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Remarks on the star product of functions on finite and compact groups

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Abstract

Using the formalism of quantizers and dequantizers, we show that the characters of irreducible unitary representations of finite and compact groups provide kernels for star products of complex-valued functions of the group elements. Examples of permutation groups of two and three elements, as well as the $SU(2)$ group, are considered. The k-deformed star products of functions on finite and compact groups are presented. The explicit form of the quantizers and dequantizers, and the duality symmetry of the considered star products are discussed.

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1 Introduction

Traditionally the finite symmetry groups and their irreducible representations are used to describe the properties of crystals and electrons in solids. Also for the description of phase transitions, one needs to know the change of symmetry structure and corresponding group representation properties. In the last years, there is a growing interest in constructing quantum mechanics of finite or discrete phase spaces [1, 2, 3]. Besides, in the context of quantum computations [4] one often considers finite-dimensional Hilbert spaces associated with qubit or, more generally, qudit states. Therefore, it is quite natural to study the phase-space realizations of quantum systems suitably associated with representations of finite or compact groups. The description of states of a quantum system by means of Wigner (quasi-)distributions in the case of continuous position and momentum variables has been considered in a wide variety of contexts [5, 6, 7, 8]. It relies on considering the phase space (a symplectic vector space) as a quotient of the Heisenberg–Weyl group by its centre. Thus a unitary representation of this group — or, equivalently, a projective unitary representation of the vector group (the symplectic vector space) — may be regarded as an immersion of this space as a smooth submanifold of the Weyl algebra generated by the unitary operators. This particular immersion allows to pull-back the $\mathbf{C}^*$-algebra of operators and, therefore, allows to induce a star product on the functions on the phase space (see, e.g., [9, 10, 11]). This product may be expressed by means of a kernel function constructed out of the unitary operators associated with points of the symplectic vector space. This procedure may be generalized to any manifold as long as a suitable orthonormality condition is implemented. We have considered several instances of this procedure in the past [12, 13, 14, 15]. The Weyl–Wigner approach we have described has been considered very often also for quantum systems with a finite-dimensional carrier Hilbert space [1, 2, 3].

The aim of this paper is to find the connection of the properties of quantum systems, which have symmetries described by finite or compact symmetry groups (crystallographic groups, permutation groups, rotation group), with the star-product quantization approach. In the context of mathematical formulation, the aim of this paper is to consider the immersion of finite or compact groups in the space of unitary operators acting on some Hilbert space and pull-back the $\mathbf{C}^*$-algebra of operators to describe nonlocal and noncommutative products on the space of functions. Such mathematical construction provides the possibility to discuss the properties of quantum systems associated with physical observables identified with operators using a classical-like approach where the observables are identified with functions on phase space. But the multiplication rule for these functions is determined by a specific star-product procedure. Some examples of crystallographic and permutation groups are presented to illustrate the procedure.

The paper is organized as follows.

In Sect. 2, we recall some basic facts about star products. In Sect. 3, we focus on the special case of finite groups. Next, in Sect. 4, some formulae involving characters are derived, and we illustrate our results by means of examples in Sect. 5. Finally, in Sect. 6, conclusions are drawn.

2 General aspects of star products

The construction of a Weyl system, when considered from the point of view of the immersion of the symplectic vector space into the group of unitary operators $\mathcal{U}(\mathcal{H})$ acting in some Hilbert space $\mathcal{H}$, may be described in the following way. We consider a manifold M and a couple of maps $U\colon M \to \mathcal{U}(\mathcal{H})$ and $D\colon M \to \mathcal{U}(\mathcal{H})$ usually called *dequantizer* and *quantizer*, respectively, with the following property:

$$\operatorname{Tr} \hat{U}(\vec{x})\hat{D}(\vec{x}') = \delta(\vec{x} - \vec{x}'). \tag{1}$$

With any operator $\hat{A}$ acting in the Hilbert space $\mathcal{H}$ we can associate a function on M by setting:

$$f_A(\vec{x}) = \operatorname{Tr}\left(\hat{A}\hat{U}(\vec{x})\right). \tag{2}$$

Conversely, with each function one associates an operator by setting:

$$\hat{A} = \int f_A(\vec{x})\hat{D}(\vec{x})\, d\vec{x}. \tag{3}$$

The role the operator-valued maps $\hat{U}$ and $\hat{D}$ (that we will also call ‘basic operators’ in the following) play in these formulae explains their names.

The star product of functions induced by the operator product is defined by

$$\left(f_A \star f_B\right)(\vec{x}) := f_{AB}(\vec{x}). \tag{4}$$

The kernel function or 'structure constants' implementing the associative product has the following expression involving the couple quantizer–dequantizer:

$$K(\vec{x},\vec{y},\vec{z}) = \mathrm{Tr}\left(\hat{D}(\vec{y})\hat{D}(\vec{z})\hat{U}(\vec{x})\right). \tag{5}$$

hence:

$$\left(f_A \star f_B\right)(\vec{x}) = \iint K(\vec{x},\vec{y},\vec{z}) f_A(\vec{y}) f_B(\vec{z})\, d\vec{y} d\vec{z}. \tag{6}$$

From the definition we find that the associativity condition is trivially satisfied:

$$\left(f_A \star f_B\right) \star f_C = f_A \star \left(f_B \star f_C\right). \tag{7}$$

At this point, the skew-symmetrization provides us, in a natural way, with a Lie algebra of functions on M defined by means of the integral kernel

$$C\left(\vec{x},\vec{y},\vec{z}\right) = \mathrm{Tr}\left(\left[\hat{D}(\vec{y}), \hat{D}(\vec{z})\right] \hat{U}(\vec{x})\right), \tag{8}$$

which, in a synthetic way, may be written as

$$C\left(\vec{x},\vec{y},\vec{z}\right) \to C^{\vec{x}}_{\vec{y}\vec{z}}. \tag{9}$$

It should be noticed that, restricting to real part of the algebra, the symmetrized product provides us with a Jordan algebra. The compatibility condition between the two products would then provide us with a Lie–Jordan algebra.

3 The case of finite groups

Let us restrict now our attention to finite groups. Let G be a group with N elements, $G = \{g_1, g_2, \ldots, g_N\}$. It is well known that all the irreducible representations of such a group are finite-dimensional and unitarizable and satisfy the orthogonality conditions (see, for example [16, 17])

$$\sum_{k=1}^{N} u^{(s)}_{mn}(g_k) u^{*(p)}_{\alpha\beta}(g_k) = \delta_{m\alpha}\delta_{n\beta}\frac{N}{N_s}\delta_{sp}, \tag{10}$$

where N_s is the dimension of the representation $u^{(s)}$ (the dimension of the vector space where $u^{(s)}$ acts). We may replace previous association $\vec{x} \to \hat{U}(\vec{x})$, $\vec{x} \to \hat{D}(\vec{x})$ with maps $G \to \mathcal{U}(\mathcal{H})$ given by

$$g_k \to u(g_k), \qquad g_k \to u^{-1}(g_k)\frac{N}{N_s}. \tag{11}$$

With the help of these maps, we can define complex-valued functions on G, forming the group algebra (see, e.g., [16, 17]), associated with operators (matrices) on $\mathcal{H}^{(s)}$ by setting

$$f^{(s)}_A(g_k) = \mathrm{Tr}\left(Au^{(s)}(g_k)\right) = \sum_{m=1}^{N_s}\left(Au^{(s)}(g_k)\right)_{mm} \tag{12}$$

or

$$f_A^{(s)}(g_k) = \sum_{m,j=1} A_{mj} u^{(s)}(g_k)_{jm}. \tag{13}$$

Again the "reconstruction" of the matrix from the function is provided by

$$\begin{aligned} A_{nj} &= \frac{N_s}{N} \sum_{k=1}^{N} f_A^{(s)}(g_k) u_{jn}^*(g_k) \\ &= \frac{N_s}{N} \sum_{k=1}^{N} f_A^{(s)}(g_k) u_{nj}^{-1}(g_k) \\ &= \frac{N_s}{N} \sum_{k=1}^{N} f_A^{(s)}(g_k) u_{nj}(g_k^{-1}). \end{aligned} \tag{14}$$

This shows that there is a one-to-one correspondence between complex-valued functions on G and operators in $\mathcal{H}^{(s)}$. We should stress that the operator associated with a given function depends on the chosen representation.

The kernel of the star product corresponding to the operator product is given by

$$K(g_1, g_2, g_3) = \mathrm{Tr} \left\{ \left(\frac{N_s}{N} \right)^2 u^{-1}(g_2) u^{-1}(g_3) u(g_1) \right\}. \tag{15}$$

Recalling the definition of characters of a representation, we find that up to normalization the kernel function is represented by a character of the group G that we are considering.

The associative algebra generated by these basic operators has structure constants given by

$$a_{as}^c \equiv \frac{\mathrm{Tr}\left(u(g_a)u(g_s)u^{-1}(g_c)\right)}{\mathrm{Tr}\, \mathbf{1}_k}. \tag{16}$$

If we use a deformed product [13] by means of a fixed unitary transformation k, we have

$$a_{as}^c(k) \equiv \frac{\mathrm{Tr}\left(u(g_a)ku(g_s)ku^{-1}(g_c)\right)}{\mathrm{Tr}\, \mathbf{1}_k}. \tag{17}$$

Following [15] we can also define the dual star product by exchanging the role of quantizers and dequantizers

$$\hat{U}_d(\vec{x}) \to \frac{N_s}{N} u^{-1}(g_k) \tag{18}$$

and

$$\hat{D}_d(\vec{x}) \to u(g_k). \tag{19}$$

Thus, our main result amounts to say that structure constants of the associative product induced on finite groups by operators acting on some Hilbert space carrying an irreducible representation are given by characters of the group representation we are considering.

Therefore, the tables of characters available in the literature allows us to construct in explicit manner families of associative products on $\mathcal{F}(G, \mathbf{C})$.

The construction we have considered shows, very clearly that it is not necessary that the map $G \to \mathcal{U}(\mathcal{H})$ be a group representation. Indeed we may consider a set S with a measure ds and an algebra $\mathcal{A}$ of operators and require that

$$\mathrm{Tr}\,\hat{D}(s)\hat{U}(s') = \delta(s, s'),$$

where $\delta(s, s')$ stays for a Kronecker δ or a Dirac delta as the case may be.

Then out of the basic operators we may define

$$f_A(s) = \mathrm{Tr}\left(A\hat{U}(s)\right)$$

along with

$$\left(f_a \star f_b\right)(s) = \mathrm{Tr}\,\hat{A}\hat{B}\hat{U}(s).$$

The reconstruction of $\hat{A}$ is again permitted by using $\hat{D}(s)$, it is

$$\hat{A} = \int f_A(s)\hat{D}(s)\,ds.$$

Of course, when S is finite-dimensional the measure will be a concentrated measure and the integral is replaced by a sum.

Elsewhere [13] we have considered deformations of the operator product by setting

$$\hat{A} \cdot_{\hat{K}} \hat{B} = \hat{A} \cdot \hat{K} \cdot \hat{B}.$$

If we use this deformed product on the space of operators, we induce a deformed product also on the associated functions on S as given in (17).

These observations should be kept in mind when we want to classify associative products on $\mathcal{F}(G, \mathbf{C})$. We should stress, however, that the identification of the structure constants with characters requires that we consider a group G and its unitary representations.

4 Some formulae for characters of finite groups

It is now possible to use the identification of structure constants with characters to derive easily some identities that characters must satisfy. The function associated with the unity operator will be just the character of irreducible representation

$$f_{\mathbf{I}}(g_k) = \mathrm{Tr}\left(\mathbf{I}u(g_k)\right) = \chi(g_k). \tag{20}$$

The relation $\mathbf{I} \cdot \mathbf{I} = \mathbf{I}$ implies

$$\frac{N_s^2}{N^2} \sum_{k,s=1}^{N} \chi(g_k)\chi(g_s)\chi\left(g_3 g_k^{-1} g_s^{-1}\right) = \chi(g_3) \tag{21}$$

and similarly for the dual star product.

Thus one has identity (21) which must be satisfied by characters of irreducible representations. On the other hand, the dual star-product scheme yields

$$\left(\frac{N_s}{N}\right)^2 \sum_{k,k'=1}^{N} \chi(g_k^{-1})\chi(g_{k'}^{-1})\chi\left(g_k g_{k'} g_s^{-1}\right) = \chi(g_s^{-1}). \tag{22}$$

Since

$$\mathbf{1}\cdot g = g, \tag{23}$$

one has

$$\frac{N_s^2}{N^2}\sum_{g_k g_{k'}=1}^{N}\chi(g_k)\chi(gg_{k'})\chi\left(g_s g_k^{-1} g_{k'}^{-1}\right) = \chi(gg_s). \tag{24}$$

Another composition formula for characters of finite (or compact) groups

$$\chi(g_s)\chi(g_t) = \frac{N}{N_s}\sum_r \chi\left(g_s g_r^{-1} g_t g_r\right) \tag{25}$$

is presented, for example, in [16, 17]. One can see that our formulae (22) and (24) are consistent with (25).

5 Examples

5.1 A group with two elements

Let us consider the reflection group $G = \{\mathbf{I}, P\}$ containing the identity and the reflection.

The group can be realized as group of two matrices

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The space of functions

$$f : \{\mathbf{I}, P\} \to \mathbf{C}$$

is isomorphic to $\mathbf{C}^2$, therefore the associative products on these functions can be considered as products on vectors of a two-dimensional complex vector space. If we use Dirac notation, we find

$$\mid f\rangle = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}. \tag{26}$$

The product of two functions f_1, f_2 is given by

$$\left(f_1 \star f_2\right)(k) = \sum_{k_a,k_b=1}^{2} K(k_a,k_b,k) f_1(k_a) f_2(k_b), \tag{27}$$

the kernel function (or the structure constants) is

$$K(k_a,k_b,k) = \chi\left(g_a g_b g_k^{-1}\right). \tag{28}$$

The reflection group G contains two elements — identity $I = g_1$ and reflection $P = g_2$. There are two irreducible one-dimensional representations $R^{(s)}$ given in table

$$\begin{array}{cc} g_1 & g_2 \\ 1 & 1 \end{array} \tag{29}$$

and

$$\begin{array}{cc} g_1 & g_2 \\ 1 & -1 \end{array}. \tag{30}$$

Thus this group can create two-dimensional Lie algebra. Also this group can be considered as permutation group of two elements

$$g_1 = 12, \qquad g_2 = 21. \tag{31}$$

Any function on the group G has two values

$$f(1) \equiv f(g_1), \qquad f(2) \equiv f(g_2). \tag{32}$$

The characters in (29) and (30) coincide with matrix elements. So our kernel being a function of three variables each having two values reads for representation (29)

$$K(g_{k_1}, g_{k_2}, g_k) = 1. \tag{33}$$

Thus in this case, one has

$$f_1(k) \star f_2(k) = f_1(g_1)f_2(g_1) + f_1(g_1)f_2(g_2) + f_1(g_2)f_2(g_1) + f_1(g_2)f_2(g_2). \tag{34}$$

In the vector form, we get the result of star-product of two vectors

$$\mid \psi\rangle_1 \star \mid \psi\rangle_2 = \mid \psi\rangle, \tag{35}$$

where vector $\mid \psi\rangle$ has equal components

$$\mid \psi\rangle = \begin{pmatrix} f \\ f \end{pmatrix} \tag{36}$$

and

$$f = f_1(1)f_2(1) + f_1(1)f_2(2) + f_2(1)f_1(2) + f_2(1)f_1(2). \tag{37}$$

For the case of representation (30), one has the kernel

$$K(g_1, g_1, g_1) = 1, \quad K(g_1, g_1, g_2) = -1, \quad K(g_1, g_2, g_1) = -1, \quad K(g_1, g_2, g_2) = 1. \tag{38}$$

This kernel provides the result of the product

$$\mid \tilde{\psi}\rangle = \mid \tilde{\psi}_1\rangle \star \mid \tilde{\psi}_2\rangle, \tag{39}$$

where vector $\mid \tilde{\psi}\rangle$ has two components

$$\mid \tilde{\psi}\rangle = \begin{pmatrix} \tilde{f} \\ -\tilde{f} \end{pmatrix}, \tag{40}$$

where

$$\tilde{f} = f_1(1)f_2(1) - f_1(1)f_2(2) - f_1(2)f_2(1) + f_1(2)f_2(2). \tag{41}$$

The structure constants of Lie algebra obtained by means of the kernels of the associative products are equal to zero for both kernels

$$C_{\alpha\beta}^{\gamma} = 0. \tag{42}$$

Thus we got Abelian algebras of dimension two.

We give explicit forms of these products, point-wise product, and standard convolution product.

For example, point-wise product of two vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix} \tag{43}$$

and convolution product gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 + x_2 y_2 \\ x_2 y_1 + x_1 y_2 \end{pmatrix}. \tag{44}$$

The constructed products give correspondingly

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 \\ x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 \end{pmatrix} \tag{45}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 - x_1 y_2 - x_2 y_1 + x_2 y_2 \\ -x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2 \end{pmatrix}. \tag{46}$$

One can see that product (44) is compatible[1] with the convolution product (45) provided that the vectors satisfy the condition: $x_1 = x_2$ and $y_1 = y_2$. Analogously, for vectors under the condition $x_1 = -x_2$ and $y_1 = -y_2$ product (45) is compatible with (44). This means that the two products (44) and (46) are compatible. One can take superposition of two kernels $K^{(s)}(g_1, g_2, g_3)$ corresponding to both different irreducible representations $s = 1, 2$ given by (33) and (38). This superposition satisfies the associativity equation being determined by the convolution product (44).

5.2 Quaternionic group

We consider a group G with eight elements in the representation provided by the Pauli matrices. If we properly redefine the trace, they are orthonormal. We have

$$\begin{aligned} &E = \sigma_0, \qquad P = -\sigma_0, \qquad K = i\sigma_1, \qquad L = i\sigma_2, \\ &M = i\sigma_3, \qquad K' = -i\sigma_1, \qquad L' = -i\sigma_2, \qquad M' = -i\sigma_3, \end{aligned} \tag{47}$$

where the space of functions $f : \{0, 1, 2, \ldots, 7\} \to \mathbf{C}$ is represented by $\mathbf{C}^8$ and we find the kernel function given below.

The abstract group multiplication table obtained using explicit form of the Pauli matrices

$$\begin{aligned} &\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{48}$$

[1] We recall that two associative products are said to be *compatible* if a linear combination of their structure constants defines again an associative product [18].

reads

$$
\begin{aligned}
&P^2 = E, \quad K^2 = L^2 = M^2 = K'^2 = L'^2 = M'^2 = P, \quad KL = M, \\
&LM = K, \quad MK = L, \quad K'L' = M', \quad L'M' = K', \quad M'K' = L', \\
&K'K = LL' = MM' = E, \quad LK = M', \quad ML = K', \quad KM = L'.
\end{aligned}
\tag{49}
$$

Other products of the group elements follow from this table easily. The group has five irreducible representations due to the decomposition

$$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2.$$

There are four one-dimensional and one two-dimensional representations. The representations are unitary ones. The characters of the representations are given in the following table of characters:

E	P	L	K	M	L'	K'	M'
1	1	1	1	1	1	1	1
1	1	1	−1	−1	1	−1	−1
1	1	−1	1	−1	−1	1	−1
1	1	−1	−1	1	−1	−1	1
2	−2	0	0	0	0	0	0

The first four rows in the table are characters of the one-dimensional representations satisfying the rules of group-element multiplication given in table (49). The last row contains characters of two-dimensional irreducible representation given by (47).

Now one can construct kernel of star product for a given group following the method described and using explicitly the properties of the Pauli matrices.

If we denote the elements in (47) as

$$
\begin{aligned}
&E = g_1, \qquad P = g_{-1}, \qquad K = g_2, \qquad L = g_3, \\
&M = g_4, \qquad K' = g_{-2}, \qquad L' = g_{-3}, \qquad M' = g_{-4}
\end{aligned}
\tag{50}
$$

and apply for the two-dimensional representation the formula for characters, we obtain the star-product-structure constants which are nonzero of the form

$$K_{mn}^s = \frac{1}{4}\left[\chi\left(g_m g_n g_s^{-1}\right)\right]. \tag{51}$$

Here only elements $K_{mn}^{\pm 1}$, $K_{\pm 1n}^m$, and $K_{n\pm 1}^m$ with $n = \pm m$ differ from zero, the other structure-constant elements are zero. For example,

$$K_{11}^1 = -K_{11}^{-1} = K_{-11}^{-1} = -K_{1-1}^1 = 1/2. \tag{52}$$

Since the antisymmetric part of the kernel of the star product is nonzero, the corresponding Lie algebra structure constants read

$$C_{mn}^s = K_{mn}^s - K_{nm}^s = \frac{1}{4}\mathrm{Tr}\left(\left[g_m, g_n\right] g_s^{-1}\right). \tag{53}$$

The obtained Lie algebra is a subalgebra of the eight complex linear transformations acting on $\mathbf{C}^2$ which define a four-dimensional complex vector space. By construction, the Lie algebra coincides with the Lie algebra of the $GL(2,C)$ group, i.e., the complexification of the Lie algebra of $U(2)$.

There exists another finite group with 8 elements. This group is the symmetry group of the square on a plane with four vertices at $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. This group contains four reflections with respect to lines which are axes of the Cartesian coordinates and the axes rotated by angle $2\pi/4$. There are also three rotations by angles $2\pi/4$, $2\pi/2$, and $3\pi/2$ denoted, respectively, as C_4, C_4^2, and C_4^2 and the identity element E. We denote these elements as E, C_4, C_4^2, C_4^3, Σ_1, Σ_2, σ_{13}, and σ_{24}. The elements Σ_1 and Σ_2 are reflections in the ordinate and abscissa lines, respectively, and the elements σ_{13} and σ_{24} are reflections with respect to bisectrices connecting vertices of the square. The group has a two-dimensional representation realized by Pauli matrices of the form

$$\begin{array}{cccccccc} E & C_4 & C_4^2 & C_4^3 & \Sigma_1 & \Sigma_2 & \sigma_{13} & \sigma_{24} \\ \sigma_0 & i\sigma_3 & -\sigma_0 & -i\sigma_3 & -\sigma_2 & \sigma_1 & -\sigma_1 & \sigma_2 \end{array}$$

We denote the elements as $E = g_1$, $C_4 = g_2$, $C_4^2 = g_3$, $C_4^2 = g_4$, $\Sigma_1 = g_5$, $\Sigma_2 = g_6$, $\sigma_{13} = g_7$, and $\sigma_{24} = g_8$. Then the multiplication table for this group is

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_2	g_2	g_3	g_4	g_1	g_8	g_7	g_5	g_6
g_3	g_3	g_4	g_1	g_2	g_6	g_5	g_8	g_7
g_4	g_4	g_1	g_2	g_3	g_7	g_8	g_6	g_5
g_5	g_5	g_7	g_6	g_8	g_1	g_3	g_2	g_4
g_6	g_6	g_8	g_5	g_7	g_3	g_1	g_4	g_2
g_7	g_7	g_6	g_8	g_5	g_4	g_2	g_1	g_3
g_8	g_8	g_5	g_7	g_6	g_2	g_4	g_3	g_1

The table of characters of the unitary irreducible representations is given below

E	C_4	C_4^2	c_4^3	Σ_1	Σ_2	σ_{13}	σ_{24}
2	0	−2	0	0	0	0	0
1	1	1	1	1	1	1	1
1	−1	1	−1	−1	−1	1	1
1	−1	1	−1	1	1	−1	−1
1	1	1	1	−1	−1	−1	−1

One can see that tables of characters of quaternionic group and symmetry group of square are identical. This means that star products described by the characters of irreducible representations are also the same. It is not trivial and intuitively not obvious why two different groups provide the same structure constants. In fact, these two finite groups of order eight can be mapped one into the other to be considered as different realizations of the same abstract group. The map consists of the shift of the group elements by left or right multiplication by another element of the group.

Let us describe the procedure in detail.

Given a group G with N elements $g_1, g_2, \ldots, g_N$. Let g_1 be identity element in the group $g_1 = E$. Let us have table of multiplication in the given group like

$$g_k g_j = g_s.$$

With respect to given identity element g_1, one has inverse element $g_1^{-1}, g_2^{-1}, \ldots, g_N^{-1}$. Let us consider the set $\tilde{G} = \tilde{g}_1, \tilde{g}_2, \ldots \tilde{g}_N$ with $\tilde{g}_k = g_k g_0$, where g_0 is chosen as one of N elements of the group G. Now one can introduce a new (with respect to the initial one) multiplication table in the set $\tilde{G}$ using the following rules ($\star$ rules)

$$\tilde{g}_k \star \tilde{g}_j = (g_k g_0)\, g_0^{-1}\, (g_j g_0) = \tilde{g}_s = g_s g_0. \tag{54}$$

The structure constants of the new product coincide with those of the previous one. The new rule uses idea of the so-called k-product of matrices as mentioned in Sect. 3 and considered, e.g., in [13] (or k-deformed product of matrices) where the rule of product – row by column – is modified by inserting a chosen matrix k when one multiplies two matrices a and b. This means that

$$a \cdot b \to a \cdot_k b = akb. \tag{55}$$

This matrix product is associative. The new group multiplication (54) just uses the analog of the rule (55) where the element g_0^{-1} plays the role of matrix k. This means that in terms of the initial identity element g_1 and the "deforming" shift element g_0 the new identity in $\tilde{G}$ reads

$$\tilde{E} = g_1 g_0 = g_0 = \tilde{g}_1. \tag{56}$$

In fact,

$$\tilde{g}_k \cdot \tilde{g}_1 = g_k g_0 g_0^{-1} g_1 g_0 = g_k g_0 = \tilde{g}_k. \tag{57}$$

Also

$$\tilde{E} \star \tilde{g}_k = g_0 g_0^{-1} g_k g_0 = \tilde{g}_k.$$

Thus with the new deformed multiplication rule one has the new identity element and reproduces the multiplication table of the initial group. Exactly this happens in the case of the quaternionic group and the symmetry group of square. Nevertheless, the realization of symmetry operations physically is quite different in both cases. For example, the identity element for symmetry of square means that one is doing no operation with square. The identity element which in the deformed group (quaternionic group) is reflection, physically differs from 'doing-no-operation'. These findings are in line with our general considerations at the end of Sect. 3.

5.3 Example of C_{3v} group

The group of permutations of three elements is the group of symmetry of the equilateral triangle. The elements are:

$$g_1 = 1, \quad g_2 = u_1, \quad g_3 = u_2, \quad g_4 = u_3, \quad g_5 = C_3, \quad g_6 = C_3^2, \tag{58}$$

here C_3 is a cyclic permutation and u_1, u_2, u_3 are permutations, odd ones. There are three irreducible representations with table of characters of the form

$$\begin{array}{cccccc} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 2 & 0 & 0 & 0 & -1 & -1 \end{array} \tag{59}$$

Let us discuss one-dimensional representations.

Given a 1×1 matrix A which is a number. The symbol of this operator reads

$$f_A(g) = Au(g) = \mathrm{Tr}\left(Au(g)\right). \tag{60}$$

In the case of identity representation, the reconstruction formula reads

$$A = \frac{1}{6}\sum_{k=1}^{6} f_A(g_k) = \frac{1}{6}6A = A. \tag{61}$$

Analogously reconstruction formula can be obtained for the second one-dimensional representation with characters given in the second line of (59).

The considered operators A and B, acting in a one-dimensional Hilbert space, are numbers. The product of two operators AB is just product of these numbers. The star-product of the symbols in this case reads

$$f_{AB}(g) = \mathrm{Tr}\left(ABu(g)\right) = ABu(g) = f_A(g) \star f_B(g). \tag{62}$$

Let us check that this formula is coherent with the formula with the star-product kernel

$$\begin{aligned} f_A(g) \star f_B(g) &= \sum_{g_1 g_2}\left[\mathrm{Tr}\left(\frac{1}{N}u^{-1}(g_1)\frac{1}{N}u^{-1}(g_2)u(g)\right)\right] f_A(g_1) f_B(g_2) \\ &= \frac{1}{N^2}\sum_{g_1 g_2} Au(g_1)Bu(g_2)u^{-1}(g_1)u^{-1}(g_2)u(g) \\ &= ABu(g) \end{aligned} \tag{63}$$

Thus we checked that the formula yields the result shown in (62). Now one can apply the same kernel to use the star-product of functions on the whole group. In this case, a function on the group can be considered as a 6-vector. The product of the functions is equivalent to the star-product of two 6-vectors. If one uses as a kernel of the star-product the character of identity representation, one gets

$$\vec{f}_1 \star \vec{f}_2 = \vec{f}, \tag{64}$$

where the 6-vector $\vec{f}$ has all six components equal to x and this number x is expressed in terms of the components of the vectors f_{1s} and f_{2k} as

$$f = \frac{1}{36}\sum_{k=1}^{6}\sum_{s=1}^{6} f_{1k} f_{2s}. \tag{65}$$

In the case of antisymmetric representation, the components of the vector $\vec{f}$ have different signs corresponding to even and odd permutations. One can check that these two star-products are not equivalent.

The Lie algebra structure constants are equal to zero if characters of irreducible representations have the properties

$$\chi\left(g_1 g_2 g_3^{-1}\right) = \chi\left(g_2 g_1 g_3^{-1}\right) \quad \text{or} \quad \chi\left(g_1^{-1} g_2^{-1} g_3\right) = \chi\left(g_2^{-1} g_1^{-1} g_3\right) \tag{66}$$

for all the elements g_1, g_2, g_3.

In the case of C_{3v} group, for all its one-dimensional irreducible representations, equality (66) holds. In view of this, the Lie algebras associated with these irreducible representations of this group are Abelian ones.

In the case of k-deformed star-product, one has deformed kernels

$$K_k(g_1, g_2, g_3) = \frac{N_s^2}{N^2}\mathrm{Tr}\left(ku(g_2^{-1}g_3g_1^{-1})\right) = \frac{N_s^2}{N^2} f_k(g_2^{-1}g_3g_1^{-1}) \tag{67}$$

and

$$K_k^d(g_1, g_2, g_3) = \frac{N_s}{N}\mathrm{Tr}\left(ku(g_2g_3^{-1}g_1)\right). \tag{68}$$

Let us consider in detail the two-dimensional representation of the group consisting of 6 elements g_k (or group C_{3v}). Its table of multiplication reads

$$\begin{array}{llllll}
g_1g_1 = g_1, & g_1g_2 = g_2, & g_1g_3 = g_3, & g_1g_4 = g_4, & g_1g_5 = g_5, & g_1g_6 = g_6, \\
g_1g_2 = g_2, & g_2g_2 = g_3, & g_2g_3 = g_1, & g_2g_4 = g_5, & g_2g_5 = g_6, & g_2g_6 = g_4, \\
g_3g_1 = g_3, & g_3g_2 = g_1, & g_3g_3 = g_2, & g_3g_4 = g_6, & g_3g_5 = g_4, & g_3g_6 = g_5, \\
g_4g_1 = g_4, & g_4g_2 = g_6, & g_4g_3 = g_5, & g_4g_4 = g_1, & g_4g_5 = g_3, & g_4g_6 = g_2, \\
g_5g_1 = g_5, & g_5g_2 = g_4, & g_5g_3 = g_6, & g_5g_4 = g_2, & g_5g_5 = g_1, & g_5g_6 = g_3, \\
g_6g_1 = g_6, & g_6g_2 = g_5, & g_6g_3 = g_4, & g_6g_4 = g_3, & g_6g_5 = g_2, & g_6g_6 = g_1.
\end{array}$$

The matrices of two-dimensional irreducible representation read

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}, \qquad g_3 = \begin{pmatrix} \varphi^2 & 0 \\ 0 & \varphi^{-2} \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_5 = \begin{pmatrix} 0 & \varphi \\ \varphi^{-1} & 0 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} 0 & \varphi^2 \\ \varphi^{-2} & 0 \end{pmatrix},$$

where $\varphi = e^{2\pi i/3}$ corresponds to rotation by $2\pi/3$.

According to the construction of symbol of the operator $\hat{A}$ with the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{69}$$

one has the following values of the function $f(g)$ on the permutation group

$$f_A(g_k) = \frac{1}{3}\mathrm{Tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} g_k^{-1}. \tag{70}$$

The reconstruction formula for the matrix A can be written in terms of quantizer operator and it reads

$$\begin{aligned} A \quad = \quad & \frac{1}{3}\left[(a_{11}+a_{22})\,g_1 + \left(a_{11}\varphi + a_{22}\varphi^{-1}\right) g_3 + \left(a_{11}\varphi^{-1} + a_{22}\varphi\right) g_2\right. \\ & \left.(a_{12}+a_{21})\,g_4 + \left(a_{12}\varphi^{-1} + a_{21}\varphi\right) g_5 + \left(a_{12}\varphi + a_{21}\varphi^{-1}\right) g_2\right]. \end{aligned} \tag{71}$$

We used dual formula for star-product. The Lie algebra constructed by means of the structure constants obtained with the character formula yields in the basis

$$\begin{aligned} & y_1 = g_2 - g_3, \qquad y_2 = g_4, \qquad y_3 = g_5, \\ & y_4 = g_1, \qquad y_5 = g_2 + g_3, \qquad y_6 = g_4 + g_5 + g_6 \end{aligned} \tag{72}$$

the following commutation relations:

$$[y_1, y_2] = 2y_2 + 4y_3 - 2y_6, \qquad [y_2, y_3] = -y_5,$$
$$[y_3, y_1] = 2y_3 + 4y_2 - 2y_6, \qquad [y_1, y_6] = [y_2, y_6] = [y_3, y_6] = 0. \tag{73}$$

The operators y_4, y_5, and y_6 commute with all the other operators. The Lie algebra obtained is a direct sum of a five-dimensional Lie algebra and a one-dimensional one. The five-dimensional Lie algebra is an extension of the Heisenberg–Weyl algebra defined by (y_2, y_3, y_5). The nontrivial structure constants are

$$C_{12}^2 = 2, \quad C_{12}^3 = 4, \quad C_{23}^5 = -1, \quad C_{31}^2 = 4, \quad C_{31}^3 = 2, \quad C_{12}^6 = -2, \quad C_{31}^6 = -2. \tag{74}$$

Analogously, k-deformed kernel can be constructed for star product. Also k-deformed Lie-algebra structure constants can be expressed in terms of the characters.

5.4 Example of $SU(2)$ group

The construction can be applied also for a compact group G. One has only to make change in (11) since instead of sum one has integral over compact group, i.e.,

$$\int d\mu(g)\, u^{(s)}(g)_{mn} u^{\nu(p)}_{\alpha\beta}(g) = \delta_{m\alpha}\delta_{u\beta}\frac{V}{N_s}\delta_{sp}, \tag{75}$$

where $d\mu(g)$ is invariant Haar measure and

$$\int d\mu(g) = V \tag{76}$$

is group volume. Superindices (s) and (p) describe the Casimir operators eigenvalues distinguishing different irreducible representations of the compact groups.

For spin $j = 1/2$ (defining representation), one has that

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \alpha = \cos\frac{\theta}{2}e^{i(\varphi+\psi)/2}, \quad \beta = \sin\frac{\theta}{2}e^{i(\varphi-\psi)/2}. \tag{77}$$

The symbol of a spin operator (matrix A) reads

$$f_A(g) = \mathrm{Tr}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}. \tag{78}$$

One has inverse relation using the quantizer

$$A = \int \frac{2}{V} d\mu(g)\, f_A(g) g^{-1}. \tag{79}$$

The kernel of star-product reads

$$K(g_1, g_2, g_3) = \frac{4}{V^2}\mathrm{Tr}\left[\begin{pmatrix} \alpha_3^* & -\beta_3 \\ \beta_3^* & \alpha_3 \end{pmatrix}\begin{pmatrix} \alpha_2^* & -\beta_2 \\ \beta_2^* & \alpha_2 \end{pmatrix}\begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1^* & \alpha_1^* \end{pmatrix}\right]. \tag{80}$$

The dual kernel reads

$$K^{(d)}(g_1,g_2,g_3)=\frac{2}{V}\mathrm{Tr}\left[\left(\begin{array}{cc}\alpha_1 & \beta_1\\ -\beta_1^* & \alpha_1^*\end{array}\right)\left(\begin{array}{cc}\alpha_2 & \beta_2\\ -\beta_2^* & \alpha_2^*\end{array}\right)\left(\begin{array}{cc}\alpha_3^* & -\beta_3\\ \beta_3^* & \alpha_3\end{array}\right)\right]. \qquad (81)$$

Analogous explicit formulae can be given using known matrix elements of irreducible representations of $SU(2)$ group in terms of Euler angles. The star-product of two functions on $SU(2)$ group given as the kernel (80) reads

$$\left(f_1\star f_2\right)(g)=\int K(g_1,g_2,g)f_1(g_1)f_2(g_2)\,d\mu(g_1)\,d\mu(g_2), \qquad (82)$$

where the $SU(2)$-group elements g_1, g_2, g are labelled by Euler angles. The measures $d\mu(g_1)$ and $d\mu(g_2)$ are known Haar measures of the $SU(2)$ group. The kernel can be taken from (80) and (81).

One can easily calculate the Lie algebra structure constants by calculating traces

$$C(g_1,g_2,g_3)=\frac{4}{V^2}\mathrm{Tr}\left[\left[\left(\begin{array}{cc}\alpha_3^* & -\beta_3\\ \beta_3^* & \alpha_3\end{array}\right),\left(\begin{array}{cc}\alpha_2^* & -\beta_2\\ \beta_2^* & \alpha_2\end{array}\right)\right]\left(\begin{array}{cc}\alpha_1 & \beta_1\\ -\beta_1^* & \alpha_1^*\end{array}\right)\right] \qquad (83)$$

and

$$C^{(d)}(g_1,g_2,g_3)=\frac{2}{V}\mathrm{Tr}\left[\left[\left(\begin{array}{cc}\alpha_1 & \beta_1\\ -\beta_1^* & \alpha_1^*\end{array}\right),\left(\begin{array}{cc}\alpha_2 & \beta_2\\ -\beta_2^* & \alpha_2^*\end{array}\right)\right]\left(\begin{array}{cc}\alpha_3^* & -\beta_3\\ \beta_3^* & \alpha_3\end{array}\right)\right]. \qquad (84)$$

The structure constant are not zero.

In fact, we present Lie group structure constants (84) in explicit form

$$C^{(d)}\left(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3\right)=\frac{2}{V}\left\{\left(\beta_2\beta_1^*-\beta_1\beta_2^*\right)\alpha_3^*+\left(\alpha_1\beta_2-\alpha_2\beta_1+\beta_1\alpha_2^*-\beta_2\alpha_1^*\right)\beta_3^*\right.$$
$$\left.-\beta_3\left(\beta_2^*\alpha_1-\alpha_2^*\beta_1^*+\alpha_1^*\beta_2^*-\beta_1^*\alpha_2\right)+\alpha_3\left(\beta_2^*\beta_1-\beta_1^*\beta_2\right)\right\}. \qquad (85)$$

The structure constants both for star product and Lie product are strongly related to the used irreducible two-dimensional representation of $SU(2)$ group. If one considers an arbitrary function on $SU(2)$ group $\Phi(\alpha,\beta)$, it can be decomposed into series connecting all the irreducible representations. The star-product kernel constructed makes projection to the components in these series which belong to the chosen irreducible representation. One can see this phenomenon for the group of two elements.

Thus, the functions (vectors) of the form

$$\mid f\rangle=\left(\begin{array}{c}f\\ f\end{array}\right)$$

being multiplied by kernel induced by identity representation keep this form yielding result (45).

The functions (vectors) of the form

$$\mid\Phi\rangle=\left(\begin{array}{c}x\\ -x\end{array}\right)$$

being multiplied by the same kernel yield as result zero function

$$\mid\Phi_1\rangle\star\mid\Phi_2\rangle=0.$$

Thus the kernel provides projection on the functions corresponding to the irreducible representation.

6 Conclusions

We now point out the main results of the paper.

We have shown that there exists a star product of complex-valued functions on finite and compact groups. The kernels generating such products are expressed in terms of characters of irreducible unitary representations of these groups. Thus, the known tables of the characters induce star products on the functions over the groups (in group algebras). The relations of the kernels associated with different irreducible representations (for example, the compatibility of the structure constants) needs further study. The star-product kernels provide in the standard manner Lie algebra structure constants. Therefore, we found a relation between finite (and compact) group irreducible representations and star-product kernels along with the structure constants of the related Lie algebras. We hope to study in detail the mutual compatibility of the obtained structure constants in a future publication. In particular, we shall address the decomposition of the group algebra into minimal ideals and the structure of the space of unitary representations of a given finite group G.

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