RATIO ASYMPTOTIC OF HERMITE-PADÉ ORTHOGONAL POLYNOMIALS FOR NIKISHIN SYSTEMS

A.I. Aptekarev [1] **G. López Lagomasino** [2] **I.A. Rocha** [3]

Abstract

We prove the existence of ratio asymptotic for a sequence of multiple orthogonal polynomials which share orthogonality relations with a collection of m finite Borel measures supported on a bounded interval of the real line and constitute a so called Nikishin system of measures. When $m = 1$ our result reduces to E. A. Rakhmanov's known Theorem on ratio asymptotic for orthogonal polynomials on a segment.

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1 Introduction

Let s be a finite positive Borel measure supported on a bounded interval Δ of the real line $\mathbb{R}$ such that $s' > 0$ almost everywhere on Δ and let $\{Q_n\}, n \in \mathbb{Z}_+$, be the corresponding sequence of monic orthogonal polynomials; that is, with leading coefficients equal to one. In a series of two papers (see [19] and [20]), E. A. Rakhmanov proved that under these conditions

$$\lim_{n\in\mathbb{Z}_+} \frac{Q_{n+1}(z)}{Q_n(z)} = \frac{\varphi(z)}{\varphi'(\infty)}, \qquad K \subset \mathbb{C} \setminus \Delta \tag{1}$$

(uniformly on each compact subset of $\mathbb{C} \setminus \Delta$), where $\varphi(z)$ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta$ onto $\{w : |w| > 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. This result attracted great attention because of its theoretical interest within the general theory of orthogonal polynomials and its applications to the theory of rational approximation of analytic functions. Simplified proofs of Rakhmanov's theorem may be found in [21] and

[1] Keldysh Institute of Applied Mathematics, Moscow (RUSSIA) <aptekaa@keldysh.ru> .

[2] Dpto. de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 15, 28911 Leganés, Madrid (SPAIN) <lago@math.uc3m.es> .

[3] Dpto. de Matemática Aplicada, E.U.I.T. Telecomunicación, Universidad Politécnica de Madrid, Ctra. de Valencia Km. 7, 28031 Madrid (SPAIN) <igalvar@euitt.upm.es>

[16] and extensions to the case of orthogonal polynomials with respect to varying measures (depending on the degree of the polynomial) in [14] and [15].

In recent years, a subject of major interest has been the extension of the general theory of orthogonal polynomials to the case of polynomials verifying non-standard orthogonality relations. An example is provided by polynomials whose orthogonality relations are distributed between several measures. Such polynomials arise naturally as the common denominator of Hermite-Padé approximations of systems of Markov functions. A wide class of systems of Markov functions which has proved to be adequate for the generalization of many results from the general theory of orthogonal polynomials and Padé approximation was introduced by E. M. Nikishin in [17]. Such systems are defined as follows. We use the notation proposed in [11].

Let σ_1, σ_2 be two finite Borel measures with constant sign, whose supports $\mathrm{supp}(\sigma_1)$, $\mathrm{supp}(\sigma_2)$ are contained in non intersecting intervals Δ_1, Δ_2, respectively, of the real line $\mathbb{R}$. Set

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x)\,.$$

This expression defines a new measure with constant sign whose support coincides with that of σ_1. Whenever we find it convenient we use the differential notation of a measure.

Let $\Sigma = (\sigma_1, \ldots, \sigma_m)$ be a system of finite Borel measures on the real line with constant sign and compact support. Let $\Delta_k = [a_k, b_k]$ denote the smallest interval which contains the support of σ_k. Assume that $\Delta_k \cap \Delta_{k+1} = \emptyset,\ k = 1, \ldots, m-1$. By definition, $S = (s_1, \ldots, s_m) = \mathcal{N}(\Sigma)$ is called the *Nikishin system* generated by Σ if

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, \quad s_m = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \rangle \tag{2}$$

Fix a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. The polynomial $Q_{\mathbf{n}}(x)$ is called an **n**-th *multi-orthogonal polynomial* with respect to S if it is not identically equal to zero, $\deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \cdots + n_m$, and satisfies the orthogonality relations

$$\int Q_{\mathbf{n}}(x) x^{\nu} ds_k(x) = 0, \quad \nu = 0, \ldots, n_k - 1, \quad k = 1, \ldots, m. \tag{3}$$

It is well known and easy to verify that $Q_{\mathbf{n}}(x)$ is the common denominator of the Hermite-Padé vector rational approximant

$$\pi_{\mathbf{n}}(z) := \left(\frac{P_{\mathbf{n},1}(z)}{Q_{\mathbf{n}}(z)}, \ldots, \frac{P_{\mathbf{n},m}(z)}{Q_{\mathbf{n}}(z)} \right)$$

for the system of Markov functions

$$\widehat{s}_k(z) := \int \frac{ds_k(x)}{z-x}, \quad k = 1, \ldots, m\,.$$

These approximants are defined by the conditions:

i) $\deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \cdots + n_m\,, \quad Q_{\mathbf{n}} \not\equiv 0\,,$

ii) $Q_{\mathbf{n}}(z)\widehat{s}_k(z) - P_{\mathbf{n},k}(z) = O\left(\frac{1}{z^{n_k+1}}\right), \quad z \to \infty, \quad k = 1, \ldots, m.$

In the sequel, we assume that $Q_{\mathbf{n}}$ is monic.

For an arbitrary multi-index $\mathbf{n}$ the conditions above, in general, do not guarantee that $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. If this is the case, the multi–index $\mathbf{n}$ is said to be normal and the corresponding orthogonal polynomial is uniquely determined up to a constant factor. In addition, if the zeros of $Q_{\mathbf{n}}$ are simple and lie in the interior of Δ_1 the multi–index is said to be strongly normal. (In relation to intervals of the real line the interior refers to the Euclidean topology of $\mathbb{R}$.) Let

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : 1 \leq i < j \leq m \Rightarrow n_j \leq n_i + 1\}\,.$$

In [6] it was proved that all multi-indices in $\mathbb{Z}_+^m(\circledast)$ are strongly normal. Set

$$\mathbb{Z}_+^m(*) = \{\mathbf{n} \in \mathbb{Z}_+^m : \nexists\ 1 \leq i < j < k \leq m\,, \text{ with } n_i < n_j < n_k\}\,.$$

Recently, in [8] it was shown that all multi–indices in $\mathbb{Z}_+^m(*)$ are also strongly normal. It is easy to see that $\mathbb{Z}_+^m(\circledast) \subset \mathbb{Z}_+^m(*)$. In particular, for $m = 2$ all multi–indices are strongly normal. In [7] the authors proved that this is also the case when $m = 3$. Thus, when $m = 1, 2, 3$ all multi–indices are strongly normal. A question of theoretical and practical importance is whether or not this is true for all $m \in \mathbb{Z}_+$.

The logarithmic asymptotic behavior of $\{Q_{\mathbf{n}}\}, \mathbf{n} \in \Lambda \subset \mathbb{Z}_+^m(\circledast)$, where Λ is such that the orthogonality conditions are proportionally distributed between the different components, and $\sigma_k' > 0$ almost everywhere on $\Delta_k, k = 1, \ldots, m$, was obtained in Theorem 3 of [11]. For this type of asymptotic, the condition on the measures can be substantially relaxed to being of class **Reg** in the sense defined by H. Stahl and V. Totik in [22] (see Theorem 5 of [9], where the case $\Lambda \subset \mathbb{Z}_+^m(*)$ is also considered). For indices of the form $(n, \ldots, n), n \in \mathbb{Z}_+$ and measures $\sigma_1, \ldots, \sigma_m$ absolutely continuous and in the Szegő class, the strong asymptotic was given by A. I. Aptekarev in [1].

In this paper, we are concerned with extending Rakhmanov's Theorem on ratio asymptotic for multiple orthogonal polynomials with respect to a Nikishin system of measures. In particular, we prove the following.

Theorem 1.1. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system such that $\sigma_k' > 0$ almost everywhere on $\Delta_k = \mathrm{supp}(\sigma_k), k = 1, \ldots, m$. Consider the sequence $\mathbf{I} \subset \mathbb{Z}_+^m(*)$ given by*

$$\mathbf{I} = \{(0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, 0, \ldots), \ldots, (1, \ldots, 1), (2, 1, \ldots, 1), \ldots \ldots\}.$$

Let $(Q_{\mathbf{n}}(x))_{\mathbf{n} \in \mathbf{I}}$ be the corresponding sequence of monic multiple orthogonal polynomials. For each fixed $\mathbf{n}$, set $\mathbf{n} + \mathbf{1} = \mathbf{n} + (1, \ldots, 1)$. Then, there exists a function $\widetilde{F}_1$ holomorphic on $\mathbb{C} \setminus \Delta_1$ such that

$$\lim_{\mathbf{n} \in \mathbf{I}} \frac{Q_{\mathbf{n}+\mathbf{1}}(z)}{Q_{\mathbf{n}}(z)} = \widetilde{F_1}(z), \qquad K \subset \mathbb{C} \setminus \Delta_1 .$$

The validity of Theorem 1.1 and the description of the limiting function $\widetilde{F_1}$ follows from a more general result. Before stating it, we need to introduce additional notions related with the orthogonality properties of the denominators $Q_{\mathbf{n}}$ of the Hermite–Padé approximants of a Nikishin systems. To avoid making the notation too cumbersome, in the sequel we restrict our attention to sequences of multi-indices in $\mathbb{Z}_+^m(\circledast)$.

For each $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$, define recursively the following functions

$$\Psi_{\mathbf{n},0}(x) = Q_{\mathbf{n}}(x), \quad \Psi_{\mathbf{n},k}(x) = \int \frac{\Psi_{\mathbf{n},k-1}(t)}{x - t} d\sigma_k(t), \quad k = 1, \ldots, m.$$

In Proposition 1 of [11] it was proved that for each $k = 1, \ldots, m$

$$\int \Psi_{\mathbf{n},k-1}(t)\, t^\nu d\langle \sigma_k, \ldots, \sigma_{k+r}\rangle(t) = 0, \quad \nu = 0, \ldots, n_{k+r} - 1, \quad k \leq k + r \leq m. \tag{4}$$

From here, the authors deduce that $\Psi_{\mathbf{n},k-1}, k = 1, \ldots, m$, has exactly $N_{\mathbf{n},k} = n_k + \cdots + n_m$ zeros in $\mathbb{C} \setminus \Delta_{k-1}$, that they are all simple, and lie in the interior of Δ_k. Let $Q_{\mathbf{n},k}$ be the monic polynomial of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\Psi_{\mathbf{n},k-1}$ vanishes in Δ_k and let $Q_{\mathbf{n},m+1} \equiv 1$. In Proposition 2 (see also Proposition 3) of [11] the authors prove that

$$\int x^\nu \Psi_{\mathbf{n},k-1}(x) \frac{d\sigma_k(x)}{Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m. \tag{5}$$

Set

$$H_{\mathbf{n},k} := \frac{Q_{\mathbf{n},k-1}\Psi_{\mathbf{n},k-1}}{Q_{\mathbf{n},k}}.$$

From (5), we have that for each multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$ there exists a system of polynomials

$$\{Q_{\mathbf{n},k}\}_{k=1}^m, \quad \deg Q_{\mathbf{n},k} = \sum_{\alpha=k}^m n_\alpha := N_{\mathbf{n},k}, \qquad Q_{\mathbf{n},0} = Q_{\mathbf{n},m+1} \equiv 1, \tag{6}$$

satisfying the system of full orthogonality relations

$$\int x^{\nu} Q_{\mathbf{n},k}(x) \frac{|H_{\mathbf{n},k}(x)| d\sigma_k(x)}{|Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)|} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m\,, \tag{7}$$

with respect to a varying measure. (Notice that the functions $H_{\mathbf{n},k}$ and $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ have constant sign on Δ_k, thus we can take absolute value of these functions under the integral sign without affecting the value of the integral.) Our goal is to study the ratio asymptotic for the polynomials $\{Q_{\mathbf{n},k}\}_{k=1}^m$. In particular, $Q_{\mathbf{n}} = Q_{\mathbf{n},1}$.

The answer will be given in terms of certain algebraic functions of order $m+1$ (as in Rakhmanov's Theorem for m=1). To introduce these functions, we consider the $(m+1)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{m} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \quad \mathcal{R}_k := \overline{\mathbb{C}} \setminus \{\Delta_k \cup \Delta_{k+1}\}, \qquad k = 1, \ldots, m-1, \qquad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \Delta_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \ldots, m\}$. Let $\psi^{(l)}, l = 1, \ldots, m$, be a single valued rational function on $\mathcal{R}$ whose divisor consists of one simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and one simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. Therefore,

$$\psi^{(l)}(z) = C_1/z + \mathcal{O}(1/z^2)\,,\; z \to \infty^{(0)}\,, \qquad \psi^{(l)}(z) = C_2 z + \mathcal{O}(1)\,,\; z \to \infty^{(l)}\,, \tag{8}$$

where C_1 and C_2 are constants different from zero. Since the genus of $\mathcal{R}$ equals zero, such a single valued function on $\mathcal{R}$ exists and is uniquely determined up to a multiplicative constant. We denote the branches of the algebraic function $\psi^{(l)}$, corresponding to the different sheets $k = 0, \ldots, m$ of $\mathcal{R}$ by

$$\psi^{(l)} := \{\psi_k^{(l)}\}_{k=0}^m\,.$$

In the sequel, we fix the multiplicative constant in such a way that

$$\prod_{k=0}^{m} \psi_k^{(l)}(\infty) = 1\,. \tag{9}$$

For any fixed multi-index $\mathbf{n} = (n_1, \ldots, n_m)$, set

$$\mathbf{n}^l := (n_1, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m)\,.$$

Given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1}), C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := \frac{F}{C} \tag{10}$$

Now, we can state our general theorem about ratio asymptotic for multiple orthogonal polynomials of a Nikishin system.

Theorem 1.2. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system such that $\sigma_k' > 0$ almost everywhere on $\Delta_k = \operatorname{supp}(\sigma_k), k = 1, \ldots, m$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+^m(\circledast)$ and $n_1 - n_m \leq d$, where d is a constant. Let $\{Q_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, be the corresponding system of polynomials (6). Then for each fixed $k \in \{1, \ldots, m\}$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}^l,k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F_k^{(l)}}(z), \qquad z \in K \subset \mathbb{C} \setminus \Delta_k \tag{11}$$

where

$$F_k^{(l)} := \prod_{\nu=k}^{m} \psi_\nu^{(l)}, \tag{12}$$

and the algebraic functions $\psi_\nu^{(l)}$ are defined by (8) – (9).

A direct consequence of the theorem is

Corollary 1.1. *Under the asumptions of Theorem 1.2, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n+1},k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F_k}(z), \qquad z \in K \subset \mathbb{C} \setminus \Delta_k,$$

where

$$F_k := \prod_{\nu=k}^{m} \psi_\nu, \qquad \psi_\nu := \prod_{l=1}^{m} \psi_\nu^{(l)},$$

Setting $k = 1$ and $I = \Lambda$, we obtain the statement of Theorem 1.1. On the other hand, taking $m = 1$ we arrive to Rakhmanov's formula (1) for monic orthogonal polynomials on Δ. We wish to point out that in deducing Theorem 1.2 a version of Rakhmanov's theorem for orthogonal polynomials with respect to varying measures is used; therefore, our result does not contain an alternative proof of Rakhmanov's theorem.

The proof of Theorem 1.2 is based on the following. In Section 2 we show that the zeros of the multiple orthogonal polynomials $Q_{\mathbf{n}}$ for a Nikishin system (and of the associated polynomials $Q_{\mathbf{n},k}, k = 2, \ldots, m$) corresponding to consecutive strongly normal indices satisfy an interlacing property as that fulfilled by standard orthogonal polynomials on the real line. Therefore, for each fixed $k, l = 1, \ldots, m$, the ratios $Q_{\mathbf{n}^l,k}/Q_{\mathbf{n},k}$ form normal families of analytic functions in $\mathbb{C} \setminus \Delta_k$, respectively. Using results from the theory of orthogonal polynomials with respect to varying measures developed earlier by G. López, we prove that the limit functions of convergent subsequences satisfy a system of boundary value problems. This is done in Section 3. A similar system of equations appeared before in A. I. Aptekarev's proof of the strong asymptotic behavior of multiple orthogonal polynomials with respect to a Nikishin system. To conclude, in Section 4 we show that the system of boundary value problems has a unique solution which may be expressed by means of the algebraic functions defined above. In the sequel, we preserve the notation introduced previously.

2 Interlacing properties of zeros

In [12], D. Kershaw proved an interlacing property for the zeros of polynomials which are orthogonal to a Tchebyshev system with respect to the Lebesgue measure on an interval of the real line. The authors of [8] adapt Kershaw's arguments to show that if $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_+(\circledast)$ then the zeros of $Q_{\mathbf{n}}$ and $Q_{\mathbf{n}^l}$ interlace. We include here an alternative proof which allows to derive the interlacing property not only for the zeros of $Q_{\mathbf{n}}$ and $Q_{\mathbf{n}^l}$ but also for the zeros of $Q_{\mathbf{n},k}$ and $Q_{\mathbf{n}^l,k}$, $k = 1, \ldots, m$.

A family of N real continuous functions $\{m_1(x), \ldots, m_N(x)\}$ is said to be a *Tchebyshev system* on an interval $[a, b]$ if there do not exist constants c_k, not all equal to zero, such that $\sum_{k=1}^{N} c_k m_k(x)$ has more than $N - 1$ zeros on $[a, b]$. If for all $n, 1 \leq n \leq N$, $\{m_1(x), \ldots, m_n(x)\}$ is a Tchebychev system on $[a, b]$ then such a system of N functions is called a *Markov system* on $[a, b]$. For details on the properties of Markov and Tchebyshev systems see, for example, [2] and [13]. We call *change knot* a point of the real line where a function changes its sign.

Lemma 2.1. *Let* $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_m^+(\circledast)$ *for some fixed* $l \in \{1, \ldots, m\}$ *Then, for each* $k \in \{0, 1, \ldots, m\}$ *and for any real constants* A *and* B *such that* $|A| + |B| > 0$, *the function* $A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}^l,k}$ *has at most* $|\mathbf{n}| + 1 - (n_1 + \cdots + n_k)$ *zeros in* $\mathbb{C} \setminus \Delta_k$ *(counting multiplicities) and at least* $|\mathbf{n}| - (n_1 + \cdots + n_k)$ *change knots in the interior of* $\Delta_{k+1} (\Delta_0 = \Delta_{m+1} = \emptyset)$. *Therefore, all the zeros of* $A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}^l,k}$ *in* $\mathbb{C} \setminus \Delta_k$ *are simple and lie in* $\mathbb{R} \setminus \Delta_k$.

Proof. Fix A and B satisfying the conditions of the lemma. For simplicity, we denote

$$G_{\mathbf{n},k} = A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}^l,k}\,.$$

and $N_{\mathbf{n},k} = n_k + \cdots + n_m$. Let h_j be a polynomial of degree $\leq n_j - 1$ and $s_{i,j} = \langle \sigma_i, \ldots, \sigma_j \rangle, 1 \leq i \leq j \leq m$. From (4) it follows that

$$0 = \int G_{\mathbf{n},k}(t) \sum_{j=k+1}^{m} h_j(t) \widehat{s}_{k+2,j}(t) d\sigma_{k+1}(t) \tag{13}$$

where $\widehat{s}_{k+2,k+1} \equiv 1$. Theorem 2 in [8] gives us that the system of functions

$$\{1, \ldots, x^{n_{k+1}-1}, \widehat{s}_{k+2,k+2}, \ldots, x^{n_{k+2}-1}\widehat{s}_{k+2,k+2}, \ldots, \widehat{s}_{k+2,m}, \ldots, x^{n_m-1}\widehat{s}_{k+2,m}\}$$

forms a Markov system on Δ_{k+1}. From this and on account of (13), we obtain that $G_{\mathbf{n},k}$ has at least $N_{\mathbf{n},k+1}$ change knots in the interior of Δ_{k+1} as stated.

For $k = 0$ it is obvious that $G_{\mathbf{n},0} = AQ_{\mathbf{n},0} + BQ_{\mathbf{n}^l,0}$ has at most $|\mathbf{n}| + 1 = N_{\mathbf{n},1} + 1$ zeros in $\mathbb{C}$. Let us assume that for $k = 0, \ldots, r-1, 1 \leq r \leq m$, the function $G_{\mathbf{n},k}$ has at most $N_{\mathbf{n},k+1} + 1$ zeros, counting multiplicities, in $\mathbb{C} \setminus \Delta_k$ and $G_{\mathbf{n},r}$ has at least $N_{\mathbf{n},r+1} + 2$ zeros in $\mathbb{C} \setminus \Delta_r$ ($N_{\mathbf{n},m+1} = 0$).

By definition, the function $G_{\mathbf{n},r}$ is analytic in $\mathbb{C} \setminus \Delta_r$ and symmetric with respect to the real line (this function ant its derivatives take conjugate values at conjugate points). Therefore, there exists a polynomial $W_{\mathbf{n},r}$ with real coefficients of degree $\geq N_{\mathbf{n},r+1} + 2$ whose zeros lie in $\mathbb{C} \setminus \Delta_r$ such that

$$\frac{G_{\mathbf{n},r}}{W_{\mathbf{n},r}} \in H(\mathbb{C} \setminus \Delta_r)\,.$$

From (4) we also have that

$$0 = \int G_{\mathbf{n},r-1}(t) \frac{z^{n_r} - t^{n_r}}{z - t} d\sigma_r(t)\,.$$

Therefore,

$$G_{\mathbf{n},r}(z) = \frac{1}{z^{n_r}} \int \frac{t^{n_r} G_{\mathbf{n},r-1}(t)}{z - t} d\sigma_r(t) = \mathcal{O}\left(\frac{1}{z^{n_r+1}}\right), \qquad z \to \infty\,,$$

and taking into consideration the degree of $W_{\mathbf{n},r}$, we obtain

$$\frac{z^j G_{\mathbf{n},r}}{W_{\mathbf{n},r}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \Delta_r)\,, \qquad j = 0, \ldots, N_{\mathbf{n},r} + 1\,.$$

Let Γ be a closed Jordan curve which surrounds Δ_r and such that all the zeros of $W_{\mathbf{n},r}$ lie in the exterior of Γ. Using Cauchy's Theorem, the integral expression for $G_{\mathbf{n},r}$, Fubini's Theorem, and Cauchy's Integral Formula, for each $j = 0, \ldots, N_{\mathbf{n},r} + 1$, we have

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{z^j G_{\mathbf{n},r}(z)}{W_{\mathbf{n},r}(z)} dz = \frac{1}{2\pi i} \int_\Gamma \frac{z^j}{W_{\mathbf{n},r}(z)} \int \frac{G_{\mathbf{n},r-1}(t)}{z-t} d\sigma_r(t) dz = \int \frac{t^j G_{\mathbf{n},r-1}(t)}{W_{\mathbf{n},r}(t)} d\sigma_r(t) \, .$$

Since the measure $d\sigma_r(t)/W_{\mathbf{n},r}(t)$ has constant sign on Δ_r this implies that $G_{\mathbf{n},r-1}$ has at least $N_{\mathbf{n},r}+2$ change knots in the interior of Δ_r which contradicts our induction hypothesis since this function can have at most $N_{\mathbf{n},r} + 1$ zeros in $\mathbb{C} \setminus \Delta_{r-1}$ which contains Δ_r.

According to the statements just proved, we have that $N_{\mathbf{n},k+1}$ distinct zeros of $G_{\mathbf{n},k}$ lie on Δ_{k+1}. The only one possible not localized zero must be in $\mathbb{R} \setminus \Delta_k$ because the function is symmetric and it cannot have complex conjugate zeros without exceeding the total amount of zeros it can have in $\mathbb{C} \setminus \Delta_k$. The extra zero on the real line cannot coincide with any of the previously located zeros on Δ_{k+1} without violating the amount of change knots the function is forced to have on this interval. Therefore, all the zeros are simple which proves the lemma. □

Remark 2.1. *We wish to stress that* $A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}^l,k}$ *does not necessarily have* $N_{\mathbf{n},k+1} + 1$ *simple zeros in* $\mathbb{R} \setminus \Delta_k$*. For example, if* $B = 0, G_{\mathbf{n},k}$ *has* $N_{\mathbf{n},k+1}$ *simple zeros for all* $k = 0, \ldots, m-1$*. When* $A = 0$*, then it has* $N_{\mathbf{n},k+1} + 1$ *simple zeros for* $k = 0, \ldots, l-1$ *and* $N_{\mathbf{n},k+1}$ *for* $k = l, \ldots, m-1$*.*

Theorem 2.1. *Let* $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_m^+(\circledast)$ *for some fixed* $l \in \{1, \ldots, m\}$ *Then, for each* $k \in \{0, 1, \ldots, m-1\}$*, the functions* $\Psi_{\mathbf{n}^l,k}, \Psi_{\mathbf{n},k}$ *do not have common zeros in* $\mathbb{R} \setminus \Delta_k$ *and between two consecutive zeros of* $\Psi_{\mathbf{n}^l,k}$ *lies one zero of* $\Psi_{\mathbf{n},k}$*. Consequently, the zeros of* $Q_{\mathbf{n}^l,k}$ *and* $Q_{\mathbf{n},k}, k \in \{1, \ldots, m\}$*, interlace.*

Proof. Let us suppose that there exists a point $x_\nu \in \mathbb{R} \setminus \Delta_k$ such that $\Psi_{\mathbf{n}^l,k}(x_\nu) = \Psi_{\mathbf{n},k}(x_\nu) = 0$. As we have seen, x_ν must be a simple zero of each one of these functions. Therefore, $(\Psi_{\mathbf{n}^l,k})'(x_\nu) \neq 0 \neq (\Psi_{\mathbf{n},k})'(x_\nu)$. Thus, there must exist real constants A, B different from zero such that

$$G_{\mathbf{n},k}(x_\nu) = (A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}^l,k})(x_\nu) = (G_{\mathbf{n},k})'(x_\nu) = 0 \, .$$

This means that the function $G_{\mathbf{n},k}$ has a double zero at x_ν which is impossible due to Lemma 2.1.

Fix $y \in \mathbb{R} \setminus \Delta_k$ and set $G^y_{\mathbf{n},k}(z) = \Psi_{\mathbf{n}^l,k}(z)\Psi_{\mathbf{n},k}(y) - \Psi_{\mathbf{n}^l,k}(y)\Psi_{\mathbf{n},k}(z)$. Notice that this function was considered in Lemma 2.1 with $A = -\Psi_{\mathbf{n}^l,k}(y)$ and $B = \Psi_{\mathbf{n},k}(y)$. Let $x_\nu, x_{\nu+1}$ be two consecutive zeros of $\Psi_{\mathbf{n}^l,k}$ in $\mathbb{R} \setminus \Delta_k$ and let $y \in (x_\nu, x_{\nu+1})$. Then $|A| + |B| > 0$

The function $G^y_{\mathbf{n},k}(z)$ is a real function when it is restricted to $\mathbb{R} \setminus \Delta_k$ and analytic in $\mathbb{C} \setminus \Delta_k$. We have that $(G^y_{\mathbf{n},k})'(z) = (\Psi_{\mathbf{n}^l,k})'(z)\Psi_{\mathbf{n},k}(y) - \Psi_{\mathbf{n}^l,k}(y)(\Psi_{\mathbf{n},k})'(z)$. Let us assume that for some $y_0 \in (x_\nu, x_{\nu-1})$ we have that $(G^{y_0}_{\mathbf{n},k})'(y_0) = 0$. Since $(G^y_{\mathbf{n},k})(y) = 0$ for all $y \in (x_\nu, x_{\nu+1})$ we would have that $(G^{y_0}_{\mathbf{n},k})(z)$ has a zero of order ≥ 2 (with respect to z) at y_0 which contradicts the assertion of Lemma 2.1. Consequently,

$$(G^y_{\mathbf{n},k})'(y) = (\Psi_{\mathbf{n}^l,k})'(y)\Psi_{\mathbf{n},k}(y) - \Psi_{\mathbf{n}^l,k}(y)(\Psi_{\mathbf{n},k})'(y)$$

takes values with constant sign for all $y \in (x_\nu, x_{\nu+1})$. At the extreme points $x_\nu, x_{\nu+1}$ this function cannot be equal to zero because as we proved above $\Psi_{\mathbf{n},k}, \Psi_{\mathbf{n}^l,k}$ do not have common zeros. By continuity, $(G^y_{\mathbf{n},k})'$ preserves the same sign on all $[x_\nu, x_{\nu+1}]$ (and, consequently, on each side of the interval Δ_k). Thus

$$\operatorname{sign}(G^{x_\nu}_{\mathbf{n},k})'(x_\nu) = \operatorname{sign}((\Psi_{\mathbf{n}^l,k})'(x_\nu)\Psi_{\mathbf{n},k}(x_\nu)) =$$

$$\operatorname{sign}((\Psi_{\mathbf{n}^l,k})'(x_{\nu+1})\Psi_{\mathbf{n},k}(x_{\nu+1})) = \operatorname{sign}(G^{x_{\nu+1}}_{\mathbf{n},k})'(x_{\nu+1})\,.$$

Since

$$\operatorname{sign}(\Psi_{\mathbf{n}^l,k})'(x_\nu) \neq \operatorname{sign}(\Psi_{\mathbf{n}^l,k})'(x_{\nu+1})\,,$$

we obtain that

$$\operatorname{sign}\Psi_{\mathbf{n},k}(x_\nu) \neq \operatorname{sign}\Psi_{\mathbf{n},k}(x_{\nu+1})\,,$$

consequently, there must be an intermediate zero of $\Psi_{\mathbf{n},k}$ between x_ν and $x_{\nu+1}$. The last assertion of the theorem follows from the previous one and the definition of the polynomials $Q_{\mathbf{n},k}$. The theorem has been proved. □

3 Asymptotic properties

Let $\Lambda \subset \mathbb{Z}^m_+(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$ we have that $\mathbf{n}^l \in \mathbb{Z}^m_+(\circledast)$. From the interlacing property of the zeros of $Q_{\mathbf{n}^l,k}$ and $Q_{\mathbf{n},k}$ we have that the families of functions $\left\{\frac{Q_{\mathbf{n}^l,k}}{Q_{\mathbf{n},k}}\right\}_{\mathbf{n}\in\Lambda}$, $k = 1, \ldots, m$, are uniformly bounded on each compact subsetset of $\mathbb{C} \setminus \Delta_k$. By Montel's Theorem, there exists a subsequence of multi-indices Λ' and a family of functions, $\widetilde{F_k^{(l)}}$, $k = 1, \ldots, m$, holomorphic on $\mathbb{C} \setminus \Delta_k$ such that

$$\lim_{\mathbf{n} \in \Lambda'} \frac{Q_{\mathbf{n}^l,k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F_k^{(l)}}(z)\,, \qquad K \subset \mathbb{C} \setminus \Delta_k\,, \qquad k = 1, \ldots, m\,. \tag{14}$$

Notice that by Hurwitz' Theorem, the functions on the right hand side of these relations may not take the value zero in $\mathbb{C} \setminus \Delta_k$ since all the zeros of $Q_{\mathbf{n}^l,k}$ lie on Δ_k.

Generally speaking, the functions $\widetilde{F_k^{(l)}}, k = 1, \ldots, m$, may depend on Λ'. Our goal is to prove that in fact this is not so. For this purpose, we will show that these functions satisfy a certain system of boundary value problems which has a unique solution. In this section, using results on ratio and relative asymptotic of orthogonal polynomials with respect to varying measures we derive this system.

Lemma 3.1. *Under the assumptions of Theorem 1.2, let $\Lambda' \subset \Lambda$ be a subsequence of multi-indices such that* (14) *holds. Then, there exists a normalization $F_k^{(l)}, k = 1, \ldots, m$, for the functions $\widetilde{F_k^{(l)}}, k = 1, \ldots, m$, defined in* (14) *such that the set of functions $F_k^{(l)}, k = 1, \ldots, m$, satisfies the system of boundary value problems*

$$\{F_k^{(l)}(z)\}_{k=1}^m : \quad \begin{array}{ll} 1) & F_k^{(l)}, 1/F_k^{(l)} \in H(\mathbb{C} \setminus \Delta_k)\,, \\ 2) & (F_k^{(l)})'(\infty) > 0, \quad k = 1, \ldots, l\,, \\ 2') & F_k^{(l)}(\infty) > 0, \quad k = l+1, \ldots, m\,, \\ 3) & |F_k^{(l)}(x)|^2 \dfrac{1}{\left|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)\right|} = 1\,, \quad x \in \Delta_k\,, \end{array} \tag{15}$$

where $F_0^{(l)} \equiv F_{m+1}^{(l)} \equiv 1$.

Proof. The functions $\widetilde{F_k^{(l)}}$ defined in (14) satisfy 1), 2), and 2'). In fact, as it was mentioned above $\widetilde{F_k^{(l)}}$ cannot be equal to zero in $\mathbb{C} \setminus \Delta_k$; therefore, 1) follows. On the other hand, considering the degrees of the polynomials, for all $n \in \Lambda'$ the rational functions on the left of (14) at infinity are either equal to 1 when $k = l+1, \ldots, m$, or their derivative equals 1 for $k = 1, \ldots, l$, hence the limit functions must satisfy either 2) or 2') depending on k. Thus, any normalization of these functions obtained by means of a multiplication by positive constants also satisfies 1), 2), and 2')

Consider the orthogonality relations (7) verified by the polynomials $Q_{\mathbf{n},k}(x)$ with respect to the varying measures

$$\frac{|H_{\mathbf{n},k}(x)| d|\sigma_k|(x)}{|Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)|}\,, \qquad H_{\mathbf{n},k} := \frac{Q_{\mathbf{n},k-1} \Psi_{\mathbf{n},k-1}}{Q_{\mathbf{n},k}}\,.$$

By $|\sigma_k|$ we denote the total variation of σ_k. Since this measure has constant sign, then $|\sigma_k| = \sigma_k$ or $|\sigma_k| = -\sigma_k$, depending on whether σ_k is a positive or a negative measure, respectively. Obviously, in either cases the orthogonality relations remain valid.

For each $k = 1, \ldots, m$, set

$$K_{\mathbf{n},k} = \left(\int Q^2_{\mathbf{n},k}(x) \left| \frac{Q_{\mathbf{n},k-1}(x)\Psi_{\mathbf{n},k-1}(x)}{Q_{\mathbf{n},k}(x)} \right| \frac{d|\sigma_k|(x)}{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|} \right)^{-1/2} . \tag{16}$$

Take

$$K_{\mathbf{n},0} = 1 \,, \quad \kappa_{\mathbf{n},k} = \frac{K_{\mathbf{n},k}}{K_{\mathbf{n},k-1}} \,, \quad k = 1, \ldots, m \,.$$

Define

$$q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k} \,, \quad h_{\mathbf{n},k}(z) = K^2_{\mathbf{n},k-1} H_{\mathbf{n},k} \,, \quad k = 1, \ldots, m \,. \tag{17}$$

From (7)

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{|h_{\mathbf{n},k}(x)| d|\sigma_k|(x)}{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m \,,$$

and with the notation introduced above it follows that $q_{n,k}$ is orthonormal with respect to the varying measure

$$\frac{|h_{\mathbf{n},k}(x)| d|\sigma_k|(x)}{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|} = d\rho_{\mathbf{n},k}(x) \,. \tag{18}$$

Let $l \in \{1, \ldots, m\}$ be as indicated in Theorem 1.2. Reasoning as before, we obtain that $Q_{\mathbf{n}^l,k}$ and $q_{\mathbf{n}^l,k}$ are the monic orthogonal and the orthonormal polynomials, respectively, with respect to the varying measure

$$\frac{|h_{\mathbf{n}^l,k}(x)| d|\sigma_k|(x)}{|Q_{\mathbf{n}^l,k-1}(x)Q_{\mathbf{n}^l,k+1}(x)|} = \frac{|h_{\mathbf{n}^l,k}(x)|}{|h_{\mathbf{n},k}(x)|} \frac{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|}{|Q_{\mathbf{n}^l,k-1}(x)Q_{\mathbf{n}^l,k+1}(x)|} d\rho_{\mathbf{n},k}(x) \,. \tag{19}$$

Take $\Lambda' \subset \Lambda$ such that (14) takes place. Then

$$\lim_{\mathbf{n} \in \Lambda'} \frac{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|}{|Q_{\mathbf{n}^l,k-1}(x)Q_{\mathbf{n}^l,k+1}(x)|} = \frac{1}{|\widetilde{F^{(l)}_{k-1}}(x)\widetilde{F^{(l)}_{k+1}}(x)|} \,, \qquad k = 1, \ldots, m \,. \tag{20}$$

uniformly on Δ_k ($\widetilde{F^{(l)}_0} = \widetilde{F^{(l)}_{m+1}} = 1$). On the other hand, from (50) in [4] we have that

$$h_{\mathbf{n},k+1}(z) = \int \frac{|q_{\mathbf{n},k}(x)|^2}{z - x} \frac{h_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)} \,, \qquad k = 1, \ldots, m \,,$$

and using Theorem 9 of [4] it follows that

$$\lim_{n \in \Lambda} |h_{\mathbf{n},k+1}(z)| = \frac{1}{|\sqrt{(z - b_k)(z - a_k)}|} \,, \qquad K \subset \mathbb{C} \setminus \Delta_k \,, \qquad k = 1, \ldots, m,$$

where $\Delta_k = [a_k, b_k]$ (notice that from the definition we have that $h_{\mathbf{n},1} \equiv h_{\mathbf{n}^l,1} \equiv 1$). In particular,

$$\lim_{n\in\Lambda} \frac{|h_{\mathbf{n}^l,k}(x)|}{|h_{\mathbf{n},k}(x)|} = 1\,, \qquad k = 1,\ldots,m\,,$$

uniformly on Δ_k. In applying Theorem 9 of [4], the condition $n_1 - n_m \leq d$ plays a key role. From what was proved above, it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{|h_{\mathbf{n}^l,k}(x)|}{|h_{\mathbf{n},k}(x)|} \frac{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|}{|Q_{\mathbf{n}^l,k-1}(x)Q_{\mathbf{n}^l,k+1}(x)|} = \frac{1}{|\widetilde{F^{(l)}_{k-1}}(x)\widetilde{F^{(l)}_{k+1}}(x)|}\,, \tag{21}$$

uniformly on Δ_k. The function on the right hand side of this relation is continuous and different for zero on Δ_k.

Fix $k \in \{l+1,\ldots,m\}$. Notice that for this selection of k we have that $\deg Q_{\mathbf{n}^l,k} = \deg Q_{\mathbf{n},k} = N_{\mathbf{n},k}$. From (14), (18), (19), and (21), by the theorem on relative asymptotic (see Theorem 3.2 on page 293 of [5]), we conclude that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{S_k(z)}{S_k(\infty)} = \widetilde{S_k}(z) = \widetilde{F^{(l)}_k}(z)\,, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_k\,, \qquad k = l+1,\ldots,m\,, \tag{22}$$

where S_k is the Szegő function on $\overline{\mathbb{C}}\setminus\Delta_k$ with respect to the weight $|\widetilde{F^{(l)}_{k-1}}(x)\widetilde{F^{(l)}_{k+1}}(x)|^{-1}$, $x \in \Delta_k$. (We point out that the function which plays the role of the weight multiplying the measure $\rho_{\mathbf{n},k}$ in (19) remains fixed in Theorem 3.2 of [5]; nevertheless, it is easy to verify that the proof follows through under (21).) The function S_k, is uniquely determined by the following conditions:

$$S_k(z): \quad \begin{array}{ll} 1) & S_k, 1/S_k \in H(\overline{\mathbb{C}} \setminus \Delta_k)\,, \\ 2) & S_k(\infty) > 0, \\ 3) & |S_k(x)|^2 \dfrac{1}{\left|(\widetilde{F^{(l)}_{k-1}}\widetilde{F^{(l)}_{k+1}})(x)\right|} = 1, \quad x \in \Delta_k. \end{array} \tag{23}$$

Now, fix $k \in \{1,\ldots,l\}$. In this situation $\deg Q_{\mathbf{n}^l,k} = \deg Q_{\mathbf{n},k} + 1 = N_{\mathbf{n},k} + 1$. Let $Q^*_{\mathbf{n},k}(x)$ be the monic polynomial of degree $N_{\mathbf{n},k}$ orthogonal with respect to the varying measure in (19). Using the same arguments as above, we have

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q^*_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{S_k(z)}{S_k(\infty)} = \widetilde{S_k}(z)\,, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_k\,, \qquad k = 1,\ldots,l\,. \tag{24}$$

As before, the Szegő function S_k is defined by (23) with $k = 1,\ldots,l$. On the other hand, since $\deg Q_{\mathbf{n}^l,k} = \deg Q^*_{\mathbf{n},k} + 1$ and both of these polynomials are orthogonal with respect to the same varying weight, then by the ratio asymptotic theorem for varying measures (see Theorem 6 on page 567 of [4]) we have

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,k}(z)}{Q^*_{\mathbf{n},k}(z)} = \frac{\varphi_k(z)}{\varphi_k'(\infty)}\,, \qquad K\subset\mathbb{C}\setminus\Delta_k\,, \tag{25}$$

where φ_k denotes the conformal representation of $\overline{\mathbb{C}}\setminus\Delta_k$ onto $\{w: |w|>1\}$ such that $\varphi_k(\infty)=\infty$ and $\varphi_k'(\infty)>0$. The function φ_k is uniquely determined by the conditions:

$$\varphi_k(z): \quad \begin{array}{ll} 1) & \varphi_k, 1/\varphi_k \in H(\mathbb{C}\setminus\Delta_k) \\ 2) & \varphi_k'(\infty)>0 \\ 3) & |\varphi_k(x)|=1, \quad x\in\Delta_k. \end{array} \tag{26}$$

From (24) and (25), we obtain

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{(S_k\varphi_k)(z)}{(S_k\varphi_k')(\infty)} = \widetilde{(S_k\varphi_k)}(z) = \widetilde{F_k^{(l)}}(z)\,, \quad K\subset\mathbb{C}\setminus\Delta_k\,, \quad k=1,\ldots,l\,. \tag{27}$$

Thus,

$$\widetilde{F_k^{(l)}} = \begin{cases} \widetilde{S_k\varphi_k}\,, & k=1,\ldots,l\,, \\ \widetilde{S_k}\,, & k=l+1,\ldots,m\,, \end{cases} \tag{28}$$

and from (23) and (28) it follows that

$$|\widetilde{F_k^{(l)}}(x)|^2 \frac{1}{\left|(\widetilde{F_{k-1}^{(l)}}\widetilde{F_{k+1}^{(l)}})(x)\right|} = \frac{1}{\omega_k}\,, \qquad x\in\Delta_k\,, \qquad k=1,\ldots,m\,, \tag{29}$$

where

$$\omega_k = \begin{cases} (S_k\varphi_k')^2(\infty)\,, & k=1,\ldots,l\,, \\ S_k^2(\infty)\,, & k=l+1,\ldots,m\,. \end{cases} \tag{30}$$

Now, let us show that there exist positive constants $c_k, k=1,\ldots,m$, such that the functions $F_k^{(l)} = c_k\widetilde{F_k^{(l)}}$ satisfy (15). In fact, according to (29) for any such constants c_k we have that

$$|F_k^{(l)}(x)|^2 \frac{1}{\left|(F_{k-1}^{(l)}F_{k+1}^{(l)})(x)\right|} = \frac{c_k^2}{c_{k-1}c_{k+1}\omega_k}\,, \qquad x\in\Delta_k\,, \qquad k=1,\ldots,m\,,$$

where $c_0=c_{m+1}=1$. The problem reduces to finding appropriate constants c_k such that

$$\frac{c_k^2}{c_{k-1}c_{k+1}\omega_k} = 1\,, \qquad k=1,\ldots,m\,. \tag{31}$$

Taking logarithm, we obtain the linear system of equations

$$2\log c_k - \log c_{k-1} - \log c_{k+1} = \log\omega_k\,, \qquad k=1,\ldots,m \tag{32}$$

($c_0=c_{m+1}=1$) on the unknowns $\log c_k$. This system has a unique solution with which we conclude the proof. □

Lemma 3.2. *Let* $F_k^{(l)}, k = 1, \dots, m$, *be a system of functions which satisfies* (15) *and let* $\mathbb{S}_k$ *be the Szegő function on* $\overline{\mathbb{C}} \setminus \Delta_k$ *which satisfies the boundary conditions*

$$|\mathbb{S}_k(x)|^2 \frac{1}{\left|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)\right|} = 1 \,, \qquad x \in \Delta_k \,, \tag{33}$$

where $F_0^{(l)} \equiv F_{m+1}^{(l)} \equiv 1$. *Then*

$$F_k^{(l)} = \begin{cases} \mathbb{S}_k \varphi_k \,, & k = 1, \dots, l \,, \\ \mathbb{S}_k \,, & k = l+1, \dots, m \,. \end{cases} \tag{34}$$

Proof. From the conditions $1), 2)$ and $2')$ in (15) we have that the functions $F_k^{(l)}, k = 1, \dots, l$, have a pole of first order at ∞ and the functions $F_k^{(l)}, k = l+1, \dots, m$, are analytic at infinity. Consequently, if we set

$$\mathbb{S}_k = \begin{cases} F_k^{(l)} / \varphi_k \,, & k = 1, \dots, l \,, \\ F_k^{(l)} \,, & k = l+1, \dots, m \,, \end{cases}$$

from (15) we see that these functions satisfy the system

$$\mathbb{S}_k(z) : \quad \begin{array}{ll} 1) & \mathbb{S}_k, 1/\mathbb{S}_k \in H(\overline{\mathbb{C}} \setminus \Delta_k) \\ 2) & \mathbb{S}_k(\infty) > 0, \\ 3) & |\mathbb{S}_k(x)|^2 \dfrac{1}{\left|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)\right|} = 1, \quad x \in \Delta_k. \end{array} \tag{35}$$

which defines uniquely the Szegő function specified in (33) and (34) follows. □

4 Limiting functions

To conclude the proof of the main Theorem 1.2, we have to verify that the system (15) has a unique solution and that the algebraic functions defined in the introduction (see the statement of Theorem 1.2) satisfy this system.

The existence and uniqueness of the solution of (15) with arbitrary right hand side, in the boundary conditions 3) of (15), belonging to the Szegő class, was proved in [1]. Here, for completeness of the presentation we give an independent proof. The existence of a solution is not a problem because Lemma 3.1 guarantees existence. For the proof of uniqueness we need an auxiliary result.

Let $\Omega_k = \overline{\mathbb{C}} \setminus \Delta_k, \; k = 1, \ldots, m$, and let $\mathrm{Harm}(\overline{\Omega_k})$ be the Banach space of all harmonic functions on Ω_k which are continuous up to the boundary $\Delta_k, \; k = 1, \ldots, m$, endowed with the supremum norm

$$\|u\|_k = \max\{|u(x)| : x \in \Delta_k\} \quad k = 1, \ldots, m.$$

Let $T_{i,j}$ be the linear operator $T_{i,j} : \mathrm{Harm}(\overline{\Omega_j}) \longrightarrow \mathrm{Harm}(\overline{\Omega_i})$ defined, solving the Dirichlet problem, by the boundary condition

$$T_{i,j}(u_j)(x) = u_j(x), \;\; x \in \partial(\Omega_i).$$

Consider the Banach space $H = \mathrm{Harm}(\overline{\Omega_1}) \times \cdots \times \mathrm{Harm}(\overline{\Omega_k})$ with the norm

$$\|(u_1, \ldots, u_m)\| = \max\{\|u_k\|_k : k = 1, \ldots, m\}$$

and the linear operator $T : H \longrightarrow H$ defined as follows

$$T = \begin{pmatrix} 0 & T_{1,2} & 0 & \cdots & 0 \\ T_{2,1} & 0 & T_{2,3} & \cdots & 0 \\ 0 & T_{3,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{36}$$

Lemma 4.1. *We have that* $(2I - T)(u) = 0$ *if and only if* $u = 0$. (*Here,* $I : H \longrightarrow H$ *denotes the identity operator.*)

Proof. Obviously, $u = 0$ implies that $(2I - T)(u) = 0$. Let us assume that $(2I - T)(u) = 0$. Then, $T(u) = 2u$ and $\|T(u)\| = 2\|u\|$, where $u = (u_1, \ldots, u_m)$ and $u_i \in \mathrm{Harm}(\overline{\Omega_i}), \;\; i = 1, \ldots, m$. From the definition of the operators $T_{i,j}$ and the maximum principle for harmonic functions, we have that

$$\|T_{i,j}(u_j)\|_i = \|u_j\|_i \leq \|u_j\|_j \, .$$

According to our assumptions, consecutive intervals Δ_i do not intersect; therefore, for $j = i - 1$ or $j = i + 1$

$$\|T_{i,j}(u_j)\|_i = \|u_j\|_j \Rightarrow u_j = \text{constant} \, .$$

Since $\|T(u)\| = 2\|u\|$ there exists $i \in \{1, \ldots, m\}$ such that

$$2\|u\| = \|T_{i,i-1}(u_{i-1}) + T_{i,i+1}(u_{i+1})\|_i \leq \|T_{i,i-1}(u_{i-1})\|_i + \|T_{i,i+1}(u_{i+1})\|_i \leq$$

$$\|u_{i-1}\|_{i-1} + \|u_{i+1}\|_{i+1} \leq 2\|u\|$$

(where $u_0 \equiv u_{m+1} \equiv 0$) and, consequently, all the previous inequalities become equalities. From this we have that if $i = 1$ or $i = m$ then necessarily u must be the null vector. If $i = 2, \ldots, m-1$ these relations imply that

$$\|T_{i,i-1}(u_{i-1})\|_i = \|u_{i-1}\|_{i-1} = \|u\| = \|T_{i,i+1}(u_{i+1})\|_i = \|u_{i+1}\|_{i+1}$$

and by the previous remark it follows that u_{i-1} and u_{i+1} are constant. Consequently, the same is true for $T_{i,i-1}(u_{i-1})$ and $T_{i,i+1}(u_{i+1})$. Moreover, it is easy to see that there are only two possibilities either $u_{i-1} \equiv u_{i+1} \equiv \|u\|$ or $u_{i-1} \equiv u_{i+1} \equiv -\|u\|$. Since $T_{i,i-1}(u_{i-1}) + T_{i,i+1}(u_{i+1}) = 2u_i$, in the first case we obtain that $u_i \equiv \|u\|$ and in the second $u_i \equiv -\|u\|$. Now, using $T_{i-1,i-2}(u_{i-2}) + T_{i-1,i}(u_i) = 2u_{i-1}$ and $T_{i+1,i}(u_i) + T_{i+1,i+2}(u_{i+2}) = 2u_{i+1}$, in the first case we obtain that $u_{i-2} = u_{i+2} \equiv \|u\|$ and in the second $u_{i-2} = u_{i+2} \equiv -\|u\|$. Iterating the process on the index i we conclude that either u has all its components constantly equal to $\|u\|$ or $-\|u\|$. This again leads to $\|u\| = 0$ because otherwise the first and last boundary conditions would fail. $\square$

Lemma 4.2. *The system of boundary value problems* (15) *has a unique solution which may be expressed by means of* (12).

Proof. Using (33) and (34) from Lemma 3.2, we obtain

$$\mathbb{S}_k(z): \quad \begin{aligned} &1) \quad \mathbb{S}_k, 1/\mathbb{S}_k \in H(\overline{\mathbb{C}} \setminus \Delta_k) \\ &2) \quad \mathbb{S}_k(\infty) > 0 \\ &3) \quad |\mathbb{S}_k(x)|^2 \frac{1}{\left|(\mathbb{S}_{k-1}\mathbb{S}_{k+1})(x)\right|} = W_k(x), \quad x \in \Delta_k \end{aligned} \tag{37}$$

where $k = 1, \ldots, m$ and

$$W_k := \left|\varphi_{k-1}^{(l)}\varphi_{k+1}^{(l)}\right|, \quad \varphi_k^{(l)} = \begin{cases} \varphi_k, & k \in \{1, \ldots, l\}, \\ 1, & k \in \{l+1, \ldots, m\}. \end{cases}$$

Denote

$$u_k := \ln|\mathbb{S}_k|, \quad w_k := \ln W_k, \quad k = 1, \ldots, m. \tag{38}$$

Notice that $u_k \in \mathrm{Harm}(\overline{\Omega_k}), k = 1, \ldots, m$. The function $\mathbb{S}_k, k = 1, \ldots, m$, is different from zero on Ω_k and positive at ∞; therefore, $\ln \mathbb{S}_k$ is holomorphic on this region. Consequently, u_k determines $\mathbb{S}_k$ uniquely. From (37) it follows that

$$2u_k(x) - u_{k-1}(x) - u_{k+1}(x) = w_k(x), \qquad x \in \Delta_k, \tag{39}$$

for $k = 1, \ldots, m$, where $u_0 \equiv u_{m+1} \equiv 0$. With the notation introduced in Lemma 4.1 equation (39) may be rewritten in matrix form as

$$(2I - T)(u) = w\,,$$

where $u = (u_1, \ldots, u_m)^t$ and $w = (w_1, \ldots, w_m)^t$. From Lemma 4.1 it follows that $u = (2I - T)^{-1}(w)$ is unique and thus there exists a unique system of functions $F_k^{(l)}, k = 1, \ldots, m$, satisfying (15).

It remains to check that the functions in (12) from the statement of Theorem 1.2

$$F_k^{(l)} = \prod_{\nu=k}^{m} \psi_\nu^{(l)},$$

where the algebraic functions $\psi_\nu^{(l)}$ are defined in (8), satisfy the system of boundary value problems (15). The validity of the conditions $1), 2)$ and $2')$ in (15) follow directly from the definition of the functions $\psi_\nu^{(l)}$. Let us check the boundary conditions 3) in (15)

$$|F_k^{(l)}(x)|^2 \frac{1}{\left|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)\right|} = 1, \quad x \in \Delta_k\,.$$

In fact, from the definition of $F_k^{(l)}$ above, we have

$$\frac{F_k^{(l)}}{F_{k-1}^{(l)}} = \frac{1}{\psi_{k-1}^{(l)}}\,. \tag{40}$$

Moreover, the symmetry (with respect to the real line) of the construction of the Riemann surface $\mathcal{R}$ of $\psi^{(l)}$ gives us

$$\psi_k^{(l)}(x) = \overline{\psi_{k-1}^{(l)}(x)}\,, \qquad x \in \Delta_k\,. \tag{41}$$

Therefore, from (40) and (41) we obtain the boundary conditions for $k = 2, 3, \ldots, m$. In fact,

$$|F_k^{(l)}(x)|^2 \frac{1}{\left|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)\right|} = \left|\frac{\psi_k^{(l)}(x)}{\psi_{k-1}^{(l)}(x)}\right| = 1, \qquad x \in \Delta_k\,.$$

For $k = 1$, the relation (40) gives

$$\frac{|F_1^{(l)}|^2}{|F_2^{(l)}|} = |\psi_1^{(l)}|^2 \Big|\prod_{\nu=2}^{m} \psi_\nu^{(l)}\Big|.$$

On account of (41), the right hand side of this expression on Δ_k becomes $\left|\prod_{\nu=0}^{m} \psi_\nu^{(l)}\right|$. However, the product of all branches of the algebraic function $\psi^{(l)}$ is a single valued function in $\overline{\mathbb{C}}$ and by their construction (see (8) and (9)), the product has no poles in $\overline{\mathbb{C}}$ and at infinity it is equal to 1; therefore, by Liouville's theorem the product is 1 everywhere in $\overline{\mathbb{C}}$. Thus, the boundary condition for $k = 1$

$$\frac{|F_1^{(l)}(x)|^2}{|F_2^{(l)}(x)|} = 1, \qquad x \in \Delta_1 \,,$$

is also verified. Therefore, this Lemma and with it Theorem 1.2 have been proved. □

The following Corollary complements Theorem 1.2.

Corollary 4.1. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system such that $\sigma_k' > 0$ almost everywhere on $\Delta_k = \operatorname{supp}(\sigma_k), k = 1, \ldots, m$. Let $\Lambda \subset \mathbb{Z}_+(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+(\circledast)$ and $n_1 - n_m \le d$, where d is a constant. Let $\{q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, be the system of orthonormal polynomials as defined in* (17) *and $\{K_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, the values given by* (16)*. Then, for each fixed $k = 1, \ldots, m$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}^l,k}}{\kappa_{\mathbf{n},k}} = \kappa_k^{(l)} \,, \tag{42}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{K_{\mathbf{n}^l,k}}{K_{\mathbf{n},k}} = \kappa_1^{(l)} \cdots \kappa_k^{(l)} \,, \tag{43}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,k}(z)}{q_{\mathbf{n},k}(z)} = \kappa_k^{(l)} \widetilde{F_k^{(l)}}(z), \qquad z \in K \subset \mathbb{C} \setminus \Delta_k \,, \tag{44}$$

where

$$\kappa_k^{(l)} = \frac{c_k^{(l)}}{\sqrt{c_{k-1}^{(l)} c_{k+1}^{(l)}}} \,, \qquad c_k^{(l)} = \begin{cases} (F_k^{(l)})'(\infty) \,, & k = 1, \ldots, m \,, \\ F_k^{(l)}(\infty) \,, & k = l+1, \ldots, m \,, \end{cases} \tag{45}$$

and the $F_k^{(l)}$ are defined by (12)*.*

Proof. By the proved Theorem 1.2, we have limit in (20) along the whole sequence Λ. Reasoning as in the deduction of formulas (22) and (27), but now in connection to orthonormal polynomials (see [4] and [5]), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,k}(z)}{q_{\mathbf{n},k}(z)} = \begin{cases} (S_k \varphi_k)(z) \,, & k = 1, \ldots, l \,, \\ S_k(z) \,, & k = l+1, \ldots, m \,, \end{cases} \qquad K \subset \mathbb{C} \setminus \Delta_k \,,$$

where S_k is defined in (23). This formula, divided by (22) or (27) according to the value of k gives

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}^l,k}}{\kappa_{\mathbf{n},k}} = \sqrt{\omega_k} = \frac{c_k}{\sqrt{c_{k-1} c_{k+1}}} \,,$$

where ω_k is defined in (30), and the c_k are the normalizing constants found in Lemma 3.1 solving the linear system of equations (32) which ensure that

$$F_k^{(l)} \equiv c_k \widetilde{F_k^{(l)}}\,, \qquad k = 1, \ldots, m\,,$$

with $F_k^{(l)}$ satisfying (15) and thus given by (12). Since $(\widetilde{F_k^{(l)}})'(\infty) = 1, k = 1, \ldots, l$, and $(\widetilde{F_k^{(l)}})(\infty) = 1, k = l+1, \ldots, m$, formula (42) immediately follows with $\kappa_k^{(l)}$ as in (45).

From the definition of $\kappa_{\mathbf{n},k}$, we have that

$$K_{\mathbf{n},k} = \kappa_{\mathbf{n},1} \cdots \kappa_{\mathbf{n},k}\,.$$

Taking the ratio of these constants for the multi-indices $\mathbf{n}$ and $\mathbf{n}^l$ and using (42), we get (43). Formula (44) is an immediate consequence of (42) and (11). □

Remark 4.1. *From* (11), (42), *and* (43), *it follows that for each* $k = 1, \ldots, m$

$$\lim_{\mathbf{n}\in\Lambda} \kappa_{\mathbf{n},k}^{1/|\mathbf{n}|} = \kappa_k^{(l)}\,, \qquad \lim_{\mathbf{n}\in\Lambda} K_{\mathbf{n},k}^{1/|\mathbf{n}|} = \kappa_1^{(l)} \cdots \kappa_k^{(l)}\,,$$

and

$$\lim_{\mathbf{n}\in\Lambda} |Q_{\mathbf{n},k}(z)|^{1/|\mathbf{n}|} = |\widetilde{F_k^{(l)}}(z)|\,, \qquad K \subset \mathbb{C} \setminus \Delta_k\,.$$

Formulas (20) *and* (31) *in [11], allow to express the limits above in terms of the solution of a vector valued equilibrium problem for the logarithmic potential. The connection is straightforward and we do not go into details to avoid introducing at this point new notations and concepts.*

Remark 4.2. *When the measures* $\sigma_k, k = 1, \ldots, m$, *are in the Szegő class, strong asymptotic of multi-orthogonal polynomials for a Nikishin system was obtained in [1]. Our present results may be derived from Theorem 1 in [1] under these more restrictive assumptions.*

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