Approximation of Transfer Functions of Unstable Infinite-Dimensional Control Systems by Rational Interpolants with Prescribed Poles

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Abstract

Rational interpolants, some of the poles of which are left free, are used to approximate the transfer functions of a large class of possibly unstable infinite-dimensional control systems. The free poles are used to detect the singularities of the transfer function which are responsible for the instability of the system.

1 Introduction

In [11], we studied the approximation of transfer functions corresponding to a large class of stable infinite-dimensional control systems using rational interpolants with prescribed poles. Here, we extend those results in order to cover unstable systems whose transfer functions have a finite number of poles. To this end, some poles are left free in the interpolating rational function whose task is to locate the singularities which cause the instability of the system. This leads to the construction of so-called generalized Padé approximants, in the terminology used by A.A. Gonchar in [5]. In recent years (see [1, 2]) the name Padé-type approximant (PTA) has been frequently used to refer to these interpolants. We prefer the more illustrative term of generalized rational interpolants with partially prescribed poles (GRIP^3) suggested in [13].

These interpolation schemes have a long history and had already been considered by Cauchy and Jacobi. In [5], a Montessus de Ballore type theorem (see [6]) was proved for GRIP^3. Here we weaken the conditions imposed in [5] obtaining convergence for more general interpolation schemes. A price is paid for this, in the estimate of the rate of convergence. Our final goal is to approximate meromorphic functions on unbounded sets with an essential singularity at infinity. In this case the restrictions imposed in [5] on the table of interpolation points and the table of fixed poles cannot be achieved.

A large class of transfer functions corresponding to time-delay systems satisfy the conditions we impose. It is well known (see [4]) that any function of the form

$$G(z) = \frac{\sum_1^Q q_j(z)e^{-\alpha_j z}}{p_0(z) + \sum_1^P p_i(z)e^{-\gamma_i z}},$$

where p_i, q_j are polynomials with $\deg p_0 > \deg p_i$, $\deg q_j$, for all $i \neq 0$ and all j, and $\gamma_i, \alpha_j \geq 0$, has only a finite number of poles on any half plane of the form $\{\Re z \geq \rho\}$, $\rho \in \mathcal{R}$.

The paper is divided as folloes. Section 2 contains all basic elements regarding the construction of our interpolation schemes and the different convergence criteria employed. Section 3 is dedicated to the study of convergence on compact subsets of the complex plane whereas in Section 4 the unbounded case is treated. In the final section, we illustrate our method with some numerical examples.

2 Some preliminary notions

2.1 Construction of the rational approximants

Let $\mathcal{P}_n$ be the space of all polynomials of degree at most n. By $\mathcal{R}_{n,m}$ we denote the set of all rational functions $r = p/q$, where $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$.

Let a system of interpolation points $\alpha = \{\alpha_{n,j}\}$, $j = 1, \ldots, n$, $n = 1, 2, \ldots$, and a system of fixed poles $\beta = \{\beta_{n,j}\}$, $j = 1, \ldots, n$, $n = 1, 2, \ldots$, be given. For classical Padé approximants $\alpha_{n,j} \equiv 0$ and $\beta_{n,j} \equiv \infty$. We assume that the tables α and β have no common elements; it is convenient to assume that $\alpha \subset E^o$, $\beta \subset F^o$, where E and F are compact subsets of the extended complex plane $\overline{\mathcal{C}}$ whose interior E^o and F^o satisfy $E^o \cap F^o = \emptyset$. Set

$$a_n(z) = \prod_{j=1}^{n}(z - \alpha_{n,j}), \qquad b_n(z) = \prod_{j=1}^{n}(z - \beta_{n,j}), \qquad n = 1, 2, \ldots.$$

If $\beta_{n,j} = \infty$, the corresponding factor in $b_n(z)$ is substituted by 1. For simplicity we do not allow $\alpha_{n,j} = \infty$.

Let f be holomorphic on E^o. It is easy to see that for each $(n, m, k) \in \mathcal{N}^3$, there exist polynomials p and q ($q \not\equiv 0$) satisfying

$$\deg p \leq n, \quad \deg q \leq m, \qquad \frac{q b_k f - p}{a_{n+m+1}} \in \mathcal{H}(E^o). \tag{2.1}$$

The existence of p and q reduces to the solution of a homogeneous system of $n + m + 1$ linear equations on $n + m + 2$ unknowns (the coefficients of p and q). Therefore, a non-trivial solution exists. Any non-trivial solution corresponds to a $q \not\equiv 0$ (see (2.1)). It is immediate also that any solution of (2.1) defines a unique rational function

$$r_{n,m,k} = \frac{p}{q b_k} \in \mathcal{R}_{n,m+k}. \tag{2.2}$$

In fact, assume that p' and q' ($q' \not\equiv 0$) are such that

$$\deg p' \leq n, \quad \deg q' \leq m, \qquad \frac{q' b_k f - p'}{a_{n+m+1}} \in \mathcal{H}(E^o). \tag{2.3}$$

Multiplying (2.1) times q', (2.3) times q, and deleting one expression thus obtained from the other, it follows

$$\frac{p'q - pq'}{a_{n+m+1}} \in \mathcal{H}(E^o), \qquad \deg(p'q - pq') \leq n + m.$$

But this is possible only if $p'q - pq' \equiv 0$; or what is the same $p/q \equiv p'/q'$, which is what we needed to prove.

Usually the parameters are related. The most common cases are when $m + k = n$ in which case $r_{n,m,k} \in \mathcal{R}_{n,n}$, and $m + k = n + 1$, then $r_{n,m,k} \in \mathcal{R}_{n,n+1}$.

Let D be a region in the complex plane. We denote by $\mathcal{M}_\mu(D)$ the set of all meromorphic functions on D which have μ poles in D (counting their order). If additionally f can be extended continuously to ∂D, we write $f \in \mathcal{CM}_\mu(D)$. Let E be a compact subset of $\overline{D}$ and $f \in \mathcal{H}(E^o) \cap \mathcal{CM}_\mu(D)$; that is, f is holomorphic in E^o, meromorphic in D (with μ poles), and continuous on the boundary of D.

Let us obtain an integral representation for the error $(f - r_{n,m,k})$. Assume that D is a bounded region and its boundary ∂D is formed by a finite number of non-intersecting closed rectifiable Jordan curves. We also denote by ∂D the positively oriented contour which these curves define with respect to D. Take $F = \mathcal{C} \setminus D$, $\alpha \subset E^o$, $\beta \subset F^o$, as above, and $r_{n,m,k} \in \mathcal{R}_{n,m+k}$ defined by (2.1) and (2.2), where $\mu \leq m$. Let $\{z_1, \ldots, z_\mu\}$ be the set of poles of f in D (repeated according to their order), and denote

$$Q(z) = \prod_{i=1}^{\mu} (z - z_i).$$

We have

$$(f - r_{n,m,k})(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{a_{n+m+1}(z)}{a_{n+m+1}(\zeta)} \frac{b_k(\zeta)}{b_k(z)} \frac{q(\zeta)}{(Qq)(z)} \frac{(Qf)(\zeta) d\zeta}{\zeta - z}, \qquad z \in D. \tag{2.4}$$

In this formula, equality is understood as such for $z \in D \setminus \{z : (Qq)(z) = 0\}$. When $z \in D \cap \{z : (Qq)(z) = 0\}$ both sides are equal to infinity. This integral expression is very useful. It is a direct consequence of Cauchy's integral formula. To see this, notice that according to the assumptions on f and the definition of $r_{n,m,k}$, we have that

$$\frac{qb_kQf - Qp}{a_{n+m+1}} \in \mathcal{H}(D) \cap \mathcal{C}(\overline{D}), \qquad \deg Qp \leq n + \mu < n + m + 1.$$

Hence, from Cauchy's integral formula

$$\begin{aligned} \left(\frac{qb_kQf - Qp}{a_{n+m+1}}\right)(z) &= \frac{1}{2\pi i} \int_{\partial D} \frac{(qb_kQf - Qp)(\zeta)}{a_{n+m+1}(\zeta)} \frac{d\zeta}{\zeta - z} \qquad (2.5) \\ &= \frac{1}{2\pi i} \int_{\partial D} \frac{(qb_kQf)(\zeta)}{a_{n+m+1}(\zeta)} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\partial D} \frac{(Qp)(\zeta)}{a_{n+m+1}(\zeta)} \frac{d\zeta}{\zeta - z}, \qquad z \in D. \end{aligned}$$

The boundary of D, is formed by an outer boundary curve Γ_0 and possibly some inner boundary curves $\Gamma_1, \ldots, \Gamma_N$ (there may be no inner boundary curves if D is simply connected). It is easy to see that

$$\frac{1}{2\pi i} \int_{\Gamma_i} \frac{(Qp)(\zeta)}{a_{n+m+1}(\zeta)} \frac{d\zeta}{\zeta - z} = 0, \qquad i = 0, 1, \ldots, N. \tag{2.6}$$

In fact, for $i = 0$, we have that $(Qp)(\zeta)/a_{n+m+1}(\zeta)(\zeta - z)$, $z \in D$ fixed, is holomorphic with respect to ζ in the unbounded component of $\mathcal{C} \setminus \Gamma_0$ and at infinity has a zero of degree at least two (since $\deg Qp < n + m + 1$). Therefore, by Cauchy's theorem, (2.6) takes place when $i = 0$. If $i = 1, \ldots, N$, then $(Qp)(\zeta)/a_{n+m+1}(\zeta)(\zeta - z)$, $z \in D$, is holomorphic in ζ in the bounded component of $\mathcal{C} \setminus \Gamma_i$, and Cauchy's theorem applied in this case on a bounded region gives us again (2.6). Thus, the second integral in the right-hand of (2.5) equals zero. Rearranging what is left of formula (2.5), we get (2.4).

For more information on these generalized rational interpolants with partially prescribed poles we recommend [13].

2.2 Types of convergence

2.2.1 Convergence in capacity

We are concerned with the approximation of f by sequences of rational functions of the form p/qb_k. Our main instrument of work is formula (2.4). We are interested in proving uniform convergence on compact subsets of $\overline{D} \setminus \{z : Q(z) = 0\}$, but in order to achieve this, in estimating from above the integral in (2.4), one runs into trouble with the zeros of the polynomials q whose location is not known in advance and may fall on the set one desires to approximate. The procedure used is the following. First, one proves convergence in a weaker sense; namely, local uniform convergence in capacity on the region. This type of convergence allows to determine the asymptotic behavior of the free poles of our approximants (zeros of q). Once the free poles are under control one proceeds to prove uniform convergence on compact subsets of the closed region (see Lemmas 2.3 and 2.6 below).

There are many equivalent ways of defining the logarithmic capacity of a compact set, the most commonly used are through the transfinite diameter (see [10], § 5.5), the Tchebyshev constant ([10], § 5.5), the Robin constant ([10], § 5.1), and minimal energy ([7], § 5.3). For definiteness, we give one definition and urge the reader to look in the references above if more details are needed. The properties we use of the capacity are all proved in [10].

Definition 2.1 *Let K be a compact subset of $\mathcal{C}$, and for each $n \geq 1$, define*

$$m_n(K) = \inf \{\|q\|_K : q(z) = z^n + \ldots\}.$$

The capacity of K is defined as

$$\mathtt{cap}(K) = \inf_{n \geq 1} m_n^{1/n}(K).$$

Mainly, we need two properties of the capacity. First, its monotonicity; that is, $K_1 \subset K_2$, implies that $\mathtt{cap}(K_1) \leq \mathtt{cap}(K_2)$, which is immediate from the definition above. Secondly, the capacity of a lemniscate. Let $h(z) = z^m + \ldots$ be a monic polynomial of degree $m \geq 1$ and $R > 0$, then

$$\mathtt{cap}\{z : |h(z)| \leq R\} = R^{1/m}. \tag{2.7}$$

The proof of this is easiest using the definition of capacity through Robin's constant and Green's function (see [10], § 5.2). Nevertheless, for our purpose, the inequality $\mathtt{cap}\{z : |h(z)| \leq R\} \leq R^{1/m}$ is sufficient. Using the definition above we give a simple proof of this inequality.

In fact, let $n = lm + s,\ s \in \{0, \ldots, m-1\}$. Then, for $K = \{z : |h(z)| \leq R\}$

$$m_n(K) \leq \sup_{z \in K} \left| h^l(z)(z - z_0)^s \right| \leq R^l \sup_{z \in K} |z - z_0|^s,$$

where z_0 is any fixed point of K. Obviously, since K is a compact subset of the complex plane and s only takes a finite number of values, there exists a constant C such that

$$\sup_{z \in K} |z - z_0|^s \leq C, \qquad s = 0, \ldots, m-1.$$

Therefore,

$$m_n(K)^{1/n} \leq R^{l/n} C^{1/n}, \qquad n = 1, 2, \ldots.$$

Since $\lim_{n\to\infty} R^{l/n} C^{1/n} = R^{1/m}$, we get

$$\mathtt{cap}(K) = \inf_{n \geq 1} m_n(K)^{1/n} \leq R^{1/m}.$$

Definition 2.2 *Let $K \subset \mathcal{C}$ be a compact set. A sequence of functions $\{f_n\}$ is said to converge in capacity to f on K if for each $\epsilon > 0$,*

$$\mathtt{cap}\{z \in K : |(f - f_n)(z)| \geq \epsilon\} \to 0 \qquad \text{as } n \to \infty.$$

The sequence $\{f_n\}$ is said to converge locally uniformly in capacity to f on a region D if it converges in capacity on each compact subset of D.

The following Lemma of A.A. Gonchar (see [5], Lemma 1) is most useful for studying the convergence of sequences of interpolating rational functions.

Lemma 2.3 *Suppose that the sequence $\{f_n\}$ converges locally uniformly in capacity to f in a region D. We have:*

1. *If the functions f_n are holomorphic in D (for all sufficiently large n) then the sequence $\{f_n\}$ converges to f uniformly on each compact subset of D and consequently the function f is holomorphic in D (more precisely, it is equivalent in capacity to such a function).*

2. *If each of the functions f_n is meromorphic in D and has at most μ ($< \infty$) poles there, then the limit function is also meromorphic with at most μ poles in D.*

3. *If each of the functions f_n is meromorphic in D and has at most μ poles in D and f has exactly μ poles in D, then for all sufficiently large n, the functions f_n also have exactly μ poles in D, the poles of the functions f_n tend to the poles $z_1, \ldots, z_\mu$ of f (taking account of their multiplicities) and the sequence $\{f_n\}$ converges uniformly to f on compact subsets of $D \setminus \{z_1, \ldots, z_\mu\}$.*

From the second statement of this Lemma it is immediate that if $z_0 \in D$ is a pole of f of order μ_0 and $\{f_n\}$ converges to f locally uniformly in capacity to f then for each $\epsilon > 0$, there exists an n_0 such that for all $n \geq n_0$, the number of poles of f_n in $\{z : |z - z_0| < \epsilon\}$ is at least μ_0. If this was not so there would be a set of indexes Λ, such that for all $n \in \Lambda$ the number of poles of f_n would be at most $\mu_0 - 1$ and according to the second statement applied to $\{f_n\}$, $n \in \Lambda$, on $D = \{z : |z - z_0| < \epsilon\}$, the function f could have at most a pole of order $\mu_0 - 1$ at z_0. For short, we say that each pole of f "attracts" at least as many poles of f_n as its order. Under the restrictions of the third statement we have that each pole of f "attracts" exactly as many poles of f_n as its order.

2.2.2 Convergence in the spherical metric

In the study of the convergence of sequences of meromorphic functions it is more natural to consider a topology in the Riemann sphere in order to define a (finite) distance to the point infinity. On one hand, this allows us to include the infinity in the domain of definition of our functions (thus consider unbounded domains) and on the other we can define convergence on compact sets which include the poles of the limiting function. In this regard, the spherical metric plays a central role.

Given two points $a,\ b \in \hat{\mathcal{C}}$, denote by $\kappa(a,b)$ the *spherical* (also called *chordal*) distance between a and b:

$$\begin{aligned}\kappa(a,b) &= \frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}, \qquad a,\ b \text{ bothfinite},\\ \kappa(a,\infty) &= \kappa(\infty,a)=\frac{1}{\sqrt{1+|a|^2}}.\end{aligned}$$

We denote by $(\hat{\mathcal{C}},\kappa)$ the set $\mathcal{C}\cup\{\infty\}$, equipped with the spherical metric.

Definition 2.4 *The spherical distance between two functions f and g defined on a set $K\subset\hat{\mathcal{C}}$ with image in $(\hat{\mathcal{C}},\kappa)$ is given by*

$$d_K(f,g)=\sup_{z\in K}\{\kappa(f(z),g(z))\}.$$

A sequence of functions $\{f_n\}$ is said to converge uniformly to f on K in the spherical metric, if

$$\lim_{n\to\infty} d_K(f_n,f)=0.$$

The sequence is said to converge locally uniformly to f on D in the spherical metric if it converges uniformly to f in the spherical metric on each compact subset K of D.

Lemma 2.5 *Let $\{f_n\}$ converge uniformly to f on K with respect to the spherical metric, and let $0<r<\infty$. Then*

1. *$\{f_n\}$ converges uniformly to f on $\{z\in K:|f(z)|\le r\}$.*
2. *$\{1/f_n\}$ converges uniformly to $1/f$ on $\{z\in K:|f(z)|\ge r\}$.*

Conversely, if there exist $A,\ B\subset K$ such that $\{f_n\}$ converges uniformly to f on A, $\{1/f_n\}$ converges uniformly to $1/f$ on B, and $K\subset A\cup B$, then $\{f_n\}$ converges uniformly to f on K with respect to the spherical metric.

Proof: If $r_1>r$, then $r_1/\sqrt{1+r_1^2}>r/\sqrt{1+r^2}$. We have that for all sufficiently large n, $d_K(f_n,f)<r_1/\sqrt{1+r_1^2}-r/\sqrt{1+r^2}$. Thus, it follows that

$$d_K(f_n,0)\le d_K(f_n,f)+d_K(f,0)\Rightarrow\frac{|f_n(z)|}{\sqrt{1+|f_n(z)|^2}}\le\frac{r_1}{\sqrt{1+r_1^2}},\qquad z\in\{|f(z)|\le r\},$$

that is, $|f_n(z)|\le r_1$ on $\{z\in K:|f(z)|\le r\}$. Then Lemma 2.5(1) follows from the inequality

$$\kappa(w_1,w_2)\ge\frac{|w_1-w_2|}{1+a^2}\qquad\text{for}\quad |w_1|\le a,\ |w_2|\le a.$$

Analogously, it can be shown that if $r_2 < r$ then $|f_n| \geq r_2$ on $\{|f(z)| \geq r\}$ for all sufficiently large n, and Lemma 2.5(2) follows from

$$\kappa(w_1, w_2) \geq \frac{a^2}{1+a^2}\left|\frac{1}{w_1} - \frac{1}{w_2}\right| \qquad \text{for} \quad |w_1| \geq a,\ |w_2| \geq a.$$

The other statement follows from the inequality

$$\kappa(w_1, w_2) \leq \min\left(|w_1 - w_2|, \left|\frac{1}{w_1} - \frac{1}{w_2}\right|\right).$$

♠

It is obvious that convergence with respect to the spherical metric of meromorphic functions implies convergence in capacity. The following converse is also true:

Lemma 2.6 *Suppose that the sequence $\{f_n\}$ converges locally uniformly in capacity to f in the region D. If each of the functions f_n is meromorphic in D and has at most μ poles in D and f has exactly μ poles in D, then for all sufficiently large n, the functions f_n also have exactly μ poles in D, the poles of the functions f_n tend to the poles $z_1, \ldots, z_\mu$ of f (taking account of their multiplicities) and the sequence $\{f_n\}$ converges locally uniformly to f in D in the spherical metric.*

Proof: In view of Lemmas 2.3 and 2.5, all that we need to prove is that $\{1/f_n\}$ converges uniformly to $1/f$ in some neighborhood of $\{z_1, \ldots, z_\mu\}$.

Let $Q(z) = \prod_{j=1}^{\mu}(z - z_j)$. From the conditions on f we know that Qf is holomorphic in D and different from zero at each point $\{z_1, \ldots, z_\mu\}$. Thus there exists a constant $c > 0$ and a compact neighborhood V of $\{z_1, \ldots, z_\mu\}$ contained in D such that

$$|Q(z)f(z)| > c, \qquad z \in V. \tag{2.8}$$

Let $Q_n(z) = \prod_{j=1}^{\mu_n}(z - z_v)$, where $\{z_{n,1}, \ldots, z_{n,\mu_n}\}$ is the set of poles of f_n in D (repeated according to their order). By Lemma 2.3, we have that the zeros of Q_n converge to the zeros of Q. Since all these polynomials are of degree $\leq n$, and in a space of finite dimension all the norms are equivalent, it follows that

$$\lim_n Q_n(z) = Q(z) \tag{2.9}$$

locally uniformly on $\mathcal{C}$.

Since ∂V is a compact subset of $D \setminus \{z_1, \ldots, z_\mu\}$, Lemma 2.3 also indicates that $\lim_{n\to\infty} f_n = f$ uniformly on ∂V. Therefore, using (2.9) we have that

$$\lim_n (Q_n f_n)(z) = (Qf)(z) \tag{2.10}$$

uniformly on ∂V. Since the functions $Q_n f_n$ and Qf are holomorphic and not equal to zero on V, from (2.10) and (2.8) we obtain that $\lim_n 1/(Q_n f_n)(z) = 1/(Qf)(z)$ uniformly on ∂V and by use of the maximum principle

$$\lim_n \frac{1}{(Q_n f_n)(z)} = \frac{1}{(Qf)(z)} \tag{2.11}$$

uniformly on V. From (2.9) and (2.11) we conclude that $\{1/f_n\}$ converges uniformly to $1/f$ on V as we needed to prove. ♠

2.3 Conditions on the tables α and β

Looking at (2.4) it is obvious that the key element we can rely on for proving convergence to zero of the integrals, say as $n \to \infty$, is that given by the factor $a_{n+m+1}(z)/a_{n+m+1}(\zeta) \cdot b_k(\zeta)/b_k(z)$. We distinguish two cases. The first when the compact set K on which we want to prove convergence is at a distance greater than zero from ∂D (that is $K \subset D$). The other is when $K \cap \partial D \neq \emptyset$. The first case is fairly easy to treat. Let us illustrate with some examples.

Let $D = [|z| < 1]$ and $K \subset [|z| \leq r]$, $r < 1$. Take $\alpha_{n,j} = 0$, $\beta_{n,j} = \infty$ for each $n = 1, \ldots,$ and $j = 1, \ldots, n$. Then

$$\left| \frac{a_{n+m+1}(z)}{a_{n+m+1}(\zeta)} \frac{b_k(\zeta)}{b_k(z)} \right| = \left| \frac{z^{n+m+1}}{\zeta^{n+m+1}} \right| \leq r^{n+m+1} \to 0, \qquad \text{as } n \to \infty$$

for $z \in K$ and $\zeta \in \partial D$. A more elaborated example is the following. Take $\alpha \subset D$ such that

$$\lim_n \sum_{j=1}^{n} (1 - |\alpha_{n,j}|) = \infty. \tag{2.12}$$

That is, the points $\alpha_{n,j}$ are even allowed to approach the boundary of D but not "too fast on the average". Take $k = n - m$ and

$$\beta_{n-m,j} = \frac{1}{\overline{\alpha}_{n+m+1,j}}, \qquad j = 1, \ldots, n - m. \tag{2.13}$$

If $|\alpha_{n+m+1,j}| \leq r' < 1$, $j = n - m + 1, \ldots, n + m + 1$, then

$$\left| \frac{a_{n+m+1}(z)}{a_{n+m+1}(\zeta)} \frac{b_k(\zeta)}{b_k(z)} \right| \leq \frac{(r + r')^{2m}}{(1 - r')^{2m}} \prod_{j=1}^{n-m} \left| \frac{z - \alpha_{n+m+1,j}}{1 - \overline{\alpha}_{n+m+1,j} z} \right| \rightrightarrows 0, \qquad \text{as } n \to \infty$$

because of (2.12) (see [14], § 9.6). (Notice that $|(\zeta - \alpha_i)/(1 - \overline{\alpha}_i \zeta)| = 1$, for $|\zeta| = 1$.)

In the sequel m will be fixed greater or equal to μ (the number of poles of f in D). The parameter k will be taken dependent of n so that $n - m - k$ is constant. If $n - m - k = 0$ or < 0 and all $\beta_{n,k} \neq \infty$, then the rational functions $r_{n,m,k}$ are proper or strictly proper respectively. Take

$$A_n(z, \zeta) = \frac{a_{n+m+1}(z)}{a_{n+m+1}(\zeta)} \frac{b_k(\zeta)}{b_k(z)}.$$

We say that $(\alpha, \beta) \in \mathcal{R}(D)$ if additional to the restrictions imposed on the tables in Subsection 2.1 they satisfy that

$$A_n(z, \zeta) \rightrightarrows_n 0, \qquad \text{as } n \to \infty$$

uniformly for $(z, \zeta) \in K \times \partial D$ where K is an arbitrary compact subset of D. Different constructions of tables α, β such that $(\alpha, \beta) \in \mathcal{R}(D)$ may be found in [14], §§ 8.4, 8.7-8.8, and [3].

A more delicate situation arises when the compact K intersects ∂D. The problem here is two-fold. On one hand, the kernel $(\zeta - z)^{-1}$ in (2.4) has a singularity on ∂D if $z \in \partial D$, on the other it is not easy to obtain tables (α, β) such that $A_n(z, \zeta) \rightrightarrows_n 0$, as $n \to \infty$, on $K \times \partial D$ if $K \cap \partial D \neq \emptyset$. These problems are handled assuming that f is sufficiently "weak" on $K \cap \partial D$. More precisely, in this situation we will consider that f is such that for each $\delta > 0$, there exists a neighborhood V_δ of $K \cap \partial D$ such that

$$\frac{1}{2\pi} \int_{V_\delta \cap \partial D} \left| \frac{f(\zeta)}{\zeta - z} \right| |d\zeta| < \delta, \qquad z \in K. \tag{2.14}$$

Concerning the tables α, β, here we will assume not only that $(\alpha, \beta) \in \mathcal{R}(D)$ but also that the sequence $\{A_n(z, \zeta)\}$ converges uniformly to zero on $K \times (\partial D \setminus V_\delta)$ for all sufficiently small $\delta > 0$ and that it be uniformly bounded on $K \times \partial D$. We resume these conditions on f and (α, β), given K such that $K \cap \partial D \neq \emptyset$, saying that $(f; \alpha, \beta) \in \mathcal{R}(K, D)$. For examples where $(f; \alpha, \beta) \in \mathcal{R}(K, D)$ see [11]; the following example was taken from [11].

Let K be the closed unit disk $\{z : |z| \leq 1\}$, let D be a region containing $K \setminus \{-1\}$ in its interior (with $-1 \in \partial D$) such that ∂D is non-tangential to K at $z = -1$; more precisely, there exists a neighborhood V of $\{-1\}$ such that

$$\partial D \cap V \subset \{z : | \arg(-(z+1)) | \leq \theta\} \tag{2.15}$$

for some $\theta \in [0, \pi/2)$. Assume that $\alpha \subset \{|z| < 1\}$ and that (α, β) satisfies (2.12) and (2.13). If $f \in \mathcal{CM}(D)$ is such that $|f(z)| = O((z+1)^\omega)$ for some $\omega > 0$, then $(f; \alpha, \beta) \in \mathcal{R}(K, D)$.

To see this, first note that (2.15) and the above condition on f imply (2.14). Because of $|(z - \alpha_i)/(1 - \overline{\alpha}_i z)| = 1$, for $|z| = 1$, and $|(\zeta - \alpha_i)/(1 - \overline{\alpha}_i \zeta)| = 1$, for $|\zeta| = 1$, we have that $\{A_n(z, \zeta)\}$ is uniformly bounded on $K \times \partial D$, and uniformly convergence to zero on $K \times (\partial D \setminus V_\delta)$ follows as before because of (2.12) (see [14], § 9.6).

3 Convergence results

3.1 Generalized rational interpolants with partially prescribed poles (GRIP^3)

As pointed out in Subsection 2.2, our first objective is to prove convergence in capacity.

Theorem 3.1 *Let D be a bounded region in the complex plane such that ∂D consists of a finite number of closed rectifiable Jordan curves. Let $f \in \mathcal{CM}_\mu(D) \cap H(E^o)$ and $(\alpha, \beta) \in \mathcal{R}(D)$, $\alpha \subset E^o$. Fix $m \geq \mu$ and let k be such that $n - m - k$ is constant for each $n = 1, 2, \ldots$. Then the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$, converges locally uniformly in capacity to f on D. Moreover, if K is a compact subset of $\overline{D}$ such that $K \cap \partial D \neq \emptyset$ and $(f; \alpha, \beta) \in \mathcal{R}(K, D)$, then the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$ converges uniformly in capacity to f on K.*

Proof: For each n, m, k the polynomials p and q appearing in the numerator and denominator of $r_{n,m,k}$ both depend on the three indexes, but m is fixed (for all n) and $n - m - k$ is constant for each n; therefore, the main index is n and to simplify the notation we write $r_{n,m,k} = p_n / b_k q_n$ (notice that $\deg p_n \leq n$, $\deg q_n \leq m$, for each n). We normalize q_n to have leading coefficient equal to 1.

Let us prove the first statement. We have that $(\alpha, \beta) \in \mathcal{R}(D)$. Fix $K \subset D$. Our main concern is with the zeros of q_n which may fall on the compact set K. Let $R > 0$ be sufficiently large so that $\overline{D} \subset \{z : |z| < R\}$. Let us write

$$q_n(z) = \prod^{\sim}(z - z_{n,j}) \prod^{\approx}(z - z_{n,j}) = \widetilde{q}_n(z) \,\widetilde{\widetilde{q}}_n(z),$$

where $\prod^{\sim}$ is the product taken over all the zeros of q_n that have module less than $2R$ and $\prod^{\approx}$ is the product over all the zeros of q_n of module greater or equal to $2R$ (thus, cannot lie on K).

For $z \in K$ and $\zeta \in \partial D$, we have

$$\left|\frac{\widetilde{\widetilde{q}}_n(\zeta)}{\widetilde{\widetilde{q}}_n(z)}\right| = \prod^{\approx}\left|\frac{\frac{\zeta}{z_{n,j}} - 1}{\frac{z}{z_{n,j}} - 1}\right| \leq \prod^{\approx}\frac{1 + \frac{R}{2R}}{1 - \frac{R}{2R}} \leq 3^m \tag{3.1}$$

since $\deg \widetilde{\widetilde{q}}_n \leq m$. On the other hand,

$$|\widetilde{q}_n(\zeta)| \leq \prod^{\sim}(|\zeta| + |z_{n,j}|) \leq \prod^{\sim}(3R) \leq \max(1, (3R)^m). \tag{3.2}$$

Therefore, using (3.1) and (3.2), we see that there exists a constant C_1 independent of n and $K \subset D$ such that

$$\left|\frac{q_n(\zeta)}{q_n(z)}\right| \leq \frac{C_1}{|\widetilde{q}_n(z)|}, \qquad z \in K,\ \zeta \in \partial D. \tag{3.3}$$

We wish to point out, for future use, that (3.3) also holds in the case when $K \cap \partial D \neq \emptyset$.

Using (2.4), we find that

$$|(f - r_{n,m,k})(z)| \leq \frac{1}{2\pi}\int_{\partial D} |A_n(z,\zeta)| \left|\frac{q_n(\zeta)}{q_n(z)Q(z)}\right| \frac{|(Qf)(\zeta)|}{|\zeta - z|}|d\zeta|, \qquad z \in D. \tag{3.4}$$

The distance from ∂D to K is positive, the length of ∂D is finite, and Qf is continuous on ∂D. From all this and (3.3), we find that there exists a finite constant C_2 such that for all $n = 1, 2, \ldots$,

$$|(f - r_{n,m,k})(z)| \leq \frac{C_2}{|(Q\,\widetilde{q}_n)(z)|}\sup\{|A_n(z,\zeta)| : z \in K,\ \zeta \in \partial D\} = \frac{C_2\theta_n}{|(Q\,\widetilde{q}_n)(z)|}, \tag{3.5}$$

where $\lim_n \theta_n = 0$, since $(\alpha, \beta) \in \mathcal{R}(D)$. If for a certain set of indexes $\Lambda \subset \mathcal{N}$, $\deg(Q\,\widetilde{q}_n) = 0$, $n \in \Lambda$, then (3.5) already implies uniform convergence on K for $n \in \Lambda$. Therefore, we can assume in the following that $\deg(Q\,\widetilde{q}_n) \geq 1$.

Because of (3.5), the monotonicity of the capacity, and (2.7), we have that for each $\epsilon > 0$

$$\begin{aligned}
\mathsf{cap}\{z \in K : |(f - r_{n,m,k})(z)| \geq \epsilon\} &\leq \mathsf{cap}\left\{z \in \mathcal{C} : \frac{C_2\theta_n}{|Q\,\widetilde{q}_n(z)|} \geq \epsilon\right\} \\
&= \mathsf{cap}\left\{z \in \mathcal{C} : |Q\,\widetilde{q}_n(z)| \leq \frac{C_2\theta_n}{\epsilon}\right\} \\
&= \left(\frac{C_2\theta_n}{\epsilon}\right)^{1/\deg(Q\widetilde{q}_n)}.
\end{aligned} \tag{3.6}$$

Using the fact that $1 \leq \deg Q \stackrel{\sim}{q}_n \leq 2m$, that $\lim_n \theta_n = 0$, and the arbitrariness of $K \subset D$, we obtain the local uniform convergence in capacity of the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$, to f in D.

Now, let $K \cap \partial D \neq \emptyset$, $K \subset \overline{D}$, and $(f; \alpha, \beta) \in \mathcal{R}(K, D)$. Fix $\delta > 0$ and take V_δ a neighborhood of $K \cap \partial D$ such that (2.14) takes place. We can rewrite (3.4) as follows:

$$\begin{aligned} |(f - r_{n,m,k})(z)| \;&\leq\; \frac{1}{2\pi} \int_{\partial D \cap V_\delta} |A_n(z,\zeta)| \left| \frac{q_n(\zeta)}{q_n(z) Q(z)} \right| \frac{|(Qf)(\zeta)|}{|\zeta - z|} |d\zeta| + \\ &+\; \frac{1}{2\pi} \int_{\partial D \setminus V_\delta} |A_n(z,\zeta)| \left| \frac{q_n(\zeta)}{q_n(z) Q(z)} \right| \frac{|(Qf)(\zeta)|}{|\zeta - z|} |d\zeta|. \end{aligned}$$

Using the same arguments we used to derive (3.5), and that $(f; \alpha, \beta) \in \mathcal{R}(K, D)$, we obtain

$$|(f - r_{n,m,k})(z)| \leq \frac{C_3(\theta_n(\delta) + \delta)}{|Q \stackrel{\sim}{q}_n(z)|}, \qquad z \in K, \tag{3.7}$$

where the bound on the first integral is obtained from (2.14) and the uniform boundedness of $\{A_n(z,\zeta)\}$ on $K \times \partial D$, and the estimate on the second integral is based on the uniform convergence of $\{A_n(z,\zeta)\}$ to zero on $K \times (\partial D \setminus V_\delta)$. Here, $\theta_n(\delta) = \sup\{|A_n(z,\zeta)| : (z,\zeta) \in K \times (\partial D \setminus V_\delta)\}$.

If for some subsequence of indexes Λ, $\deg(Q \stackrel{\sim}{q}_n) = 0$, then (3.7) indicates that

$$\limsup_{n \in \Lambda} \|(f - r_{n,m,k})(z)\|_K \leq \delta$$

for each $\delta > 0$, where $\|\cdot\|_K$ denotes the supremum norm on K. Since $\delta > 0$ is arbitrary there is uniform convergence to zero of this subsequence. We can assume then that $\deg(Q \stackrel{\sim}{q}_n) \geq 1$. Fix $\epsilon > 0$, from the monotonicity of the capacity, (2.7), and (3.7), we obtain

$$\mathtt{cap}\,\{z \in K : |(f - r_{n,m,k})(z)| \geq \epsilon\} \leq \left(\frac{C_3(\theta_n(\delta) + \delta)}{\epsilon} \right)^{1/\deg(Q \stackrel{\sim}{q}_n)}.$$

Since $\lim_n \theta_n(\delta) = 0$, it follows that

$$\limsup_n \mathtt{cap}\,\{z \in K : |(f - r_{n,m,k})(z)| \geq \epsilon\} \leq \left(\frac{C_3 \delta}{\epsilon} \right)^{1/2m}$$

for each $\delta > 0$ such that $C_3\delta/\epsilon < 1$. The left hand does not depend on δ; therefore, making $\delta \to 0$, we have

$$\limsup_n \mathtt{cap}\,\{z \in K : |(f - r_{n,m,k})(z)| \geq \epsilon\} = 0$$

for each $\epsilon > 0$. Thus, there is uniform convergence in capacity on K. ♠

We wish to point out that in the proof of Theorem 3.1 no use is made of the fact that the sequence of indexes n runs through the whole set of natural numbers. Therefore, the same results remain in force if the conditions on the tables (α, β) hold true for $n \in \Lambda \subset \mathcal{N}$ and the limit is taken along the sequence of indexes Λ. Moreover, we may allow the parameter m to vary between μ and $\mu + c_1$, where c_1 is a fixed natural number, as well as k to be such that $c_2 \leq n - m - k \leq c_3$, where c_2 and c_3 are two fixed integers.

The proof reduces to applying Theorem 3.1 (taking account of the previous remark) to a finite number of subsequences of indexes n.

The first statement of Theorem 3.1 is mainly due to A.A. Gonchar. In [5], Theorems 1 and 3, he proved an analogous result for so called extremal pairs of tables (α, β). The extremality of (α, β) implies that $(\alpha, \beta) \in \mathcal{R}(D)$. In essence, extremality means that the pair (α, β) is such that

$$\lim_{n\to\infty} \left|\frac{a_n(z)}{b_n(z)}\right|^{1/n} = \lambda \exp\left(\frac{H(z)}{c}\right)$$

locally uniformly on $D \setminus E$, where $\lambda > 0$, $H(z)$ is the harmonic function on $D \setminus E$ which takes the constant value 0 on ∂E and the value 1 on ∂D (also known as the harmonic measure of $D \setminus E$), and $c > 0$ is the so called Green capacity of E with respect to D.

For extremal pairs of tables (α, β) the value θ_n which appears in (3.5) satisfies that $\limsup_n \theta_n^{1/n} \leq q < 1$. This allows to consider sequences of GRIP[3] such that the number of free poles m_n tends to infinity as long as $\lim_n m_n/n = 0$ (see (3.6)).

Results such as Theorem 3.1 and those in [5] are known as Montessus de Ballore type theorems (see [6]). They extend in one way or another the original result of Montessus de Ballore relative to the convergence of rows of classical Padé approximants where interpolation is carried at zero, the fixed poles are taken at infinity and only a fixed finite number of poles of the approximants are left free.

Now we are ready to prove convergence in the spherical metric.

Corollary 3.2 *Let D be a bounded region in the complex plane such that ∂D consists of a finite number of closed rectifiable Jordan curves. Let $f \in \mathcal{CM}_\mu(D) \cap H(E^o)$ and $(\alpha, \beta) \in \mathcal{R}(D)$, $\alpha \subset E^o$. Fix $m = \mu$, and take k such that $n - m - k$ is constant for each $n = 1, 2, \ldots$. Then, the sequence $\{r_{n,\mu,k}\}$, $n = 1, 2, \ldots$ converges locally uniformly to f on D in the spherical metric. Moreover, if K is a compact subset of $\overline{D}$ such that $K \cap \partial D \neq \emptyset$ and $(f; \alpha, \beta) \in \mathcal{R}(K, D)$, then the sequence $\{r_{n,\mu,k}\}$, $n = 1, 2, \ldots$, converges uniformly to f on K in the spherical metric.*

Proof: The first statement follows directly from Theorem 3.1 and Lemma 2.6. In order to prove the second statement we proceed as follows. Let V be a compact neighborhood of the set $\{z_1, \ldots, z_\mu\}$ made up of the poles of f in D. Take V contained in D. Then $K = (K \cap V) \cup (K \setminus V^o)$. The compact set $K \cap V$ is contained in D; therefore, by the first part of the theorem, we have that on $K \cap V$ there is uniform convergence to f in the spherical metric. We must show that on $K \setminus V^o$ there is also uniform convergence to f in the spherical metric. Since $K \setminus V^o$ does not contain poles of f, on this set uniform convergence in the spherical metric is equivalent to uniform convergence. Therefore, it rests to show that this last assertion is true.

Take n_0 such that for $n \geq n_0$ the distance d from $K \setminus V^o$ to all the poles of $r_{n,\mu,k}$ is greater than half the distance between $K \setminus V^o$ and the set $\{z_1, \ldots, z_\mu\}$. The existence of such n_0 is guaranteed by Lemma 2.6. With this information at hand and (3.7), we get

$$|(f - r_{n,m,k})(z)| \leq \frac{C_3(\theta_n(\delta) + \delta)}{(d/2)^{\deg(Q\widetilde{q}_n)}}, \qquad z \in K \setminus V^o,\ n \geq n_0.$$

Now proceeding as in the proof of Theorem 3.1, making $n \to \infty$ and then $\delta \to 0$, we obtain

$$\limsup_n \|f - r_{n,m,k}\|_K = 0$$

as we needed to prove. ♠

3.2 Some computational hints

A negative element in the use of GRIP^3, fixing the number of free poles, is that in order to obtain local uniform convergence in the spherical metric one has to choose m equal to the exact number of poles that the function has in the region, counting their order. Even if one knows that this number is finite, it is usually difficult to determine a priori their exact amount.

In practice one proceeds as follows. You choose m and calculate some approximants along with the error they produce. If m is too small, this is evidenced by a globally bad behavior of the error. This means you must increase m. If m is too large there are two possibilities. The first is that the poles in excess leave the region where convergence is required. In this case there is no problem because convergence in capacity implies uniform convergence in the spherical metric if there are no loose poles. When some of the loose poles wander within the region without any definite limit (these are called spurious poles) one notices a generally good behavior of the approximation and occasional jumps in the neighborhood of the spurious poles, this is easily corrected decreasing the index m in a number equal to the amount of loose poles one observes in the calculations.

3.3 Averaged GRIP^3

There is another way to proceed, without varying m, in the case when m is chosen too large and spurious poles appear. Convergence in capacity indicates that quasi everywhere there is convergence. Therefore, it is to be expected that if one takes the average of the values of the approximants the resulting function behaves well. This *averaged GRIP*^{3} happens to be the regular part of the GRIP^3 obtained deleting from the rational function the singular part associated with the spurious poles.

Lemma 3.3 *Let r be a rational function and*

$$r(z) = p(z) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{A_{i,j}}{(z - z_i)^j}$$

its decomposition in simple fractions (p is a polynomial and $\{z_1, \ldots, z_n\}$ is the set of distinct poles of r in $\mathcal{C}$). Let Γ be a contour which consists of a finite number of non-intersecting closed rectifiable Jordan curves and D the region delimited by Γ. Assume that non of the poles of r lie on Γ and let the poles of r be indexed so that $\{z_1, \ldots, z_{n'}\} \subset \mathcal{C} \setminus \overline{D}$ while $\{z_{n'+1}, \ldots, z_n\} \subset D$. Denote by

$$\tilde{r}(z) = p(z) + \sum_{i=1}^{n'} \sum_{j=1}^{m_i} \frac{A_{i,j}}{(z - z_i)^j}$$

(the regular part of r in D). Then

$$\tilde{r}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{r(\zeta) d\zeta}{\zeta - z}, \qquad z \in D. \tag{3.8}$$

Proof: In fact, from Cauchy's integral formula we get

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{r(\zeta)d\zeta}{\zeta - z} &= \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{r}(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_\Gamma \frac{(r-\tilde{r})(\zeta)}{\zeta - z}d\zeta \\ &= \tilde{r}(z) + \frac{1}{2\pi i}\int_\Gamma \frac{(r-\tilde{r})(\zeta)}{\zeta - z}d\zeta.\end{aligned}$$

Therefore, it is sufficient to show that

$$\int_\Gamma \frac{(r-\tilde{r})(\zeta)}{\zeta - z}d\zeta = 0. \tag{3.9}$$

The arguments leading to the proof of (3.9) are analogous to those employed in proving (2.6). Notice that $r - \tilde{r}$ (as was Qp/a_{n+m+1} in the proof of (2.6)) is strictly proper and all its poles lie in D. We leave the details to the reader. ♠

Given any rational function r, the rational function $\tilde{r}$ obtained through an integration process of type (3.8) is said to be its average. Of course, $\tilde{r}$ will depend on the contour Γ which is used for integration. For simplicity, we state the following result as an existence theorem of sequences of converging averaged GRIP^3. Nevertheless, the proof is constructive and shows how such sequences may be obtained. Therefore, from the computational point of view the proof contains useful information which is not reflected in the statement.

Theorem 3.4 *Let D be a bounded region in the complex plane such that ∂D consists of a finite number of closed rectifiable Jordan curves. Let $f \in \mathcal{CM}_\mu(D) \cap \mathcal{H}(E^o)$ and $(\alpha, \beta) \in \mathcal{R}(D)$, $\alpha \subset E^o$. Fix $m \geq \mu$ and let k be such that $n-m-k$ is constant for each $n = 1, 2, \ldots$. Let K be a compact subset of D which contains no poles of f. Then, there exists a sequence $\{\tilde{r}_{n,m,k}\}$, $n = 1, 2, \ldots$ of averaged GRIP^3 which converges uniformly to f on K.*

Proof: Our first goal is to describe the curves on which we will take average. Because of the local uniform convergence in capacity of the sequence $\{r_{n,m,k}\}$, it is possible to assure the existence of a curve independent of n contained in D which surrounds K on which there is uniform convergence. But in practice such a curve is not always easy to find. Therefore, we will proceed differently showing how to choose a proper curve for each n.

Let $\{z_1, \ldots, z_{\mu'}\}$ be the set of distinct poles of f in D. By $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, we denote the different closed rectifiable Jordan curves which compose ∂D (as in Subsection 2.1 Γ_0 is the outer boundary of D). The compact sets $\{z_1\}, \ldots, \{z_{\mu'}\}, \Gamma_0, \ldots, \Gamma_N$, are all mutually disjoint; therefore, the distance between any two of them is strictly positive. Let d be a positive number smaller than any one of these mutual distances.

To fix the notation, as in the proof of Theorem 3.1, we write $r_{n,m,k} = p_n/b_k q_n$, where q_n is the polynomial corresponding to the free poles of $r_{n,m,k}$.

First let us construct adequate curves surrounding each of the distinct poles $z_1, \ldots, z_{\mu'}$ of f. For each z_j, $j = 1, \ldots, \mu'$, and each $n = 1, 2, \ldots$, we can find a circle $c_j(n)$ contained in the annulus $A_j(d) = \{z : d/6 \leq |z - z_j| \leq d/3\}$ and centered at z_j whose distance to each one of the zeros of q_n is greater or equal to $d/24m$.

In fact, consider the half line $l_j = \{z : z = z_j + it,\ t \geq 0\}$. The circular projection (centered at z_j) of A_j on l_j is the interval $I_j = \{z : z = z_j + it,\ d/6 \leq t \leq d/3\}$ whose length is $d/6$. Around each zero of q_n (there are at most m such zeros) draw a disk of radius $d/24m$ and take their circular projection on l_j. The projection of each disk is an interval of length not greater than the diameter of the disk; that is, $d/12m$. Therefore, the total length (Lebesgue measure) of the projection of all the disks is at most $d/12$. Thus, the interval I_j cannot be covered by the projection of the disks. Let z_j^* be a point of I_j not contained in the circular projections of the disks. Obviously, the circle

$$c_j(n) = \{z : |z - z_j| = |z_j^* - z_j|\}$$

does not intersect any of the disks constructed which surround each zero of q_n. Since the radius of those disks is $d/24m$, the distance of $c_j(n)$ to the set of zeros of q_n is greater or equal to $d/24m$ as claimed.

Let us make a similar construction for each $\Gamma_j,\ j = 0, \ldots, N$. Consider the set

$$B_j(d) = \left\{ z \in D : \frac{d}{6} \leq \mathtt{d}(z, \Gamma_j) \leq \frac{d}{3} \right\}$$

where $\mathtt{d}(z, \Gamma_j) = \inf\{|z - \zeta| : \zeta \in \Gamma_j\}$. The number $d > 0$ may be taken sufficiently small (on account of the analytic properties of Γ_j) so that B_j is annular; that is, doubly connected. In the sequel, we only consider such d's. For each $n = 1, 2, \ldots$ we can find a closed rectifiable Jordan curve $\Gamma_j(n)$ contained in B_j whose distance to each one of the zeros of q_n is greater or equal to $d/24m$. Additionally, $\Gamma_j(n)$ is chosen so that Γ_j lies in a different connected component from all the other $\Gamma_i,\ i = 0, \ldots, N,\ i \neq j$, of the two connected components in which $\Gamma_j(n)$ separates the complex plane. In most applications the curves Γ_j are formed by arcs of circles and segments for which the statements above are easy to verify reasoning more or less as we did with the points $z_1, \ldots, z_\mu$, so we will not dwell into details.

Since the sets $\{z_1\}, \ldots, \{z_{\mu'}\}, \Gamma_0, \ldots, \Gamma_N$, are all at distance one from the other greater than d, the curves $c_1(n), \ldots, c_{\mu'}(n),\ \Gamma_0(n), \ldots, \Gamma_N(n)$ are all mutually disjoint. By $\Gamma_0(n)$, we also denote the corresponding positively oriented curve. On $c_1(n), \ldots, c_{\mu'}(n),\ \Gamma_1(n), \ldots, \Gamma_N(n)$ we take a negative orientation. Thus

$$\Gamma(n) = \Gamma_0(n) \cup \ldots \cup \Gamma_N(n) \cup c_1(n) \cup \ldots \cup c_{\mu'}(n)$$

defines a contour which surrounds the compact set

$$K(d) = \{z \in D : \mathtt{d}(z, \partial D \cup \{z_1, \ldots, z_{\mu'}\}) > d\}$$

and is contained in

$$A(d) = \left(\cup_{j=1}^{\mu'} A_j(d)\right) \cup \left(\cup_{j=1}^{N} B_j(d)\right).$$

Notice that the set $A(d)$ does not depend on n, and it is a compact subset of D.

Let us prove that the sequence $\{\tilde{r}_{n,m,k}\},\ n = 1, 2, \ldots$, of averaged GRIP[3], where

$$\tilde{r}_{n,m,k}(z) = \frac{1}{2\pi i} \int_{\Gamma(n)} \frac{r_{n,m,k}(\eta) d\eta}{\eta - z}, \tag{3.10}$$

converges uniformly to f on $K(d)$.

In fact, from Cauchy's integral formula and (3.10), we have that

$$f(z) - \tilde{r}_{n,m,k}(z) = \frac{1}{2\pi i}\int_{\Gamma(n)} \frac{f(\eta) - r_{n,m,k}(\eta)}{\eta - z} d\eta, \qquad z \in K(d). \tag{3.11}$$

According to (3.5)

$$|f(\eta) - r_{n,m,k}(\eta)| \le \frac{C_2\theta_n}{|(Q\,\tilde{q}_n)(\eta)|}, \qquad \eta \in \Gamma(n) \subset A(d), \tag{3.12}$$

where

$$\theta_n = \sup\left\{|A_n(\eta,\zeta)| : \eta \in A(d),\ \zeta \in \partial D\right\}.$$

But, from the construction of $\Gamma(n)$, for $\eta \in \Gamma(n)$

$$\left|\left(Q\,\tilde{q}_n\right)(\eta)\right| \ge \left(\frac{d}{6}\right)^{\deg Q}\left(\frac{d}{24m}\right)^{\deg \tilde{q}_n}. \tag{3.13}$$

Therefore, using (3.11), (3.12), and (3.13), we find that there exists a constant C_3 such that

$$|f(z) - \tilde{r}_{n,m,k}(z)| \le C_3\theta_n, \qquad z \in K(d).$$

Since $(\alpha,\beta) \in \mathcal{R}(D)$, we have that $\lim_n \theta_n = 0$, and the assertion of the theorem follows for $K(d)$.

Finally, notice that $K(d_1) \supset K(d_2)$ for $d_1 < d_2$ and $\cup_{0<d<d_0} K(d) = D \setminus \{z_1,\ldots,z_{\mu'}\}$. Hence, for any compact subset K of $D \setminus \{z_1,\ldots,z_{\mu'}\}$, we can find a sufficiently small $d > 0$ such that on $K(d)$ there is uniform convergence of $\{\tilde{r}_{n,m,k}\}$, $n = 1,2,\ldots$, and $K(d) \supset K$. In particular, we obtain uniform convergence on K as desired. ♠

The sequence of averaged GRIP^3 which we constructed depends on the compact set on which we wish to approximate f. Through a diagonalization procedure, we can also construct sequences of averaged GRIP^3 which converge uniformly on each compact subset of $K \subset D \setminus \{z_1,\ldots,z_{\mu'}\}$. If $K \cap \partial D \ne \emptyset$, $K \subset \overline{D}$, and the spurious poles do not approach the set $K \cap \partial D$, then too sequences of convergent averaged GRIP^3 may be constructed combining the ideas expressed in the proofs of Theorem 3.4 and the second part of Theorem 3.1.

4 Extensions to the half plane

4.1 Invariance of GRIP^3 and connection with the bounded case

Under suitable conditions on the function f and the tables α, β, the results of Section 3 can be extended to unbounded regions. We restrict our attention to half-planes of the form $D_\rho = \{z : \Re z > \rho\}$, $\rho \in \mathcal{R}$, since it is the case of most interest in dealing with continuous time control systems. For $\rho = 0$, we adopt the standard notation $\mathcal{C}_+$ for the right-half plane.

As in Section 2, the notation $f \in \mathcal{CM}_\mu(D_\rho)$ means that f is meromorphic in D_ρ, has in that region μ poles counting their multiplicities, and can be extended continuously to the boundary $\partial D_\rho = \{z : \Re z = \rho\}$. Let E be a closed subset of D_ρ such that f is holomorphic on E^o; that is, $f \in \mathcal{CM}_\mu(D_\rho) \cap H(E^o)$.

Take tables of points $\alpha \subset E^o$ and $\beta \subset F^o = \{z : \Re z < \rho\}$ and with them construct the polynomials $a_n(z)$, $b_n(z)$. The arguments employed in Subsection 2.1 allow to assert that for $f \in H(E^o)$ and for each $(n, m, k) \in \mathcal{N}^3$ there exists a unique rational function $r_{n,m,k}$ defined by (2.1) and (2.2).

For sequences of GRIP^3, we can reduce the problem of convergence to f of $\{r_{n,m,k}\}$, $n = 1, 2, \ldots,$ $r_{n,m,k} = p_n/q_n b_k$, on compact subsets of D_ρ to the case of bounded regions. Let us formally transform one case into the other and see what we obtain.

Let $\zeta = T(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$, denote a fractional linear transformation. If $c = 0$, then $T(z)$ is an affine transformation and we can consider that $T(z) = az + b$ (with a, b new constants). Given a polynomial h, we denote $h^*(z) = h \circ T$. Let $f^* = f \circ T$. Substituting T in (2.1), we obtain

$$\deg p_n^* \leq n, \quad \deg q_n^* \leq m, \qquad \frac{q_n^* b_k^* f^* - p_n^*}{a_{n+m+1}^*} \in \mathcal{H}(E_1^o),$$

where $q_n^* \not\equiv 0$ and $E_1^o = \{z \in \mathcal{C} : T(z) \in E^o\}$. Therefore, $r_{n,m,k}^* = p_n^*/q_n^* b_k^*$ is the (n, m, k) GRIP^3 of f^* which interpolates this function at the zeros of a_{n+m+1}^* and has fixed poles at the zeros of b_k^*. This means that GRIP^3 are invariant under affine transformations. Thus, in the sequel, we can assume without loss of generality that $\rho = 0$ and $D_\rho = \mathcal{C}_+$.

The fractional linear transformation $T(z) = (1 - z)/(1 + z)$ transforms the unit disk $B = [|z| < 1]$ one to one onto $\mathcal{C}_+$. For this new mapping, set $f^* = f \circ T$ as before, but $q_n^* = (1 + z)^m q_n \circ T$, $p_n^* = (1 + z)^n p_n \circ T$, $a_{n+m+1}^* = (1 + z)^{n+m+1} a_{n+m+1} \circ T$ and $b_k^* = (1 + z)^k b_k \circ T$ in order that they remain being polynomials of degree not greater that the (formal) degree prescribed initially for q_n, p_n, a_{n+m+1}, and b_k, respectively.

Let us substitute $T(z) = (1 - z)/(1 + z)$ in (2.1) again. In order to interpret the result, it is best to consider two cases. Assume that $m + k \leq n$, then

$$\deg p_n^* \leq n, \quad \deg q_n^* \leq m, \qquad \frac{(1 + z)^{n-m-k} q_n^* b_k^* f^* - p_n^*}{a_{n+m+1}^*} \in \mathcal{H}(E_1^o). \tag{4.1}$$

When $m + k > n$, we obtain

$$\deg p_n^* \leq n, \quad \deg q_n^* \leq m, \qquad \frac{q_n^* b_k^* f^* - (1 + z)^{m+k-n} p_n^*}{a_{n+m+1}^*} \in \mathcal{H}(E_1^o). \tag{4.2}$$

(Notice that $z = -1$ does not belong to E_1^o, since $\infty \notin E^o$; therefore, we can cancel out any integer power of $(1 + z)$ multiplying the functions on the right of (4.1) and (4.2) without altering the analyticity of the resulting function on E_1^o.)

From (4.1) we see that $r_{n,m,k}^* = p_n^*/(1+z)^{n-m-k} q_n^* b_k^*$ is the $(n, m, n-m)$ GRIP^3 of f^* which interpolates this function at the zeros of a_{n+m+1}^* and has fixed poles at the zeros of $(1+z)^{n-m-k} b_k^*$. It seems that we have increased the number of fixed poles, but notice that if $m + k < n$ then in the original rational function $r_{n,m,k}$, we had already fixed $n-m-k$ poles at infinity. This corresponds to taking $\beta_{n,j} = \infty$ for some j (see sentence above (2.1)). Therefore, it is natural that those poles fixed at infinity now appear in $r_{n,m,k}^*$ at $z = -1$ which is the point which is transformed into infinity through T.

In (4.2), we have an apparently more strange situation. Formally speaking, the rational function $r_{n,m,k}^* = (1 + z)^{m+k-n} p_n^*/q_n^* b_k^*$ is not a GRIP^3 since a fixed zero of multiplicity $m + k - n$ has appeared in the numerator. It seems that we had not considered this construction before; but if in $r_{n,m,k}$, we have $m + k > n$, then in fact a zero of multiplicity $m+k-n$ has been fixed at infinity. Naturally, this zero appears at $z=-1$ under the effect of the transformation T. Therefore, in some sense $r^*_{n,m,k}$ is also an (n,m,k) GRIP[3] of f^* with some zeros fixed at $z=-1$ instead of ∞.

4.2 Convergence on compact subsets of D_ρ

First, let us state the conditions which we will require on the tables α and β. Given D_ρ, let E and F be as described in the beginning of Subsection 4.1. Let $\alpha\subset E^o$, $\beta\subset F^o$. Denote $T_1(z)=z-\rho$ an affine transformation which transforms D_ρ into $\mathcal{C}_+$, and $T_2(z)=(1-z)/(1+z)$ a fractional linear transformation which represents $\mathcal{C}_+$ onto $[|z|<1]$ (this transformation and its inverse considered in Subsection 4.1 have the same form). Associated with tables $\alpha=\{\alpha_{n,j}\}$ and $\beta=\{\beta_{n,j}\}$ are the tables $\alpha^*=\{\alpha^*_{n,j}\},\ \beta^*=\{\beta^*_{n,j}\},\ j=1,\dots,n,\ n=1,2,\dots$, where

$$\alpha^*_{n,j}=(T_2\circ T_1)(\alpha_{n,j}),\qquad \beta^*_{n,j}=(T_2\circ T_1)(\beta_{n,j}).$$

As before, m will be fixed greater or equal to μ (the number of poles of f in D_ρ). The parameter k will be taken so that for each $n,\ n-m-k$ is constant. Let

$$A^*_n(z,\zeta)=\frac{a^*_{n+m+1}(z)}{a^*_{n+m+1}(\zeta)}\frac{b^*_k(\zeta)}{b^*_k(z)}$$

where

$$a^*_n(z)=\prod_{j=1}^n(z-\alpha^*_{n,j}),\qquad b^*_n(z)=\prod_{j=1}^n(z-\beta^*_{n,j}).$$

Except for a constant multiple (depending on n), it is easy to see that a^*_n and b^*_n as defined in this Section coincide with a^*_n and b^*_n as introduced in subsection 4.1. Since these polynomials appear once in the numerator and once in the denominator of A^*_n, it does not matter how we normalize the polynomials a^*_n and b^*_n. Any normalization on them leads to the same A^*_n.

We say that $(\alpha,\beta)\in\mathcal{R}(D_\rho)$ if and only if $(\alpha^*,\beta^*)\in\mathcal{R}([|z|<1])$. That is to say if

$$A^*_n(z,\zeta)\overrightarrow{\rightarrow}_n 0,\qquad n\to\infty,$$

on each compact set of the form $K\times[|\zeta|=1]$, where K is an arbitrary compact subset of $[|z|<1]$.

For $\mathcal{C}_+$ and $k=n-m$, from (2.12) it is easy to see that $(\alpha,\beta)\in\mathcal{R}(\mathcal{C}_+)$ if

$$\lim_n\sum_{j=1}^n\left(1-\left|\frac{1-\alpha_{n,j}}{1+\alpha_{n,j}}\right|\right)=\infty, \tag{4.3}$$

$|(1-\alpha_{n+m+1,j})/(1+\alpha_{n+m+1,j})|\le r'<1,\ j=n-m+1,\dots,n+m+1$, and $\beta_{n-m,j}=-\overline{\alpha}_{n+m+1,j},\ j=1,\dots,n-m,\ n\ge m+1$.

The values of f on the boundary of D_ρ are of importance in order to obtain an integral remainder formula of type (2.4); therefore, we need to have some control on the growth of f. In the sequel, $\eta\in\mathcal{R}$ and

$$|f(z)|\le O\left(\frac{1}{|z|^\eta}\right),\qquad z\in\overline{D}_\rho,\ z\to\infty. \tag{4.4}$$

We have

Theorem 4.1 *Consider the half-plane D_ρ and let $f \in \mathcal{CM}_\mu(D_\rho) \cap H(E^o)$ satisfy (4.4) for some $\eta \in \mathcal{R}$. Let $(\alpha, \beta) \in \mathcal{R}(D_\rho)$, $\alpha \subset E^o$. Fix $m \geq \mu$, let k be such that $n - m - k = r$ takes a constant value for each $n = 1, 2, \ldots$, where $r + \eta > -1$. Then, the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$, converges locally uniformly in capacity to f on D_ρ. Moreover, for $m = \mu$, the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$, converges locally uniformly to f on D_ρ in the spherical metric.*

Proof: From the invariance of (n, m, k) GRIP^3 under affine transformation and the way in which $\mathcal{R}(D_\rho)$ is defined, clearly it is sufficient to consider the case when $D_\rho = \mathcal{C}_+$.

Relations (4.1) and (4.2) indicate that

$$\frac{(1+z)^{n-m-k} f^* q_n^* b_k^* - p_n^*}{a_{n+m+1}^*} \in \mathcal{H}(E_1^o), \tag{4.5}$$

where $\deg p_n^* \leq n$ and $\deg q_n^* \leq m$. On the other hand, from the conditions on f we have that the function $(1+z)^{n-m-k} f^*$ is continuous on $|z| = 1$ except possibly at the point $z = -1$, where

$$|(1+z)^{n-m-k} f^*(z)| = O\left(|1+z|^{r+\eta}\right), \qquad |z| \leq 1, \; z \to -1, \tag{4.6}$$

and $r + \eta > -1$. Therefore, this function is integrable on $[|z| = 1]$. Arguing as in the deduction of the integral formula (2.4), from (4.5), we obtain

$$(f^* - r_{n,m,k}^*)(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(1+\zeta)^r}{(1+z)^r} \frac{(q_n^* Q^* f^*)(\zeta)}{(q_n^* Q^*)(z)} A_n^*(z, \zeta) \frac{d\zeta}{\zeta - z}, \qquad |z| < 1, \tag{4.7}$$

where Q^* is the monic polynomial of degree μ whose zeros are the poles of $f^*(z) = f((1-z)/(1+z))$ in $[|z| < 1]$, and $r_{n,m,k}^* = p_n^*/q_n^* b_k^* (1+z)^r$.

The only difference between (4.7) and (2.4) is the ratio $(1+\zeta)^r/(1+z)^r$. The factor $1/|1+z|^r$ can obviously be bounded above on any compact subset K of $[|z| < 1]$ and we have (4.6); therefore, it is easy to obtain the following bound reasoning as in the deduction of (3.5).

$$|(f^* - r_{n,m,k}^*)(z)| \leq C \frac{\theta_n}{|(Q^* \tilde{q}{}_n^*)(z)|}, \qquad z \in K. \tag{4.8}$$

Here C is a constant independent of n, $\tilde{q}{}_n^*$ is the monic polynomial whose zeros are the zeros of q_n^* of module less than 2 and

$$\theta_n = \sup\{|A_n^*(z, \zeta)| : z \in K, \; |z| = 1\}.$$

With (4.8) one proves the local uniform convergence in capacity of $\{r_{n,m,k}^*\}$, $n = 1, 2, \ldots$, to f^* as is done in Theorem 3.1 and also the local uniform convergence in the spherical metric when $\mu = m$. Changing variables one obtains the corresponding statements for the sequence $\{r_{n,m,k}\}$, $n = 1, 2, \ldots$, on $D_\rho = \mathcal{C}_+$. ♠

Reducing the problem to the unit disk, we can also construct sequences of averaged GRIP^3 which converge to f on compact subsets of D_ρ minus the set of poles of f in D_ρ. We limit ourselves to stating the result. As in theorem 4.1 the proof consists in reducing the problem to the unit disk, employing the same arguments as in Theorem 3.4 and then returning to the initial problem by the inverse mapping.

Theorem 4.2 *Under the assumptions of Theorem 4.1 regarding f and (α,β), assume additionally that K is a compact subset of D_ρ which contains no pole of f. Then, there exists a sequence $\{\hat{r}_{n,m,k}\}$, $n = 1, 2, \ldots$, of averaged GRIP^3 which converges to f uniformly on K.*

4.3 Convergence on unbounded subsets of D_ρ

We are interested in proving convergence of GRIP^3 on unbounded subsets of D_ρ of the form $\overline{D}_{\rho'}$, where $\rho' > \rho$. A typical situation in infinite dimensional control systems is that when $\overline{D}_{\rho'} = \overline{\mathcal{C}}_+$ and D_ρ is such that $\rho < 0$. Because of formula (4.7), we prefer to reduce the study of the general case to that when $D_\rho = \mathcal{C}_+$ and approximation is required on $\overline{D}_{\rho'}$, $\rho' > 0$. As we saw in Section 4.1, this reduction is always possible by an affine transformation.

Since we will be approximating near infinity, all the more there is reason to control the growth of f near that point. Hence, in this Section (4.4) also takes place.

As in the proof of the second part of Theorem 3.1 one must impose additional conditions on the tables (α,β). Let T_1 and T_2 be the transformations introduced in Section 4.2. Set

$$D^*_{\rho'} = \{z \in \mathcal{C} : z = T_2 \circ T_1(w),\ w \in D_{\rho'}\}.$$

Obviously, $D^*_{\rho'}$ is a disk contained in $[|z| < 1]$ which is tangent to $[|z| = 1]$ at $z = -1$. Let $\{A^*_n(z,\zeta)\}$, $n = 1, 2, \ldots$, be the sequence introduced in Section 4.2. We say that $(\alpha,\beta) \in \mathcal{R}(\overline{D}_{\rho'}, D_\rho)$ if $(\alpha,\beta) \in \mathcal{R}(D_\rho)$, for each $\delta > 0$ sufficiently small, $A^*_n(z,\zeta)$ converges uniformly to zero on $\{\overline{D}^*_{\rho'} \setminus \{-1\}\} \times \{[|\zeta| = 1] \setminus [|\zeta + 1| < \delta]\}$ and is uniformly bounded on $\overline{D}^*_{\rho'} \times [|\zeta| = 1]$. Based on known results from Blaschke products, pairs of tables (α,β) satisfying the properties indicated are not hard to construct. For example, for $k = n - m$, $\overline{D}_{\rho'} = \overline{\mathcal{C}}_+$ and D_ρ such that $\rho < 0$, we can take α and β as indicated in (4.3) with the additional restriction that $\Re\alpha_{n,j} > -\rho$ for all $n = 1, 2, \ldots$, and $j = 1, \ldots, n$ (so that $\beta_{n,j} \subset F^o \subset \{z : \Re z < \rho\}$).

In the proof of the next result, we use the following lemma.

Lemma 4.3 *Let D^* be an open disk strictly contained in $[|z| < 1]$ which is tangent to $[|z| = 1]$ at $z = -1$. Let $\gamma > 0$ and $0 < \delta < \pi$, then*

$$\int_{\pi-\delta}^{\pi+\delta} \left|\frac{1+z}{1+\zeta}\right| \frac{|1+\zeta|^\gamma}{|z-\zeta|} d\theta = O\left(\delta^\gamma |\log \delta|\right), \qquad \delta \to 0, \quad \zeta = e^{i\theta},$$

uniformly with respect to $z \in \overline{D^*} \setminus \{-1\}$.

Proof: We make use of an inequality which was proved in [11], Lemma 4.4. For each fixed $a > 0$

$$\int_{|t|>R} \frac{dt}{|t|^\gamma |it - u|} = O\left(\frac{\log R}{R^\gamma}\right), \qquad R \to \infty, \tag{4.9}$$

uniformly with respect to u in $\{u : \Re u \geq a\}$.

The circumference ∂D^* cuts the real line at -1 and at another point $x_0 \in (-1, 1)$ (since $D^* \neq [|z| < 1]$). Set $a = (1 - x_0)/(1 + x_0)$. Due to known properties of Moebius transformations it is easy to see that the application $u = (1 - z)/(1 + z)$ transforms $\overline{D}^* \setminus \{-1\}$ onto $\{\Re u \geq a\}$ and $[|z| < 1]$ onto $\{\Re u > 0\}$.

Set $\tau = (1-\zeta)/(1+\zeta)$. For $\tau = it$, $t \in \mathcal{R}$, and $\zeta = e^{1\theta}$, $\theta \in [0, 2\pi)$, we have that $it = (1-e^{i\theta})/(1+e^{i\theta}) = -i \tan\theta/2$. Therefore, making the change of variables $z = (1-u)/(1+u)$ and $\zeta = (1-\tau)/(1+\tau)$, we obtain using (4.9)

$$\begin{aligned}\int_{\pi-\delta}^{\pi+\delta} \left|\frac{1+z}{1+\zeta}\right| \frac{|1+\zeta|^\gamma}{|z-\zeta|} d\theta &= 2^\gamma \int_{|t|>\tan\frac{\pi-\delta}{2}} \frac{dt}{|1+it|^\gamma |it-u|} \\ &\leq 2^\gamma \int_{|t|>\tan\frac{\pi-\delta}{2}} \frac{dt}{|t|^\gamma |it-u|} = O\left(\frac{\log\tan[(\pi-\delta)/2]}{\tan^\gamma[(\pi-\delta)/2]}\right), \quad \delta \to 0,\end{aligned}$$

uniformly with respect to $z \in \overline{D^*}\setminus\{-1\}$. It rests to notice that the expression $\log\tan[(\pi-\delta)/2]/\tan^\gamma[(\pi-\delta)/2]$ is equivalent to $\delta^\gamma|\log\delta|$ as $\delta \to 0$. ♠

Theorem 4.4 *Consider the half-plane D_ρ and let $f \in \mathcal{CM}_\mu(D_\rho) \cap \mathcal{H}(E^o)$ satisfy* (4.4). *Let $\rho' > \rho$ and $(\alpha, \beta) \in \mathcal{R}(\overline{D}_{\rho'}, D_\rho)$, $\alpha \subset E^o$. Fix $m = \mu$, and let k be such that $n - \mu - k = r$ takes a constant value for each $n = 1, 2, \ldots$, where $r + \eta > -1$. Then, the sequence $\{(f - r_{n,\mu,k})(u)/(u-\rho)^{r+1}\}$, $n = 1, 2, \ldots$, converges uniformly to zero on each compact set of the form $\overline{D}_{\rho'} \setminus U_\epsilon$, where U_ϵ is an ϵ-neighborhood of the set of poles of f in D_ρ. In particular, if $r \leq -1$, we obtain that $\{r_{n,\mu,k}\}$, $n = 1, 2, \ldots$, converges to f uniformly in the spherical metric on $\overline{D}_{\rho'}$.*

Proof: We can assume without loss of generality that $D_\rho = \mathcal{C}_+$. The conditions of Theorem 4.1 are satisfied; therefore, formula (4.7) takes place with f^*, $r^*_{n,m,k}$, q^*_n, Q^*, and $A^*_n(z,\zeta)$ as defined there.

The first statement of the theorem is equivalent to proving that the sequence $\{(1+z)^{r+1}(f^* - r^*_{n,m,k})(z)\}$, $n = 1, 2, \ldots$, converges uniformly to zero on each compact subset of the form $\overline{D^*} \setminus U^*_\epsilon$, where U^*_ϵ is an ϵ-neighborhood of the poles of f^* in $[|z| < 1]$ and $D^* = \{z = (1-u)/(1+u),\ \Re u \geq \rho' > 0\}$. Fix $\epsilon > 0$.

From Theorem 4.1, we already know that the zeros of q^*_n converge to the zeros of Q^* according to multiplicity; that is, there are no spurious poles since $m = \mu$ (see Lemma 2.3). In particular, for all sufficiently large n, the zeros of q^*_n all lie in an $\epsilon/2$-neighborhood of the poles of f^* in $[|z| < 1]$. From this, it immediately follows that

$$\left|\frac{(q^*_n Q^*)(\zeta)}{(q^*_n Q^*)(z)}\right| \leq C, \qquad |\zeta| = 1, \quad z \in \overline{D^*} \setminus U^*_\epsilon, \quad n \geq n_0. \tag{4.10}$$

In the sequel, we only consider indexes $n \geq n_0$.

Choose δ such that $0 < \delta < \pi$. From the conditions on (α, β), (4.4), (4.7), (4.10) and Lemma 4.3, we obtain

$$\begin{aligned}\left|(1+z)^{r+1}(f^* - r^*_{n,\mu,k})(z)\right| &= \left|\frac{1}{2\pi i}\int_{[|\zeta|=1]} A^*_n(z,\zeta)\frac{(q^*_n Q^*)(\zeta)}{(q^*_n Q^*)(z)} \frac{(1+\zeta)^{r+1} f^*(\zeta)}{\zeta - z} \frac{1+z}{1+\zeta} d\zeta\right| \\ &\leq C_1 \int_{\pi-\delta}^{\pi+\delta} \left|\frac{1+z}{1+\zeta}\right| \frac{|1+\zeta|^{r+\eta+1}}{|\eta - z|} d\theta + C_2\theta_n(\delta) \\ &\leq O(\delta^\gamma|\log\delta|) + C_2\theta_n(\delta), \zeta = e^{i\theta}, z \in \overline{D^*} \setminus (U^*_\epsilon \cup \{-1\}), \end{aligned} \tag{4.11}$$

where $\gamma = r + \eta + 1 > 0$ and

$$\theta_n(\delta) = \sup\left\{|A_n(z,\zeta)| : z \in \overline{D^*} \setminus -1;\ \zeta = e^{i\theta},\ \theta \in [0, 2\pi] \setminus (\pi - \delta, \pi + \delta)\right\}.$$

For each $\epsilon' > 0$, we can now select $\delta > 0$ sufficiently small so that $C_1 O(\delta^\gamma |\log \delta|) < \epsilon'/2$, and then $n_0' \geq n_0$ such that for all $n \geq n_0'$, $C_2\theta_n(\delta) < \epsilon'/2$. Therefore, from (4.11),

$$\left|(1+z)^{r+1}(f^* - r^*_{n,\mu,k})(z)\right| < \epsilon', \qquad n \geq n_0', \quad z \in \overline{D^*} \setminus (U^*_\epsilon \cup \{-1\}).$$

Taking the limit $z \to -1$, the inequality also holds at $z = -1$. This proves the first statement.

To prove the second statement, first notice that if $r \leq -1$, then $1/|u-\rho|^{r+1}$ is bounded below on $\{\Re u \geq \rho' > \rho\}$ by a positive number, thus from the first statement we get that $\{r_{n,\mu,k}\}$ converges uniformly to f on each compact subset of the form $\overline{D}_{\rho'} \setminus U_\epsilon$. On the other hand, from Theorem 4.1, on U_ϵ there is convergence in the spherical metric for all sufficiently small $\epsilon > 0$; therefore, applying Lemma 2.5 we obtain the desired result. ♠

5 Numerical results

Example 1: Consider the transfer function

$$G(z) = \frac{20(6e^{-2z} + 2e^{-z} - 6)}{6z^2 + (6e^{-2z} - 2e^{-z} - 66)z - (2e^{-3z} + 30e^{-2z} - 12e^{-z} - 180)}.$$

This function was used in [8] to illustrate another method of approximation of unstable systems. The method used in [8] consists in substituting each term of the form e^{-kz} by the k^{th}-power of the $[m,m]$ Padé approximant of e^{-z} for a given $m \in \mathcal{N}$. We call this method the *Padé exponential* method.

We compute rational approximations of G of order $[n-1/n]$ for $n = 5, 8, 11, 14$ and 17, using GRIP^3. As it was stated in [8], G has exactly two unstable poles, thus we fix $n-2$ poles and choose $n+2$ points of interpolation. The poles and interpolation points were selected as the poles and zeros respectively of the $[n-2/n-2]$ Padé approximants of e^{-2z}. The remaining interpolation points were selected at 1, 3, 5 and 7. Known properties of the distribution of zeros and poles of the Padé approximants of e^{-z} (see [12], pag. 6) guarantee that with this selection $(\alpha, \beta) \in \mathcal{R}(\overline{\mathcal{C}}_+, D_{-1})$.

The results are given in Table 1 (errors of the approximations of the unstable poles) and Figure 1 (L_∞ errors of the approximations on the imaginary axis). For comparison, we include also the numerical results obtained by applying the method of [8].

Order	5	8	11	14	17
Padé exp.	1.4e-1	3.3e-2	4.2e-3	3.5e-4	1.9e-5
	2.4e-1	2.0e-2	4.5e-4	2.0e-5	9.6e-7
GRIP^3	3.9e-5	9.0e-8	5.6e-9	4.0e-9	4.2e-9
	2.1e-2	9.8e-5	1.9e-7	1.9e-7	2.0e-7

Table 1: Errors of the approximations of the unstable poles.

As can be seen, in this example the GRIP^3 give better numerical results for the approximation of G on the imaginary axis and locates the poles of G more rapidly.

Figure 1: L_∞ errors of the approximations on the imaginary axis.

Example 2: Although the results given here guarantee convergence of GRIP^3 for a large class of tables α and β, the particular choice of these tables is important to obtain good low-order approximations. Our numerical experiments indicate that the zeros and poles of diagonal Pade approximants of the exponential function are a good choice for the tables of interpolation points and of fixed poles respectively when dealing with transfer functions of time delay systems. We have also observed that the selection of the extra interpolation points does not have too much effect on the error on the imaginary axis but choosing some of them in the neighborhood of the unstable poles greatly increases the convergence of the method in approximating the unstable poles.

The problem is that apriori we do not always have an idea of where the unstable poles lie. We have experimented with the following idea which has shown to be effective in numerical calculations. If the number of free poles is equal to the number of unstable poles ($\mu = m$) then the number of extra interpolation points is twice the number of unstable poles. We use these points as free parameters to make corrections on each iteration. On the first iteration the 2μ extra interpolation points are chosen more or less arbitrarily. On the second iteration we take the μ free poles of the first iteration and leave μ of the extra interpolation points better located. From the third iteration on, we take the free poles of the two previous iterations.

We illustrate this idea in the next example, taken from [9]. Consider the transfer function

$$G(z) = \frac{2e^{-z}}{2z^2 + e^{-z}}.$$

We compute rational approximations of G of order $[n-1/n]$ for $n = 4, 5, 6,$ and 7, using GRIP^3. Using the argument principle on the denominator of G, it can be shown that G has two unstable poles which are complex conjugated one to the other. Thus, we fix $n-2$ poles and choose $n+2$ interpolation points. For each n, the $n-2$ fixed poles and $n-2$ interpolation points were selected as the poles and zeros respectively of the $[n-2/n-2]$ Padé approximants of e^{-z}. The remaining interpolation points were taken as the free poles computed for the approximations of the two previous values considered of n. As initial interpolation points for the first iteration, we took $1,\ 3,\ 5$ and 7.

The results are given in Table 2 (error of the approximations of the unstable poles) and Table 3 (L_∞ errors of the approximations on the imaginary axis). For comparison, we include also the results obtained when the correction is not made. In this example, the unstable poles and the free poles of the approximations are complex conjugated, thus the error of the approximations are the same for both unstable poles.

Order	4	5	6	7
with correction	3.6e-1	7.9e-3	4.9e-5	5.3e-9
without correction	3.6e-1	1.7e-1	6.0e-2	2.4e-2

Table 2: Errors of the approximations of the unstable poles.

Order	4	5	6	7
with correction	2.8	3.0e-1	1.3e-2	1.0e-2
without correction	2.8	2.1	1.0	7.7e-1

Table 3: L_∞ errors of the approximations on the imaginary axis.

As can be seen, the correction on each iteration in the selection of the interpolation poles suggested above increases considerably the accuracy of the method, mainly in the localization of the unstable poles, but it also has some positive effect in the L_∞ approximation on the imaginary axis.

Ejemplo 3: In time-delay transfer functions the exponential may appear with several different exponents. Each such exponential has a different period of oscillation on the imaginary axis. In the denominator, there is a polynomial term which dominates that part of the function on the imaginary axis for large values of the variable, so we are mainly concerned with the exponentials which appear in the numerator. Therefore, in constructing our GRIP^3, we choose (whenever possible) the zeros and poles of the Padé approximants of an exponential function whose period of oscillation on the imaginary axis divides the period of oscillation of each exponential appearing in the given transfer function.

Consider the transfer function

$$G(z) = \frac{e^{-z/3} + e^{-z/2}}{z - 6 - 6e^{-z/6}}.$$

This function has a real simple pole in the positive half plane. The exponentials in the numerator have oscillation periods on the imaginary axis equal to 6π and 4π respectively; therefore, we recommend to take the zeros and poles of e^{-z} whose period of oscillation on the imaginary axis is 2π.

In the table below, we give the numerical results that we obtained for the approximation on the imaginary axis of this function taking the zeros and poles of $e^{-z/3}, e^{-z/2}$, and e^{-z}, (that is, $k = 1/3, 1/2$, and 1 respectively). The extra interpolation points where taken initially in all cases at 1 and 2 for the first iteration and introduced the (corresponding) correction suggested in example 2 in the following iterations. We calculated the errors for the GRIP^3 approximants of order $[n-1, n], n = 4, 7, 10, 13$.

Order	4	7	10	13
$k = 2$	1.9	1.7	1.7	1.7
$k = 3$	2.0	1.7	1.2	1.1
$k = 6$	2.1	1.9	1.7	0.9

Table 4: L_∞ errors of the approximations on the imaginary axis.

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