**Proposition 3.6.** *Let* $\mu \in \mathcal{N}$, $p \in \mathbb{N}$, *and let* $\mu_{p+1}$ *be the measure given by* $\mathrm{d}\mu_{p+1} = |z-c|^{2(p+1)}\mathrm{d}\mu$. *Then, for the polynomials* $\varphi_n(z;\mathrm{d}\mu_{p+1})$,

$$\lim_n \frac{\varphi_n(z;\mathrm{d}\mu_{p+1})}{\varphi_{n+p+1}(z)} = \left(\frac{\bar{c}}{|c|}\frac{1}{\bar{c}z-1}\right)^{p+1}$$

*holds l. u. in* $|z|>1$ *if* $|c|\geqslant 1$, *and uniformly in* $|z|\geqslant 1$ *if* $|c|>1$. *Furthermore,*

$$\lim_n \frac{\kappa_n(\mathrm{d}\mu_{p+1})}{\kappa_{n+p+1}} = \frac{1}{|c|^{p+1}}.$$

**Proof.** Lemma 3.3 implies that $\mu_j$, given by $\mathrm{d}\mu_j = |z-c|^{2j}\,\mathrm{d}\mu$, belongs to the $\mathcal{N}$ class, for $j=0,1,\ldots,p+1$. From (3.5) and Proposition 3.5, we get

$$\lim_n \frac{\kappa_n(\mathrm{d}\mu_{p+1})}{\kappa_{n+p+1}} = \lim_n \prod_{j\ 0}^{p} \frac{\kappa_{n+j}(\mathrm{d}\mu_{p+1-j})}{\kappa_{n+j+1}(\mathrm{d}\mu_{p-j})} = \frac{1}{|c|^{p+1}}$$

and

$$\lim_n \frac{\varphi_n(z;\mathrm{d}\mu_{p+1})}{\varphi_{n+p+1}(z)} = \prod_{j\ 0}^{p} \lim_n \frac{\varphi_{n+j}(z;\mathrm{d}\mu_{p+1-j})}{\varphi_{n+j+1}(z;\mathrm{d}\mu_{p-j})} = \left[\frac{c}{|c|}\frac{1}{(\bar{c}z-1)}\right]^{p+1}. \qquad \square$$

**Corollary 3.7.** *Assume that* $\mu \in \mathcal{N}$, $|c|>1$, *and* $p\in\mathbb{N}$. *Then,*

$$\lim_n \left(1+\boldsymbol{\varphi}_{n+1}(c)\,[M(n)]^{-1}\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H\right) = \lim_n \frac{\det[M(n+1)]}{\det[M(n)]} = |c|^{2(p+1)}$$

*holds, with* $\boldsymbol{\varphi}_{n+1}(c) = \left(\varphi_{n+1}(c),\, \varphi'_{n+1}(c), \ldots, \varphi^{(p)}_{n+1}(c)\right)$.

**Proof.** It is straightforward from (3.2) and Lemma 3.1. $\square$

Now, let $\mu$ be a probability measure with $\mathbb{T}$ as support. Let us assume that $\mu\in\mathcal{N}$, $c\in\mathbb{C}$, and $p\in\mathbb{N}$. Again, we write $\varphi_n(z;\mathrm{d}\mu_j)$ the $n$th orthonormal polynomial with respect to $\mu_j$, and $K_n(z,y;\mathrm{d}\mu_j)$ the corresponding $n$th kernel, $j=0,\,1,\ldots,p+1$. For fixed $m\in\mathbb{N}$, with $m\geqslant p$, we define the linear operator $\mathscr{F}_m : \mathbb{P}_{m+1} \to \mathbb{C}^{p+1}$ as

$$\begin{aligned}\mathscr{F}_m(P) &:= \int_{|z|\ 1} P(z)\,\overline{\mathbf{K}_{m+1}(z,c)}\,\mathrm{d}\mu(z) = (P(z),\mathbf{K}_{m+1}(z,c)) = \mathbf{P}(c)\\ &= \left(P(c),\,P'(c),\ldots,P^{(p)}(c)\right).\end{aligned}$$

It is easy to prove that

$$\mathscr{F}_m[(z-c)P(z)] = \mathbf{P}(c)\,B$$

for $P \in \mathbb{P}_m$, where $B$ is the matrix

$$B := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{(p+1,\,p+1)}.$$

For each $m \geqslant p$ and each $j = 0, \dots, p$, we shall denote

$$\begin{aligned} \mathbf{v}_{m+1} &= \mathscr{F}_m\left[\varphi_{m+1}(z; \mathrm{d}\mu)\right] = \boldsymbol{\varphi}_{m+1}(c; \mathrm{d}\mu), \\ \mathbf{v}_{m-j} &= \mathscr{F}_m\left[\varphi_{m-j}(z; \mathrm{d}\mu_{j+1})\right] = \boldsymbol{\varphi}_{m-j}(c; \mathrm{d}\mu_{j+1}), \\ \mathbf{L}^{(k)}_{m-j} &= \mathscr{F}_m\left[K^{(0,k)}_{m-j}(z, c; \mathrm{d}\mu_j)\right] = \left(K^{(0,k)}_{m-j}(c, c; \mathrm{d}\mu_j), \dots, K^{(p,k)}_{m-j}(c, c; \mathrm{d}\mu_j)\right), \\ \mathbf{L}_{m-j} &= \mathbf{L}^{(0)}_{m-j}. \end{aligned}$$

Notice that $\mathbf{L}^{(k)}_m$ is the $k$th row $(0 \leqslant k \leqslant p)$ of the matrix $M(m)$.

From (3.4), for $j = 0, 1, \dots, p$, we can put

$$(z-c)\,\varphi_{m-j}(z; \mathrm{d}\mu_{j+1}) = \alpha^{(j)}_{m-j}\left[\varphi_{m-j+1}(z; \mathrm{d}\mu_j) - \beta^{(j)}_{m-j} K_{m-j}(z, c; \mathrm{d}\mu_j)\right],$$

with

$$\alpha^{(j)}_{m-j} = \left(1 + \frac{|\varphi_{m-j+1}(c; \mathrm{d}\mu_j)|^2}{K_{m-j}(c, c; \mathrm{d}\mu_j)}\right)^{-1/2}, \qquad \beta^{(j)}_{m-j} = \frac{\varphi_{m-j+1}(c; \mathrm{d}\mu_j)}{K_{m-j}(c, c; \mathrm{d}\mu_j)}.$$

Thus, by applying $\mathscr{F}_m$

$$\mathbf{v}_{m-j} B = \alpha^{(j)}_{m-j}\left(\mathbf{v}_{m-j+1} - \beta^{(j)}_{m-j} \mathbf{L}_{m-j}\right).$$

By iteration, we get

$$\mathbf{v}_{m+1} = \frac{\mathbf{v}_{m-j} B^{j+1}}{\alpha^{(j)}_{m-j}, \dots, \alpha^{(0)}_m} + \frac{\beta^{(j)}_{m-j}}{\alpha^{(j-1)}_{m-j+1}, \dots, \alpha^{(0)}_m} \mathbf{L}_{m-j} B^j + \dots + \beta^{(0)}_m \mathbf{L}_m. \tag{3.6}$$

Since $B^{p+1} = 0$, (3.6) becomes

$$\mathbf{v}_{m+1} = \frac{\beta^{(p)}_{m-p}}{\alpha^{(p-1)}_{m-p+1}, \dots, \alpha^{(0)}_m} \mathbf{L}_{m-p} B^p + \dots + \frac{\beta^{(1)}_{m-1}}{\alpha^{(0)}_m} \mathbf{L}_{m-1} B + \beta^{(0)}_m \mathbf{L}_m \tag{3.7}$$

for $j = p$.

On the other hand, for the kernel polynomials we have (see [5])

$$\begin{aligned} (z-c)\,\overline{(y-c)} K_{m-j-1}(z, y; \mathrm{d}\mu_{j+1}) &= K_{m-j}(z, y; \mathrm{d}\mu_j) \\ &\quad - \frac{K_{m-j}(z, c; \mathrm{d}\mu_j)\, K_{m-j}(c, y; \mathrm{d}\mu_j)}{K_{m-j}(c, c; \mathrm{d}\mu_j)}, \end{aligned}$$

from where, computing the $k$th derivative in $z=c$ and taking conjugate, we obtain

$$k\,(y-c)K^{(0,k-1)}_{m-j-1}(y,c;\mathrm{d}\mu_{j+1}) = K^{(0,k)}_{m-j}(y,c;\mathrm{d}\mu_j) - \frac{K_{m-j}(y,c;\mathrm{d}\mu_j)\,K^{(0,k)}_{m-j}(c,c;\mathrm{d}\mu_j)}{K_{m-j}(c,c;\mathrm{d}\mu_j)}.$$

Then, applying $\mathscr{F}_m$, we get

$$k\,\mathbf{L}^{(k-1)}_{m-j-1}\,B = \mathbf{L}^{(k)}_{m-j} - \delta^{(k)}_{m-j}\,\mathbf{L}_{m-j}, \tag{3.8}$$

with

$$\delta^{(k)}_{m-j} = \frac{K^{(0,k)}_{m-j}(c,c;\mathrm{d}\mu_j)}{K_{m-j}(c,c;\mathrm{d}\mu_j)}.$$

**Proposition 3.8.** *If $j=0,1,\dots,p$, then the following statements hold*

(i) $\mathbf{L}_{m-j}\,B^j\,[M(m)]^{-1}[\mathbf{x}B^i]^H = 0$, *for all $i \geqslant j+1$ and for all $\mathbf{x}\in\mathbb{C}^{p+1}$.*
(ii) $\mathbf{L}_{m-j}\,B^j\,[M(m)]^{-1}\left[\mathbf{L}_{m-j}\,B^j\right]^H = K_{m-j}(c,c;\mathrm{d}\mu_j)$.

**Proof.** (i) For $j=0$, we have

$$\mathbf{L}_m = \left(K_m(c,c;\mathrm{d}\mu),\dots,K^{(p,0)}_m(c,c;\mathrm{d}\mu)\right),$$

and

$$\mathbf{L}_m\,[M(m)]^{-1} = (1,0,\dots,0).$$

Thus, the statement follows immediately. Assume it is true for $0,1,\dots,j-1$. Then, if $k=1$, we obtain from (3.8)

$$\mathbf{L}_{m-j}\,B^j\,[M(m)]^{-1}[\mathbf{x}B^i]^H = \mathbf{L}^{(1)}_{m-j+1}\,B^{j-1}\,[M(m)]^{-1}[\mathbf{x}B^i]^H - \delta^{(1)}_{m-j+1}\,\mathbf{L}_{m-j+1}\,B^{j-1}\,[M(m)]^{-1}[\mathbf{x}B^i]^H.$$

The induction hypothesis gives

$$\mathbf{L}_{m-j}\,B^j\,[M(m)]^{-1}[\mathbf{x}B^i]^H = \mathbf{L}^{(1)}_{m-j+1}\,B^{j-1}\,[M(m)]^{-1}[\mathbf{x}B^i]^H,$$

and, for $k=2,\dots,j$ in (3.8),

$$\begin{aligned}\mathbf{L}_{m-j}\,B^j\,[M(m)]^{-1}[\mathbf{x}B^i]^H &= \frac{1}{2}\,\mathbf{L}^{(2)}_{m-j+2}\,B^{j-2}\,[M(m)]^{-1}[\mathbf{x}B^i]^H = \cdots\\ &= \frac{1}{j!}\,\mathbf{L}^{(j)}_m\,[M(m)]^{-1}[\mathbf{x}B^i]^H\\ &= \frac{1}{j!}\,(\overbrace{0,\dots,0}^{j-1},\,1,0,\dots,0)\left(B^{\mathrm{T}}\right)^i\mathbf{x}^H = 0\end{aligned}$$

follows, taking into account that

$$\left(B^{\mathrm{T}}\right)^{i}=\left(\begin{array}{cccccc}0 & 0 & \cdots & 0 & \cdots & 0\\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\\ 0 & 0 & \cdots & 0 & \cdots & 0\\ \hline i! & 0 & \cdots & 0 & \cdots & 0\\ 0 & \frac{(i+1)!}{1!} & \cdots & 0 & \cdots & 0\\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\\ 0 & 0 & \cdots & \frac{p!}{(p-i)!} & \cdots & 0\end{array}\right)\begin{array}{l}\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} i\\ \\ \\ \\ \\ \end{array}$$

(ii) For $j=0$, we have

$$\mathbf{L}_m\,[M(m)]^{-1}\mathbf{L}_m^H=(1,0,\ldots,0)\,\mathbf{L}_m^H=K_m(c,c;\mathrm{d}\mu).$$

Assume that our hypothesis is true for $0,1,\ldots,j-1$. From (3.8) and the first statement,

$$\begin{aligned}\mathbf{L}_{m-j}B^j\,[M(m)]^{-1}\left[\mathbf{L}_{m-j}B^j\right]^H &= \mathbf{L}_{m-j+1}^{(1)}B^{j-1}\,[M(m)]^{-1}\left[\mathbf{L}_{m-j}B^j\right]^H=\ldots\\ &= \frac{1}{j!}\,\mathbf{L}_m^{(j)}\,[M(m)]^{-1}\left[\mathbf{L}_{m-j}B^j\right]^H\\ &= \frac{1}{j!}\,(\overbrace{0,\ldots,0}^{j-1},\,1,\,0,\ldots,0)\,\left(B^{\mathrm{T}}\right)^j\mathbf{L}_{m-j}^H\\ &= K_{m-j}(c,c;\mathrm{d}\mu_j)\end{aligned}$$

follows, for $k=1,\ldots,j$. $\square$

**Corollary 3.9.** $\{\mathbf{L}_{m-j}B^j\}_{j\ 0}^{p}$ *is an orthogonal basis in* $\mathbb{C}^{p+1}$ *for the inner product* $\mathbf{x}\,[M(m)]^{-1}\mathbf{y}^H$.

Notice that (3.7) gives also an orthogonal decomposition for $\mathbf{v}_{m+1}$ (with respect to the inner product $\mathbf{x}\,[M(m)]^{-1}\mathbf{y}^H$) for $j=0,\ldots,p-1$.

**Corollary 3.10.** *If* $\mu\in\mathcal{N}$ *and* $|c|\geqslant 1$, *then*

$$\lim_m\left[\mathbf{v}_{m-i+1}B^i\,[M(m)]^{-1}[\mathbf{v}_{m-i+1}B^i]^H\right]=|c|^{2(p-i+1)}-1$$

*holds for* $i=0,\ldots,p$.

**Proof.** Because the orthogonality of the decomposition (3.6), we have

$$\mathbf{v}_{m+1}\,[M(m)]^{-1}\mathbf{v}_{m+1}^{H} = \frac{\mathbf{v}_{m-j}B^{j+1}\,[M(m)]^{-1}\left[\mathbf{v}_{m-j}B^{j+1}\right]^{H}}{\left|\alpha_{m-j}^{(j)},\ldots,\alpha_{m}^{(0)}\right|^{2}} + \frac{\left|\beta_{m-j}^{(j)}\right|^{2}\mathbf{L}_{m-j}B^{j}\,[M(m)]^{-1}\left[\mathbf{L}_{m-j}B^{j}\right]^{H}}{\left|\alpha_{m-j+1}^{(j-1)},\ldots,\alpha_{m}^{(0)}\right|^{2}} + \cdots + \left|\beta_{m}^{(0)}\right|^{2}\mathbf{L}_{m}\,[M(m)]^{-1}\mathbf{L}_{m}^{H}.$$

But, Corollary 3.7 implies that

$$\lim_{m}\left(\mathbf{v}_{m+1}\,[M(m)]^{-1}\mathbf{v}_{m+1}^{H}\right) = |c|^{2p+2} - 1.$$

On the other hand, $\lim_m \alpha_{m-k}^{(k)} = |c|^{-1}$ and, by Proposition 3.8 (ii),

$$\left|\beta_{m-k}^{(k)}\right|^{2}\mathbf{L}_{m-k}B^{k}\,[M(m)]^{-1}\left[\mathbf{L}_{m-k}B^{k}\right]^{H} = \frac{|\varphi_{m-k+1}(c;\mathrm{d}\mu_k)|^{2}}{K_{m-k}(c,c;\mathrm{d}\mu_k)},$$

$(k = 0,\ldots j)$, which tends to $|c|^2 - 1$ when $m \to \infty$ (Lemma 2.6). Thus, we have

$$|c|^{2p+2} - 1 = |c|^{2j+2}\lim_{m}\left[\mathbf{v}_{m-j}B^{j+1}\,[M(m)]^{-1}\left[\mathbf{v}_{m-j}B^{j+1}\right]^{H}\right] + |c|^{2j}\left(|c|^{2}-1\right) + \cdots + |c|^{2} - 1,$$

and

$$\lim_{m}\left[\mathbf{v}_{m-j}B^{j+1}\,[M(m)]^{-1}\left[\mathbf{v}_{m-j}B^{j+1}\right]^{H}\right] = |c|^{2(p-j)} - 1$$

holds for $j = 0,\ldots,p-1$. So, the Proposition is true for $i = 1,\ldots,p$. For $i = 0$, we recover Corollary 3.7. □

**Lemma 3.11.** *Let $\mathbf{x}_1,\ldots,\mathbf{x}_k$ be vectors in a complex vector space with an inner product $[\cdot,\cdot]$. Then,*

$$\sum_{i,j=1}^{k}\left|\left[\mathbf{x}_i,\mathbf{x}_j\right]\right| \leqslant k\sum_{i=1}^{k}[\mathbf{x}_i,\mathbf{x}_i].$$

**Proof.** By the Cauchy–Schwarz's inequality we have

$$\left|\left[\mathbf{x}_i,\mathbf{x}_j\right]\right| \leqslant \left([\mathbf{x}_i,\mathbf{x}_i]\cdot\left[\mathbf{x}_j,\mathbf{x}_j\right]\right)^{1/2} \leqslant \frac{1}{2}\left([\mathbf{x}_i,\mathbf{x}_i] + \left[\mathbf{x}_j,\mathbf{x}_j\right]\right).$$

Hence,

$$\sum_{i,j=1}^{k}\left|\left[\mathbf{x}_i,\mathbf{x}_j\right]\right| \leqslant \frac{1}{2}\sum_{i,j=1}^{k}\left([\mathbf{x}_i,\mathbf{x}_i] + \left[\mathbf{x}_j,\mathbf{x}_j\right]\right) = k\sum_{i=1}^{k}[\mathbf{x}_i,\mathbf{x}_i]. \qquad \square$$

**Lemma 3.12.** *If* $\mu \in \mathcal{N}$ *and* $|c| > 1$, *then for fixed numbers* $i \in \mathbb{Z}$ *and* $j, k, h \in \mathbb{N}$, *the following statements are fulfilled*

(i) $|\varphi_{n+i}^{(k)}(c; \mathrm{d}\mu_j)| = \mathcal{O}(|\varphi_n^{(k)}(c)|)$,

(ii) $|K_{n+i}^{(k,h)}(c, c; \mathrm{d}\mu_j)| = \mathcal{O}(|\varphi_n^{(k)}(c)\, \varphi_n^{(h)}(c)|)$.

**Proof.** (i) If $z = c$ ($|c| > 1$) in Proposition 3.6, then

$$\lim_n \frac{\varphi_n(c; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}(c)} = \left( \frac{\overline{c}}{|c|} \cdot \frac{1}{|c|^2 - 1} \right)^{j+1}.$$

On the other hand, from Lemma 2.4 we have

$$\frac{\varphi_n^{(k)}(c; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}^{(k)}(c)} = \frac{\varphi_{n+j+1}^{(k-1)}(c)}{\varphi_{n+j+1}^{(k)}(c)} \left( \frac{\varphi_n^{(k-1)}(z; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}^{(k-1)}(z)} \right)' \Bigg|_{z\ c} + \frac{\varphi_n^{(k-1)}(c; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}^{(k-1)}(c)}.$$

Since Lemma 2.5 (i) and the uniform convergence of $(\varphi_n^{(k-1)}(z; \mathrm{d}\mu_{j+1}))/\varphi_{n+j+1}^{(k-1)}(z)$, we can write

$$\lim_n \frac{\varphi_n^{(k)}(c; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}^{(k)}(c)} = \lim_n \frac{\varphi_n^{(k-1)}(c; \mathrm{d}\mu_{j+1})}{\varphi_{n+j+1}^{(k-1)}(c)} = \left( \frac{\overline{c}}{|c|} \cdot \frac{1}{|c|^2 - 1} \right)^{j+1}.$$

Furthermore, $\lim_n(\varphi_{n+i}^{(k)}(z))/\varphi_n^{(k)}(z) = z^i$ *l. u.* in $|z| > 1$. Then, the first statement follows.

(ii) Choose $z = c$ in Lemma 2.6 (ii). Then,

$$\lim_n \frac{K_{n+i}^{(k,h)}(c, c; \mathrm{d}\mu_j)}{\varphi_{n+i+1}^{(k)}(c; \mathrm{d}\mu_j)\, \overline{\varphi_{n+i+1}^{(h)}(c; \mathrm{d}\mu_j)}} = \frac{1}{|c|^2 - 1},$$

and it is enough to use (i). $\square$

**Theorem 3.13.** *If* $\mu \in \mathcal{N}$ *and* $|c| > 1$, *the spectral radius of* $[M(n)]^{-1}$, $\rho([M(n)]^{-1})$, *tends to zero when* $n \to \infty$.

**Proof.** For each $n \in \mathbb{N}$, denote $\mathbf{V}_j^{(n)} = \mathbf{L}_{n+j}\, B^{p-j}(j = 0, \ldots, p)$. Thus (Corollary 3.9), $(\mathbf{V}_j^{(n)})_{j\ 0}^p$ is an orthogonal basis in $\mathbb{C}^{p+1}$ with respect to the inner product $\mathbf{x}\,[M(n+p)]^{-1}\mathbf{y}^H$.

It is necessary to prove that $\lim_n (\mathbf{x}\,[M(n)]^{-1}\mathbf{x}^H)/||\mathbf{x}||^2 = 0$, for all $\mathbf{x} \neq 0$. However, if $(\mathbf{u}_j^{(n)})_{j\ 0}^p$ is an orthonormal basis of $\mathbb{C}^{p+1}$ for each $n$, and $\mathbf{x} = \sum_{j\ 0}^p x_{jn}\mathbf{u}_j^{(n)}$, from Lemma 3.11 we have

$$\frac{\mathbf{x}\,[M(n)]^{-1}\mathbf{x}^H}{||\mathbf{x}||^2} = \frac{\sum_{i,j} x_{in}\,\mathbf{u}_i^{(n)}\,[M(n)]^{-1}\left[x_{jn}\,\mathbf{u}_j^{(n)}\right]^H}{\sum_i |x_{in}|^2}$$

$$\leqslant (p+1)\,\frac{\sum_i |x_{in}|^2\,\mathbf{u}_i^{(n)}\,[M(n)]^{-1}\left[\mathbf{u}_i^{(n)}\right]^H}{\sum_i |x_{in}|^2}$$

$$\leqslant (p+1)\,\sum_i \mathbf{u}_i^{(n)}\,[M(n)]^{-1}\left[\mathbf{u}_i^{(n)}\right]^H.$$

Hence, it is enough to prove that $\lim_n \mathbf{u}_i^{(n)}\,[M(n)]^{-1}\left[\mathbf{u}_i^{(n)}\right]^H = 0$, or

$$\lim_n \frac{\mathbf{u}_i^{(n)}\,[M(n)]^{-1}\left[\mathbf{u}_i^{(n)}\right]^H}{\left|\left|\mathbf{u}_i^{(n)}\right|\right|^2} = 0, \tag{3.9}$$

when $(\mathbf{u}_i^{(n)})_{i\;0}^p$ is orthogonal (for each $n$).

So, we will orthogonalize $(\mathbf{V}_j^{(n)})$ by using the Gram Schmidt method, and, at once, we will study (3.9). Thus, let $(\mathbf{u}_j^{(n)})$ be the orthogonal basis such that

$$\mathbf{u}_0^{(n)} = \mathbf{V}_0^{(n)}, \mathbf{u}_j^{(n)} = \mathbf{V}_j^{(n)} - \sum_{k\;0}^{j-1} \theta_k^{(j)}\,\mathbf{u}_k^{(n)} \quad (j = 1, \ldots, p),$$

where the $\theta_k^{(j)}$'s (which depend on $n$) are given by

$$\theta_k^{(j)} = \frac{\mathbf{V}_j^{(n)}\left[\mathbf{u}_k^{(n)}\right]^H}{\left|\left|\mathbf{u}_k^{(n)}\right|\right|^2}, \quad k = 0, \ldots, j-1;\; j = 1, \ldots, p.$$

If we consider

$$\mathbf{V}_j^{(n)} = \left[\overbrace{0, \ldots, 0}^{p-j}, (p-j)!K_{n+j}(c, c; \mathrm{d}\mu_{p-j}), \ldots, \frac{(p-k)!}{(j-k)!}K_{n+j}^{(j-k,0)}(c, c; \mathrm{d}\mu_{p-j}), \ldots\right],$$

then

$$\mathbf{u}_j^{(n)} = \left[\overbrace{0, \ldots, 0}^{p-j}, (p-j)!K_{n+j}(c, c; \mathrm{d}\mu_{p-j}), 0, \ldots, 0\right]$$

follows for $j = 0, \ldots, p$. From here, we have

$$\theta_k^{(j)} = \frac{1}{(j-k)!}\,\frac{K_{n+j}^{(j-k,0)}(c, c; \mathrm{d}\mu_{p-j})}{K_{n+k}(c, c; \mathrm{d}\mu_{p-k})}, \quad k = 0, \ldots, j-1;\; j = 1, \ldots, p.$$

By using Lemma 3.12, we obtain

$$\left|\theta_k^{(j)}\right| = \mathcal{O}\left(\left|\frac{\varphi_n^{(j-k)}(c)}{\varphi_n(c)}\right|\right), \tag{3.10}$$

and $\left\|\mathbf{u}_j^{(n)}\right\| = \mathcal{O}(|\varphi_n(c)|^2)$. Also, $\lim_n(\varphi_n^{(j-k)}(c))/\varphi_n^{(j)}(c) = 0$ $(1 \leqslant k \leqslant j)$, from where $|\theta_k^{(j)}| \leqslant |\theta_0^{(j)}|$ for $k = 0, \ldots, j$ and $n$ large enough.

We will use induction to prove that

$$\frac{\mathbf{u}_j^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_j^{(n)}\right]^H}{\left\|\mathbf{u}_j^{(n)}\right\|^2} = \mathcal{O}\left(\frac{\prod_{h\ 0}^{j}\left|\varphi_n^{(j-h)}(c)\right|^2}{|\varphi_n(c)|^{2(j+2)}}\right)$$

with $j = 0, \ldots, p$, or, equivalently, taking into account that $\left\|\mathbf{u}_j^{(n)}\right\| = \mathcal{O}(|\varphi_n(c)|^2)$,

$$\mathbf{u}_j^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_j^{(n)}\right]^H = \mathcal{O}\left(\frac{\prod_{h\ 0}^{j}\left|\varphi_n^{(j-h)}(c)\right|^2}{|\varphi_n(c)|^{2j}}\right).$$

In fact, if $j = 0$,

$$\begin{aligned}\mathbf{u}_0^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_0^{(n)}\right]^H &= \mathbf{V}_0^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{V}_0^{n}\right]^H = K_{n+j}(c, c; \mathrm{d}\mu_p)\\ &= \mathcal{O}\left(|\varphi_n(c)|^2\right),\end{aligned}$$

according to Corollary 3.10. Assume that the hypothesis is true for $j-1$. Then,

$$\begin{aligned}\mathbf{u}_j^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_j^{(n)}\right]^H &= \mathbf{V}_j^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{V}_j^{(n)}\right]^H\\ &\quad + \sum_{k,i\ 0}^{j-1}\theta_k^{(j)}\,\mathbf{u}_k^{(n)}\,[M(n+p)]^{-1}\left[\theta_i^{(j)}\,\mathbf{u}_i^{(n)}\right]^H.\end{aligned}$$

Since Lemma (3.11) and (3.10),

$$\begin{aligned}\mathbf{u}_j^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_j^{(n)}\right]^H &\leqslant K_{n+j}(c, c; \mathrm{d}\mu_{p-j})\\ &\quad + j\left|\theta_0^{(j)}\right|^2\sum_{k\ 0}^{j-1}\mathbf{u}_k^{(n)}\,[M(n+p)]^{-1}\left[\mathbf{u}_k^{(n)}\right]^H\\ &= \mathcal{O}\left(|\varphi_n(c)|^2\right) + \mathcal{O}\left(\left|\frac{\varphi_n^{(j)}(c)}{\varphi_n(c)}\right|^2\right)\\ &\quad \times \sum_{k\ 0}^{j-1}\mathcal{O}\left(\frac{\prod_{h\ 0}^{k}\left|\varphi_n^{(k-h)}(c)\right|^2}{|\varphi_n(c)|^{2k}}\right).\end{aligned}$$

Keeping in mind that $\lim_n (\varphi_n^{(j-k)}(c))/\varphi_n^{(j)}(c) = 0$, for $1 \leqslant k \leqslant j$, the induction process is finished. Hence,

$$\rho([M(n+p)]^{-1}) = \mathcal{O}\left( \frac{\prod_{h\ 0}^{p} \left|\varphi_n^{(p-h)}(c)\right|^2}{|\varphi_n(c)|^{2(p+2)}} \right)$$

$$= \mathcal{O}\left( \prod_{h\ 0}^{p} \left| \frac{\varphi_n^{(p-h)}(c)}{[\varphi_n(c)]^{1+r}} \right|^2 \right), \tag{3.11}$$

with $r = 1/(p+1) > 0$, and, thus, $\rho([M(n+p)]^{-1})$ tends to zero (Lemma 3.4). □

Now, we will explain not only how to expand the polynomial $\varphi_{n-p}(z; \mathrm{d}\mu_{p+1})$ in terms of $\varphi_{n+1}(z)$ and $\varphi_{n+1}^*(z)$, but also to find several asymptotic properties for the expansion coefficients.

Let us consider (3.3)

$$(z-c)^{p+1}\varphi_{n-p}(z; \mathrm{d}\mu_{p+1}) = \frac{\kappa_{n-p}(\mathrm{d}\mu_{p+1})}{\kappa_{n+1}} \Big( \varphi_{n+1}(z) - \boldsymbol{\varphi}_{n+1}(c)\,[M(n)]^{-1}[\mathbf{K}_n(z,c)]^{\mathrm{T}} \Big), \tag{3.12}$$

where $\boldsymbol{\varphi}_{n+1}(c) := \boldsymbol{\varphi}_{n+1}(c; \mathrm{d}\mu)$, $\mathbf{K}_n(z,c) := \mathbf{K}_n(z,c; \mathrm{d}\mu)$, $\kappa_{n+1} := \kappa_{n+1}(\mathrm{d}\mu)$, and $c \in \mathbb{C}$. Define

$$\mathbf{T}_n(z) = \left[\tau_{n0}(z),\, \tau_{n1}(z), \ldots, \tau_{np}(z)\right]^{\mathrm{T}} := [M(n)]^{-1}[\mathbf{K}_n(z,c)]^{\mathrm{T}}. \tag{3.13}$$

Notice that $\frac{\partial^j}{\partial z^j}\,[\mathbf{K}_n(z,c)]^{\mathrm{T}}|_{z\ c}$ is the $j$th column of $M(n)$ $(0 \leqslant j \leqslant p)$. It follows that

$$\tau_{ni}^{(j)}(c) = \delta_{ij}, \quad i, j = 0, \ldots, p,$$

and, for $n = p$,

$$\tau_{pi}(z) = \frac{(z-c)^i}{i!}, \quad i = 0, \ldots, p.$$

Substituting (3.13) in (3.12), we have

$$(z-c)^{p+1}\,\varphi_{n-p}(z; \mathrm{d}\mu_{p+1}) = \frac{\kappa_{n-p}(\mathrm{d}\mu_{p+1})}{\kappa_{n+1}} \left[\varphi_{n+1}(z) - \boldsymbol{\varphi}_{n+1}(c)\,\mathbf{T}_n(z)\right], \tag{3.14}$$

for all $n \geqslant p$. Remark that, for $n = p$, the bracket in the second member is the difference between $\varphi_{p+1}(z)$ and his Taylor polynomial of degree $p$.

**Proposition 3.14.** *The sequence* $(\mathbf{T}_n(z))_{n \geqslant p}$ *satisfies the following recurrence relation*

$$\mathbf{T}_{n+1}(z) = \mathbf{T}_n(z) + \frac{\kappa_{n+1}(\mathrm{d}\mu)}{\kappa_{n-p}(\mathrm{d}\mu_{p+1})} (z-c)^{p+1} \varphi_{n-p}(z; \mathrm{d}\mu_{p+1}) \times [M(n+1)]^{-1} \left[\boldsymbol{\varphi}_{n+1}(c)\right]^H.$$

**Proof.** We have

$$M(n+1)\left[\mathbf{T}_{n+1}(z) - \mathbf{T}_n(z)\right] = M(n+1)\,\mathbf{T}_{n+1}(z) - \left[M(n) + \left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \boldsymbol{\varphi}_{n+1}(c)\right] \mathbf{T}_n(z).$$

Then, because (3.13), it follows

$$\begin{aligned} M(n+1)\left[\mathbf{T}_{n+1}(z) - \mathbf{T}_n(z)\right] &= \left[\mathbf{K}_{n+1}(z,c)\right]^{\mathrm{T}} - \left[\mathbf{K}_n(z,c)\right]^{\mathrm{T}} - \left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \\ &\quad \times \boldsymbol{\varphi}_{n+1}(c)\,\mathbf{T}_n(z) \\ &= \left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \left[\varphi_{n+1}(z) - \boldsymbol{\varphi}_{n+1}(c)\,\mathbf{T}_n(z)\right], \end{aligned}$$

and, from (3.14), we can conclude the proof. □

Applying the operator $\mathscr{F}_n$ in the Christoffel Darboux formula (2.1), we obtain

$$\begin{aligned} \mathscr{F}_n\left[(1-\bar{y}z)K_n(z,y)\right] &= \overline{\mathbf{K}_n(y,c)}\left[(1-\bar{y}c)I - \bar{y}B\right] \\ &= \overline{\varphi^*_{n+1}(y)}\,\boldsymbol{\varphi}^*_{n+1}(c) - \overline{\varphi_{n+1}(y)}\,\boldsymbol{\varphi}_{n+1}(c), \end{aligned}$$

where

$$\boldsymbol{\varphi}^*_{n+1}(c) := \left[\varphi^*_{n+1}(c),\, \varphi^{*'}_{n+1}(c), \ldots, \varphi^{*(p)}_{n+1}(c)\right].$$

So, if $z\bar{c} \neq 1$,

$$\overline{\mathbf{K}_n(z,c)} = \frac{\overline{\varphi^*_{n+1}(z)}\,\boldsymbol{\varphi}^*_{n+1}(c) - \overline{\varphi_{n+1}(z)}\,\boldsymbol{\varphi}_{n+1}(c)}{1-c\bar{z}} \left[I - \frac{\bar{z}}{1-c\bar{z}} B\right]^{-1}$$

is obtained. Remember that $B$ is nilpotent ($B^{p+1} = 0$). Thus,

$$\mathbf{K}_n(z,c) = \frac{\varphi^*_{n+1}(z)\,\overline{\boldsymbol{\varphi}^*_{n+1}(c)} - \varphi_{n+1}(z)\,\overline{\boldsymbol{\varphi}_{n+1}(c)}}{1-z\bar{c}} \sum_{k=0}^{p} \left(\frac{z}{1-z\bar{c}} B\right)^k. \tag{3.15}$$

**Proposition 3.15.** *If $c \neq 0$, then, for each $n \geqslant p+1$ there are two polynomials $P(z;n)$ and $Q(z;n)$, such that*

$$(1-\bar{c}z)^{p+1}(z-c)^{p+1}\varphi_{n-p-1}(z;\mathrm{d}\mu_{p+1}) = P(z;n)\,\varphi_n(z) + Q(z;n)\,\varphi^*_n(z).$$

*with* $\deg P(z;n) = p+1$ *and* $\deg Q(z;n) \leqslant p$.

**Proof.** Carrying (3.15) on (3.12), we obtain

$$
\begin{aligned}
&(z-c)^{p+1}\varphi_{n-p-1}(z;\mathrm{d}\mu_{p+1})\\
&=\frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\kappa_n}\left[\varphi_n(z)\left(1+\sum_{k\ 0}^{p}\frac{z^k}{(1-\bar{c}z)^{k+1}}\boldsymbol{\varphi}_n(c)\,[M(n-1)]^{-1}\left[\boldsymbol{\varphi}_n(c)B^k\right]^H\right)\right.\\
&\left.\quad-\varphi_n^*(z)\sum_{k\ 0}^{p}\frac{z^k}{(1-\bar{c}z)^{k+1}}\boldsymbol{\varphi}_n(c)\,[M(n-1)]^{-1}\left[\boldsymbol{\varphi}_n^*(c)B^k\right]^H\right].
\end{aligned}
$$

From here, we get

$$
\begin{aligned}
P(z;n)&=\frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\kappa_n}\left[(1-\bar{c}z)^{p+1}+\sum_{k\ 0}^{p}z^k\,(1-\bar{c}z)^{p-k}\,\boldsymbol{\varphi}_n(c)\,[M(n-1)]^{-1}\right.\\
&\qquad\left.\times\left[\boldsymbol{\varphi}_n(c)\,B^k\right]^H\right]\\
Q(z;n)&=-\frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\kappa_n}\sum_{k\ 0}^{p}z^k\,(1-\bar{c}z)^{p-k}\,\boldsymbol{\varphi}_n(c)\,[M(n-1)]^{-1}\left[\boldsymbol{\varphi}_n^*(c)B^k\right]^H. \quad\square
\end{aligned}
$$

**Theorem 3.16.** *Let* $\mu\in\mathcal{N}$ *and* $|c|>1$. *Then, for the polynomials* $P(z;n)$ *and* $Q(z;n)$ *given in Proposition* 3.15,

$$
\lim_{n} P(z;n)=\left[\frac{\bar{c}}{|c|}\,(c-z)\right]^{p+1},\quad \lim_{n} Q(z;n)=0
$$

*l. u. in* $\mathbb{C}$.

**Proof.** If $p=0$, then

$$
(1-\bar{c}z)\,(z-c)\,\varphi_{n-1}(z;\mathrm{d}\mu_1)=\hat{P}(z;n)\,\varphi_n(z)+\hat{Q}(z;n)\,\varphi_n^*(z),
$$

where

$$
\hat{P}(z;n)=\frac{\kappa_{n-1}(\mathrm{d}\mu_1)}{\kappa_n}\left[(1-\bar{c}z)+\frac{|\varphi_n(c)|^2}{K_{n-1}(c,c)}\right],
$$

$$
\hat{Q}(z;n)=-\frac{\kappa_{n-1}(\mathrm{d}\mu_1)}{\kappa_n}\,\frac{\varphi_n(c)\,\overline{\varphi_n^*(c)}}{K_{n-1}(c,c)},
$$

and $\deg\hat{P}(z;n)=1$, $\deg\hat{Q}(z;n)\leqslant 0$.

From (2.2), Lemma 2.6 (i), and Proposition 3.5, the above statement for $\hat{P}(z;n)$ and $\hat{Q}(z;n)$ can be deduced in a straightforward way.

Assume that our assertion is true for $p-1$. There are two polynomials $\tilde{P}(z;n)$ and $\tilde{Q}(z;n)$such that

$$
(1-\bar{c}z)^p\,(z-c)^p\,\varphi_{n-p}(z;\mathrm{d}\mu_p)=\tilde{P}(z;n)\,\varphi_n(z)+\tilde{Q}(z,n)\,\varphi_n^*(z), \tag{3.16}
$$

with $\deg \tilde{P}(z;n) = p$ and $\deg \tilde{Q}(z;n) \leqslant p-1$, and

$$\lim_n \tilde{P}(z;n) = \left[\frac{\overline{c}}{|c|}(c-z)\right]^p, \quad \lim_n \tilde{Q}(z;n) = 0,$$

*l. u.* in $\mathbb{C}$.

But, if $\mu \in \mathcal{N}$, then also $\mu_p \in \mathcal{N}$. Hence, there exist two polynomials $D(z;n)$ and $E(z;n)$, with $\deg D(z;n) = 1$, $\deg E(z;n) \leqslant 0$, such that

$$(1-\overline{c}z)(z-c)\,\varphi_{n-p-1}(z;\mathrm{d}\mu_{p+1}) = D(z;n)\,\varphi_{n-p}(z;\mathrm{d}\mu_p) + E(z;n)\,\varphi^*_{n-p}(z;\mathrm{d}\mu_p). \tag{3.17}$$

Then, $\lim_n D(z;n) = (\overline{c}/|c|)\,(c-z)$, $\lim_n E(z;n) = 0$ *l. u.* in $\mathbb{C}$.

Write $\tilde{P}^*(z;n) = z^p\,\overline{\tilde{P}}(1/z;n)$ and $\tilde{Q}^*(z;n) = z^{p-1}\,\overline{\tilde{Q}}(1/z;n)$. From (3.16), we have

$$(1-\overline{c}z)^p\,(z-c)^p\,\varphi^*_{n-p}(z;\mathrm{d}\mu_p) = \tilde{P}^*(z;n)\,\varphi^*_n(z) + z\,\tilde{Q}^*(z;n)\,\varphi_n(z),$$

and, from (3.17), we get

$$(1-\overline{c}z)^{p+1}\,(z-c)^{p+1}\,\varphi_{n-p-1}(z;\mathrm{d}\mu_{p+1}) = P(z;n)\,\varphi_n(z) + Q(z;n)\,\varphi^*_n(z),$$

with

$$P(z;n) = D(z;n)\,\tilde{P}(z,n) + z\,E(z;n)\,\tilde{Q}^*(z;n),$$

$$Q(z;n) = D(z;n)\,\tilde{Q}(z;n) + E(z;n)\,\tilde{P}^*(z;n),$$

and our results follow. $\square$

## 4. Orthogonal polynomials related to a Sobolev-type inner product

Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal polynomial sequence related to a probability measure $\mu$ supported in $\mathbb{T}$ which induces an inner product $(\cdot,\cdot)$ on $\mathbb{P}$. For $c \in \mathbb{C}$ and $A \in \mathbb{C}^{(p+1,p+1)}$ $(p \in \mathbb{N})$, we define in $\mathbb{P}$ the *Sobolev-type* inner product

$$\langle f,g\rangle := (f,g) + \sum_{i,j\ \ 0}^{p} f^{(i)}(c)\,a_{ij}\,\overline{g^{(j)}(c)},$$

with $A = (a_{ij})^p_{i,j\ \ 0}$. If we write $\mathbf{f}(z) = [f(z), f'(z), \ldots, f^{(p)}(z)]$, then

$$\langle f,g\rangle = (f,g) + \mathbf{f}(c)\,A\,[\mathbf{g}(c)]^H.$$

We say that $\psi_n \in \mathbb{P}_n$ is the $n$th (left) orthonormal polynomial with respect to $\langle\cdot,\cdot\rangle$ if

$$\langle \psi_n(z), z^k\rangle = 0 \quad (k = 0, 1, \ldots, n-1)$$

$$|\langle \psi_n, \psi_n\rangle| = 1.$$

The definition for the right orthonormal polynomials is analogous. Both of them are the same when $A$ is an Hermitian matrix, because $\langle\cdot,\cdot\rangle$ is Hermitian.

**Proposition 4.1.** *Assume that $\psi_n$ exists. Then,*

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\,\varphi_n(z) - \boldsymbol{\psi}_n(c)\,A\,[\mathbf{K}_{n-1}(z,c)]^{\mathrm{T}} \tag{4.1}$$

*holds, where $\gamma_n > 0$ is the leading coefficient of $\psi_n$.*

**Proof.** Notice that $\psi_n(z) - (\gamma_n/\kappa_n)\,\varphi_n(z) \in \mathbb{P}_{n-1}$. Then, we have

$$\left(\psi_n(y) - \frac{\gamma_n}{\kappa_n}\,\varphi_n(y),\ K_{n-1}(y,z)\right) = \psi_n(z) - \frac{\gamma_n}{\kappa_n}\,\varphi_n(z).$$

On the other hand,

$$\begin{aligned}\left(\psi_n(y) - \frac{\gamma_n}{\kappa_n}\,\varphi_n(y), K_{n-1}(y,z)\right) &= (\psi_n(y), K_{n-1}(y,z))\\ &= \langle\psi_n(y), K_{n-1}(y,z)\rangle - \boldsymbol{\psi}_n(c)\,A\,[\mathbf{K}_{n-1}(z,c)]^{\mathrm{T}}\\ &= -\,\boldsymbol{\psi}_n(c)\,A\,[\mathbf{K}_{n-1}(z,c)]^{\mathrm{T}}. \quad \square\end{aligned}$$

**Proposition 4.2.** *Assume that $\psi_n$ exists. Then,*

$$\boldsymbol{\psi}_n(c)\,[I + AM(n-1)] = \frac{\gamma_n}{\kappa_n}\,\boldsymbol{\varphi}_n(c) \tag{4.2}$$

*holds.*

**Proof.** Taking derivatives of order $p$ in (4.1) for $z = c$,

$$\boldsymbol{\psi}_n(c) = \frac{\gamma_n}{\kappa_n}\,\boldsymbol{\varphi}_n(c) - \boldsymbol{\psi}_n(c)\,AM(n-1). \quad \square$$

**Proposition 4.3.** *If $\psi_n$ exists, then we get*

$$\frac{\gamma_n}{\kappa_n}\,(\varphi_n, \varphi_n) = \frac{\kappa_n}{\gamma_n}\,\langle\psi_n, \psi_n\rangle - \boldsymbol{\psi}_n(c)\,A\,[\boldsymbol{\varphi}_n(c)]^{H}.$$

**Proof.** From

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\,\varphi_n(z) + (\text{lower degree terms}),$$

$$\varphi_n(z) = \frac{\kappa_n}{\gamma_n}\,\psi_n(z) + (\text{lower degree terms}),$$

we obtain

$$
\begin{aligned}
(\psi_n, \varphi_n) &= \frac{\gamma_n}{\kappa_n}(\varphi_n, \varphi_n), \\
(\psi_n, \varphi_n) &= \langle \psi_n, \varphi_n \rangle - \boldsymbol{\psi}_n(c) A [\boldsymbol{\varphi}_n(c)]^H \\
&= \left\langle \psi_n, \frac{\kappa_n}{\gamma_n} \psi_n \right\rangle - \boldsymbol{\psi}_n(c) A [\boldsymbol{\varphi}_n(c)]^H \\
&= \frac{\kappa_n}{\gamma_n} \langle \psi_n, \psi_n \rangle - \boldsymbol{\psi}_n(c) A [\boldsymbol{\varphi}_n(c)]^H. \quad \square
\end{aligned}
$$

**Theorem 4.4.** *If* $\det[I + AM(n-1)] \neq 0$, *then the nth Sobolev-type orthonormal polynomial* $\psi_n$ *exists if and only if* $\det[I + AM(n)] \neq 0$.

**Proof.** Assume that $\det[I + AM(n-1)] \neq 0$. Define the polynomial

$$
\Psi_n(z) := \varphi_n(z) - \boldsymbol{\varphi}_n(c)[I + AM(n-1)]^{-1} A [\mathbf{K}_{n-1}(z, c)]^{\mathrm{T}}. \tag{4.3}
$$

Then, if we derive $p$ times in the above expression and we evaluate it in $z = c$, we have

$$
\begin{aligned}
\boldsymbol{\Psi}_n(c) &= \boldsymbol{\varphi}_n(c) - \boldsymbol{\varphi}_n(c)[I + AM(n-1)]^{-1} AM(n-1) \\
&= \boldsymbol{\varphi}_n(c)[I + AM(n-1)]^{-1}.
\end{aligned}
$$

For $k = 0, 1, \ldots, n-1$, we get

$$
\begin{aligned}
\langle \Psi_n, \varphi_k \rangle &= (\Psi_n, \varphi_k) + \boldsymbol{\Psi}_n(c) A [\boldsymbol{\varphi}_k(c)]^H \\
&= (\varphi_n, \varphi_k) - \boldsymbol{\varphi}_n(c)[I + AM(n-1)]^{-1} A [\boldsymbol{\varphi}_k(c)]^H \\
&\quad + \boldsymbol{\Psi}_n(c) A [\boldsymbol{\varphi}_k(c)]^H.
\end{aligned}
$$

Thus, if $k < n$, $\langle \Psi_n, \varphi_k \rangle = 0$ follows immediately. For $k = n$ we have

$$
\langle \Psi_n, \varphi_n \rangle = 1 + \boldsymbol{\Psi}_n(c) A [\boldsymbol{\varphi}_n(c)]^H = 1 + \boldsymbol{\varphi}_n(c)[I + AM(n-1)]^{-1} A [\boldsymbol{\varphi}_n(c)]^H.
$$

Now, consider the matrix identities

$$
\begin{pmatrix} I & A[\boldsymbol{\varphi}_n(c)]^H \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} I + AM(n-1) & -A[\boldsymbol{\varphi}_n(c)]^H \\ \boldsymbol{\varphi}_n(c) & 1 \end{pmatrix}
= \begin{pmatrix} I + AM(n) & 0 \\ \boldsymbol{\varphi}_n(c) & 1 \end{pmatrix},
$$

$$\begin{pmatrix} I + AM(n-1) & -A\,[\boldsymbol{\varphi}_n(c)]^H \\ \boldsymbol{\varphi}_n(c) & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} I & [I + AM(n-1)]^{-1} A\,[\boldsymbol{\varphi}_n(c)]^H \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} I + AM(n-1) & 0 \\ \boldsymbol{\varphi}_n(c) & 1 + \boldsymbol{\varphi}_n(c)\,[I + AM(n-1)]^{-1} A\,[\boldsymbol{\varphi}_n(c)]^H \end{pmatrix},$$

whose determinants are equal. Then

$$\det[I + AM(n)] = \det[I + AM(n-1)] \Big(1 + \boldsymbol{\varphi}_n(c)\,[I + AM(n-1)]^{-1} \times A\,[\boldsymbol{\varphi}_n(c)]^H\Big)$$

and so

$$\langle \Psi_n, \varphi_n \rangle = 1 + \boldsymbol{\varphi}_n(c)\,[I + AM(n-1)]^{-1} A\,[\boldsymbol{\varphi}_n(c)]^H = \frac{\det[I + AM(n)]}{\det[I + AM(n-1)]}.$$

From here, the statement follows.

Besides, notice that $\psi_n(z) = \Psi_n(z)/| < \Psi_n, \Psi_n > |^{1/2} = \gamma_n z^n +$ (lower degree terms), with $\gamma_n > 0$, verifies

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n} \Big(\varphi_n(z) - \boldsymbol{\varphi}_n(c)\,[I + AM(n-1)]^{-1} A\,[\mathbf{K}_{n-1}(z, c)]^{\mathrm{T}}\Big). \quad \square \tag{4.4}$$

If $A$ is positive semidefinite, $A + [M(n-1)]^{-1}$ is positive definite ($n \geqslant p + 1$), according to Proposition 2.1. Hence, we get

$$\det[I + AM(n-1)] = \det[M(n-1)] \times \det\Big(A + [M(n-1)]^{-1}\Big) > 0$$

and, thus

**Corollary 4.5.** *Let $A$ be positive semidefinite. Then, the Sobolev-type orthonormal polynomial sequence exists for $n \geqslant p + 1$.*

**Remark.** From (4.2) and Proposition 4.3

$$\left(\frac{\kappa_n}{\gamma_n}\right)^2 = 1 + \boldsymbol{\Psi}_n(c) A\,[\boldsymbol{\varphi}_n(c)]^H = 1 + \frac{\kappa_n}{\gamma_n}\,\boldsymbol{\psi}_n(c) A\,[\boldsymbol{\varphi}_n(c)]^H \tag{4.5}$$

follows.

Now, we are going to study the behavior of the $n$th Sobolev-type orthonormal polynomial $\psi_n$ in terms of the matrix $A$ which will be considered as a parameter.

**Theorem 4.6.** *Let $A$ be a positive definite matrix. Write $\sigma = 1/(\rho(A^{-1}))$, where $\rho(M)$ is the spectral radius of $M$. Then, for fixed $n \in \mathbb{N}$,*

$$\lim_{\sigma\to\infty} \psi_{n+p+1}(z;A) = (z-c)^{p+1}\,\varphi_n(z;\mathrm{d}\mu_{p+1})$$

*holds l. u. in* $\mathbb{C}$.

[Notice that $\sigma \to \infty$ is equivalent to the fact that *all* eigenvalues of $A$ tend to $\infty$.]

**Proof.** From (4.4),

$$\psi_{n+p+1}(z;A) = \frac{\gamma_{n+p+1}(A)}{\kappa_{n+p+1}} \times \left(\varphi_{n+p+1}(z) - \boldsymbol{\varphi}_{n+p+1}(c)\,[A^{-1}+M(n+p)]^{-1}\left[\mathbf{K}_{n+p}(z,c)\right]^{\mathrm{T}}\right)$$

holds because $A$ is nonsingular. From (4.5) and (4.2), we get

$$\left(\frac{\kappa_{n+p+1}}{\gamma_{n+p+1}(A)}\right)^2 = 1 + \boldsymbol{\varphi}_{n+p+1}(c)\,[A^{-1}+M(n+p)]^{-1}\left[\boldsymbol{\varphi}_{n+p+1}(c)\right]^H.$$

Thus, since $M(n+p)$ is positive definite and $\lim_{\sigma\to\infty} A^{-1} = 0$, it follows that

$$\lim_{\sigma\to\infty}\left(\frac{\kappa_{n+p+1}}{\gamma_{n+p+1}(A)}\right)^2 = 1 + \boldsymbol{\varphi}_{n+p+1}(c)\,[M(n+p)]^{-1}\left[\boldsymbol{\varphi}_{n+p+1}(c)\right]^H = \left(\frac{\kappa_{n+p+1}}{\kappa_n(\mathrm{d}\mu_{p+1})}\right)^2.$$

Furthermore,

$$\frac{\kappa_{n+p+1}}{\gamma_{n+p+1}(A)}\,\psi_{n+p+1}(z;A) - \frac{\kappa_{n+p+1}}{\kappa_n(\mathrm{d}\mu_{p+1})}\,(z-c)^{p+1}\,\varphi_n(z;\mathrm{d}\mu_{p+1}) = \boldsymbol{\varphi}_{n+p+1}(c)\left([M(n+p)]^{-1} - [A^{-1}+M(n+p)]^{-1}\right)\left[\mathbf{K}_{n+p}(z,c)\right]^{\mathrm{T}}.$$

Let $n$ be fixed, and let us consider an arbitrary compact subset $H \subset \mathbb{C}$. Then, $\left|K_{n+p}^{(0,i)}(z,c)\right|$ is uniformly bounded in $H$. For $\sigma \to \infty$, we obtain from here that

$$\lim_{\sigma\to\infty}\left[\psi_{n+p+1}(z;A) - (z-c)^{p+1}\,\varphi_n(z;\mathrm{d}\mu_{p+1})\right] = 0$$

*l. u.* in $\mathbb{C}$. □

**Theorem 4.7.** *Let $\mu \in \mathcal{N}$ and $|c| > 1$. If $A$ is a nonsingular matrix, then there is $n_0 \in \mathbb{N}$ such that the (left) orthonormal polynomial $\psi_n$ with respect to a Sobolev-type inner product*

$$\langle f, g \rangle = \int_{|z| \ 1} f(z)\,\overline{g(z)}\,\mathrm{d}\mu(z) + \mathbf{f}(c)\,A\,[\mathbf{g}(c)]^H$$

*exists for all $n \geqslant n_0$.*

**Proof.** Since Theorem 4.4, we can guarantee that there exists $\psi_n$, for all $n \geqslant n_0$, when $\det(I + AM(n-1)) \neq 0$. Notice that

$$\det[I + AM(n-1)] = \det(A) \cdot \det[M(n-1)] \cdot \det\left(I + A^{-1}[M(n-1)]^{-1}\right).$$

As $\lim_n \rho([M(n-1)]^{-1}) = 0$, then $\lim_n \det(I + A^{-1}[M(n-1)]^{-1}) = \det(I) = 1$ (Theorem 3.13), that is, there exists $n_0 \in \mathbb{N}$ such that $\det[I + AM(n-1)] > 0$, for $n \geqslant n_0$. $\square$

By using (4.1), then

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\left(\varphi_n(z) - \boldsymbol{\varphi}_n(c)\,[I + AM(n-1)]^{-1}A\,[\mathbf{K}_{n-1}(z,c)]^{\mathrm{T}}\right).$$

If we denote

$$\mathbf{R}_n(z) := [I + AM(n)]^{-1}A\,[\mathbf{K}_n(z,c)]^{\mathrm{T}}, \tag{4.6}$$

it follows that

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\,[\varphi_n(z) - \boldsymbol{\varphi}_n(c)\,\mathbf{R}_{n-1}(z)]. \tag{4.7}$$

The $\mathbf{R}_n(z)$ are the analogous expressions to the $\mathbf{T}_n(z)$, which are defined in (3.13), and those verify a similar relationship to the one given in Proposition 3.14:

**Proposition 4.8.** *Assume that $I + AM(n+1)$ and $I + AM(n)$ are nonsingular matrices. Then,*

$$\mathbf{R}_{n+1}(z) = \mathbf{R}_n(z) + \frac{\kappa_{n+1}}{\gamma_{n+1}}\,[I + AM(n+1)]^{-1}A\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \psi_{n+1}(z)$$

*holds.*

**Proof.** The nonsingularity of $I + AM(n+1)$, and $I + AM(n)$ implies that the polynomial $\psi_{n+1}$ exists. Thus, we have

$$[I+AM(n+1)]\,[\mathbf{R}_{n+1}(z)-\mathbf{R}_n(z)] = [I+AM(n+1)]\,\mathbf{R}_{n+1}(z) - \left[I+AM(n)+A\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H\boldsymbol{\varphi}_{n+1}(c)\right]\mathbf{R}_n(z).$$

From here, according to (4.6),

$$\begin{aligned}[I + AM(n+1)][\mathbf{R}_{n+1}(z) - \mathbf{R}_n(z)] &= A\left([\mathbf{K}_{n+1}(z,c)]^{\mathrm{T}} - [\mathbf{K}_n(z,c)]^{\mathrm{T}}\right) \\ &\quad - A\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \boldsymbol{\varphi}_{n+1}(c)\,\mathbf{R}_n(z) \\ &= A\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \left[\varphi_{n+1}(z) - \boldsymbol{\varphi}_{n+1}(c)\,\mathbf{R}_n(z)\right] \\ &= A\left[\boldsymbol{\varphi}_{n+1}(c)\right]^H \frac{\kappa_{n+1}}{\gamma_{n+1}}\,\psi_{n+1}(z). \qquad \square\end{aligned}$$

**Proposition 4.9.** *Assume that* $\psi_n$ *exists. Then, there exist two polynomials* $P^A(z;n)$, *and* $Q^A(z;n)$, *with* $\deg P^A(z;n) = p+1$ *and* $\deg Q^A(z;n) \leqslant p$, *such that*

$$(1-\bar{c}z)^{p+1}\,\psi_n(z) = P^A(z;n)\,\varphi_n(z) + Q^A(z;n)\,\varphi_n^*(z)$$

*holds for* $c \neq 0$.

**Proof.** In the same way as in Proposition 3.15, we obtain

$$P^A(z;n) = \frac{\gamma_n}{\kappa_n}\left((1-\bar{c}z)^{p+1} + \sum_{k\ 0}^{p} z^k\,(1-\bar{c}z)^{p-k}\,\boldsymbol{\varphi}_n(c)\,[A^{-1}+M(n-1)]^{-1} \times \left[\boldsymbol{\varphi}_n(c)\,B^k\right]^H\right)$$

$$Q^A(z;n) = -\frac{\gamma_n}{\kappa_n}\sum_{k\ 0}^{p} z^k\,(1-\bar{c}z)^{p-k}\,\boldsymbol{\varphi}_n(c)\,[A^{-1}+M(n-1)]^{-1}\left[\boldsymbol{\varphi}_n^*(c)\,B^k\right]^H. \quad \square$$

## 5. Asymptotic behavior for $\psi_n$

We will assume that $\mu \in \mathcal{N}$, $|c| > 1$ and $A$ is a nonsingular matrix. In these conditions, Theorems 3.13 and 4.7 hold, and the existence for the $n$th Sobolev-type orthonormal polynomial $\psi_n = \gamma_n z^n + (\text{lower degree terms})$ for $n$ large enough is guaranteed.

**Proposition 5.1.** $\lim_n \kappa_n/\gamma_n = |c|^{p+1}$.

**Proof.** From (4.5), we get

$$\left(\frac{\kappa_n}{\gamma_n}\right)^2 = \frac{\det[I + AM(n)]}{\det[I + AM(n-1)]}$$
$$= \frac{\det[M(n)]}{\det[M(n-1)]} \cdot \frac{\det[I + A^{-1}[M(n)]^{-1}]}{\det\left(I + A^{-1}[M(n-1)]^{-1}\right)}.$$

Now, by using Theorem 3.13, we obtain

$$\lim_n \det\left(I + A^{-1}[M(n)]^{-1}\right) = 1$$

and, thus, we conclude that

$$\lim_n \left(\frac{\kappa_n}{\gamma_n}\right)^2 = \lim_n \frac{\det[M(n)]}{\det[M(n-1)]} = \lim_n \left(1 + \boldsymbol{\varphi}_n(c)\,[M(n-1)]^{-1}[\boldsymbol{\varphi}_n(c)]^H\right)$$
$$= |c|^{2p+2},$$

according to (3.2) and Corollary 3.7. □

**Theorem 5.2.** $\lim_n \psi_n(z)/\varphi_n(z) = (\overline{c}/|c| \cdot (z-c)/\overline{c}z - 1)^{p+1}$ *uniformly in* $|z| \geqslant 1$, *or*, *equivalently*, $\lim_n \psi_n(z)/(\varphi_{n-p-1}(z; \mathrm{d}\mu_{p+1})) = (z-c)^{p+1}$ *uniformly in* $|z| \geqslant 1$.

**Proof.** Note that the equivalence for these both conditions follows immediately from Proposition 3.6.

We shall denote $\hat{\varphi}_n(z) = (1/\kappa_n)\,\varphi_n(z)$, $\hat{\varphi}_n(z; \mathrm{d}\mu_{p+1}) = (1/\kappa_n(\mathrm{d}\mu_{p+1}))\,\varphi_n(z; \mathrm{d}\mu_{p+1})$, and $\hat{\psi}_n(z) = (1/\gamma_n)\,\psi_n(z)$ the corresponding $n$th monic orthogonal polynomials. From (3.14) and (4.7), we have

$$\frac{(z-c)^{p+1}\,\hat{\varphi}_{n-p-1}(z; \mathrm{d}\mu_{p+1})}{\hat{\varphi}_n(z)} - \frac{\hat{\psi}_n(z)}{\hat{\varphi}_n(z)} = \frac{\boldsymbol{\varphi}_n(c)[\mathbf{R}_{n-1}(z) - \mathbf{T}_{n-1}(z)]}{\hat{\varphi}_n(z)}$$
$$= \frac{\boldsymbol{\varphi}_n(c)[\mathbf{R}_{n-1}(z) - \mathbf{T}_{n-1}(z)]}{\varphi_n(z)}.$$

We prove that

$$\lim_n \left[\boldsymbol{\varphi}_n(c) \cdot \frac{\mathbf{T}_{n-1}(z) - \mathbf{R}_{n-1}(z)}{\varphi_n(z)}\right] = 0 \tag{5.1}$$

uniformly in $|z| \geqslant 1$.

As $A$ is nonsingular, we get

$$\mathbf{R}_{n-1}(z) = \left(I + [M(n-1)]^{-1}A^{-1}\right)^{-1}\mathbf{T}_{n-1}(z),$$

according to (3.13) and (4.6). Hence,

$$\boldsymbol{\varphi}_n(c) \cdot \frac{\mathbf{T}_{n-1}(z) - \mathbf{R}_{n-1}(z)}{\varphi_n(z)} = \boldsymbol{\varphi}_n(c) \left[ I - \left( I + [M(n-1)]^{-1} A^{-1} \right)^{-1} \right] \frac{\mathbf{T}_{n-1}(z)}{\varphi_n(z)}$$

$$= \boldsymbol{\varphi}_n(c) \, [M(n-1)]^{-1} A^{-1} \left( I + [M(n-1)]^{-1} A^{-1} \right)^{-1} \times \frac{\mathbf{T}_{n-1}(z)}{\varphi_n(z)}$$

holds and, thus,

$$\left| \boldsymbol{\varphi}_n(c) \cdot \frac{\mathbf{T}_{n-1}(z) - \mathbf{R}_{n-1}(z)}{\varphi_n(z)} \right| \leqslant ||\boldsymbol{\varphi}_n(c)|| \cdot \rho([M(n-1)]^{-1}) \cdot ||A^{-1}|| \cdot \left|\left| \left( I + [M(n-1)]^{-1} A^{-1} \right)^{-1} \right|\right| \cdot \frac{||\mathbf{T}_{n-1}(z)||}{|\varphi_n(z)|}.$$

Here, the matrix norm $||C|| = \sqrt{\rho(C C^H)}$ is used. Because $\rho([M(n)]^{-1})$ tends to zero, then $\lim_n \left|\left| (I + [M(n-1)]^{-1} A^{-1})^{-1} \right|\right| = 1$. Thus, to prove (5.1) is equivalent to prove

$$\lim_n ||\boldsymbol{\varphi}_n(c)|| \, \rho([M(n-1)]^{-1}) \, \frac{||\mathbf{T}_{n-1}(z)||}{|\varphi_n(z)|} = 0$$

uniformly in $|z| \geqslant 1$.

First, from (3.11) (Theorem 3.13), we get

$$\rho([M(n-1)]^{-1}) = \mathcal{O}\left( \frac{\left| \prod_{k\ 0}^{p} \varphi_n^{(p-k)}(c) \right|^2}{|\varphi_n(c)|^{2(p+2)}} \right),$$

and thus

$$||\boldsymbol{\varphi}_n(c)|| \, \rho([M(n-1)]^{-1}) = \mathcal{O}\left( \left| \varphi_n^{(p)}(c) \right| \right) \cdot \mathcal{O}\left( \frac{\left| \prod_{k\ 0}^{p} \varphi_n^{(p-k)}(c) \right|^2}{|\varphi_n(c)|^{2(p+2)}} \right)$$

$$= \mathcal{O}\left( \left| \frac{\varphi_n^{(p)}(c)}{[\varphi_n(c)]^{1+r}} \right| \right) \cdot \mathcal{O}\left( \prod_{k\ 0}^{p} \left| \frac{\varphi_n^{(p-k)}(c)}{[\varphi_n(c)]^{1+r}} \right|^2 \right)$$

follows, with $r = 1/(2p+3) > 0$. Hence $\lim_n ||\boldsymbol{\varphi}_n(c)|| \, \rho([M(n-1)]^{-1}) = 0$ (Lemma 3.4).

Furthermore, by considering (3.13) and (3.15) we can obtain the following upper bound

$$\frac{||\mathbf{T}_{n-1}(z)||}{|\varphi_n(z)|} \leqslant \frac{\rho([M(n-1)]^{-1})\,||\mathbf{K}_{n-1}(z,c)||}{|\varphi_n(z)|}$$

$$\leqslant \frac{\rho([M(n-1)]^{-1})}{|1-\bar{c}z|} \cdot \left|\left| \frac{\overline{\varphi_n^*(z)}}{\overline{\varphi_n(z)}}\, \boldsymbol{\varphi}_n^*(c) - \boldsymbol{\varphi}_n(c) \right|\right| \cdot \sum_{k=0}^{p} \left| \frac{z}{1-\bar{c}z} \right|^k ||B||^k.$$

Now,

$$\rho([M(n-1)]^{-1}) \left|\left| \frac{\overline{\varphi_n^*(z)}}{\overline{\varphi_n(z)}}\, \boldsymbol{\varphi}_n^*(c) - \boldsymbol{\varphi}_n(c) \right|\right|$$

$$\leqslant \rho([M(n-1)]^{-1}) \cdot ||\boldsymbol{\varphi}_n(c)|| \left( 1 + \left| \frac{\varphi_n^*(z)}{\varphi_n(z)} \right| \frac{||\boldsymbol{\varphi}_n^*(c)||}{||\boldsymbol{\varphi}_n(c)||} \right),$$

with

$$\left| \frac{\varphi_n^*(z)}{\varphi_n(z)} \right| \frac{||\boldsymbol{\varphi}_n^*(c)||}{||\boldsymbol{\varphi}_n(c)||} \leqslant \left| \frac{\varphi_n^*(z)}{\varphi_n(z)} \right| \left( \sum_{k=0}^{p} \left| \frac{\varphi_n^{*(k)}(c)}{\varphi_n^{(k)}(c)} \right|^2 \right)^{1/2},$$

where the first factor is uniformly bounded by 1 in $|z| \geqslant 1$, and the second one tends to zero (Lemma 2.4 and (2.2)). Also, $z/(1-\bar{c}z)$ and $1/(1-\bar{c}z)$ are uniformly bounded by $1/(|c|-1)$ in $|z| \geqslant 1$.

In these conditions,

$$\lim_{n} \frac{||\mathbf{T}_{n-1}(z)||}{|\varphi_n(z)|} \leqslant \sum_{k=0}^{p} \frac{||B||^k}{(|c|-1)^{k+1}} \cdot \lim_{n} \left[ \rho([M(n-1)]^{-1})\,||\boldsymbol{\varphi}_n(c)|| \right] = 0.$$

follows immediately.

Hence

$$\lim_{n} \frac{\hat{\psi}_n(z)}{\hat{\varphi}_n(z)} = \lim_{n} \frac{(z-c)^{p+1}\, \hat{\varphi}_{n-p-1}(z; \mathrm{d}\mu_{p+1})}{\hat{\varphi}_n(z)},$$

i.e.,

$$\lim_{n} \frac{\kappa_n}{\gamma_n} \frac{\psi_n(z)}{\varphi_n(z)} = \lim_{n} \frac{\kappa_n}{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})} \frac{(z-c)^{p+1}\, \varphi_{n-p-1}(z; \mathrm{d}\mu_{p+1})}{\varphi_n(z)}.$$

But, Propositions 3.6 and 5.1 yield

$$\lim_n \frac{\kappa_n}{\gamma_n} = \lim_n \frac{\kappa_n}{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}.$$

So, the statement is proved. □

**Corollary 5.3.** *There is $n_0 \in \mathbb{N}$ such that, for $n \geqslant n_0$, the nth Sobolev-type orthonormal polynomial $\psi_n$ has exactly $p+1$ zeros in $|z| > 1$, which accumulate in $c$, while the remaining zeros belong to $|z| < 1$.*

**Proof.** Taking into account that

$$\lim_n \frac{\psi_n(z)}{\varphi_{n-p-1}(z; \mathrm{d}\mu_{p+1})} = (z-c)^{p+1},$$

uniformly in $|z| \geqslant 1$, the result follows immediately from Hurwitz's Theorem. □

**Theorem 5.4.** *For the polynomial coefficients $P^A(z;n)$ and $Q^A(z;n)$ in Proposition 4.9,*

$$\lim_n P^A(z;n) = \left[\frac{\overline{c}}{|c|}(c-z)\right]^{p+1}, \quad \lim_n Q^A(z;n) = 0$$

*hold l. u. in $\mathbb{C}$.*

**Proof.** Consider the expressions in Propositions 3.15 and 4.9. Then, we can write

$$\frac{\kappa_n}{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})} P(z;n) - \frac{\kappa_n}{\gamma_n} P^A(z;n) = \sum_{k\ 0}^{p} z^k (1-\overline{c}z)^{p-k} \boldsymbol{\varphi}_n(c) \Big( [M(n-1)]^{-1} - [A^{-1} + M(n-1)]^{-1} \Big) \left[\boldsymbol{\varphi}_n(c) B^k\right]^H.$$

But, from

$$[M(n-1)]^{-1} - [A^{-1} + M(n-1)]^{-1} = [M(n-1)]^{-1} \Big( I + A^{-1}[M(n-1)]^{-1} \Big)^{-1} A^{-1} [M(n-1)]^{-1},$$

it follows that

$$\frac{\kappa_n}{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})} \left| P(z;n) - \frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\gamma_n} P^A(z;n) \right| \leqslant ||\boldsymbol{\varphi}_n(c)||^2 \left[\rho([M(n-1)]^{-1})\right]^2 \times \left|\left| \Big( I + A^{-1}[M(n-1)]^{-1} \Big)^{-1} \right|\right| \, ||A^{-1}|| \sum_{k\ 0}^{p} |z|^k |1-\overline{c}z|^{p-k} \, ||B^k||.$$

In Theorem 5.2, $\lim_n \left\| \left( I + A^{-1}[M(n-1)]^{-1} \right)^{-1} \right\| = 1$ and $\lim_n ||\boldsymbol{\varphi}_n(c)|| \rho([M(n-1)]^{-1}) = 0$ are stated. Besides, we have

$$\lim_n \frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\gamma_n} = \lim_n \frac{\kappa_{n-p-1}(\mathrm{d}\mu_{p+1})}{\kappa_n} \cdot \lim_n \frac{\kappa_n}{\gamma_n} = 1,$$

according to Propositions 5.1 and 3.6. Thus, Theorem 3.16 implies the statement for $P^A(z;n)$.

The proof is similar for $Q^A(z;n)$. □

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