Differential properties for Sobolev orthogonality on the unit circle ☆

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Abstract

The aim of this paper is to study differential properties of the sequence of monic orthogonal polynomials with respect to the following Sobolev inner product:

$$\langle f,g\rangle_s = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\mu(\theta) + \frac{1}{\lambda}\int_0^{2\pi} f'(e^{i\theta})\overline{g'(e^{i\theta})}\,\frac{d\theta}{2\pi},$$

where μ is a finite positive Borel measure on $[0,2\pi]$ verifying the following conditions: the Carathéodory function associated with μ has an analytic extension outside the unit disk and the induced norm is equivalent to the Lebesgue norm in the space L_2. Here $d\theta/2\pi$ is the normalized Lebesgue measure and λ is a positive real number. The nonhomogeneous second-order differential equations satisfied by the sequence of monic Sobolev orthogonal polynomials are obtained. Moreover, as an application, a sample of Dirichlet boundary value problem is solved.

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1. Introduction

One of the reasons for studying the Sobolev orthogonality is because of its connections with spectral methods for boundary value problems (see [2]). Moreover, in [4] (also appears) a general result

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about trigonometric approximation in a Sobolev space. However, on the unit circle, the connection between orthogonal polynomials and linear differential operators had not been a subject of study in the Sobolev case.

In the present paper we study the differential properties of a class of Lebesgue–Sobolev orthogonal polynomials on the unit circle. Indeed we consider an inner product of the following type:

$$\langle f,g\rangle_{\mathrm{s}} = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\mu(\theta) + \frac{1}{\lambda}\int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi}, \tag{1}$$

where μ is a finite positive Borel measure on $[0,2\pi]$ satisfying the following conditions: its Carathéodory function has an analytic extension outside the unit disk and the induced norm is equivalent to the Lebesgue norm in the space L_2. Here $\mathrm{d}\theta/2\pi$ is the normalized Lebesgue measure and λ is a positive real number.

In this situation, we introduce a differential operator closely connected with the Sobolev inner product, which allows to obtain a nonhomogeneous second-order linear differential equation satisfied by the corresponding sequence of monic Sobolev orthogonal polynomials. As a consequence, we apply the Sobolev orthogonal polynomials to solve certain type of linear differential equations that can be considered a sort of Dirichlet boundary value problem.

2. The Sobolev inner product: Differential properties

Let μ be a finite positive Borel measure on $[0,2\pi]$, and let us denote by c_n, the moments for the measure μ, that is,

$$c_n = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\mu(\theta), \quad \text{for } n\in\mathbb{Z}.$$

For simplicity, we assume that μ is a probability measure, that is, $c_0=1$.

We also consider the Carathéodory function F associated with the measure μ

$$F(z) = 1 + 2\sum_{k=1}^{\infty}\overline{c_k}z^k.$$

It is well-known that F is analytic in the unit disk D, ($F\in H(D)$), and its real part is positive in D. Moreover, if we denote the Radon–Nikodym derivative of the measure μ, by $\mu'(\theta)$ then (see [5], p. 13, formula 1.21),

$$\mu'(\theta) = \lim_{r\to 1^-}\Re F(r\mathrm{e}^{\mathrm{i}\theta}) := \Re F(\mathrm{e}^{\mathrm{i}\theta}) \quad \text{a.e. in } [0,2\pi).$$

Since $\limsup\sqrt[n]{|c_n|}\leqslant 1$, in the sequel we assume $\limsup\sqrt[n]{|c_n|}<1$, in order to select a suitable class of measures for our purposes. So, first we prove the following lemma.

Lemma 1. *Let μ be a finite positive Borel measure on $[0,2\pi]$ with Carathéodory function F. If $\limsup\sqrt[n]{|c_n|}=1/\rho<1$, then the function $G(z)=\frac{1}{2}(F(z)+\bar{F}(1/z))\in H(A_\rho)$, where A_ρ is the annulus, $A_\rho=\{z\colon 1/\rho<|z|<\rho\}$.*

Moreover, μ is an absolutely continuous measure with respect to the Lebesgue measure.

Proof. It is straightforward to see that F is analytic in the disk $D_\rho = \{z\colon |z| < \rho\}$, and therefore G is analytic in the annulus A_ρ.

Let us consider the Schur function f defined by

$$f = \frac{1-F}{1+F}.$$

It is well-known that $|f| < 1$ in D and $f \in H(D)$. Since $1 + F \in H(D_\rho)$, we can extend f to D_ρ except in those points of D_ρ where F takes the value -1, and this constitutes a set of isolated points.

Now, in order to obtain that μ is absolutely continuous, we prove that the following condition holds (see [1]):

$$\Re\left(\int_0^{2\pi} \frac{f}{1+f}\,\mathrm{d}\theta\right) = 0.$$

Indeed,

$$\Re\left(\int_0^{2\pi} \frac{f(\mathrm{e}^{\mathrm{i}\theta})}{1+f(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\theta\right) = \Re\left(\int_0^{2\pi} \frac{1-F(\mathrm{e}^{\mathrm{i}\theta})}{2}\,\mathrm{d}\theta\right).$$

Since

$$\frac{1-F(z)}{2z} \in H(D_\rho) \quad \text{then} \quad \int_{\mathbb{T}} \frac{1-F(z)}{2z}\,\mathrm{d}z = 0,$$

that is,

$$\int_0^{2\pi} \frac{1-F(\mathrm{e}^{\mathrm{i}\theta})}{2\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = 0,$$

and therefore $\Re\,(\int_0^{2\pi}\{[1-F(\mathrm{e}^{\mathrm{i}\theta})]/2\}\,\mathrm{d}\theta) = 0$. $\square$

In the conditions of the above Lemma, we can rewrite the Sobolev inner product as follows:

$$\begin{aligned}\langle f, g\rangle_{\mathrm{s}} &= \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\mu'(\theta)\,\frac{\mathrm{d}\theta}{2\pi} + \frac{1}{\lambda}\int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi}\\ &= \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}G(\mathrm{e}^{\mathrm{i}\theta})\,\frac{\mathrm{d}\theta}{2\pi} + \frac{1}{\lambda}\int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi},\end{aligned}$$

or equivalently,

$$\langle f, g\rangle_{\mathrm{s}} = \langle f, g\rangle_\mu + \frac{1}{\lambda}\langle f', g'\rangle_\theta = \langle fG, g\rangle_\theta + \frac{1}{\lambda}\langle f', g'\rangle_\theta.$$

Next, we introduce some appropriate spaces in order to define the domain of a linear differential operator. We consider the Hardy space

$$H_2 = \left\{ f \in H(D);\, f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\},$$

which is a Hilbert space with the norm $\|f\|_\theta = (\sum_{n=0}^{\infty} |a_n|^2)^{1/2}$. In fact, it is isometric to the linear subspace

$$L_2^+ = \left\{ g \in L_2\colon \int_0^{2\pi} g(e^{i\theta})e^{-in\theta}\,d\theta = 0 \text{ for } n = 1, 2, \ldots \right\}.$$

Also, we introduce the Hardy–Sobolev space

$$\mathrm{HS}_2 = \{f \in H_2\colon f' \in H_2\},$$

which is a Hilbert space with norm $\|f\|_{s_\theta}^2 = \|f\|_\theta^2 + \|f'\|_\theta^2 = \sum_{n=0}^{\infty} (n^2+1)|a_n|^2$.

Now, let us assume that the measure μ is such that the induced norm, $\|\cdot\|_\mu$, is equivalent to the Lebesgue norm, $\|\cdot\|_\theta$, in the space L_2. Therefore the norms $\|\cdot\|_s$ and $\|\cdot\|_{s_\theta}$ defined by

$$\|f\|_s^2 = \|f\|_\mu^2 + \frac{1}{\lambda}\|f'\|_\theta^2 \quad \text{and} \quad \|f\|_{s_\theta}^2 = \|f\|_\theta^2 + \|f'\|_\theta^2$$

are equivalent in HS_2.

Let us define the following linear differential operator

$$\mathscr{L}(y(z)) = \frac{1}{\lambda}(z^2 y''(z) + zy'(z)) + G(z)y(z), \quad \text{for } y \in \mathrm{HS}_2.$$

If we denote

$$\mathscr{D}(\mathscr{L}) = \{y \in \mathrm{HS}_2\colon \mathscr{L}(y) \in L_2\},$$

we obtain the following:

Theorem 2. *The linear differential operator* $\mathscr{L} : \mathscr{D}(\mathscr{L}) \to L_2$ *defined by*

$$\mathscr{L}(y(z)) = \frac{1}{\lambda}(z^2 y''(z) + zy'(z)) + G(z)y(z)$$

satisfies

(i)

$$\forall f, g \in \mathscr{D}(\mathscr{L}) \quad \langle \mathscr{L}(f), g\rangle_\theta = \langle f, g\rangle_s. \tag{2}$$

(ii) $\mathscr{L}$ *is injective.*

Proof. (i)

$$\langle \mathscr{L}(f), g\rangle_\theta = \left\langle \frac{1}{\lambda}(z^2 f'' + zf'), g\right\rangle_\theta + \langle fG, g\rangle_\theta$$
$$= \left\langle \frac{1}{\lambda}(z^2 f'' + zf'), g\right\rangle_\theta + \langle f, g\rangle_\mu.$$

Next, using integration by parts, we prove that

$$\langle f', g'\rangle_\theta = \langle z^2 f'' + zf', g\rangle_\theta,$$

which yields (i).

Indeed,

$$\begin{aligned}\langle f', g'\rangle_\theta &= \int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}}\,\frac{\mathrm{d}\theta}{2\pi}\\ &= \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\Big]_0^{2\pi} - \int_0^{2\pi} \mathrm{i}\overline{g(\mathrm{e}^{\mathrm{i}\theta})}(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\mathrm{e}^{2\mathrm{i}\theta} f''(\mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}\theta}{2\pi}\\ &= \int_0^{2\pi} \overline{g(\mathrm{e}^{\mathrm{i}\theta})}(\mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{e}^{2\mathrm{i}\theta} f''(\mathrm{e}^{\mathrm{i}\theta}))\,\frac{\mathrm{d}\theta}{2\pi} = \langle z^2 f'' + z f', g\rangle_\theta.\end{aligned}$$

(ii) It is straightforward from (i). □

Let us denote by $\{\Phi_n\}$ the sequence of monic orthogonal polynomials with respect to the Sobolev inner product $\langle\ ,\rangle_\mathrm{s}$. Next, we obtain the second-order linear differential equation satisfied by each polynomial of this sequence.

Theorem 3. *For each natural number n, the polynomial Φ_n satisfies the non homogeneous second-order linear differential equation*:

$$\frac{1}{\lambda}(z^2 y''(z) + z y'(z)) + G(z) y(z) = H_n(z),$$

that is,

$$\mathscr{L}(y)(z) = H_n(z),$$

where $H_n \in H(A_\rho)$, and H_n has the following Laurent series expansion:

$$H_n(z) = \sum_{k=1}^{\infty} B_{n,k} z^{-k} + z^n \sum_{k=0}^{\infty} A_{n,k} z^k$$

with

$$B_{n,k} = \langle z^k \Phi_n(z), 1\rangle_\mathrm{s} = \sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} c_{k+i} \quad (k \geqslant 1)$$

and

$$A_{n,k} = \langle \Phi_n(z), z^{n+k}\rangle_\mathrm{s} = \begin{cases} \|\Phi_n\|_\mathrm{s}^2 & k = 0,\\ \displaystyle\sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!}\overline{c_{n+k-i}}, & k > 0.\end{cases}$$

Proof. Let us compute $\mathscr{L}(\Phi_n)$. Since $\langle \mathscr{L}(\Phi_n), z^j\rangle_\theta = \langle \Phi_n, z^j\rangle_\mathrm{s}$ and $\langle \Phi_n, z^j\rangle_\mathrm{s} = 0$ for $j = 0, \ldots, n-1$, then in the Laurent expansion of $\mathscr{L}(\Phi_n)$, the coefficients of the powers of z from 0 up to $n-1$, are 0. Therefore, $\mathscr{L}(\Phi_n) = H_n$ with

$$H_n(z) = \sum_{k=1}^{\infty} B_{n,k} z^{-k} + z^n \sum_{k=0}^{\infty} A_{n,k} z^k.$$

The coefficients can be obtained as follows:

$$A_{n,k} = \langle \mathscr{L}(\Phi_n), z^{n+k}\rangle_\theta = \langle \Phi_n(z), z^{n+k}\rangle_\mathrm{s}$$

and

$$B_{n,k} = \langle z^k \mathscr{L}(\Phi_n), 1\rangle_\theta = \langle z^k \Phi_n(z), 1\rangle_{\mathrm{s}}.$$

Hence, for $k = 0$, $A_{n,0} = \|\Phi_n\|_{\mathrm{s}}^2$.

For $k > 0$,

$$A_{n,k} = \langle \Phi_n(z), z^{n+k}\rangle_\mu = \sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} \overline{c_{n+k-i}}$$

and

$$B_{n,k} = \langle z^k \Phi_n(z), 1\rangle_\mu = \sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} c_{k+i}.$$

Next we prove that $H_n \in H(A_\rho)$.

From the extremal property of the norms of the monic Sobolev orthogonal polynomials, we get $\|\Phi_n\|_\mu \leqslant 1$ (see [3]). Taking into account that the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\theta$ are equivalent in L_2, then $\|\Phi_n\|_\theta \leqslant M\|\Phi_n\|_\mu \leqslant M$, and we deduce that there exists a positive constant K such that

$$\max_{i=0,\dots,n} \left\{ \frac{|\Phi_n^{(i)}(0)|}{i!} \right\} \leqslant K.$$

Hence $|B_{n,k}| \leqslant (n+1)K \max_{i=0,\dots,n} \{|c_{k+i}|\}$ and $|A_{n,k}| \leqslant (n+1)K \max_{i=0,\dots,n} \{|c_{k+i}|\}$.

Moreover,

$$\limsup \sqrt[k]{|B_{n,k}|} \leqslant \limsup \sqrt[k]{(n+1)K \max_{i=0,\dots,n} \{|c_{k+i}|\}} = \frac{1}{\rho},$$

and $\limsup \sqrt[k]{|A_{n,k}|} \leqslant \frac{1}{\rho}$. Thus $H_n \in H(A_\rho)$.

Indeed $H_n \in L_2$.

$$\sum_{k=1}^{\infty} |B_{n,k}|^2 = \sum_{k=1}^{\infty} \left| \sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} c_{k+i} \right|^2$$

$$\leqslant \sum_{k=1}^{\infty} \|\Phi_n\|_\theta^2 (c_k^2 + \cdots + c_{k+n}^2) = \|\Phi_n\|_\theta^2 \sum_{k=1}^{\infty} (c_k^2 + \cdots + c_{k+n}^2) < +\infty.$$

In the same way, $\sum_{k=1}^{\infty} |A_{n,k}|^2 < +\infty$. □

Conversely,

Theorem 4. *Let us consider the linear differential equation* $\mathscr{L}(y)(z) = H(z)$, *with*

$$H(z) = \sum_{k=1}^{\infty} B_k z^{-k} + z^n \sum_{k=0}^{\infty} A_k z^k,$$

and $A_0 \neq 0$.

If a monic polynomial of degree n, P_n *verifies* $\mathscr{L}(P_n) = H$, *then the coefficients are given by* $A_0 = \|P_n\|_{\mathrm{s}}^2$, *and for* $k \geqslant 1$, $A_k = \langle P_n, z^{n+k}\rangle_{\mathrm{s}}$, $B_k = \langle z^k P_n, 1\rangle_{\mathrm{s}}$, *and* $P_n = \Phi_n$.

Proof. For $k=0,\dots,n-1$, $\langle P_n, z^k\rangle_s = \langle H, z^k\rangle_\theta = 0$ and for $k=n$ $\langle P_n, z^n\rangle_s = \langle H, z^n\rangle_\theta = A_0 \neq 0$ holds. Therefore $P_n = \Phi_n$, and the remainder coefficients are $\langle P_n, z^{n+m}\rangle_s = \langle H, z^{n+m}\rangle_\theta = A_m$ and $\langle z^m P_n, 1\rangle_s = \langle z^m H, 1\rangle_\theta = B_m$.

3. Applications

Next, we apply the Sobolev orthogonal polynomials to obtain the explicit solutions of certain type of linear differential equations. Indeed, we solve a sort of Dirichlet boundary value problem: "Given a linear differential equation on the unit circle, find a solution which is holomorphic in the unit disk".

Theorem 5. *Let $H(z) = \sum_{-\infty}^{\infty} \hat{H}(i) z^i \in L_2$, and let us consider the linear differential equation*

$$\mathscr{L}(y)(z) = H(z) \quad \textit{for } z \in \mathbb{T} = \{z\colon |z| = 1\}.$$

Then there exists $f \in \mathrm{HS}_2$, $f = \sum_{k=0}^{\infty} a_k \Phi_k$ such that $\mathscr{L}(f) = H$ a.e. in $\mathbb{T}$ if and only if the following relations hold:

$$\forall n \geqslant 0, \quad a_n \|\Phi_n\|_s^2 = \sum_{i=0}^{n} \hat{H}(i) \, \overline{\frac{\Phi_n^{(i)}(0)}{i!}} \tag{3}$$

and

$$\textit{for } k > 0, \quad \sum_{n=0}^{\infty} a_n \left(\sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} c_{k+i} \right) = \hat{H}(-k). \tag{4}$$

Proof. $(\Rightarrow)$ Since $\langle f, \Phi_n\rangle_s = \langle \mathscr{L}(f), \Phi_n\rangle_\theta$, then $a_n \|\Phi_n\|_s^2 = \langle H, \Phi_n\rangle_\theta$, that is

$$a_n \|\Phi_n\|_s^2 = \left\langle \sum_{-\infty}^{\infty} \hat{H}(i) z^i, \sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} z^i \right\rangle_\theta = \sum_{i=0}^{n} \hat{H}(i) \, \overline{\frac{\Phi_n^{(i)}(0)}{i!}}.$$

In the same way, for every $k > 0$, $\langle z^k f, 1\rangle_s = \langle z^k H, 1\rangle_\theta$. Since

$$\langle z^k f, 1\rangle_s = \langle z^k f, 1\rangle_\mu = \left\langle z^k \sum_{n=0}^{\infty} a_n \Phi_n, 1 \right\rangle_\mu = \sum_{n=0}^{\infty} a_n \left(\sum_{i=0}^{n} \frac{\Phi_n^{(i)}(0)}{i!} c_{i+k} \right),$$

we obtain (4).

$(\Leftarrow)$ Now, let us assume that relations (3) and (4) hold. We define $f = \sum_{n=0}^{\infty} a_n \Phi_n$ with a_n given by (3), that is, $a_n = \langle H, \Phi_n\rangle_\theta / \|\Phi_n\|_s^2$.

Next we prove that $f \in \mathrm{HS}_2$.

$$|a_n| \leqslant \frac{\|H\|_\theta \|\Phi_n\|_\theta}{\|\Phi_n\|_s^2} \leqslant \frac{C}{\|\Phi_n\|_s^2},$$

because of $H \in L_2$ and $\|\Phi_n\|_\theta \leqslant M \|\Phi_n\|_\mu \leqslant M$ for some positive constant M.

Besides,

$$\|f\|_s^2=\sum_{n=0}^{\infty}|a_n|^2\|\Phi_n\|_s^2\leqslant C^2\sum_{n=0}^{\infty}\frac{1}{\|\Phi_n\|_s^2}$$

and it holds that $\sum_{n=0}^{\infty}1/\|\Phi_n\|_s^2<+\infty$.

Indeed,

$$\|\Phi_n\|_s^2=\|\Phi_n\|_\mu^2+\frac{1}{\lambda}\|\Phi_n'\|_\theta^2\geqslant\|\Phi_n\|_\mu^2+\frac{n^2}{\lambda}\geqslant\frac{n^2}{\lambda}$$

which implies $1/\|\Phi_n\|_s^2\leqslant\lambda/n^2$. Hence $f\in HS_2$ and therefore there exist f and f' a.e. in $\mathbb{T}$ and there also exists f'' in D.

Finally, we prove $\mathscr{L}(f)=H$, by using again (3) and (4).

$$\begin{aligned}\langle\mathscr{L}(f)-H,\Phi_n\rangle_\theta&=\langle\mathscr{L}(f),\Phi_n\rangle_\theta-\langle H,\Phi_n\rangle_\theta\\&=\langle\mathscr{L}(f),\Phi_n\rangle_\theta-\sum_{i=0}^{n}\hat{H}(i)\frac{\overline{\Phi_n^{(i)}(0)}}{i!}=0\end{aligned}$$

and for $k<0$,

$$\begin{aligned}\langle z^k(\mathscr{L}(f)-H),1\rangle_\theta&=\langle z^k\mathscr{L}(f),1\rangle_\theta-\langle z^kH,1\rangle_\theta\\&=\langle z^k\mathscr{L}(f),1\rangle_\theta-\hat{H}(-k)=0.\end{aligned}$$

Then $\mathscr{L}(f)=H$ in $D-\{0\}$, that is,

$$\frac{1}{\lambda}z^2f''(z)=H(z)-\frac{1}{\lambda}zf'(z)-G(z)f(z)$$

and so z^2f'' is defined a.e. in $\mathbb{T}$ and therefore $\mathscr{L}(f)=H$ a.e. in $\mathbb{T}$. □

The most simple example of the preceding theorem is the following. If $G=1$, then $\Phi_n=z^n$; so if we take $H=\sum_{n=0}^{\infty}\hat{H}(n)z^n\in L_2$, there exists $f\in H(D)$ such that

$$\frac{1}{\lambda}(z^2f''+zf')+f=H$$

and f is given by

$$f=\sum_{n=0}^{\infty}\frac{\hat{H}(n)}{n^2/\lambda+1}z^n.$$

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