A matrix algorithm towards solving the moment problem of Sobolev type[☆]

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Abstract

We propose a matrix algorithm which is the first step towards considering a given matrix as a moment matrix of Sobolev type in the diagonal form of an arbitrary (not necessarily finite) order on the real line or the unit circle. This continues our recent work [Proc. Amer. Math. Soc., 128 (2000) 2309] on the moment problem of Sobolev type and gives an alternative approach to what is in [J. Approx. Theory 100 (1999) 364].

Keywords: Moment problem of Sobolev type; Hankel and subhankel matrix; Toeplitz and subtoeplitz matrix; Hankelization and toeplitzation of a matrix

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1. Introduction

For the last 10 years or so some attention has been paid to the study of polynomials orthogonal with respect to an inner product

$$\langle p, q\rangle_{\mathrm{S}} = \sum_{k=0}^{N} \int_{\mathbb{R}} p^{(k)}\overline{q^{(k)}}\mathrm{d}\mu_k, \quad p, q \in \mathbb{C}[X], \tag{1}$$

where $\{\mu_k\}_{k=0}^{\infty}$ are positive Borel measures; the same kind of inner product can be def ned when all the measures are supported on the unit circle or other sets of the complex plane. Polynomials orthogonal with respect this kind of inner product are called Sobolev orthogonal polynomials (cf. [1,3,6,10], the last two papers focus on algebraic properties, in particular, on recurrence relations). The basic fact which differs this kind of orthogonality from the standard case ($N = 0$) is that the shift operator related to the appropriate inner product is neither symmetric (the real line case) nor unitary (the unit circle case). However, as was proved in [4] as well as in [7], if the operator of multiplication by some polynomial is symmetric with respect to (1), then the sequence $\{\mu_k\}_{k=1}^{\infty}$ is composed of measures which are finit sums of point masses.

Because, in principle, orthogonality of polynomials is closely related to moment problems it is quite natural to extend the Hamburger moment problem or the trigonometric one from their classical setting to orthogonality proposed by (1). In both these classical cases the structure of the moment matrices (like being Hankel or Toeplitz) comes from the algebraic structure of the set on which the representing measures are considered (the real line or the unit circle). Thus the question appears to what extent this kind of interrelation can still be supported in the case of $N > 0$. Implementing this idea we have shown in [8] how a given bisequence $\{s_{m,n}\}_{m,n=0}^{\infty}$ can be treated as the Gram matrix of the sequence $\{X^n\}_{n=0}^{\infty}$ of monomials with respect to an inner product of a more general form than that of (2) (when each measure μ_k is replaced by a matrix of measures in a sense). On the other hand, in [2] necessary and sufficien conditions are found for $\{s_{m,n}\}_{m,n=0}^{\infty}$ to be a moment matrix with respect to the inner product (1). In fact, the matrix is decomposed in a natural way as a sum (of a f xed number) of Hankel matrices corresponding to the ingredients of the sum appearing in (1). Our contribution in this paper to the problem consists in proposing, instead of formulae, a matrix algorithm which allows us to fi d the needed decomposition as well as to determine its length. We show that even in a very simple case the sum can be inf nite and also that there may exist matrices being a Sobolev type moment ones (in the sense of [8]) for which the diagonal decomposition, like in [2] or here, do not lead to integral representation.

Convention. Assuming $N \in \{1, 2, \ldots\} \cup \{+\infty\}$ we introduce the following shorthand notation:

$$\mathbb{N}_N \overset{\mathrm{df}}{=} \begin{cases} \{0, 1, \ldots, N\} & \text{if } N \text{ is fi ite,} \\ \mathbb{N} & \text{otherwise.} \end{cases}$$

2. The real line case

2.1.

The moment problem of Sobolev type we are going to deal with here consists in f nding, for a given (bi)sequence $S \stackrel{\text{df}}{=} (s_{m,n})_{m,n=0}^{\infty}$ of real numbers, a sequence $\{\mu_k\}_{k=0}^{N}$ of positive measures such that

$$s_{m,n} = \sum_{k=0}^{N} \int_{\mathbb{R}} (x^m)^{(k)} (x^n)^{(k)} \mu_k(\mathrm{d}x), \quad m, n = 0, 1, \ldots \tag{2}$$

This is the *diagonal* form of it and appearence of the possibility of $N = +\infty$ is a newly considered case [8]. Nevertheless, since the integral involves polynomials of f xed degrees m, n, differentiation makes the above sum always f nite though its length may depend on m and n (in fact, it may increase in $m + n$). The right-hand side of (2) is an inner product in $\mathbb{C}[X]$ usually denoted by $\langle x^m, x^n \rangle$.

A glance at the right-hand side gives us immediately another form [1] of (2)

$$s_{m,n} = \sum_{k=0}^{N} k!^2 \binom{m}{k} \binom{n}{k} s_{m,n}^{(k)}, \quad m, n = 0, 1, \ldots \tag{3}$$

with

$$s_{m,n}^{(k)} \stackrel{\text{df}}{=} \begin{cases} \int_{\mathbb{R}} x^{m+n-2k} \mu_k(\mathrm{d}x) & m, n \geqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

If we think of S as well as $S^{(k)} \stackrel{\text{df}}{=} (s_{m,n}^{(k)})_{m,n=0}^{\infty}$ as of infi ite matrices, then after def ning

$$D_k \stackrel{\text{df}}{=} \operatorname{diag}(d_{k,i})_{i=0}^{\infty}, \quad \text{with } d_{k,i} \stackrel{\text{df}}{=} k! \binom{i}{k}, \quad k = 0, 1, \ldots, \tag{4}$$

we can rewrite (3) as

$$S = \sum_{k=0}^{N} D_k S^{(k)} D_k. \tag{5}$$

The matrices $S^{(k)}$ are Hankel ones after removing f rst k rows and columns (which are apparently zero). The question we would like to answer here is: *can we always decompose a given* (*by necessity*) *symmetric matrix as in* (5) *with some N* (*which has to be determined as well*). In this paper we propose a matrix algorithm which yields a positive answer to this question.

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[1] We have to assume $\binom{i}{j} = 0$ if $i < j$.

2.2.

All the matrices considered in this paper are *inf nite* dimensional and these considered in this section are *real*. We just refer in the sequel to them simply as to matrices (occasionally one may try to look at some of them as properly define operators in ℓ^2). So, if we defin the matrices

$$V \overset{\text{df}}{=} \begin{pmatrix} 0 & 0 & \dots \\ 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad P_0 \overset{\text{df}}{=} \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

the following relations [2] hold:

$$V^{\mathrm{t}}V = I \overset{\text{df}}{=} \text{the identity matrix}, \quad VV^{\mathrm{t}} = I - P_0, \quad P_0V = 0. \tag{6}$$

Denote by $\boldsymbol{M}$ the algebra of all matrices and defin now mappings $\boldsymbol{r}_i$ and $\boldsymbol{l}_i$, $i = 1, 2, \dots,$ acting on matrices as

$$\boldsymbol{r}_i(A) \overset{\text{df}}{=} (V^{\mathrm{t}})^i A V^i, \quad \boldsymbol{l}_i(A) \overset{\text{df}}{=} V^i A (V^{\mathrm{t}})^i, \qquad A \in \boldsymbol{M}.$$

Notice that they are linear mappings and $\boldsymbol{r}_i = \boldsymbol{r}_1^i$ and $\boldsymbol{l}_i = \boldsymbol{l}_1^i$.

The very f rst use we make of these mappings is to def ne $\boldsymbol{M}_i^0$ as the subalgebra of all the elements of $\boldsymbol{M}$ which are of the form

$$\boldsymbol{l}_i(A), \quad A \in \boldsymbol{M}.$$

In other words, $\boldsymbol{M}_i^0$ is composed of all the matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

where A belongs to $\boldsymbol{M}$ and the 0 in the top left corner of the above matrix being an $i \times i$-dimensional matrix. Then, because

$$\boldsymbol{r}_i : \boldsymbol{M}_i^0 \ni \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \mapsto A \in \boldsymbol{M},$$

$\boldsymbol{r}_i$ restricted to such matrices is a bijection with the inverse being equal to $\boldsymbol{l}_i$. Moreover, the mappings $\boldsymbol{l}_i$ and $\boldsymbol{r}_i : \boldsymbol{M}_i^0 \mapsto \boldsymbol{M}$ are multiplicative.

Denote by $\boldsymbol{H}$ the linear space of all Hankel matrices and call a matrix A *subhankel* (of degree i, say) if $\boldsymbol{r}_i(A) \in \boldsymbol{H}$. Denoting all subhankel matrices of degree i by $\boldsymbol{H}_i$ we come to our principal class of matrices, that is to

$$\boldsymbol{H}_i^0 \overset{\text{df}}{=} \boldsymbol{H}_i \cap \boldsymbol{M}_i^0$$

(the matrices $S^{(k)}$ of the preceding subsection belong to this class); going on with terminology we call the elements of $\boldsymbol{H}_i^0$ *subhankel matrices of width of nullity equal to i*.

[2] The superscript t is to transpose matrices.

We need a kind of partial inverse to the matrix D_i define as

$$D_i' \stackrel{\text{df}}{=} \boldsymbol{l}_i((\boldsymbol{r}_i(D_i))^{-1}) \tag{7}$$

just leaves the f rst i elements of the diagonal of D_i, which are already 0, and for the rest puts its inverses instead; this is nothing else than the generalized inverse of Moore–Penrose of a diagonal matrix, cf. [5, p. 243].

2.3.

Let e_n, $n = 0, 1, \dots$, stand for the canonical zero–one basis in ℓ^2. Then

$$\begin{aligned} &Ve_n = e_{n+1}, \quad n = 0, 1, \dots, \\ &V^{\mathrm{t}}e_0 = 0, \quad V^{\mathrm{t}}e_n = e_{n-1}, \ \ n = 1, 2, \dots \end{aligned} \tag{8}$$

which give for any matrix A

$$\langle V^i P_0 A V^i e_m, e_n\rangle = \langle Ae_{m+i}, P_0 e_{n-i}\rangle = \begin{cases} 0, & n \neq i, \\ \langle Ae_{m+i}, e_0\rangle, & n = i. \end{cases} \tag{9}$$

This causes the defi ition

$$\boldsymbol{h}(A) \stackrel{\text{df}}{=} \sum_{k=0}^{\infty} V^k P_0 A V^k \tag{10}$$

to make sense as well as shows immediately that $\boldsymbol{h}(A)$ is a Hankel matrix with the same fi st row as that of A provided the latter is symmetric; think of $\boldsymbol{h}(A)$ as the hankelization of A. More formally, given a symmetric matrix $A = (a_{m,n})_{m,n=0}^{\infty}$, a matrix $B = (b_{m,n})_{m,n=0}^{\infty}$ is said to be a *hankelization* of A if it is Hankel and $a_{m,n} = b_{m,n}$ for m and n such that $mn = 0$. Thus, what we have just done enables us to state the following proposition.

Proposition 1. *Formula* (10) *define the unique hankelization of a given symmetric matrix A. Consequently, A is Hankel if and only if $A = \boldsymbol{h}(A)$.*

We now have all the necessary ingredients done to perform the algorithm.

2.4.

One more notation: $\boldsymbol{h}_i \stackrel{\text{df}}{=} \boldsymbol{l}_i \circ \boldsymbol{h} \circ \boldsymbol{r}_i$, so $\boldsymbol{h}_0 = \boldsymbol{h}$. Def ne the sequence of pairs $\{S^{(i)}, R^{(i)}\}_{i=0}^{\infty}$ of matrices as follows:

$$\begin{aligned} &S^{(0)} \stackrel{\text{df}}{=} \boldsymbol{h}_0(S) = \boldsymbol{h}(S), \quad R^{(0)} \stackrel{\text{df}}{=} S - S^{(0)}, \\ &S^{(i)} \stackrel{\text{df}}{=} \boldsymbol{h}_i(D_i' R^{(i-1)} D_i'))), \quad R^{(i)} \stackrel{\text{df}}{=} R^{(i-1)} - D_i S^{(i)} D_i, \quad n = 1, 2, \dots \end{aligned} \tag{11}$$

Theorem 2. *Given a symmetric matrix S, the following decomposition holds*:

$$S = \sum_{k=0}^{N} D_k S^{(k)} D_k, \tag{12}$$

where $S^{(k)} \in \boldsymbol{H}_k^0$ and D_k are as in (4). *In case $N = +\infty$ the appearing series is entrywise finite.*[3]

The above decomposition is unique provided $S^{(k)} \in \boldsymbol{H}_k^0$.

Proof. The uniqueness is clear if one notices that the above decomposition is linear as well as that the class $\boldsymbol{H}_i^0$ is a linear space: yet the zero matrix decomposes uniquely.

The construction gives us immediately that $S^{(k)} \in \boldsymbol{H}_k^0$. Because $D_k \in \boldsymbol{H}_k^0$, the matrix $D_k S^{(k)} D_k$ belongs to $\boldsymbol{H}_k^0$ too. This yields $\langle D_k S^{(k)} D_k e_m, e_n\rangle = 0$ if either $k > m$ or $k > n$. Thus the sum of (12) is entrywise finite, that is

$$\sum_{k=0}^{N} \langle D_k S^{(k)} D_k e_m, e_n\rangle = \sum_{k=0}^{\min\{m,n\}} \langle D_k S^{(k)} D_k e_m, e_n\rangle. \tag{13}$$

We now want to show that $R^{(k-1)} \in \boldsymbol{M}_k^0$.

With the convention $e_n = 0$ for $n < 0$ notice first that, after specifying the matrix D'_k as $\operatorname{diag}(d'_{k,i})_{i=0}^{\infty}$,

$$\langle S^{(k)} e_m, e_n\rangle = d'_{k,m+n-k} d'_{k,k} \langle R^{(k-1)} e_{m+n-k}, e_k\rangle. \tag{14}$$

Indeed, by (9),

$$\begin{aligned}
\langle S^{(k)} e_m, e_n\rangle &= \langle \boldsymbol{l}_k(\boldsymbol{h}(\boldsymbol{r}_k(D'_k R^{(k-1)} D'_k)))e_m, e_n\rangle \\
&= \langle V^k \boldsymbol{h}((V^{\mathrm{t}})^k D'_k R^{(k-1)} D'_k V^k)(V^{\mathrm{t}})^k e_m, e_n\rangle \\
&= \langle \boldsymbol{h}((V^{\mathrm{t}})^k D'_k R^{(k-1)} D'_k V^k e_{m-k}, e_{n-k}\rangle \\
&= \langle (V^{\mathrm{t}})^k D'_k R^{(k-1)} D'_k V^k e_{m+n-2k}, e_0\rangle \\
&= d'_{k,m+n-k} d'_{k,k} \langle R^{(k-1)} e_{m+n-k}, e_k\rangle.
\end{aligned}$$

First we show that

$$\begin{aligned}
\langle R^{(k)} e_m, e_n\rangle = \langle R^{(k-1)} e_m, e_n\rangle& \\
-d_{k,m} d_{k,n} d'_{k,m+n-k} d'_{k,k} \langle R^{(k-1)} e_{m+n-k}, e_k\rangle&.
\end{aligned} \tag{15}$$

From (14), we have

$$\begin{aligned}
\langle R^{(k)} e_m, e_n\rangle &= \langle R^{(k-1)} e_m, e_n\rangle - \langle D_k S^{(k)} D_k e_m, e_n\rangle \\
&= \langle R^{(k-1)} e_m, e_n\rangle - d_{k,m} d_{k,n} \langle S^{(k)} e_m, e_n\rangle \\
&= \langle R^{(k-1)} e_m, e_n\rangle - d_{k,m} d_{k,n} d'_{k,m+n-k} d'_{k,k} \langle R^{(k-1)} e_{m+n-k}, e_k\rangle.
\end{aligned}$$

[3] This means that the (m, n)th entry is equal to 0 for k sufficiently large.

Because $R^{(0)} \in \boldsymbol{M}_1^0$, the induction goes as follows: assuming $R^{(k-1)} \in \boldsymbol{M}_k^0$, the only thing one has to check is that $\langle R^{(k)} e_m, e_n\rangle = 0$ for $m = k$ or $n = k$. This we can get inserting any of these two cases in (15).

The proof of (12) goes as follows. Using (13), the right-hand side of (12) can be written as

$$\begin{aligned}\sum_{k=0}^{N}\langle D_k S^{(k)} D_k e_m, e_n\rangle &= \sum_{k=0}^{\min\{m,n\}}\langle D_k S^{(k)} D_k e_m, e_n\rangle \\ &= S^{(0)} + \sum_{k=1}^{\min\{m,n\}}\langle (R^{(k-1)} - R^{(k)}) e_m, e_n\rangle \\ &= \langle (S^{(0)} + R^{(0)}) e_m, e_n\rangle - \langle R^{(\min\{m,n\})} e_m, e_n\rangle \\ &= \langle S e_m, e_n\rangle - \langle R^{(\min\{m,n\})} e_m, e_n\rangle \\ &= \langle S e_m, e_n\rangle. \end{aligned} \tag{16}$$

The last equality in the above holds because $R^{(k-1)} \in \boldsymbol{H}_k^k$ implies straightforwardly $\langle R^{(\min\{m,n\})} e_m, e_n\rangle = 0$. □

Remark 3. Algorithm (11) terminates, that is, N in Theorem 2 is fi ite if and only if $\boldsymbol{h}_k(S^{(k)}) = \boldsymbol{h}_k(D_k' R^{(i-1)} D_k'))$ for some k. If this happen, then there is a k for which $S^{(k)}$ satisfying this condition is not zero and that k is equal to N. This leads us to the situation which can be compared with that worked out in [2, Theorem 2]. However, the important difference is that, while N in [2] is f xed from the beginning, in our situation it is free (and it has to be determined in the process as well.

2.5.

Let us take the identity matrix I as S and try to decompose it according to Theorem 2. It is a matter of direct verificatio to check that then $S^{(k)} = D_k' P_k D_k'$, where $P_k \stackrel{\mathrm{df}}{=} \operatorname{diag}(\delta_{i,k})_{i=0}^{\infty}$, and

$$s_{m,n} = \delta_{m,n} = \sum_{k=0}^{\infty}\int_{\mathbb{R}} (x^m)^{(k)} (x^n)^{(k)} \mathrm{d}((k!)^{-2}\delta_0).$$

On the other hand, let S be

$$\begin{pmatrix} 1 & 1 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Because it is symmetric we can decompose it according to Theorem 2. Thus we get

$$S^{(0)} = \begin{pmatrix} 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad S^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $S^{(k)} = 0$, $k = 2, 3, \dots$ Because $S^{(0)}$ is not positive definite, S is *not* a Sobolev moment sequence in the diagonal form of any order. [4] However,

$$s_{m,n} = \sum_{i,j=0}^{1} \int_{\mathbb{R}} (x^m)^{(i)} (x^n)^{(j)} \mathrm{d}\delta_0,$$

which means that $S = (s_{m,n})_{m,n=0}^{\infty}$ is a Sobolev moment sequence of order 1 in non-diagonal form, cf. [8].

In conclusion, while, according to Theorem 2, a moment sequence of Sobolev type of any order can always be decomposed as in (12), it need not be a moment sequence of Sobolev type in the diagonal form.

3. The unit circle case

3.1.

Suppose a sequence of positive measures $\{\mu_k\}_{k=0}^{\infty}$ on the unit circle $\mathbb{T}$ is given. Then the sequence $\{s_{m,n}\}_{m,n}$ given by

$$s_{m,n} = \sum_{k=0}^{N} \int_{\mathbb{T}} \frac{\mathrm{d}^k}{\mathrm{d}z^k} z^m \frac{\mathrm{d}^k}{\mathrm{d}\bar{z}^k} \bar{z}^n \mu_k(\mathrm{d}z), \quad \text{for any } m, n, \tag{17}$$

can be viewed as the moment sequence of Sobolev type on the unit circle. So (a diagonal form of) the moment problem of Sobolev type consists in finding, for a given sequence $\{S_{m,n}\}_{m,n=0}^{\infty} \subset \mathbb{C}$, a sequence of positive measures $\{\mu_k\}_{k=0}^{\infty}$ on $\mathbb{T}$ such that the representation (17) holds. However, we have now two possibilities:

(a) $\{s_{m,n}\}_{m,n} = \{s_{m,n}\}$ and (17) holds for $m, n = 0, 1, \dots$;
(b) $\{s_{m,n}\}_{m,n} = \{s_{m,n}\}$ and (17) holds for $m, n \in \mathbb{Z}$.

When $N = 0$, which is not our case, each of these cases determines the other, for $N > 0$ this does not seem to be so.

In this paper we consider case (a). The right-hand side of (17) can be written as

$$s_{m,n} = \sum_{k=0}^{N} k!^2 \binom{m}{k} \binom{n}{k} s_{m,n}^{(k)}, \quad m, n = 0, 1, \dots \tag{18}$$

[4] Notice that, because $s_{m,n} = \langle x^m x^n, \delta_0 - \delta_0' \rangle + \langle (x^m)'(x^n)', \delta_0 \rangle$, the matrix S can be represented as a Sobolev moment sequence in the diagonal form in the sense of linear (not necessarily positive) functionals on polynomials, cf. [9].

with

$$s_{m,n}^{(k)} \overset{\mathrm{df}}{=} \begin{cases} \int_{\mathbb{T}} z^{m-n}\mu_k(\mathrm{d}z) & m,n \geqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

With notation of (4) we can rewrite (18) as

$$S = \sum_{k=0}^{N} D_k S^{(k)} D_k, \tag{19}$$

where $S^{(k)} \overset{\mathrm{df}}{=} (s_{m,n}^{(k)})_{m,n=0}^{\infty}$, $k \in \mathbb{N}_N$. Now the matrices $S^{(k)}$ are Toeplitz ones after removing f rst k rows and columns (which are apparently zero). Again the question is: *can we always decompose a given* (*by necessity*) *Hermitian matrix as in* (19) *with some* N (*which has to be determined as well*).

3.2.

Denote by $\boldsymbol{T}$ the linear space of all Toeplitz matrices. Call a matrix A *subtoeplitz* (of degree i) if $\boldsymbol{r}_i(A) \in \boldsymbol{T}$ and denote all subtoeplitz matrices of degree i by $\boldsymbol{T}_i$ as well as set

$$\boldsymbol{T}_i^0 \overset{\mathrm{df}}{=} \boldsymbol{T}_i \cap \boldsymbol{M}_i^0$$

The elements of $\boldsymbol{T}_i^0$ are called *subtoeplitz matrices of width of nullity equal to* i.

Notice that because

$$\langle V^i(-P_0AP_0 + P_0A + AP_0)(V^{\mathrm{t}})^i e_m, e_n\rangle = \begin{cases} \langle Ae_0, e_0\rangle, & m = n = i, \\ \langle Ae_0, e_{n-m}\rangle, & m = i,\ n \geqslant m, \\ \langle Ae_{m-n}, e_0\rangle, & n = i,\ m \geqslant n, \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

the matrix $\boldsymbol{t}(A)$ define as

$$\boldsymbol{t}(A) \overset{\mathrm{df}}{=} \sum_{k=0}^{\infty} V^k(-P_0AP_0 + P_0A + AP_0)(V^{\mathrm{t}})^k, \quad A \in \boldsymbol{M}, \tag{21}$$

is a Toeplitz matrix generated by the fi st row and the f rst column of A. So it can be viewed as a *toeplization* of A.

Proposition 4. *Formula* (21) *define the unique toeplization of a given matrix* A. *Consequently*, A *is Toeplitz if and only if* $A = \boldsymbol{t}(A)$.

With notation $\boldsymbol{t}_i \overset{\mathrm{df}}{=} \boldsymbol{l}_i \circ \boldsymbol{t} \circ \boldsymbol{r}_i$ ($\boldsymbol{t}_0 = \boldsymbol{t}$), def ne the sequence of pairs $\{S^{(i)}, R^{(i)}\}_{i=0}^{\infty}$ of matrices as follows:

$$S^{(0)} \overset{\text{df}}{=} \boldsymbol{t}_0(S) = \boldsymbol{t}(S), \quad R^{(0)} \overset{\text{df}}{=} S - S^{(0)};$$
$$S^{(i)} \overset{\text{df}}{=} \boldsymbol{t}_i(D_i' R^{(i-1)} D_i'))), \quad R^{(i)} \overset{\text{df}}{=} R^{(i-1)} - D_i S^{(i)} D_i, \qquad n = 1, 2, \dots \tag{22}$$

Theorem 5. *Given a matrix S, the following decomposition holds*:

$$S = \sum_{k=0}^{N} D_k S^{(k)} D_k, \tag{23}$$

where $S^{(k)} \in \boldsymbol{T}_k^0$ *and* D_k *are as in* (4). *The matrices* $S^{(k)}$ *are Hermitian provided so is S. In case* $N = +\infty$ *the appearing series is entrywise fin te. The above decomposition is unique provided* $S^{(k)} \in \boldsymbol{H}_k^0$.

The proof of Theorem 5 goes in the same way as that of Theorem 2 using (20) instead of (9).

Additional remark. The algorithm we propose in this paper results in shaping a given matrix in a way which makes it suitable for being represented as a Sobolev type moment one. Another question, which we leave apart, is numerical implementation of this algorithm and, in particular, its stability.

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