the simultaneous rational approximation, the asymptotic behavior deduced from the Apéry's difference equation is used instead. This fact makes the irrationality proof of $\zeta(3)$ much more elementary. Lastly, Section 4 comprises some known results mainly regarding the Apéry's difference equation (6) satisfied by the sequences of numbers (numerators and denominators of the rational approximants) that approach $\zeta(3)$. Thus, a more selfconsisting character is given to the paper. In this section, the method to deduce that the remainder sequence $(r_n)_{n\geq 0}$ verifies Apéry's difference equation follows the ideas exposed in [19] and [20], and serves to double-check the results obtained in previous sections. Finally, a relation between triple and contour integrals are established. Consequently, Beukers's integral (59) verifies the Apéry's difference equation.

2 Preliminary notions

Simultaneous rational approximation also known as Hermite-Padé approximation (see [13], [17], and [18] for a detailed explanation of the concept) is a quite useful approach in proving irrationality of some *notable* constants like $\zeta(2)$ and $\zeta(3)$, being

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad s > 1, \tag{2}$$

the Riemann zeta-function. This kind of approximation appears naturally when the analyzed constants are the values of the Cauchy transforms $\hat{\mu}(z)$ of certain measures, for $z \in \mathbb{C}$ outside the support of the measure. Indeed, this approach to the study of the irrationality has its historical roots in the Hermite's proof of the transcendence of the number e. A particular well-known statement that characterizes the irrational numbers is the following lemma:

Lemma 2.1 *Let x be a real number, and let $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ be two sequences of integers such that $|q_n x - p_n| > 0$, for all positive integers $n \in \mathbb{N}$, and $\lim_{n\to\infty} q_n x - p_n = 0$; then x is irrational.*

In section 3 and subsection 3.2 we will implicitly use the above lemma 2.1. Furthermore, for the real number $x = \zeta(3)$, mainly based on the orthogonality of some forms (see relation (18)) we will find two sequences $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ that form the rational approximants p_n/q_n to $\zeta(3)$. In addition, these sequences verify a second order difference equation with variable coefficients (Apéry's difference equation [1]).

For $k \in \mathbb{N}$, consider the following recurrence relation of order k

$$u_n = \alpha_{n,1} u_{n-1} + \cdots + \alpha_{n,k} u_{n-k}, \quad n = 1, 2, \ldots, \tag{3}$$

for which the variable coefficients have limits

$$\lim_{n\to\infty} \alpha_{n,i} = a_i, \quad i = 1, 2, \ldots, k. \tag{4}$$

The polynomial

$$\lambda^k - a_1 \lambda^{k-1} - \cdots - a_k = \prod_{i=1}^{k} (\lambda - \lambda_i), \tag{5}$$

is called the characteristic polynomial of the recurrence relation (3). The recurrence relation (3) is said to be non-degenerate if $\alpha_{n,i} \neq 0$ for $n = 1, 2, \ldots$

Theorem 2.1 *(Perron's theorem [14]) Suppose that for non-degenerate recurrence relation (3), the condition (4) is verified, and all the zeros of the characteristic polynomial are ordered in modulus* $|\lambda_1| < \cdots < |\lambda_k|$*. Then for each zero* λ *of the characteristic polynomial (5) there exists a solution* $(u_n)_{n\geq 1-k}$ *of (3) such that* $\lim_{n\to\infty} u_{n+1}/u_n = \lambda$.

Notice that the general solution $(u_n)_{n\geq 1-k}$ can be expressed as a linear combination of the k-linearly independent solutions $(u_{n,i})_{n\geq 1-k}$ of (3) such that $\lim_{n\to\infty} u_{n+1,i}/u_{n,i} = \lambda_i$, $i = 1, 2, \ldots, k$.

Perron's theorem requires that the relation (3) be non-degenerate. Consequently, some authors [6] interprets it as a refinement of the following classical theorem of Poincaré.

Theorem 2.2 *(Poincaré's theorem [15]) Let assume for the recurrence relation (3), that the condition (4) is verified, and all the zeros of the characteristic polynomial are ordered in modulus* $|\lambda_1| < \cdots < |\lambda_k|$*. Then for each solution* $(u_n)_{n\geq 1-k}$ *of (3), either* $u_n = 0$, $\forall n > N$ $(N \in \mathbb{Z})$, *or* $\lim_{n\to\infty} u_{n+1}/u_n$ *converges to one of the roots* λ *of the characteristic polynomial (5).*

Poincaré's theorem does not answer a priori to which of the roots λ does converge the ratio u_{n+1}/u_n while Perron's theorem does.

A typical application of the previous results is the Apéry's recurrence relation (also known as Apéry's difference equation)

$$(n+1)^3 y_{n+1} - (2n+1)(17n^2+17n+5)y_n + n^3 y_{n-1} = 0, \quad n = 1, 2, \ldots, \tag{6}$$

which gives two linearly independent solutions $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$, with the initial conditions

$$p_0 = 0, \quad p_1 = 6, \quad q_0 = 1, \quad q_1 = 5,$$

respectively. Indeed, (see expressions (3) in [10])

$$q_n = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2, \quad p_n = \sum_{k=1}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \gamma_{k,n},$$

$$\gamma_{k,n} = \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3} \binom{n+j}{j}^{-1} \binom{n}{j}^{-1}.$$

However, the search of the above explicit expressions for the sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ that verify the Apéry's recurrence relation becomes a difficult problem.

A natural question appears: How to obtain constructively the Apéry's difference equation (6)? In addition, it will be desired to develop an approach that gives the recurrence relation as well as nice representations of the solutions, i.e., in terms of their quantitative and qualitative behavior. Also, it will be interesting to relate one of the independent solutions of the above difference equation with the complex contour integrals given in [10] viewed from a simultaneous rational approximation problem. Below, the reader will find answers to these questions.

3 Mixed type I and type II multiple orthogonal polynomials

For all positive integers $k \in \mathbb{N}$, the Riemann zeta-function (2) can be expressed as

$$\zeta(k+1) = \frac{(-1)^k}{k!} \int_0^1 \frac{\log^k x}{1-x} dx.$$

This function is related up to a multiplicative factor with two of the following three functions

$$f_k(z) = \frac{1}{(k-1)!} \int_0^1 \frac{(\log x)^{k-1}}{z-x}\, dx, \quad k = 1, 2, 3. \tag{7}$$

In particular, $f_1(1)$ is the Cauchy transform of the Lebesgue measure supported on the interval $(0,1)$, $f_2(1) = -\zeta(2)$, and $f_3(1) = \zeta(3)$. The above system $\{f_1(z), f_2(z), f_3(z)\}$ slightly differs from those used in [18] for constructing a rational approximation problem; specifically, there is a sing change in $f_2(z)$ when comparing both systems.

For the system (7) and polynomials $A_n(z)$, $B_n(z)$, $C_n(z)$, and $D_n(z)$ we pose the following simultaneous rational approximation problem near infinity

$$A_n(z)f_1(z) + B_n(z)f_2(z) - C_n(z) = O(z^{-n-1}), \tag{8}$$

$$A_n(z)f_2(z) + 2B_n(z)f_3(z) - D_n(z) = O(z^{-n-1}), \tag{9}$$

$$A_n(1) = 0, \tag{10}$$

being $A_n(z) = \sum_{k=0}^{n} a_k^{(n)} z^k$, and $B_n(z) = \sum_{k=0}^{n} b_k^{(n)} z^k$ polynomials of degree exactly n.

For the linear space of polynomials with complex coefficients we will use the notation $\mathbb{P}$, while $\mathbb{P}_n$ will denote the corresponding n-dimensional subspace. Indeed, $A_n(z), B_n(z), C_n(z), D_n(z) \in \mathbb{P}_n$; below we will find explicitly each of these polynomials.

The equations (8) and (9) define a type I rational approximation problem for the systems (f_1, f_2) and (f_2, f_3), respectively (see [13]). The combination (8)-(9) is a vector type II rational approximation problem with common vector denominator (A_n, B_n), see [17] and [18]. This fact motivates the denomination of the problem (8)-(10) as a mixed type I and type II rational approximation problem.

Notice that this approximation problem leads to a linear system of equations for determining $(4n+3)$-unknowns (the coefficients of the unknown polynomials $A_n(z), B_n(z), C_n(z), D_n(z) \in \mathbb{P}_n$). However, one can find a smaller linear system of equations for $(2n+1)$-unknowns that produces the polynomial solutions $A_n(z)$, $B_n(z)$, $C_n(z)$, and $D_n(z)$, instead. This is due to the fact that $C_n(z)$, and $D_n(z)$ will depend on $A_n(z)$ and $B_n(z)$ (see equations (11) and (13) below). In addition, the aforementioned linear system of equations for determining the $(2n+1)$-unknown coefficients of $A_n(z)$ and $B_n(z)$ derives from two orthogonality conditions of certain forms (see relations (12) and (14)). Indeed, a detailed explanation of these facts is included here.

Substituting $f_1(z)$ and $f_2(z)$ into the equation (8) give

$$\int_0^1 \frac{A_n(z)+B_n(z)\log x}{z-x}\,dx - C_n(z) = O\left(z^{-n-1}\right).$$

Observe that $\frac{A_n(z)-A_n(x)}{z-x}$ and $\frac{B_n(z)-B_n(x)}{z-x}$ are polynomials of degree $n-1$, then one selects

$$\int_0^1 \frac{A_n(z)-A_n(x)}{z-x}\,dx + \int_0^1 \frac{B_n(z)-B_n(x)}{z-x}\log x\,dx = C_n(z) \in \mathbb{P}_n. \quad (11)$$

Consequently, the remainder term is given by

$$\begin{aligned} r_{n,1}(z) &= \int_0^1 \frac{A_n(x)+B_n(x)\log x}{z-x}\,dx \\ &= \sum_{k=n}^{\infty} \frac{1}{z^{k+1}} \int_0^1 x^k (A_n(x)+B_n(x)\log x)dx = O\left(z^{-n-1}\right), \end{aligned}$$

where the orthogonality condition

$$\int_0^1 x^k (A_n(x)+B_n(x)\log x)dx = 0, \qquad k=0,\ldots,n-1, \quad (12)$$

guarantees the prescribed order near infinity imposed by equation (8).

Analogously, one proceeds with the equation (9) of the simultaneous approximation problem (8)-(10). Indeed, one gets

$$\int_0^1 \frac{A_n(z)-A_n(x)+(B_n(z)-B_n(x))\log x}{z-x}\log x\,dx = D_n(z) \in \mathbb{P}_n. \quad (13)$$

Thus, a second orthogonality condition

$$\int_0^1 \left(\left(A_n(x) + B_n(x)\log x\right)\log x\right) x^k dx = 0, \qquad k = 0, \ldots, n-1, \tag{14}$$

guarantees the prescribed order near infinity of the second remainder term

$$\begin{aligned} r_{n,2}(z) &= \int_0^1 \frac{(A_n(x) + B_n(x)\log x)\log x}{z - x}\, dx \qquad (15) \\ &= \sum_{k=n}^{\infty} \frac{1}{z^{k+1}} \int_0^1 x^k (A_n(x) + B_n(x)\log x)\log x\, dx = O\left(z^{-n-1}\right). \end{aligned}$$

Accordingly, the initial approximation problem for determining $4n+3$ unknown coefficients is reduced to $2n+1$ coefficients since $C_n(z)$ and $D_n(z)$ depend on $A_n(z)$ and $B_n(z)$. In fact, we have found the orthogonality conditions (12) and (14) for the forms

$$F_n(x) = (A_n(x) + B_n(x)\log x) \quad \text{and} \quad G_n(x) = F_n(x)\log x, \tag{16}$$

respectively, i.e.,

$$\int_0^1 F_n(x)x^k dx = \int_0^1 G_n(x)x^k dx = 0, \quad k = 0, \ldots, n-1. \tag{17}$$

Notice that (17) gives $2n$ linear equations with $(2n+1)$-unknowns (underdetermined linear system of equations). Furthermore, the following expression

$$\int_0^1 F_n(x)\left(P(x) + Q(x)\log x\right) dx = 0, \quad \forall P(x), Q(x) \in \mathbb{P}_{n-1}, \tag{18}$$

yields. This relation can be interpreted as an orthogonality condition among two different forms

$$\int_0^1 F_n(x)F_m(x)dx = 0, \quad \text{whenever } n \neq m.$$

Another consequence of the orthogonality relations (17) are the following formulas

$$\int_0^1 p(x)\frac{F_n(x)}{z-x}dx = p(z)\int_0^1 \frac{F_n(x)}{z-x}dx, \tag{19}$$

$$\int_0^1 q(x)\frac{G_n(x)}{z-x}dx = q(z)\int_0^1 \frac{G_n(x)}{z-x}dx, \tag{20}$$

being $p(z)$ and $q(z)$ arbitrary polynomials of degree at most n.

The coefficients for polynomials $A_n(x)$ and $B_n(x)$ are determined from the orthogonality conditions (17). Indeed, one gets the rational functions

$$\begin{aligned} R_{n,1}(t) &= \sum_{k=0}^{n}\left[\frac{a_k^{(n)}}{t+k+1} - \frac{b_k^{(n)}}{(t+k+1)^2}\right], \\ R_{n,2}(t) &= -\sum_{k=0}^{n}\left[\frac{a_k^{(n)}}{(t+k+1)^2} - 2\frac{b_k^{(n)}}{(t+k+1)^3}\right]. \end{aligned} \tag{21}$$

The linear system of equations (17) is an underdetermined one, so the coefficients $a_k^{(n)}$, and $b_k^{(n)}$ are determined up to a constant factor κ_n. Indeed,

$$R_{n,1}(t) = \kappa_n \frac{t^2(t-1)^2\cdots(t-n+1)^2}{(t+1)^2\cdots(t+n+1)^2}, \tag{22}$$

and

$$R_{n,2}(t) = \frac{d}{dt}R_{n,1}(t) = 2R_{n,1}(t)\left[\sum_{k=0}^{n-1}\frac{1}{t-k} - \sum_{k=1}^{n+1}\frac{1}{t+k}\right]. \tag{23}$$

With the choice $\kappa_n = -1$ from (21) one obtains

$$b_k^{(n)} = -\lim_{t\to-(k+1)}(t+k+1)^2 R_{1,n}(t) = \binom{n+k}{k}^2\binom{n}{k}^2, \quad k=0,1,\ldots,n, \tag{24}$$

and

$$a_k^{(n)} = \operatorname*{Res}_{t=-k-1} R_{n,1}(t) = 2\left(H_{n+k} - 2H_k + H_{n-k}\right) b_k^{(n)}, \quad k=0,1,\ldots,n, \tag{25}$$

where $H_k^{(r)}$ denotes the Harmonic Number k of order r ($H_k^{(1)} = H_k$ and $H_0 = 0$). Accordingly, the following expressions

$$
\begin{aligned}
B_n(1) &= \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2, \quad B_0(x) = 1, \\
C_n(1) &= \sum_{k=1}^{n} \left(a_k^{(n)} H_k - b_k^{(n)} H_k^{(2)} \right), \quad C_0(x) = 0, \\
D_n(1) &= \sum_{k=1}^{n} \left(2b_k^{(n)} H_k^{(3)} - a_k^{(n)} H_k^{(2)} \right), \quad D_0(x) = 0,
\end{aligned} \tag{26}
$$

are easily calculated by taking into account the formulas (11) and (13) as well as (24) and (25).

In particular, for $n \in \mathbb{N} \cup \{0\}$, the simultaneous rational approximation problem (8)-(10) by setting $z = 1$ gives

$$B_n(1) f_2(1) - C_n(1) = r_{n,1}(1) = \int_0^1 \frac{F_n(x)}{1-x}\, dx, \tag{27}$$

$$2B_n(1) f_3(1) - D_n(1) = r_{n,2}(1) = \int_0^1 \frac{G_n(x)}{1-x}\, dx = \sum_{t=n}^{\infty} R_{n,2}(t). \tag{28}$$

Notice that (15) is valid for $|x/z| < 1$, and we have written in (28) the relation

$$r_{n,2}(1) = \sum_{t=n}^{\infty} R_{n,2}(t). \tag{29}$$

Apparently, for such a limit situation at the singular point $z = 1$, the performed change of integration and summation in (15) has no sense, however a rigorous justification of (29) is given in lemma 3.4 below. In what follows in this section regarding the irrationality proof of $\zeta(3)$, relation (29) is irrelevant, while orthogonal forms $F_n(x)$, $n = 0, 1, \ldots$, play a fundamental role. However, section 4 makes use of relation (29) aimed to prove the irrationality of $\zeta(3)$ base on Apéry's recurrence relation as well as to connect the triple integral and complex contour integral approaches.

Now, the main goal will be the study of the asymptotic behavior of $r_{n,2}(1)$, when n goes to infinity. Observe that according with (28), the opposite situation when n is identically equal to zero leads to the identities $f_2(1) = -\zeta(2)$ and $f_3(1) = \zeta(3)$, respectively.

For brevity, let us denote the sequences involved in (27)-(28) as

$$\begin{gathered}(q_n)_{n\geq 0} = \{B_n(1)\}_{n=0}^{\infty}, \quad (p_n)_{n\geq 0} = \{D_n(1)\}_{n=0}^{\infty}, \quad (r_n)_{n\geq 0} = \{r_{n,2}(1)\}_{n=0}^{\infty}, \\ (\tilde{q}_n)_{n\geq 0} = -(q_n)_{n\geq 0}, \quad (\tilde{p}_n)_{n\geq 0} = \{C_n(1)\}_{n=0}^{\infty}, \quad (\tilde{r}_n)_{n\geq 0} = \{r_{n,1}(1)\}_{n=0}^{\infty}.\end{gathered} \tag{30}$$

Hence, by using (3)-(7) the expressions (27)-(28) can be rewritten as

$$\tilde{q}_n\zeta(2) - \tilde{p}_n = \tilde{r}_n, \quad 2q_n\zeta(3) - p_n = r_n, \quad n \geq 0. \tag{31}$$

Regardless of what method is used for the approximation of $\zeta(3)$, the above expression (31) must containt good rational approximants p_n/q_n such that $|\zeta(3) - p_n/q_n| = o(q_n^{-1})$, then based on lemma 2.1 $\zeta(3)$ has to be irrational, provided that $|q_n\zeta(3) - p_n| > 0$, for all positive integers $n \in \mathbb{N}$. This condition is crucial. Indeed, one of the main difficulties in the study of the irrationality of $\zeta(3)$ is to determine whether or not the terms of the remainder sequence $(r_n)_{n\geq 0}$ vanish. Essentially, two different approaches have been suggested for answering this question (see [3], [17] and [18] where tripe integrals are used for such purpose, and [10] where complex variable techniques are applied). Here, we will obtain that the sequences involved in (31), i.e., $(q_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ verify the Apéry's difference equation, and two of them are linearly independent.

The following lemma gives a simple answer to the above question on the non-vanishing terms of $(r_n)_{n\geq 0}$.

Lemma 3.1 *Suppose that $A_n(x), B_n(x) \in \mathbb{P}_n \setminus \{0\}$ are the solutions of the rational approximation problem (8)-(10), i.e., with coefficients determined by*

(24) and (25), respectively. Then, for $n = 0, 1, 2, \ldots$, the remainder function $r_{n,2}(z)$ does not vanish at $z = 1$.

Proof. Suppose on the contrary that $r_{n,2}(1) = 0$, i.e.,

$$r_n = \int_0^1 \frac{(A_n(x) + B_n(x)\log x)\log x}{1-x}\,dx = 0.$$

Let us consider the auxiliary function $f(x) = \dfrac{(A_n(x) + B_n(x)\log x)^2}{1-x}$, and $f(1) = 0$, on $(0, 1]$, where polynomials $A_n(x)$ and $B_n(x)$ are solutions of the approximation problem (8)-(10). Notice that $f(x) \geq 0$, and $\displaystyle\int_0^1 f(x)dx > 0$.

On the other hand, one has

$$\int_0^1 f(x)dx = \int_0^1 \left(\frac{q_n G_n(x)}{1-x} - \left(\tilde{A}_{n-1}(x) + \tilde{B}_{n-1}(x)\log x\right) F_n(x)\right) dx, \quad (32)$$

being $\tilde{A}_{n-1}(x) = \frac{A_n(1) - A_n(x)}{1-x}$ and $\tilde{B}_{n-1}(x) = \frac{B_n(1) - B_n(x)}{1-x}$. From the orthogonality conditions (18) and equation (32) one gets

$$0 < \int_0^1 f(x)dx = q_n r_n.$$

Hence, r_n cannot vanish. This contradiction proves our statement.

As a consequence of the above lemma the terms of the sequence $(r_n)_{n\geq 0}$, where $r_n = 2q_n\zeta(3) - p_n$, are strictly positive. It follows from being $(q_n)_{n\geq 0}$ a strictly positive sequence -see (24)-.

3.1 Apéry's difference equation and irrationality of $\zeta(3)$

From the simultaneous rational approximation problem (8)-(10) we know that the remainder function $r_{n,2}(z)$ decreases at infinity while the polynomial $B_n(z)$ obviously grows, so hopefully at the singular point $z = 1$, the numerical sequences formed by $B_n(1)$ and $r_{n,2}(1)$, $n = 0, 1, \ldots$, will conserve such a behavior and be linearly independents. The next lemmas give a positive answer to these questions.

Lemma 3.2 *The sequences* $(q_n)_{n\geq 0}$ *and* $(r_n)_{n\geq 0}$ *are linearly independent.*

Proof. Suppose on the contrary that there exists nonzero constants α_1 and α_2 (not all zero) such that

$$\alpha_1 q_n + \alpha_2 r_n = 0, \quad n = 0, 1, \ldots$$

Equivalently, the analog of the Wronskian determinant for q_n and r_n vanishes,

$$W(q_n, r_n) = \det \begin{pmatrix} q_n & r_n \\ q_{n+1} & r_{n+1} \end{pmatrix} = 0.$$

Let us consider the integral

$$I_n = \int_0^1 \frac{F_n(x)F_{n+1}(x)}{1-x} dx.$$

Using formulas (19)-(20) one gets

$$I_n = \int_0^1 \frac{F_n(x)F_{n+1}(x)}{1-x} dx = q_n \int_0^1 \frac{G_{n+1}(x)}{1-x} = q_n r_{n+1}. \tag{33}$$

On the other hand,

$$\begin{aligned} I_n &= \int_0^1 \frac{F_n(x)F_{n+1}(x)}{1-x} dx = \int_0^1 \frac{xF_n(x)F_{n+1}(x)}{1-x} dx \\ &= q_{n+1} \int_0^1 \frac{xG_n(x)}{1-x} - a_{n+1}^{(n+1)} \int_0^1 x^n F_n(x) dx - b_{n+1}^{(n+1)} \int_0^1 x^n G_n(x) dx \\ &= q_{n+1} r_n - \frac{12}{(n+1)^3}. \end{aligned} \tag{34}$$

Equating (33) and (34) one obtains that $W(q_n, r_n) \neq 0$. This contradiction proves our statement.

Notice that the preceding lemma makes use of an *ad hoc* property

$$a_{n+1}^{(n+1)} \int_0^1 x^n F_n(x) dx + b_{n+1}^{(n+1)} \int_0^1 x^n G_n(x) dx \neq 0,$$

which follows by straightforward calculations from the expressions (22)-(25).

Lemma 3.3 *The sequences* $(q_n)_{n\geq 0}$ *and* $(r_n)_{\geq 0}$ *verify the following second order difference equation*

$$\alpha_n y_{n+1} + \beta_n y_n + \gamma_n y_{n-1} = 0, \quad n = 1, 2, \dots \tag{35}$$

where $\alpha_n = (n+1)^3$, $\beta_n = -(2n+1)(17n^2+17n+5)$, *and* $\gamma_n = n^3$.

Proof. From lemma 3.2, and equations (33)-(34), we have

$$q_n r_{n+1} = q_{n+1} r_n - \frac{12}{(n+1)^3}, \quad \text{or equivalently,} \quad q_{n-1} r_n = q_n r_{n-1} - \frac{12}{n^3}.$$

Multiplying the first equation by a non-zero factor α, the second equation by a non-zero factor $-\gamma$, and adding both equations one gets

$$q_n \left(\alpha r_{n+1} + \gamma r_{n-1}\right) + 12 \left(\frac{\alpha}{(n+1)^3} - \frac{\gamma}{n^3} \right) = r_n \left(\alpha q_{n+1} + \gamma q_{n-1}\right).$$

Hence,

$$q_n \left(\alpha r_{n+1} + \gamma r_{n-1}\right) = r_n \left(\alpha q_{n+1} + \gamma q_{n-1}\right), \tag{36}$$

yields, when $\alpha = \left(1 + \frac{1}{n}\right)^3 \gamma$. Equivalently,

$$q_n \left(\alpha_n r_{n+1} - b_n r_n + \gamma_n r_{n-1}\right) = r_n \left(\alpha_n q_{n+1} - b_n q_n + \gamma_n q_{n-1}\right), \tag{37}$$

where α_n and γ_n are given in (35), and b_n is chosen arbitrarily.

Clearly, from equation (37) one has the following equivalency: If there exist a coefficient b_n such that

$$\alpha_n q_{n+1} - b_n q_n + \gamma_n q_{n-1} = 0, \quad n = 1, 2, \dots, \tag{38}$$

then,

$$\alpha_n r_{n+1} - b_n r_n + \gamma_n r_{n-1} = 0, \quad n = 1, 2, \dots, \tag{39}$$

and vice versa.

Since the sequences $(q_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ are linearly independent, identity (37) implies that the above second order homogeneous difference equations are the only possible choice -of minimal length- that these sequences might satisfy. Notice that if equations (38) and (39) are non-homogeneous, then the sequences $(q_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ were linearly dependent, which contradices lemma 3.3. Now, let us find the coefficient b_n such that the sequences $(q_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ verify the recurrence relation

$$\alpha_n y_{n+1} - b_n y_n + \gamma_n y_{n-1} = 0, \quad n = 1, 2, \dots \tag{40}$$

Observe that b_n can not vanish. On the contrary if b_n vanishes, then from (36) one has

$$\alpha_n r_{n+1} + \gamma_n r_{n-1} = 0 \iff \alpha_n q_{n+1} + \gamma_n q_{n-1} = 0,$$

and none of these degenerate equations are fulfilled.

Accordingly, from equations (38)-(39) we have

$$(n+1)^3 \frac{q_{n+1}}{q_n} + n^3 \frac{q_{n-1}}{q_n} = (n+1)^3 \frac{r_{n+1}}{r_n} + n^3 \frac{r_{n-1}}{r_n} = b_n. \tag{41}$$

Now, aimed to use Perron's theorem 2.1 one seeks coefficient b_n such that when n goes to infinity the ratio b_n/n^3 behaves like a constant $a \in (-\infty, 2) \cup (2, \infty)$. This last requirement implies that the roots t_1 and t_2 of the characteristic equation of (40) can be ordered in modulus since

$$t_1 = \frac{a - \sqrt{a^2-4}}{2}, \quad \text{and} \quad t_2 = \frac{a + \sqrt{a^2-4}}{2}.$$

So let us formally write $b_n = an^3 + bn^2 + cn + d$. Thus, from (26) and (41) one gets a linear system of equations for determining the unknown coefficients a, b, c, and d. In particular, for $n = 1, 2, 3, 4$ the following linear system

$$\begin{cases} 117 = a + b + c + d, \\ 535 = 8a + 4b + 2c + d, \\ 1463 = 27a + 9b + 3c + d, \\ 3105 = 64a + 16b + 4c + d, \end{cases}$$

yields, being its solution $a = 34$, $b = 51$, $c = 27$, and $d = 5$. Hence, $b_n = \beta_n$, and the lemma is completely proved.

In [9] a lower bound for the minimal length of the polynomial recurrence of a binomial sum is studied. Here, specifically for the Apéry's sequence $(q_n)_{n\geq 0}$ the previous lemma gives answer to this question based on some properties of the forms defined by (17).

Remark 3.1 *The equation (40) verifies the hypotheses of Perron's theorem; consequently*

$$\frac{q_{n+1}}{q_n} = \mathcal{O}(t_1), \quad \frac{r_{n+1}}{r_n} = \mathcal{O}(t_2),$$

where t_1 and t_2 denote the roots of the characteristic equation of (40). Therefore, multiplying both side of (41) by $1/(n+1)^3$ and taking limits $n \to \infty$, yields the following relation

$$t_1 + \frac{1}{t_1} = t_2 + \frac{1}{t_2} = 34.$$

Now, t_1 and t_2 are solutions of the above equation. Indeed,

$$(\sqrt{2}+1)^4 + \frac{1}{(\sqrt{2}+1)^4} = (\sqrt{2}-1)^4 + \frac{1}{(\sqrt{2}-1)^4} = 34.$$

Finally, observe that from (37) sequence $(p_n)_{n\geq 0}$ also solves (35).

As consequence of elementary previous lemmas 3.2 and 3.3 as well as Perron's theorem the irrationality of $\zeta(3)$ can be easily proved.

Theorem 3.2 *(Apéry [1]) The real number $\zeta(3)$ is irrational.*

Proof. The characteristic equation for (35) is $t^2 - 34t + 1$ and their zeros are: $t_1 = (\sqrt{2}+1)^4$ and $t_2 = (\sqrt{2}-1)^4$, respectively. Hence, from Perron's theorem 2.1 and lemma 3.2 one has the behavior $q_n = \mathcal{O}\left(t_1^n\right)$ and $r_n = \mathcal{O}\left(t_2^n\right)$, as n goes to infinity, for the two linearly independent solutions, respectively. Thus, from (28) one writes

$$2q_n\zeta(3) - p_n = r_n = \mathcal{O}\left((\sqrt{2}-1)^{4n}\right).$$

Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{l_n^3(2q_n\zeta(3)-p_n)} = \limsup_{n\to\infty} \sqrt[n]{l_n^3 r_n} \le e^3(\sqrt{2}-1)^4 < 1,$$

where l_n denotes the least common multiple of $\{1, 2, \ldots, n\}$. This inequality proves the irrationality of $\zeta(3)$.

3.2 Contour integral representation for $(r_n)_{n\geq 0}$

An important fact in the context of our simultaneous rational approximation problem remains to be addressed; specifically, the equality: $r_n = \sum_{t=n}^{\infty} R_{n,2}(t)$, $n = 0, 1, \ldots,$ whose proof is included here. Accordingly, the goal now is to prove it, and show later how different *mathematical technologies* (different approaches) fit together (see subsection 4.1). Immediately, afther being this proof done, base on some known procedures [20] the irrationality of $\zeta(3)$ is again proved.

Lemma 3.4 *The following relations for the remainder term (15) at $z = 1$ are valid:*

$$r_n = \int_0^1 \frac{G_n(x)}{1-x}dx = \sum_{t=n}^{\infty} R_{n,2}(t) = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} R_{n,1}(\nu)\left(\frac{\pi}{\sin \pi\nu}\right)^2 d\nu, \quad (42)$$

where $G_n(x)$ and $R_{n,1}(\nu)$ are given in (16) and (22), respectively.

Proof. The Cauchy kernel $1/(z-x)$ in the integral expression for the remainder term (15) can be substituted by the expression

$$\frac{1}{x-z} = \frac{1}{2iz}\int\limits_{\Re\nu=-\frac{1}{2}} \left(-\frac{x}{z}\right)^{\nu}\left(\frac{1}{\sin \pi\nu}\right)d\nu, \quad \left|\frac{x}{z}\right| < 1, \quad x \neq 0, \qquad (43)$$

where the vertical line $\Re z = -1/2$ is oriented from top to bottom. Indeed, the right-hand side can be directly computed by expressing conveniently the integrand function, i.e., the product of the complex power $(-x/z)^\nu$ by $1/\sin\pi\nu$, for $\nu = -\frac{1}{2}+ib$, $b \in (-\infty, \infty)$. Observe that the integral converges uniformly for $\epsilon \le x \le 1$, and $z < -1-\epsilon$, $\epsilon \in (0,1)$. Thus,

$$\begin{aligned} r_{n,2}(z) &= -\int_0^1 \left(\frac{1}{2iz} \int\limits_{\Re\nu=-\frac{1}{2}} \left(-\frac{x}{z}\right)^\nu \frac{d\nu}{\sin\pi\nu} \right) G_n(x)dx, \\ &= -\frac{1}{2iz} \int\limits_{\Re\nu=-\frac{1}{2}} \left(\int_0^1 x^\nu G_n(x)dx \right) \left(-\frac{1}{z}\right)^\nu \frac{d\nu}{\sin\pi\nu} \\ &= -\frac{1}{2z\pi i} \int\limits_{\Re\nu=-\frac{1}{2}} R_{n,2}(\nu) \left(-\frac{1}{z}\right)^\nu \left(\frac{\pi d\nu}{\sin\pi\nu}\right), \end{aligned} \tag{44}$$

yields, since $z < -1$ and $0 < x \le 1$.

Now, accomplishing the integrand function in (44) by using the differentiation formula (23) and taking limit when z goes to 1, one gets

$$2q_n\zeta(3) - p_n = r_n = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} R_{n,1}(\nu) \left(\frac{\pi}{\sin\pi\nu}\right)^2 d\nu. \tag{45}$$

Finally, let us show that $r_n = \sum_{t=n}^{\infty} R_{n,2}(t)$, which will complete the proof. Thus, for the evaluation of the above integral one expresses it as a limit of contour integrals along the contour Ω_n that goes along the imaginary line from $-\frac{1}{2}+iL_n$ to $-\frac{1}{2}-iL_n$ and then counterclockwise along a semicircle centered at $-1/2$ from $-\frac{1}{2}+iL_n$ to $-\frac{1}{2}-iL_n$, where the semicircle radius $L_n > n+\frac{1}{2}$. We have taken L_n to be greater than $n+\frac{1}{2}$, so that n singularities of the integrand function are enclosed within the curve. The rational function $R_{n,1}(z) = \mathcal{O}(L_n^{-2})$ on the arc of Ω_n, while the function $(\sin\pi z)^{-1}$ is bounded.

Now, by residue theorem one can compute (45). Indeed,

$$\underset{z=n}{Res}\left(R_{n,1}(z)\left(\frac{\pi}{\sin\pi z}\right)^2\right) = R_{n,2}(n), \quad n = 0, 1, 2\ldots,$$

which can be easily checked by considering the following expansions at the integers

$$R_{n,1}(z) = R_{n,1}(n) + R_{n,2}(n)(z-n) + \mathcal{O}\left((z-n)^2\right),$$
$$\left(\frac{\pi}{\sin\pi z}\right)^2 = \frac{1}{(z-n)^2} + \mathcal{O}(1).$$

Therefore,

$$2q_n\zeta(3) - p_n = r_n = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} R_{n,1}(\nu)\left(\frac{\pi}{\sin\pi\nu}\right)^2 d\nu = \sum_{t=n}^{\infty} R_{n,2}(t),$$

holds. The lemma is completely proved.

Observe that the summation starts at $t = n$ since the rational function $R_{n,2}(t)$ vanishes at $n = 0, 1, \ldots, n-1$.

The above lemma is presented in a constructive way. We have started from expression (15) and concluded with the verification of the right-hand side of expression (28).

Notice that the integration in (42) over a shifted imaginary line $\Re z = -1/2+k$, $k = 1, \ldots, n$, does not alter the result. Indeed, if one integrates over a contour Ω_n shifted to the right along the real axis whenever the imaginary line does not pass by zero and does not exceed the singularity at $z = n+1$ the same result for (42) yields. Thus, lemma 3.4 is in accordance to lemma 1.1.

4 Comparison with some known results

This section comprises some known results leading to the irrationality of $\zeta(3)$. Mostly, these results are related with the Apéry's difference equation

(6) satisfying by the sequence of numbers (numerators and denominators of the rational approximants) that approach $\zeta(3)$ (see (28)-(30)) as well as the remainder sequence $(r_n)_{n\geq 0}$. The method to deduce that sequence $(r_n)_{n\geq 0}$ verifies Apéry's difference equation follows the ideas exposed in [19] and [20]; however the computations below might slightly differ from those.

Observe that from (36) the following relation involving the polynomial $B_n(z)$ and the remainder term $r_{n,2}(z)$ at $z = 1$, holds:

$$\begin{aligned} B_n(1) \int_0^1 \frac{a_n G_{n+1}(x) + c_n G_{n-1}(x)}{1-x} \\ = (a_n B_{n+1}(1) + c_n B_{n-1}(1)) \int_0^1 \frac{G_n(x)}{1-x} dx, \end{aligned} \tag{46}$$

being $a_n = \left(1 + \frac{1}{n}\right)^3 c_n$, and c_n an arbitrary constant.

Clearly, from lemma 3.1 and (46) the verification of the second order difference equation (35) for the sequence $(q_n)_{n\geq 0}$ will implies the verification of the same difference equation for the remainder sequence $(r_n)_{n\geq 0}$, and vice versa. Therefore, if $(q_n)_{n\geq 0}$ satisfies (35) then, $(r_n)_{n\geq 0}$ also verifies it (see equation (37) of lemma 3.3).

Remark 4.1 *If there exists nonzero constants α_n, β_n, γ_n, such that*

$$\alpha_n q_{n+1} + \beta_n q_n + \gamma_n q_{n-1} = 0, \quad \textit{then} \quad \alpha_n r_{n+1} + \beta_n r_n + \gamma_n r_{n-1} = 0,$$

holds. The converse is also true.

Accordingly, it is enough to find the difference equation that is fulfilled by $(q_n)_{n\geq 0}$ or $(r_n)_{n\geq 0}$, indistinctly. Let us firstly prove the verification of (35) for the sequence $(q_n)_{n\geq 0}$ in a more standard way (see [19]), and lastly prove independently of the previous remark that $(r_n)_{n\geq 0}$ also fulfilled Apéry's difference equation.

Proposition 4.1 *There exists non-zero constants α_n, β_n, γ_n, such that*

$$\alpha_n q_{n+1} + \beta_n q_n + \gamma_n q_{n-1} = 0, \quad n = 1, 2, \ldots, \tag{47}$$

holds.

Proof. The equation (47) is equivalent to

$$\sum_{k=0}^{n+1} \left(\alpha_n b_k^{(n+1)} + \beta_n b_k^{(n)} + \gamma_n b_k^{(n-1)} \right) = 0,$$

since $b_j^{(i)} = 0$, for $i < j$. Hence,

$$\alpha_n \binom{n+k+1}{k}^2 \binom{n+1}{k}^2 + \beta_n \binom{n+k}{k}^2 \binom{n}{k}^2 + \gamma_n \binom{n+k-1}{k}^2 \binom{n-1}{k}^2 = f_n(k+1) - f_n(k), \tag{48}$$

such that $f_n(0) = f_n(n+2) = 0$. Indeed, one can define

$$f_n(k) = \frac{k^4 \pi_{2,n}(k) \binom{n+k}{k}^2 \binom{n}{k}^2}{(k-n-1)^2 (k+n)^2}, \tag{49}$$

being $\pi_{2,n}(k) = a_n k^2 + b_n k + c_n$, a polynomial of degree 2 in k, and coefficients depending on n.

From (48) and (49) the following equation

$$\begin{gathered}\alpha_n (n+k+1)^2 (n+k)^2 + \beta_n (n-k-1)^2 (n+k)^2 + \gamma_n (n-k+1)^2 (n-k)^2 \\ = (k-n-1)^2 (k+n)^2 \pi_{2,n}(k+1) - k^4 \pi_{2,n}(k),\end{gathered}$$

yields. This expression leads to a linear system of equations (5 equations for the 6-unknowns: α_n, β_n, γ_n, a_n, b_n, c_n). A particular solution to this system is given by

$$\begin{aligned} &\alpha_n = (n+1)^3, \quad \beta_n = -(2n+1)(17n^2 + 17n + 5), \quad \gamma_n = n^3, \\ &a_n = 8(2n+1), \quad b_n = -12(2n+1), \quad c_n = -16n(2n^2 + 3n + 1), \end{aligned} \tag{50}$$

which proves (47).

In the same fashion that proposition 4.1, one can prove the verification of (35) for the sequence $(r_n)_{n \geq 0}$, see [20].

Proposition 4.2 *There exists non-zero constants* α_n, β_n, γ_n, *such that*

$$\alpha_n r_{n+1} + \beta_n r_n + \gamma_n r_{n-1} = 0, \quad n = 1, 2, \ldots, \tag{51}$$

holds.

Recall that the remainder function $r_{n,2}(z)$ is given by (15), so relation (51) can be written as

$$\int_0^1 \frac{\alpha_n G_{n+1}(x) + \beta_n G_n(x) + \gamma_n G_{n-1}(x)}{1-x} dx = 0,$$

where the constant coefficients are given in (50). For proving the above proposition one needs to use lemma 3.4 and the following result (see [20]):

Lemma 4.1 *There exists non-zero constants* α_n, β_n, γ_n, *such that*

$$\alpha_n R_{n+1,1}(t) + \beta_n R_{n,1}(t) + \gamma_n R_{n-1,1}(t) = F_n(t+1) - F_n(t), \tag{52}$$

where $F_n(t) = 4(2n+1)(-2t^2 - 3t + 4(n^2+n))R_{n,1}(t)$, *holds.*

Proof. Let us assume that

$$\alpha_n R_{n+1,1}(t) + \beta_n R_{n,1}(t) + \gamma_n R_{n-1,1}(t) = F_n(t+1) - F_n(t). \tag{53}$$

For finding the coefficients $\alpha_n, \beta_n, \gamma_n$ as well as function $F_n(t)$ we divide both side of (53) by $R_{n,1}(t)$ and use

$$\frac{R_{n+1,1}(t)}{R_{n,1}(t)} = \frac{(n-t)^2}{(t+n+2)^2}, \quad \frac{R_{n-1,1}(t)}{R_{n,1}(t)} = \frac{(t+n+1)^2}{(t-n+1)^2}.$$

Hence,

$$\alpha_n \frac{(n-t)^2}{(t+n+2)^2} + \beta_n + \gamma_n \frac{(t+n+1)^2}{(t-n+1)^2} = \frac{F_n(t+1) - F_n(t)}{R_{n,1}(t)}.$$

Now, assume that $F_n(t)$ is equal to $P_n(t)R_{1,n}(t)$, where $P_n(t)$ must be determined. Thus,

$$\begin{aligned}&[\alpha_n(n-t)^2+\beta_n(t+n+2)^2]\,(t-n+1)^2+\gamma_n(t+n+1)^2(t+n+2)^2\\&\quad=(t+1)^4P_n(t+1)-(t-n+1)^2(t+n+2)^2P_n(t),\end{aligned}$$

yields, being $P_n(t)=a_nt^2+b_nt+c_n$. This expression leads to a linear system of equations (5-equations for the 6-unknowns coefficients $\alpha_n,\beta_n,\gamma_n,a_n,b_n,$ and c_n). Hence, by solving this system one obtains the unknown coefficients

$$\alpha_n=(n+1)^3,\quad \beta_n=-(2n+1)(17n^2+17n+5),\quad \gamma_n=n^3,$$

as well as the polynomial $P_n(t)=4(2n+1)(-2t^2-3t+4(n^2+n))$. Therefore, equation (52) holds, and the lemma is completely proved.
Proof of proposition 4.2 (Zudilin [20]). Now, differentiating expression (52) and summing the result over $t=0,1,\ldots,$ one gets (51) since for $n\geq 1$ the rational function $R_{n,1}(t)$ has a zero of order 2, and consequently $F_n'(0)=0$.

Remark 4.2 *Using Perron's theorem 2.1 the irrationality of* $\zeta(3)$ *holds (see theorem 3.2 above).*

The converse result of remark 4.1 immediately follows if one multiplies both sides of (52) by $(t+n)^2$, substitutes $t=-n$, and sum over all integers n, i.e., the sequence $(q_n)_{n\geq 0}$ also satisfies the Apéry's difference equation. Consequently, the sequence $(p_n)_{n\geq 0}$ (where $p_n=D_n(1)$, see (26)) also verifies it. However, the direct implication of remark 4.1 only follows from lemma 3.3, therefore $(p_n)_{n\geq 0}$ also satisfies Apéry's difference equation.

In closing, it is enough to indicate the difference equation that satisfies the sequence $(q_n)_{n\geq 0}$ to have the irrationality of $\zeta(3)$ as an intrinsic property of the simultaneous rational approximation problem (lemma 3.3), provided that the roots of the characteristic equation for the aforementioned second order difference equation gives a good approximation. Apéry was totally right when first announced his proof [1] of the irrationality of $\zeta(3)$ by showing the difference equation satisfied by the sequence of integers $\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2$, $n=0,1,\dots$.

4.1 Relation between triple and contour integrals

The relation between the two approaches, namely, complex contour integrals introduced by Nesterenko [10], and triple integrals involving Legendre polynomials considered by Beukers [3] can be easily obtained by using the Mellin convolution

$$(f*g)(x)=\int_1^x f(t)g\left(\frac{x}{t}\right)\frac{dt}{t}.$$

Let us prove that the expression of this convolution coincides with the form $F_n(x)$ given in (16) if one takes $f(z)=g(z)=L_n(z)$, being $L_n(z)$ the n-th Legendre polynomial

$$L_n(z)=\frac{1}{n!}\frac{d^n}{dz^n}z^n(1-z)^n=\sum_{k=0}^{n}l_k^{(n)}x^k,\quad l_k^{(n)}=(-1)^k\binom{n+k}{k}\binom{n}{k},\tag{54}$$

orthogonal with respect to the Lebesgue measure on $(0,1)$, i.e.,

$$F_n(x)=A_n(x)+B_n(x)\log x=(L_n*L_n)(x)=\int_1^x L_n(x/t)L_n(t)\frac{dt}{t}.\tag{55}$$

Indeed,

$$\begin{aligned}(L_n*L_n)(x)&=\int_1^x\left(\sum_{j=0}^{n}l_j^{(n)}\left(\frac{x}{t}\right)^j\right)\left(\sum_{k=0}^{n}l_k^{(n)}t^k\right)\frac{dt}{t}\\&=\sum_{j,k=0}^{n}l_j^{(n)}l_k^{(n)}x^j\int_1^x t^{k-j-1}dx\\&=\sum_{\substack{j,k=0\\k\neq j}}^{n}l_j^{(n)}l_k^{(n)}\frac{x^k-x^j}{k-j}+\log x\sum_{k=0}^{n}\left(l_k^{(n)}\right)^2x^k\\&=\bar{A}_n(x)+B_n(x)\log x.\end{aligned}\tag{56}$$

Now, let us prove that $\bar{A}_n(x) = A_n(x)$, where the coefficients of $A_n(x)$ are given in (25). Thus,

$$\begin{aligned}\int_0^1 (\bar{A}_n(x) + B_n(x)\log x)x^k dx &= \int_0^1 x^k \int_1^x L_n(x/t)L_n(t)\frac{dt}{t}dx \\ &= \int_0^1 L_n(t)\int_t^0 L_n(x/t)x^k dx\frac{dt}{t}.\end{aligned}$$

Changing the variable x by yt one gets the following relation

$$\int_0^1 (\bar{A}_n(x) + B_n(x)\log x)x^k dx = -\int_0^1 L_n(t)t^k dt \int_0^1 L_n(y)y^k dy, \tag{57}$$

which vanishes for $k = 0, 1, \ldots, n-1$, in virtue of the orthogonality of Legendre polynomials (54). Hence, if one subtracts equation (57) from (12)

$$\int_0^1 (A_n(x) - \bar{A}_n(x))x^k dx = 0, \quad k = 0, \ldots, n-1,$$

yields. Since $A_n(x) - \bar{A}_n(x) \in \mathbb{P}_n$ two possible situations appear; either $A_n(x) - \bar{A}_n(x) \equiv 0$, or expression $A_n(x) - \bar{A}_n(x)$ must be equal to $L_n(x)$ up to a multiplicative factor. From (25), and (56) by comparing the leading coefficients follows the first situation. Accordingly, the relation (55) is valid.

Analogously, for getting the second orthogonality condition (14) one uses

$$\int_0^1 x^k \log x \int_1^x L_n(x/t)L_n(t)\frac{dt}{t}dx = \int_0^1 L_n(t)\int_t^0 L_n(x/t)x^k \log x\, dx\frac{dt}{t}.$$

Now, replacing x by yt the following relation

$$\int_0^1 G_n(x)x^k dx = -\int_0^1 L_n(t)t^k \int_0^1 L_n(y)y^k(\log y + \log t)dy\, dt,$$

yields. Since the above integral is symmetric with respect to the variables y and t one has

$$\int_0^1 G_n(x)x^k dx = -2\int_0^1 L_n(t)t^k \log t dt \int_0^1 L_n(y)y^k dy.$$

From the orthogonality relation of the Legendre polynomials when $k = 0, 1, \ldots, n-1$, relation (14) holds.

Notice that from (42) we have

$$\int_0^1 \frac{G_n(x)}{1-x}dx = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} R_{n,1}(\nu)\left(\frac{\pi}{\sin\pi\nu}\right)^2 d\nu.$$

Now, let us transform the left-hand side of the above expression. Thus, from (55) one gets

$$\begin{aligned}\int_0^1 \frac{G_n(\xi)}{1-\xi}d\xi = \int_0^1 \frac{F_n(\xi)\log\xi}{1-\xi}d\xi &= \int_0^1\int_y^0 \frac{\log\xi}{1-\xi}L_n(\xi/y)L_n(y)d\xi\frac{dy}{y}\\ &= -\int_0^1\int_0^1 \frac{\log xy}{1-xy}L_n(x)L_n(y)dxdy,\end{aligned}$$

where variable ξ has been changed by xy.

Considering the following well-known relation

$$\int_0^1 \frac{dv}{1-(1-xy)v} = -\frac{\log xy}{1-xy},$$

one obtains

$$\int_0^1 \frac{G_n(\xi)}{1-\xi}d\xi = \int_0^1\int_0^1\int_0^1 \frac{L_n(x)L_n(y)}{1-(1-xy)v}dxdydv.$$

Using expression (54) in terms of the nth derivative, and integrating by parts n times with respect to x

$$\int_0^1 \frac{G_n(\xi)}{1-\xi}d\xi = \int_0^1\int_0^1\int_0^1 \frac{(xyv)^n(1-x)^nL_n(y)}{(1-(1-xy)v)^{n+1}}dxdydv,$$

yields. Substitute $v = (1-z)/(1-(1-xy)z)$, on the right-hand side, one gets

$$\int_0^1 \frac{G_n(\xi)}{1-\xi}d\xi = \int_0^1\int_0^1\int_0^1 \frac{(1-x)^n(1-z)^nL_n(y)}{1-(1-xy)z}dxdydz.$$

Using once more expression (54), and integrating by parts with respect to y

$$\int_0^1 \frac{G_n(\xi)}{1-\xi}d\xi = \int_0^1\int_0^1\int_0^1 \frac{(xyz(1-x)(1-y)(1-z))^n}{(1-(1-xy)z)^{n+1}}dx\,dy\,dz.$$

yields.

Finally, from lemma 3.4 relation

$$\begin{aligned} &\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} R_{n,1}(\nu)\left(\frac{\pi}{\sin\pi\nu}\right)^2 d\nu \\ &= \int_0^1\int_0^1\int_0^1 \frac{(xyz(1-x)(1-y)(1-z))^n}{(1-(1-xy)z)^{n+1}}dx\,dy\,dz, \end{aligned} \tag{58}$$

holds.

As a consequence the Beukers's integral [3]

$$\mathcal{B}_n = \int_0^1\int_0^1\int_0^1 \frac{(xyz(1-x)(1-y)(1-z))^n}{(1-(1-xy)z)^{n+1}}dx\,dy\,dz, \tag{59}$$

verifies the Apéry's recurrence relation (35), i.e.,

$$(n+1)^3\mathcal{B}_{n+1} - (2n+1)(17n^2+17n+5)\mathcal{B}_n + n^3\mathcal{B}_{n-1} = 0,$$

which allows to compute numerically (59) for n large enough more faster than a naive direct integration.

In closing, notice that from Perron's theorem $\mathcal{B}_n = \mathcal{O}\left((\sqrt{2}-1)^{4n}\right)$. Again, this result is in accordance with the fact that

$$\max_{0\le x,y,z\le 1} \frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z} \le (\sqrt{2}-1)^4.$$

Therefore

$$\begin{aligned} \int_0^1 \frac{G_n(x)}{1-x}dx \le& (\sqrt{2}-1)^{4n}\int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-xy)z}dxdydz \\ &\le (\sqrt{2}-1)^{4n}2\zeta(3), \end{aligned}$$

which was used by Beukers in [3] to prove the irrationality of $\zeta(3)$.

Acknowledgments

The author is grateful to Professors P. Deift and J. Geronimo for several helpful discussions on an earlier draft of this paper. This work was started while the author visited the School of Mathematics at Georgia Institute of Technology (Gatech) and the Courant Institute of Mathematical Sciences at New York University for part of the academic years 2003 and 2007 (during the latest year the author was hired as visiting professor at Gatech). He gratefully acknowledges the aforementioned colleagues for their kind hospitality and numerous mathematical discussions during his stay as well as for stimulating about a Riemann-Hilbert approach for the study of this irrationality problem [2].

References

[1] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque **61** (1979), 11–13.

[2] J. Arvesú, P. Deift, and J. Geronimo, *On a Riemann-Hilbert approach and irrationality of* $\zeta(3)$, manuscript (2008).

[3] F. Beukers, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. London Math. Soc. **11** (1979), 268–272.

[4] F. Beukers, *Legendre polynomials in irrationality proofs*, Bull. Austral. Math. Soc. **22** (1980), 431–438.

[5] F. Beukers, *Padé approximations in number theory*, Padé approximation and its applications, (Amsterdam, 1980), 90–99, Lecture Notes in Math., **888**, Springer, Berlin-New York, 1981.

[6] V. I. Buslaev, *Poincare's theorem and its applications to the convergence of continued fractions*, Sbornik Math. **189** (1998), 1749–1764.

[7] L. A. Gutnik, *On the irrationality of certain quantities involving* $\zeta(3)$, Acta Arith. **42** (1983), 255–264.

[8] M. Hata, *A new irrationality measure for* $\zeta(3)$, Acta Arith. **92** (2000), 47–57.

[9] Y. Jin, and H. Dickinson, *Apéry sequences and Legendre transforms*, J. Austral. Math. Soc. **68** (2000), 349–356.

[10] Yu. V. Nesterenko, *A few remarks on* $\zeta(3)$, Math. Notes **59**, No. 6, (1996), 625–636.

[11] Yu. V. Nesterenko, *Integral identities and constructions of approximations to zeta values*, Journal de Théorie des Nombres de Bordeaux **15** (2003), 535–550.

[12] A. F. Nikiforov, V. B. Uvarov, *Special Functions in Mathematical Physics*, Birkhauser Verlag, Basel, (1988).

[13] E. M. Nikishin, V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, vol. 92, Amer. Math. Soc., Providence, RI, 1991.

[14] O. Perron, *Über ein Satz des Herrn Poincaré*, J. Reine Angew. Math., **136** (1909), 17–37; *Über die Poincarésche lineare Differenzgleichung*, **137** (1910), 6–64.

[15] H. Poincaré, *Sur les équations linéaires aux différentielles et aux différences finies*, Amer. J. Math., **7** (1885), 203–258.

[16] G. Rhin, C. Viola. *The Group Structure for* $\zeta(3)$, Acta Arith. **97** (2001), 269–293.

[17] V. N. Sorokin, *Hermite-Padé approximations for Nikishin systems and the irrationality of* $\zeta(3)$, Communications of the Moscow Math. Soc. (1993), 176–177.

[18] W. Van Assche, *Multiple orthogonal polynomials, irrationality and transcendence*, Contemporary Mathematics **236** (1999), 325–342.

[19] A. Van der Poorten, A proof that Euler missed... Apérys proof of the irrationality of $\zeta(3)$, An informal report, Math. Intelligencer 1 (1978/79), no. 4, 195–203.

[20] W. Zudilin, *An elementary proof of Apéry's theorem*, E-print math. NT/0202159, Moscow Lomonosov State University (2002), 1–8.

[21] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, Journal de Théorie des Nombres de Bordeaux **16** (2004), 251–291.