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Asymptotic of extremal polynomials in the complex plane

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Abstract

We study the zero location and asymptotic zero distribution of sequences of polynomials which satisfy an extremal condition with respect to a norm given on the space of all polynomials.

Keywords and phrases: Extremal polynomials, multiplication operator, asymptotic distribution of zeros.

1 Introduction.

Sobolev orthogonal polynomials play a central role in the extension of the general theory of orthogonal polynomials. Let $\mu_0, \ldots, \mu_N$ be finite positive Borel measures in the complex plane such that the support $S(\mu_0)$ of μ_0 contains infinitely many points and all polynomials are integrable. Let $\mathcal{P}$ denote the vector space of all polynomials. On $\mathcal{P}$ we define the norm

$$\|q\|_S = \left(\sum_{j=0}^{N} \int |q^{(j)}|^2 d\mu_j \right)^{1/2}, \qquad q \in \mathcal{P}, \tag{1}$$

where $q^{(j)}$ denotes the jth derivative of q. The nth monic Sobolev polynomial with respect to this norm is the unique polynomial q_n of degree n and leading coefficient equal to 1 such that

$$\|q_n\|_S = \inf\{\|q\|_S : q = z^n + \cdots\} .$$

The existence and uniqueness of q_n is easily guaranteed. In fact, q_n is the monic polynomials of degree n that satisfies the orthogonality relations

$$0 = \sum_{j=0}^{N} \int (z^\nu)^{(j)} \overline{q_n^{(j)}} d\mu_j , \qquad \nu = 0, \dots, n-1$$

Some of the most relevant results in the asymptotic theory of Sobolev orthogonal polynomials may be found in [7], [11] for strong asymptotic behavior, and [6], [10] for weak asymptotic. There are no results specific for ratio asymptotic; that is, except when strong asymptotic takes place. A key problem in the study of the asymptotic behavior of Sobolev orthogonal polynomials is the question regarding the location of the zeros of such polynomials. Unlike the case of standard orthogonality, the zeros of Sobolev orthogonal polynomials can abandon the support of the measures involved in the inner product. An approach which allows to deal with this problem, in terms of a bound of the multiplication operator defined on $\mathcal{P}$, was introduced in [9] using the notion of a sequentially dominated family of measures. This approach was also used in [10].

The family $\mu_0, \dots, \mu_N$ of measures is said to be sequentially dominated if

$$d\mu_j = f_j d\mu_{j-1} , \qquad j = 1, \dots, N , \tag{2}$$

where the f_j are bounded positive Borel measurable functions. In [10] the authors prove that if $S(\mu_0)$ is compact and the family of measures is sequentially dominated then the multiplication operator is bounded in the normed space $(\mathcal{P}, \|\cdot\|_S)$ and the zeros of the Sobolev orthogonal polynomials lie in the disk centered at the origin and radius equal to twice the norm of the operator. Set $f_0 \equiv 1$. Notice that sequential domination allows to write (1) in the form

$$\|q\|_S = \left(\sum_{j=0}^{N} \int f_0 \cdots f_j |q^{(j)}|^2 d\mu_0 \right)^{1/2} , \qquad q \in \mathcal{P} . \tag{3}$$

In Theorem 4.1 of [12], the author proves, for Sobolev inner products supported on the real line, that the boundedness of the multiplication operator implies that the corresponding Sobolev norm is essentially sequentially dominated. Essential sequential domination means that the given Sobolev norm is equivalent to another Sobolev norm which is sequentially dominated. Following basically the same arguments, in [1] the authors prove a similar result for measures supported in the complex plane. Therefore, sequential domination is a natural restriction if we are concerned with finding bounds for the multiplication operator.

Let μ be a finite Borel measure with compact support consisting of infinitely many points in $\mathbb{C}$, $\Lambda = \mathrm{diag}(\lambda_j), 0 \le j \le N$, is a diagonal matrix of bounded positive μ almost everywhere measurable functions, and $U = (u_{j,k}), 0 \le j,k \le N$, is a square matrix of bounded Borel measurable functions such that the matrix

$$U(x) = (u_{j,k}(x))\,, \quad 0 \le j,k \le N\,,$$

is unitary μ almost everywhere. We say that U is unitary. Set

$$W = U\Lambda U^*\,, \tag{4}$$

where U^* denotes the transpose conjugate of U.

Let $T = (T_0, \ldots, T_N)$, where $T_j : \mathcal{P} \longrightarrow \mathcal{P}, j = 0, \ldots, N$, are linear applications. We assume that T is injective. Fix $p, 1 \le p < \infty$. Set

$$\|q\|_1 = \left(\int \left[T(q) W^{2/p} T(q)^* \right]^{p/2} d\mu \right)^{1/p} = \left(\int \left[T(q) U \Lambda^{2/p} U^* T(q)^* \right]^{p/2} d\mu \right)^{1/p}. \tag{5}$$

It is not difficult to verify that under the assumptions imposed, $\|\cdot\|_1$ defines a norm on $\mathcal{P}$. For $p = 2$ this norm coincides with the one introduced in [5]. Moreover, (3) can be expressed in the form (5) taking $T(q) = (q, \ldots, q^{(N)})$ and U the identity matrix. A more general case is when U is an arbitrary unitary matrix with constant coefficients. In this case you obtain a generalized Sobolev norm in which the product of derivatives of different order appears.

For simplicity, in defining $\|\cdot\|_1$ we have decided to start out from the decomposition (4). One can begin at an earlier stage from a μ almost everywhere positive definite matrix W made up of bounded Borel measurable functions, or even from a matrix of measures which evaluated on each Borel set is positive semi-definite (see, for example, [3, Lemma 11, page 1341]). Under general assumptions, on W or on the matrix of measures, the existence of a (non constructive) decomposition of type (4) can be guaranteed but this is a delicate matter which we prefer to avoid in order to preserve the constructiveness of our arguments. A simple case in which there are no difficulties in carrying out the decomposition is when W is a positive definite matrix with constant entries or, more generally, with continuous entries.

We say that $q_n = z^n + \cdots$ is an nth monic extremal polynomial with respect to (5) if

$$\|q_n\|_1 = \inf\{\|q\|_1 : q = z^n + \cdots\}\,.$$

The existence of q_n is easy to prove. When $1 < p < \infty$ the norm $\|\cdot\|_1$ is strictly convex and thus q_n is uniquely determined. For the definition of a strictly convex norm and its connection with the uniqueness property see pp. 22-23 of [2].

One of the basic results of this paper states the following.

Theorem 1 *Let $S(\mu)$ be compact, $T(q) = (q, \ldots, q^{(N)})$, and U unitary. Assume that*

$$\lambda_j/\lambda_k \leq C\,, \qquad 0 \leq j,k \leq N\,, \tag{6}$$

μ almost everywhere. Let $\{q_n\}, n \in \mathbb{Z}_+$, be a sequence of extremal polynomials with respect to (5). Then the zeros of the polynomials in $\{q_n\}, n \in \mathbb{Z}_+$, are uniformly bounded in the complex plane.

The radius of a disk centered at the origin containing the zeros of all the q_n can be determined in terms of C. When (6) takes place we say that the family of measures $\lambda_0 d\mu, \ldots, \lambda_N d\mu$, is totally dominated.

Section 2 is dedicated to the proof of a general result on the uniform bound of the zeros of sequences of extremal polynomials from which Theorem 1 follows directly. In Section 3 we study the zero distribution of extremal polynomials for the case considered in Theorem 1.

To state the result on the zero distribution of extremal polynomials we need some concepts. In [15], the authors introduce the class **Reg** of regular measures. For measures supported on a compact set of the complex plane, they prove that (see Theorem 3.1.1) $\mu \in \mathbf{Reg}$ if and only if

$$\lim_{n\to\infty} \|Q_n\|_{L_2(\mu)}^{1/n} = \operatorname{cap}(S(\mu))\,.$$

Here, Q_n denotes the nth monic orthogonal polynomial (in the standard sense) with respect to μ, $\|\cdot\|_{L_2(\mu)}$ is the usual norm in the space $L_2(\mu)$ of square integrable functions with respect to μ, and $\operatorname{cap}(S(\mu))$ denotes the logarithmic capacity of $S(\mu)$. If $S(\mu)$ is a regular compact set with respect to the solution of the Dirichlet problem on the unbounded connected component of the complement of $S(\mu)$ in the extended complex plane and $1 \leq p < \infty$, we have (see Theorem 3.4.3 in [15]) that $\mu \in \mathbf{Reg}$ if and only if

$$\lim_{n\to\infty} \left(\frac{\|\widetilde{q}_n\|_{S(\mu)}}{\|\widetilde{q}_n\|_{L_p(\mu)}} \right)^{1/n} = 1\,, \tag{7}$$

where $\{\widetilde{q}_n\}, n \in \mathbb{Z}_+$, is any sequence of polynomials such that $\deg \widetilde{q}_n = n, n \in \mathbb{Z}_+$. Here and in the following, $\|\cdot\|_{S(\mu)}$ denotes the sup norm on $S(\mu)$.

For any polynomial q of exact degree n, let us define

$$\nu(q) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}\,,$$

where $z_1, \ldots, z_n$ are the zeros of q repeated according to their multiplicity, and δ_{z_j} is the Dirac measure with mass one at the point z_j. This is the so called normalized zero counting measure associated with q. By $\omega_{S(\mu)}$ we denote the equilibrium measure on $S(\mu)$. We have

Theorem 2 *Let us assume that $\lambda_0 d\mu \in$ **Reg**, $S(\mu)$ is regular with respect to the Dirichlet problem, (6) takes place, $T(q) = (q, \dots, q^{(N)})$, and $\{q_n\}, n \in \mathbb{Z}_+$, is the sequence of extremal polynomials with respect to the corresponding $\|\cdot\|_1$. Then*

$$\lim_{n\to\infty} \|q_n^{(j)}\|_{S(\mu)}^{\frac{1}{n}} = \operatorname{cap}(S(\mu)) \,, \qquad j \in \mathbb{Z}_+ \,. \tag{8}$$

Furthermore, if $S(\mu)$ has empty interior and its complement is connected, then

$$\lim_{n\to\infty} \nu(q_n^{(j)}) = \omega_{S(\mu)} \,, \qquad j \in \mathbb{Z}_+ \,, \tag{9}$$

in the weak star topology of measures.

2 Bound of M on the space $(\mathcal{P}, \|\cdot\|_1)$.

Let $M : \mathcal{P} \longrightarrow \mathcal{P}$ be the multiplication operator; that is $M(q) = xq$. We are interested in finding sufficient conditions which guarantee that the multiplication operator is bounded on $(\mathcal{P}, \|\cdot\|_1)$. The reason for our interest comes from the following result which extends Theorem 2 of [10] to any norm on $\mathcal{P}$.

Theorem 3 *Let $(\mathcal{P}, \|\cdot\|)$ be a normed space and assume that*

$$\|M\| = \sup_{\|q\|=1} \|M(q)\| < +\infty \,.$$

Let $q_n = z^n + \cdots, n = 1, 2, \dots,$ be such that

$$\|q_n\| = \inf\{\|q\| : q = z^n + \cdots\} \,. \tag{10}$$

Then the zeros of q_n lie in the bounded disk $\{z : |z| \le 2\|M\|\}$.

Proof. Let z_0 be a zero of q_n. Then, there exists a monic polynomial q of degree $n-1$ such that $q_n = (z - z_0)q$. Since q_n satisfies (10), we have

$$|z_0|\|q\| - \|zq\| \le \|z_0 q - zq\| = \|q_n\| \le \|zq\|.$$

Then,

$$|z_0|\|q\| \le 2\|zq\| \le 2\|M\|\|q\|.$$

Since $\|q\| \ne 0$, the conclusion readily follows. ∎

Is is easy to see that in Theorem 2 the norm may not be substituted by a semi norm. In fact, suppose that there exists a polynomial $q, q \not\equiv 0$, such that $\|q\| = 0$. Obviously, for any constant $c \ne 0$, the polynomial cq satisfies the same conditions. Let $n > \deg q$ and q_n be an extremal monic polynomial of degree n. It is easy to see that $q_n + cq$ is also a monic extremal polynomial for all c. Taking c sufficiently large we can have zeros of $q_n + cq$ as large as we want.

Let $U = [u_0, \ldots, u_n]$ where $u_j, 0 \leq j \leq N$, are the column vectors of U. Notice that

$$T(q)W^{2/p}T(q)^* = \sum_{j=0}^{N} \lambda_j^{2/p} |T(q)u_j|^2 \,,$$

μ almost everywhere. It is well known (see [8] Theorem 27 and pages 71–72) that for $x_j \geq 0, j = 0, 1, \ldots, N$,

$$\sum_{j=0}^{N} x_j^\alpha \leq \left(\sum_{j=0}^{N} x_j \right)^\alpha \leq (N+1)^{\alpha-1} \sum_{j=0}^{N} x_j^\alpha, \quad \alpha \geq 1,$$

and

$$(N+1)^{\alpha-1} \sum_{j=0}^{N} x_j^\alpha \leq \left(\sum_{j=0}^{N} x_j \right)^\alpha \leq \sum_{j=0}^{N} x_j^\alpha, \quad 0 < \alpha \leq 1.$$

Using theses inequalities, it follows that for $p \geq 2$

$$\sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \leq \left| T(q)W^{2/p}T(q)^* \right|^{p/2} \leq (N+1)^{(p-2)/2} \sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \,,$$

μ almost everywhere and if $1 \leq p < 2$

$$(N+1)^{(p-2)/2} \sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \leq \left| T(q)W^{2/p}T(q)^* \right|^{p/2} \leq \sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \,.$$

Consequently, for all $p \geq 1$ there exist positive constants C_1, C_2 such that

$$C_1 \sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \leq \left| T(q)W^{2/p}T(q)^* \right|^{p/2} \leq C_2 \sum_{j=0}^{N} \lambda_j |T(q)u_j|^p \,.$$

Set

$$\|q\|_2 = \left(\sum_{j=0}^{N} \int \lambda_j |T(q)u_j|^p d\mu \right)^{1/p} , \qquad q \in \mathcal{P} \,.$$

It follows that (see (5))

$$C_1^{1/p} \|q\|_2 \leq \|q\|_1 \leq C_2^{1/p} \|q\|_2 \,, \qquad q \in \mathcal{P}. \tag{11}$$

Because of this, it is equivalent to prove the boundedness of the multiplication operator with respect to $\| \cdot \|_1$ and $\| \cdot \|_2$.

We will also consider the norm on $\mathcal{P}$ given by

$$\|q\|_3 = \left(\sum_{j=0}^{N} \int \lambda_j |T_j(q)|^p d\mu \right)^{1/p}, \qquad q \in \mathcal{P}\,. \tag{12}$$

The $\|\cdot\|_2$ norm reduces to the $\|\cdot\|_3$ norm when $U = I$. Let us prove some properties of these three norms.

Lemma 1 *For $1 < p < \infty$, all three norms $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_3$ are strictly convex on $\mathcal{P}$.*

Proof. In order to prove that $\|\cdot\|_1$ is strictly convex it is sufficient to show that if q, r are polynomials, not identically equal to zero, and $\|q+r\|_1 = \|q\|_1 + \|r\|_1$, then there exists $\alpha > 0$ such that $r \equiv \alpha q$.

Let $\|\cdot\|_{euc}$ denote the Euclidean norm on $\mathbb{C}^{N+1}$ and $\|\cdot\|_\mu$ the usual norm on $L_p(\mu), 1 < p < \infty$. Obviously,

$$\|q\|_1 = \| \|T(q)U\Lambda^{1/p}\|_{euc} \|_\mu\,.$$

For short, let us denote $\widetilde{q} = T(q)U\Lambda^{1/p}$. Then, using the triangular inequality and the monotonicity of the integral, we have

$$\|q+r\|_1 = \| \|\widetilde{q+r}\|_{euc}\|_\mu = \| \|\widetilde{q} + \widetilde{r}\|_{euc} \cdot \|_\mu \leq \| \|\widetilde{q}\|_{euc} + \|\widetilde{r}\|_{euc}\|_\mu \leq$$

$$\| \|\widetilde{q}\|_{euc}\|_\mu + \| \|\widetilde{r}\|_{euc}\|_\mu = \|q\|_1 + \|r\|_1\,.$$

If $\|q+r\|_1 = \|q\|_1 + \|r\|_1$, we must have equality on each step above. It follows (see p. 63 in [13]) that there exists an $\alpha > 0$ such that

$$\|\widetilde{r}\|_{euc} = \alpha \|\widetilde{q}\|_{euc}\,,$$

μ almost everywhere. Also,

$$\|\widetilde{q} + \widetilde{r}\|_{euc} = \|\widetilde{q}\|_{euc} + \|\widetilde{r}\|_{euc}$$

μ almost everywhere. Let $x \in S(\mu)$, the last equality yields that μ almost everywhere there exists $\alpha(x) > 0$ such that

$$\widetilde{r}(x) = \alpha(x)\widetilde{q}(x)\,.$$

Then

$$\|\widetilde{r}(x)\|_{euc} = \alpha(x)\|\widetilde{q}(x)\|_{euc}\,.$$

Since $q \not\equiv 0$, $\|\widetilde{q}(x)\|_{euc} \neq 0$, and $\alpha(x) = \alpha$, μ almost everywhere. Therefore,

$$\widetilde{r} = T(r)U\Lambda^{1/p} = \alpha\widetilde{q} = T(\alpha q)U\Lambda^{1/p}$$

μ almost everywhere. Since $U\Lambda^{1/p}$ is injective μ almost everywhere and T is injective, it follows that $r = \alpha q$, μ almost everywhere, and thus $r \equiv \alpha q$ as we needed to prove.

The $\|\cdot\|_3$ norm is a special case of the $\|\cdot\|_2$ norm, so to conclude the proof it is sufficient to show that the $\|\cdot\|_2$ norm is strictly convex. In order to prove this, one can follow essentially the previous arguments since

$$\|q\|_2 = \left(\sum_{k=0}^{N} \|T(q)u_j\|^p_{L_p(\lambda_j d\mu)} \right)^{1/p},$$

and the p norm on $\mathbb{C}^{N+1}$ is also strictly convex. We leave the details to the reader. □

Lemma 2 *Assume that* (6) *takes place. There exist positive constants* C_3, C_4, C_5, C_6, *such that*

$$C_3\|q\|_3 \leq \|q\|_2 \leq C_4\|q\|_3\,, \qquad q \in \mathcal{P}\,. \tag{13}$$

and

$$C_5\|q\|_3 \leq \|q\|_1 \leq C_6\|q\|_3\,, \qquad q \in \mathcal{P}\,. \tag{14}$$

Proof. We already know that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (regardless of (6)), so it suffices to prove (13). Let us prove the first inequality in (13), the second one is obtained analogously but easier. Let e_j denote the unitary column vector with 1 in jth position and 0 in the rest. Let $v_j = (u_{j,0}, \ldots, u_{j,N})^*$ be the jth column of U^*. Since U is unitary, v_j is the transpose conjugate of the jth row of U. Suppose that $p > 1$, using Holder's inequality and (6), we have

$$\lambda_j|T_j(q)|^p = \lambda_j|T(q)e_j|^p = \lambda_j|T(q)Uv_j|^p = \lambda_j|\sum_{k=0}^{N} T(q)u_k u^*_{j,k}|^p$$

$$\leq \lambda_j \left(\sum_{k=0}^{N} |T(q)u_k|^p \right) \left(\sum_{k=0}^{N} |u^*_{j,k}|^r \right)^{p/r} \leq C(N+1)^{p/r} \sum_{k=0}^{N} \lambda_k |T(q)u_k|^p\,, \tag{15}$$

μ almost everywhere, where $\frac{1}{p} + \frac{1}{r} = 1$ (notice that $|u^*_{j,k}| \leq 1, \mu$ almost everywhere). For $p = 1$ the inequality above is even easier to obtain with the constant C on the right hand. Therefore,

$$\left(\sum_{j=0}^{N} \int \lambda_j |T_j(q)|^p d\mu \right)^{1/p} \leq C^{1/p}(N+1)^{1/r} \left(\sum_{k=0}^{N} \int \lambda_k |T(q)u_k|^p d\mu \right)^{1/p}$$

as needed. □

Theorem 4 *Let* $T(q) = (q, q^{(1)}, \ldots, q^{(N)})$ *and* (6) *take place. Then the multiplication operator is bounded on* $(\mathcal{P}, \|\cdot\|_3)$ *and, consequently, with respect to the norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *in* $\mathcal{P}$. *In all three spaces the zeros of the extremal polynomials are uniformly bounded.*

Proof. The last statement is a consequence of Theorem 3. From (11) and (13) the three norms are equivalent so it is sufficient to show that the operator is bounded with respect to $\|\cdot\|_3$.

Notice that

$$(xq)^{(j)} = xq^{(j)} + jq^{(j-1)}\,, \qquad k = 0, \dots, N\,.$$

Therefore,

$$\|xq\|_3 = \left(\sum_{j=0}^{N} \int \lambda_j |xq^{(j)} + jq^{(j-1)}|^p d\mu\right)^{1/p}$$

$$\leq 2^{(p-1)/p}\left(\sum_{j=0}^{N} \int \lambda_j (|xq^{(j)}|^p + |jq^{(j-1)}|^p) d\mu\right)^{1/p} \leq C_7 \|q\|_3$$

for an appropriate constant C_7. In the last step one uses a bound for $|x|$ on $S(\mu)$ and (6) in order to correct the measure which multiplies $|q^{(j-1)}|$ plus obvious details. □

Theorem 1 is Theorem 4 as applied to $\|\cdot\|_1$. Notice that in deducing the last inequality in the proof of Theorem 4 it is only required that the functions λ_j be sequentially dominated. When $U = I$, since $\|\cdot\|_2$ and $\|\cdot\|_3$ coincide, Lemma 2 is not needed and, therefore, the theorem remains valid under the weaker assumption of sequential domination.

Another application is produced taking

$$T_i(q) = \sum_n \frac{q^{(n(N+1)+i)}(0)}{(n(N+1)+i)!}\, x^n\,, \qquad i = 0, \dots, N. \tag{16}$$

These operators appear in [4] in connection with the study of sequences of polynomials on the real line that satisfy recurrence relations with $2N+3$ terms. It is well known that there exists a close relation between polynomials satisfying recurrence relations of higher order and matrix orthogonal polynomials.

Theorem 5 *Let $T = (T_0, T_1, \dots, T_N)$, where $T_k, k = 0, \dots, N$, is defined according to* (16), *and* (6) *take place. Then the multiplication operator is bounded on $(\mathcal{P}, \|\cdot\|_3)$ and, consequently, with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathcal{P}$. In all three spaces the zeros of the extremal polynomials are uniformly bounded.*

Proof. It is easy to verify that

$$\begin{aligned} T(xq) &= (T_0(xq), \dots, T_N(xq)) \\ &= (xT_N(q), T_0(q), \dots, T_{N-1}(q)). \end{aligned}$$

Thus

$$\|M(q)\|_3 = \left(\int \lambda_0 |xT_N(q)|^p d\mu + \sum_{j=1}^{N} \int \lambda_j |T_{j-1}(q)|^p d\mu\right)^{1/p} \leq C_8 \|q\|_3\,,$$

where C_8 is the product of the constant C in (6) times the maximum between the sup norm of $|x|$ on $S(\mu)$ and 1. The rest of the proof is a direct consequence of Theorem 3 and Lemma 2. $\Box$

3 Asymptotic distribution of zeros.

We are ready for the

Proof of Theorem 2. Let $\mathcal{T}_n$ be the n-th monic Tchebychev polynomial of degree n with respect to the compact set $S(\mu)$, and q_n the n-th monic extremal polynomial with respect to $\|\cdot\|_1$. Denote $d\nu = \lambda_0 d\mu$. From the extremal property of q_n, (6), and (14), we have

$$\begin{aligned} C_5\|q_n\|_{L_p(\nu)} &\le C_5\|q_n\|_3 \le \|q_n\|_1 \le \|\mathcal{T}_n\|_1 \le C_6\|\mathcal{T}_n\|_3 \\ &\le (|\nu|(1+NC))^{1/p} C_6 \max_{0\le k\le N} \|\mathcal{T}_n^{(k)}\|_{S(\mu)}\,, \end{aligned} \tag{17}$$

where $|\nu| = \nu(S(\mu))$.

It is well known that $\lim_{n\to\infty} \|\mathcal{T}_n\|_{S(\mu)}^{\frac{1}{n}} = \operatorname{cap}(S(\mu))$. By [10, Lemma 3.1] applied to $\mathcal{T}_n$, it follows that

$$\limsup_{n\to\infty} \|\mathcal{T}_n^{(j)}\|_{S(\mu)}^{\frac{1}{n}} \le \operatorname{cap}(S(\mu))\,, \qquad j \in \mathbb{Z}_+. \tag{18}$$

From (17) and (18), we obtain

$$\limsup_{n\to\infty} \|q_n\|_{L^p(\nu)}^{\frac{1}{n}} \le \operatorname{cap}(S(\mu))\,.$$

This together with (7) imply

$$\limsup_{n\to\infty} \|q_n\|_{S(\mu)}^{\frac{1}{n}} \le \operatorname{cap}(S(\mu))\,,$$

and using again [10, Lemma 3.1], we obtain

$$\limsup_{n\to\infty} \|q_n^{(j)}\|_{S(\mu)}^{\frac{1}{n}} \le \operatorname{cap}(S(\mu))\,, \qquad j \in \mathbb{Z}_+\,.$$

On the other hand,

$$\liminf_{n\to\infty} \|q_n^{(j)}\|_{S(\mu)}^{\frac{1}{n}} \ge \operatorname{cap}(S(\mu))\,, \qquad j \in \mathbb{Z}_+\,.$$

since this inequality holds for any sequence of polynomials such that $\deg q_n = n$. Hence, (8) takes place. If $S(\mu)$ has empty interior and connected complement, according to Corollary III.4.8 in [14], (8) implies (9). With this we conclude the proof. $\Box$

Notice that from the proof also follows that

$$\lim_{n\to\infty} \|q_n\|_1^{1/n} = \lim_{n\to\infty} \|q_n\|_{L^p(\nu)}^{1/n} = \mathrm{cap}(S(\mu))\,.$$

Let $g_\Omega(z;\infty)$ denote Green's function for the unbounded component Ω of the complement of $S(\mu)$ with logarithmic singularity at ∞. We will assume that $S(\mu)$ is regular with respect to the Dirichlet problem. Then, $g_\Omega(z;\infty)$ is continuous up to the boundary and we extend it continuously to all $\mathbb{C}$ assigning it the value zero on the complement of Ω.

Theorem 6 *Let us assume that $\lambda_0 d\mu \in \mathbf{Reg}$, $S(\mu)$ is regular with respect to the Dirichlet problem,* (6) *takes place, $T(q) = (q, \ldots, q^{(N)})$, and $\{q_n\}, n \in \mathbb{Z}_+$, is a sequence of extremal polynomials with respect to $\|\cdot\|_1$. Then, for each $j \in \mathbb{Z}_+$*

$$\limsup_{n\to\infty} |q_n^{(j)}(z)|^{\frac{1}{n}} \leq \mathrm{cap}(S(\mu))e^{g_\Omega(z;\infty)}\,, \tag{19}$$

uniformly on compact subsets of $\mathbb{C}$. Furthermore,

$$\lim_{n\to\infty} |q_n^{(j)}(z)|^{\frac{1}{n}} = \mathrm{cap}(S(\mu))e^{g_\Omega(z;\infty)}, \tag{20}$$

uniformly on each compact subset of $\{z : |z| > 2\|M\|_1\} \cap \Omega$. Finally, if the interior of $S(\mu)$ is empty and its complement connected, we have equality in (19) *for all $z \in \mathbb{C}$ except on a set of capacity zero, $S(\omega_{S(\mu)}) \subset \{z : |z| \leq 2\|M\|_1\}$, and*

$$\lim_{n\to\infty} \frac{q_n^{(j+1)}(z)}{nq_n^{(j)}(z)} = \int \frac{d\omega_{S(\mu)}(x)}{z-x},$$

uniformly on each compact subset of $\{\, z \ : \ |z| > 2\|M\|_1\}$.

Proof. Fix $j \in \mathbb{Z}_+$ and set

$$v_n(z) = \frac{1}{n-j} \log \frac{|q_n^{(j)}(z)|}{\|q_n^{(j)}\|_{S(\mu)}} - g_\Omega(z;\infty).$$

Let us show that

$$v_n(z) \leq 0, \quad z \in \mathbb{C} \cup \{\infty\}. \tag{21}$$

This function is subharmonic in $\Omega \cup \infty$ and on the boundary of Ω it is ≤ 0. By the maximum principle for subharmonic functions it is ≤ 0 on all $\Omega \cup \{\infty\}$. On the complement of Ω, by the maximum principle of analytic functions, we have that $|q_n^{(j)}(z)|/\|q_n^{(j)}\|_{S(\mu)} \leq 0$ and $g_\Omega(z,\infty) = 0$ by definition. Therefore, (21) takes place. Taking upper limit in (21) and using (8) we get (19).

From Theorem 3, we have that for all $n \in \mathbb{Z}_+$, the zeros of the extremal polynomials are contained in the disc $\{z : |z| \leq 2\|M\|_1\}$. It is well known that the zeros of the derivative of a polynomial lie in the convex hull of the zeros of the polynomial itself. Therefore, for all $j \in \mathbb{Z}_+$, the zeros of $q_n^{(j)}$ for all $n \in \mathbb{Z}_+$ lie in $\{z : |z| \leq 2\|M\|_1\}$. Using this, we have that $\{v_n\}_{n\in\mathbb{Z}_+}$, forms a sequence of harmonic functions in $\Omega' = \{z : |z| > 2\|M\|_1\} \cap (\Omega \cup \{\infty\})$ uniformly bounded on each compact subset of Ω'. Take a sequence of indices Λ such that $\{v_n\}_{n\in\Lambda}$ converges uniformly on each compact subset of Ω'. Let v_Λ denote its limit. Obviously, v_Λ is harmonic and ≤ 0 in Ω'. Because of (8), $v_\Lambda(\infty) = 0$. Therefore, $v_\Lambda \equiv 0$ in Ω'. Since this is true for every convergent subsequence of $\{v_n\}_{n\in\mathbb{Z}_+}$, we get that the whole sequence converges to zero uniformly on each compact subset of Ω' which is equivalent to (20).

If the interior of $S(\mu)$ is empty and its complement connected, we can use (9). The measures $\nu_{n,j} = \nu(q_n^{(j)}), n \in \mathbb{Z}_+$, and $\omega_{S(\mu)}$ have their support contained in a compact subset of $\mathbb{C}$. Using this and (9), from the Lower Envelope Theorem (see [15, page 223]), we obtain

$$\liminf_{n\to\infty} \int \log \frac{1}{|z-x|} d\nu_{n,j}(x) = \int \log \frac{1}{|z-x|} d\omega_{S(\mu)}(x)\,,$$

for all $z \in \mathbb{C}$ except on a set of zero capacity. This is equivalent to having equality in (19) except on a set of zero capacity, because (see [15, page 10])

$$g_\Omega(z;\infty) = \log \frac{1}{\mathrm{cap}(S(\mu))} - \int \log \frac{1}{|z-x|} d\omega_{S(\mu)}(x)\,.$$

Let $x_{n,i}^j, i = 1, \cdots, n-j$, denote the $n-j$ zeros of $q_n^{(j)}$. As indicated, all these zeros are contained in $\{z : |z| \leq 2\|M\|_1\}$. From (9), each point of $S(\mu)$ must be a limit point of zeros of $\{q_n^{(j)}\}$; therefore, $S(\omega_{S(\mu)}) \subset \{z : |z| \leq 2\|M\|_1\}$. Decomposing into simple fractions and using the definition of $\nu_{n,j}$, we obtain

$$\frac{q_n^{(j+1)}(z)}{nq_n^{(j)}(z)} = \frac{1}{n}\sum_{i=1}^{n-j} \frac{1}{z - x_{n,i}^j} = \frac{n-j}{n} \int \frac{1}{z-x} d\nu_{n,j}(x). \tag{22}$$

Therefore, for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{ \frac{q_n^{(j+1)}(z)}{nq_n^{(j)}(z)} \right\}, \quad n \in \mathbb{Z}_+, \tag{23}$$

is uniformly bounded on each compact subset of $\{z : |z| > 2\|M\|_1\}$.

On the other hand, all the measures $\nu_{n,j}, n \in \mathbb{Z}_+$, are supported in $\{z : |z| \leq 2\|M\|_1\}$ and for $z, |z| > 2\|M\|_1$, fixed, the function $(z-x)^{-1}$ is continuous with respect to x on $\{x : |x| \leq 2\|M\|_1\}$. Therefore, from (9) and (22), we find that any subsequence of (23) uniformly convergent on compact subsets of $\{z : |z| > 2\|M\|_1\}$, converges pointwise to $\int (z-x)^{-1} d\omega_{S(\mu)}(x)$. Thus, the whole sequence converges uniformly to this function on compact subsets of $\{z : |z| > 2\|M\|_1\}$ and we are done. □

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