



q -Coherent pairs and q -orthogonal polynomials

I. Area ^{a,*}, E. Godoy ^a, F. Marcellán ^b

^a *Departamento de Matemática Aplicada, E.T.S.I. Industriales y Minas, Universidad de Vigo, Campus Lagoas Marcosende, 36200 Vigo, Spain*

^b *Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, c/Butarque 15, 28911 Leganés (Madrid), Spain*

Abstract

In this paper we introduce the concept of q coherent pair of linear functionals. We prove that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q coherent pair of linear functionals, then at least one of them has to be a q classical linear functional. Moreover, we present the classification of all q coherent pairs of positive definite linear functionals when \mathbf{u}_0 or \mathbf{u}_1 is either the little q Jacobi linear functional or the little q Laguerre/Wall linear functional. Finally, by using limit processes, we recover the classification of coherent pairs of linear functionals stated by Meijer.

Keywords: q coherence; q Jacobi polynomials; q Laguerre/Wall polynomials; q orthogonal polynomials

1. Introduction

The concepts of coherent pair and symmetric coherent pair have been introduced by Iserles et al. in [1] in the framework of the study of orthogonal polynomials associated with the Sobolev inner product

$$\langle f, g \rangle_S = \int_{\mathbb{R}} fg \, d\mu_0 + \lambda \int_{\mathbb{R}} f'g' \, d\mu_1, \quad (1)$$

* Corresponding author.

E mail address: pacomarc@ing.uc3m.es (I. Area).

where μ_0 and μ_1 are non-atomic positive Borel measures on the real line such that

$$\left| \int_{\mathbb{R}} x^k d\mu_i(x) \right| < \infty, \quad k \geq 0, \quad i = 0, 1.$$

In fact, coherence means that a relation between the monic orthogonal polynomial sequence (MOPS) $\{P_n\}_n$ and $\{T_n\}_n$, associated with the measures μ_0 and μ_1 , respectively,

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad n \geq 1, \quad (2)$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers, is satisfied.

Hahn [2] seems to have been the first to realize that the characterizations of classical orthogonal polynomial sequences based on derivatives and differential equations are too much restrictive [3]. He used a more general operator, the so-called q -difference operator defined by [2, Eq. (2.3)]

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad q \neq 1,$$

and $(D_q f)(0) := f'(0)$ by continuity, provided $f'(0)$ exists. Note that $\lim_{q \uparrow 1} (D_q f)(x) = f'(x)$ if f is a differentiable function.

The Askey tableau of hypergeometric orthogonal polynomials contains the classical orthogonal polynomials which can be written in terms of a hypergeometric function, starting at the top with Wilson and Racah polynomials and ending at the bottom with Hermite polynomials [4,5]. Hahn [2] studied the q -analogue of this classification. So, there are q -analogues of all the families in the Askey tableau, often several q -analogues for one classical family. The most general sets of these q -analogues are the Askey Wilson polynomials [4] and the q -Racah polynomials [6], which contain all other families as special or limit cases [7]. In [8, p. 115] Koornwinder presented a q -Hahn tableau: a q -analogue of that part of the Askey tableau which is dominated by Hahn polynomials. Basic hypergeometric functions and q -orthogonal polynomials for arbitrary (including complex) values of q are connected with quantum algebras and groups [9].

The aim of this paper is to extend the recent study on coherent pairs of linear functionals [10] and Δ -coherent pairs of linear functionals [11,12] to q -coherent pairs. More concretely, we characterize the sequences of orthogonal polynomials $\{P_n\}_n$ and $\{T_n\}_n$ such that

$$T_n(x) = \frac{(D_q P_{n+1})(x)}{[n+1]} - \sigma_n \frac{(D_q P_n)(x)}{[n]}, \quad n \geq 1,$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers and

$$[n] = \frac{q^n - 1}{q - 1}, \quad n \geq 1, \quad q \neq 1$$

(see [2, p. 5]). Moreover, we determine all q -coherent pairs of linear functionals when dealing with little q -Jacobi and little q -Laguerre/Wall linear functionals. By using limit properties for linear functionals the classification given by Meijer in the continuous case [10] is reached. In this way, an interesting direction of research can be open. If $(\mathbf{d}\mu_0, \mathbf{d}\mu_1)$ is a q -coherent pair of positive measures, then the study of q -Sobolev orthogonal polynomials seems to be very natural. A particular case of q -coherent pairs has been developed in [13]. On the other hand, in the Doctoral Dissertation by Koekoek [14] and a subsequent paper [15] it was studied a Sobolev type inner product

$$(f, g)_S = \int_0^\infty fg \mathbf{d}\mu_0 + \sum_{k=0}^N M_k (D_q^k f)(0) (D_q^k g)(0),$$

$$\mathbf{d}\mu_0 = \frac{x^z}{(-(1-q)x; q)_\infty} dx,$$

as a generalization of a q -analogue of the classical Laguerre polynomials.

The outline of the paper is as follows. In Section 2, we give basic definitions and results which will be helpful in the following sections. In Section 3, we present the q -classical linear functionals. In Section 4, we introduce the concept of q -coherent pair of linear functionals, and we prove that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair, then both \mathbf{u}_0 and \mathbf{u}_1 are q -semiclassical linear functionals. In Section 5, we prove that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair of linear functionals, then at least one of them must be a q -classical linear functional. In Section 6, we give the classification of all q -coherent pairs of positive-definite linear functionals $(\mathbf{u}_0, \mathbf{u}_1)$ when \mathbf{u}_0 or \mathbf{u}_1 is either the little q -Jacobi linear functional or the little q -Laguerre linear functional. Finally, in Section 7, by using limit relations we recover the classification of all coherent pairs of positive definite linear functionals.

2. Notations and basic results

Let \mathbb{P} be the linear space of complex polynomials and let \mathbb{P}' be its algebraic dual space. We denote by $\langle \mathbf{u}, f \rangle$ the duality bracket for $\mathbf{u} \in \mathbb{P}'$ and $f \in \mathbb{P}$, and we denote by $(\mathbf{u})_n = \langle \mathbf{u}, x^n \rangle$, with $n \geq 0$, the canonical moments of \mathbf{u} .

Definition 1. A linear functional $\mathbf{u} : \mathbb{P} \rightarrow \mathbb{C}$ is said to be quasi-definite if all the principal submatrices of the infinite Hankel matrix $H = [(\mathbf{u})_{i+j}]_{i,j=0}^\infty$ are non-singular.

It is known [16] that a linear functional \mathbf{u} is quasi-definite if and only if there exists an MOPS $\{P_n\}_n$ orthogonal with respect to \mathbf{u} , i.e.,

1. $P_n(x) = x^n + \text{terms of lower degree}$ for every $n \geq 0$,
2. $\langle \mathbf{u}, P_n(x)P_m(x) \rangle = \Gamma_n \delta_{nm}$ ($\Gamma_n \neq 0$) for every $n, m \geq 0$.

An MOPS $\{P_n\}_n$ with respect to \mathbf{u} satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \quad (3)$$

with $\gamma_n \neq 0$ for $n \geq 0$ and initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.

Definition 2. Given a complex number c , the Dirac functional δ_c is defined by $\langle \delta_c, p(x) \rangle := p(c)$ for every $p \in \mathbb{P}$.

Definition 3. Given a functional \mathbf{u} , we define, for each polynomial p , the linear functional $p\mathbf{u}$ as follows: $\langle p\mathbf{u}, r(x) \rangle := \langle \mathbf{u}, p(x)r(x) \rangle$ for every $r \in \mathbb{P}$. For each complex number c the functional $(x - c)^{-1}\mathbf{u}$ is given by $\langle (x - c)^{-1}\mathbf{u}, r(x) \rangle := \langle \mathbf{u}, (r(x) - r(c))/(x - c) \rangle$ for every $r \in \mathbb{P}$.

Note that

$$(x - c)^{-1}((x - c)\mathbf{u}) = \mathbf{u} - (\mathbf{u})_0 \delta_c \quad \text{for every } \mathbf{u} \in \mathbb{P}', \quad (4)$$

while $(x - c)((x - c)^{-1}\mathbf{u}) = \mathbf{u}$.

Definition 4. Let q be a complex number, $q \neq 1$ and $q \neq 0$. The q -difference operator D_q is defined by

$$(D_q p)(x) = \frac{p(qx) - p(x)}{(q - 1)x}, \quad x \neq 0 \quad \text{for every } p \in \mathbb{P}, \quad (5)$$

and $(D_q p)(0) := p'(0)$ by continuity.

In what follows we shall always assume that $0 < q < 1$. The action of the D_q operator on a monomial $f(x) = x^n$ gives us

$$(D_q f)(x) = \begin{cases} \frac{(q^n - 1)x^{n-1}}{(q - 1)} = [n]x^{n-1}, & n > 0, \\ 0, & n = 0, \end{cases} \quad (6)$$

where the numbers $[n]$ defined as

$$\begin{cases} [n] = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}, & n > 0, \\ [0] = 0 \end{cases} \quad (7)$$

are useful in the theory of representations of quantum groups and algebras. Let p and r be two polynomials. Then

$$(D_q(pr))(x) = r(x)(D_q p)(x) + p(qx)(D_q r)(x). \quad (8)$$

It is clear that the q -difference operator D_q defined in (5) converges to the derivative operator $\mathcal{D} = d/dx$ when $q \uparrow 1$. Given a linear functional \mathbf{u} , the linear functional $\mathcal{D}\mathbf{u}$ is defined [17] as $\langle \mathcal{D}\mathbf{u}, p \rangle = -\langle \mathbf{u}, \mathcal{D}p \rangle$ for every $p \in \mathbb{P}$.

Definition 5. A sequence of linear functionals $\{\mathbf{u}_n\}_n$ converges to $\mathbf{u} \in \mathbb{P}'$ if and only if $\{\langle \mathbf{u}_n, p \rangle\}_n$ converges to $\langle \mathbf{u}, p \rangle$ for every $p \in \mathbb{P}$.

Definition 6. For $\mathbf{u} \in \mathbb{P}'$, we introduce the functional $D_q \mathbf{u}$ such that $\langle D_q \mathbf{u}, p(x) \rangle = -\langle \mathbf{u}, (D_q p)(x) \rangle$ for every $p \in \mathbb{P}$.

Note that from the above definition, $D_q \mathbf{u}$ converges to $\mathcal{D}\mathbf{u}$ when $q \uparrow 1$. We shall also need the following properties of the D_q operator:

Proposition 1. For $\mathbf{u} \in \mathbb{P}'$ and for $p \in \mathbb{P}$ we have

$$D_q[p(x)\mathbf{u}] = p(q^{-1}x)D_q \mathbf{u} + (D_q p)(q^{-1}x)\mathbf{u}. \quad (9)$$

Proof. Let us define $\tilde{p}(x) := p(q^{-1}x)$. Then we obtain

$$\begin{aligned} \langle D_q[p(x)\mathbf{u}], r(x) \rangle &= -\langle p(x)\mathbf{u}, (D_q r)(x) \rangle \\ &= -\langle \mathbf{u}, p(x)(D_q r)(x) \rangle \\ &= -\langle \mathbf{u}, \tilde{p}(qx)(D_q r)(x) \rangle \\ &= -\langle \mathbf{u}, (D_q \tilde{p}r)(x) - r(x)(D_q \tilde{p})(x) \rangle \\ &= \langle p(q^{-1}x)D_q \mathbf{u} + (D_q p)(q^{-1}x)\mathbf{u}, r(x) \rangle \end{aligned}$$

for every $r \in \mathbb{P}$. \square

3. q -Classical linear functionals

Definition 7. A functional \mathbf{u} is said to be a q -classical linear functional if \mathbf{u} is quasi-definite and there exist polynomials ϕ and ψ with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$ such that

$$D_q[\phi(x)\mathbf{u}] = \psi(x)\mathbf{u}. \quad (10)$$

The corresponding MOPS associated with \mathbf{u} is said to be a q -classical MOPS.

In [2,8] we can find the families of q -classical polynomial sequences namely: Big q -Jacobi, little q -Jacobi, big q -Laguerre, q -Meixner, alternative q -Charlier, little q -Laguerre/Wall, Moak, Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, discrete q -Hermite I and discrete q -Hermite II. In Table 1 the polynomials $\phi(x)$ and $\psi(x)$ appearing in the distributional equation (10) are given for each q -classical family, presented according to their representation as basic hypergeometric series (equivalently, to their situation in the q -Hahn tableau) [8, p. 115].

Table 1
Polynomials in the distributional Eq. (10) for each q classical family

$P_n(x)$	$\phi(x)$	$\psi(x)$
Big q Jacobi	$aq(x-1)(bx-c)$	$\frac{cq-x+aq(1-(b+c)q+bqx)}{(q-1)q}$
Little q Jacobi	$abqx\left(x-\frac{1}{bq}\right)$	$\frac{aq-1}{(1-q)q} + \frac{(1-abq^2)x}{(1-q)q}$
Big q Laguerre	$abq(x-1)$	$\frac{a+b-abq}{1-q} - \frac{x}{(1-q)q}$
q Meixner	$(x-1)(bc+x)$	$\frac{c+q-bcq-qx}{q-q^2}$
Alternative q Charlier	ax^2	$\frac{1+x+aqx}{(q-1)q}$
Little q Laguerre/Wall	ax	$\frac{x+aq-1}{(q-1)q}$
Moak	$q^z\left(\frac{1}{q-1}-x\right)x$	$\frac{1+q^{1+z}(1+(q-1)x)}{(q-1)^2q}$
Al Salam Carlitz I	a	$\frac{1+a-x}{q-1}$
Al Salam Carlitz II	$(x-1)(x-a)$	$\frac{1+a}{1-q} + \frac{x}{q-1}$
Stieltjes Wigert	x^2	$\frac{qx-1}{(q-1)q}$
Discrete q Hermite I	1	$\frac{x}{1-q}$
Discrete q Hermite II	x^2-1	$\frac{x}{q-1}$

The q -classical orthogonal polynomials can be characterized in the following proposition (see [2,19]).

Proposition 2. *Let $\{P_n\}_n$ be an MOPS associated with a linear functional \mathbf{u} . The following statements are equivalent:*

1. $\{P_n\}_n$ is a q -classical MOPS.
2. $\{Q_n\}_n$ defined by

$$Q_n(x) = \frac{(D_q P_{n+1})(x)}{[n+1]}, \quad n \geq 0, \quad (11)$$

is also an MOPS. Furthermore, if \mathbf{u} satisfies $D_q[\phi(x)\mathbf{u}] = \psi(x)\mathbf{u}$, then $\{Q_n\}_n$ is orthogonal with respect to the functional $\tilde{\mathbf{u}} = \phi(x)\mathbf{u}$.

Next we introduce the concept of q -semiclassical linear functional, as suggested in [20, p. 128], and some properties for these functionals (see [19]).

Definition 8. A linear functional \mathbf{u} is said to be q -semiclassical if \mathbf{u} is quasi-definite and there exist two polynomials ϕ and ψ such that

$$D_q[\phi(x)\mathbf{u}] = \psi(x)\mathbf{u}, \quad (12)$$

where $\deg(\phi) = t \geq 0$ and $\deg(\psi) = p \geq 1$. An MOPS with respect to a q -semiclassical functional \mathbf{u} is called a q -semiclassical MOPS.

It is possible to associate with (12) a non-negative integer s as follows: $s = \max\{\deg(\psi) - 1, \deg(\phi) - 2\}$ but a q -semiclassical functional \mathbf{u} satisfies an infinite number of equations as (12). It is enough to multiply both sides of Eq. (12) by a monic polynomial f with $\deg(f) = l$ and from Proposition 1 we have $D_q[f(qx)\phi(x)\mathbf{u}] = (\phi(x)(D_q f)(x) + f(x)\psi(x))\mathbf{u}$. So \mathbf{u} fulfills also $D_q[\phi_1(x)\mathbf{u}] = \psi_1(x)\mathbf{u}$ where now $\phi_1(x) = f(qx)\phi(x)$ and $\psi_1(x) = \phi(x)(D_q f)(x) + f(x)\psi(x)$.

From Eq. (12) we have $s_1 = \max\{p_1 - 1, t_1 - 2\} = s + l$. Hence we can associate with a q -semiclassical functional \mathbf{u} a set of non-negative integer numbers $h(\mathbf{u})$.

Definition 9. Let \mathbf{u} be a q -semiclassical functional. The minimum of the set $h(\mathbf{u})$ is called the class of \mathbf{u} . When s is the class of \mathbf{u} then the sequence $\{P_n\}_n$ orthogonal with respect to \mathbf{u} is said to be of class s .

4. q -Coherent pairs

Let us introduce the concept of q -coherent pair of linear functionals, as a q -analogue of coherent pair of linear functionals, i.e., when $q \rightarrow 1$ we recover the concept of coherent pair of linear functionals used in [10,21,22].

Definition 10. Let \mathbf{u}_0 and \mathbf{u}_1 be two quasi-definite linear functionals, whose MOPS are $\{P_n\}_n$ and $\{T_n\}_n$, respectively. We define $(\mathbf{u}_0, \mathbf{u}_1)$ as a q -coherent pair of linear functionals if

$$T_n(x) = \frac{(D_q P_{n+1})(x)}{[n+1]} - \sigma_n \frac{(D_q P_n)(x)}{[n]}, \quad n \geq 1, \quad (13)$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers and the numbers $[n]$ are defined in (7).

Example. Let us consider the little q -Laguerre/Wall linear functional $\mathbf{u}^{(a)}$ (see [23])

$$\langle \mathbf{u}^{(a)}, p \rangle = \sum_{k=0}^{\infty} p(q^k) \frac{(aq)^k}{(q; q)_k}, \quad 0 < aq < 1 \text{ for every } p \in \mathbb{P}, \quad (14)$$

where the q -shifted factorials are given by

$$(c; q)_0 = 1, \quad (c; q)_j = (1-c)(1-cq) \cdots (1-cq^{j-1}), \quad j \geq 1 \quad (15)$$

(see [18]). The MOPS associated with this linear functional is the MOPS of little q -Laguerre/Wall polynomials $\{p_n(x; a|q)\}_n$. They are particular little q -Jacobi polynomials and q -analogues of Laguerre polynomials (see [8, p. 117]). They are related with monic orthogonal Wall polynomials $\{W_n(x; b, q)\}_n$ (see [16, p. 198] and [18, p. 196]) by means of

$$p_n(x; a|q) = \frac{W_n(qx; aq, q)}{q^n}, \quad n \geq 0.$$

Since little q -Laguerre/Wall polynomials $f_n(x) = p_n(x; a|q)$ satisfy

$$p_n(x; a|q) = \frac{(D_q f_{n+1})(x)}{[n+1]} - \sigma_n \frac{(D_q f_n)(x)}{[n]}, \quad \text{where } \sigma_n = aq^n(q^n - 1), \quad (16)$$

we deduce that $(\mathbf{u}^{(a)}, \mathbf{u}^{(a)})$ is a q -coherent pair of linear functionals.

Definition 11. Let $\{P_n\}_n$ be an MOPS associated with the functional \mathbf{u} . The sequence of linear functionals $\{\alpha_n\}_n$ defined by $\langle \alpha_n, P_m(x) \rangle = \delta_{nm}$, $n, m \geq 0$, is called the dual basis of $\{P_n\}_n$.

In fact,

$$\alpha_n = \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x) \rangle} \mathbf{u}. \quad (17)$$

An immediate consequence of the above equation can be stated as follows.

Proposition 3. *Let $\{P_n\}_n$ be an MOPS associated with the linear functional \mathbf{u} and let $\{Q_n\}_n$ as in (11). If we denote by $\{\alpha_n\}_n$ and $\{\check{\alpha}_n\}_n$ the corresponding dual bases, then $D_q \check{\alpha}_n = -[n+1]\alpha_{n+1}$.*

Proposition 4. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair and let $\{\alpha_n^{(0)}\}_n$ and $\{\alpha_n^{(1)}\}_n$ be the dual bases of \mathbf{u}_0 and \mathbf{u}_1 , respectively. If we denote by $\{\check{\alpha}_n^{(0)}\}_n$ the dual basis corresponding to $\{Q_n\}_n$ defined in (11), then we have*

$$\check{\alpha}_n^{(0)} = \alpha_n^{(1)} - \sigma_{n+1}\alpha_{n+1}^{(1)}, \quad (18)$$

$$[n+1]\alpha_{n+1}^{(0)} = \sigma_{n+1}D_q\alpha_{n+1}^{(1)} - D_q\alpha_n^{(1)}, \quad n \geq 0. \quad (19)$$

Proof. Since $\{\alpha_n^{(1)}\}_n$ is a basis of \mathbb{P}' we can write

$$\check{\alpha}_n^{(0)} = \sum_{m \geq 0} \lambda_{m,n} \alpha_m^{(1)},$$

where

$$\begin{aligned} \lambda_{m,n} &= \langle \check{\alpha}_n^{(0)}, T_m(x) \rangle = \langle \check{\alpha}_n^{(0)}, Q_m(x) - \sigma_m Q_{m-1}(x) \rangle \\ &= \begin{cases} 1 & \text{if } m = n, \\ -\sigma_{n+1} & \text{if } m = n + 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and then (18) holds. Applying the D_q operator to (18) and using Proposition 3, Eq. (19) is obtained. \square

We conclude this section proving that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair of linear functionals both \mathbf{u}_0 and \mathbf{u}_1 are q -semiclassical linear functionals.

Theorem 1. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of linear functionals and let $\{P_n\}_n$, $\{T_n\}_n$ be the corresponding MOPS associated with \mathbf{u}_0 and \mathbf{u}_1 , respectively. Then,*

(i) *The functional \mathbf{u}_1 is a q -semiclassical linear functional of class at most 1. That is, there exist two polynomials ϕ_1 and ψ_1 of degree at most 3 and 2, respectively, such that*

$$D_q[\phi_1(x)\mathbf{u}_1] = \psi_1(x)\mathbf{u}_1. \quad (20)$$

Their explicit expressions are

$$\phi_1(x) = [2] \frac{P_2(qx)}{p_2} c_1(x) - \frac{P_1(qx)}{p_1} c_2(x), \quad (21)$$

$$\psi_1(x) = \frac{P_1(x)}{p_1} (D_q c_2)(q^{-1}x) - [2] \frac{P_2(x)}{p_2} (D_q c_1)(q^{-1}x) + (D_q \phi_1)(q^{-1}x), \quad (22)$$

where

$$c_{n+1}(x) = \sigma_{n+1} \frac{T_{n+1}(x)}{t_{n+1}} - \frac{T_n(x)}{t_n}, \quad n \geq 0, \quad (23)$$

and $p_n := \langle \mathbf{u}_0, P_n^2 \rangle$, $t_n := \langle \mathbf{u}_1, T_n^2 \rangle$.

(ii) There exist polynomials A_3 and B_2 of degree at most 3 and 2, respectively, such that

$$A_3(x)\mathbf{u}_0 = B_2(x)\mathbf{u}_1, \quad (24)$$

where

$$A_3(x) = \phi_1(q^{-1}x), \quad (25)$$

$$B_2(x) = c_1(q^{-1}x)(D_q c_2)(q^{-1}x) - c_2(q^{-1}x)(D_q c_1)(q^{-1}x). \quad (26)$$

(iii) The functional \mathbf{u}_0 is a q -semiclassical linear functional of class at most 6 since it verifies the distributional equation

$$D_q[\phi_0(x)\mathbf{u}_0] = \psi_0(x)\mathbf{u}_0, \quad (27)$$

where

$$\phi_0(x) = \phi_1(x)\phi_1(q^{-1}x)B_2(x), \quad (28)$$

$$\begin{aligned} \psi_0(x) = & \{B_2(q^{-1}x)\psi_1(x) + (D_q B_2)(q^{-1}x)\phi_1(x)\}A_3(q^{-1}x) \\ & + \phi_1(x)(D_q B_2)(q^{-1}x)A_3(x), \end{aligned} \quad (29)$$

are polynomials of degree at most 8 and 7, respectively.

Proof. Let us write (19) using (17)

$$[n+1] \frac{P_{n+1}(x)}{p_{n+1}} \mathbf{u}_0 = D_q[c_{n+1}(x)\mathbf{u}_1], \quad n \geq 0. \quad (30)$$

For $n = 0$ and $n = 1$, Eq. (30) can be written

$$\begin{aligned} \frac{P_1(x)}{p_1} \mathbf{u}_0 &= D_q[c_1(x)\mathbf{u}_1] = c_1(q^{-1}x)D_q \mathbf{u}_1 + (D_q c_1)(q^{-1}x)\mathbf{u}_1, \\ [2] \frac{P_2(x)}{p_2} \mathbf{u}_0 &= D_q[c_2(x)\mathbf{u}_1] = c_2(q^{-1}x)D_q \mathbf{u}_1 + (D_q c_2)(q^{-1}x)\mathbf{u}_1. \end{aligned} \quad (31)$$

(i) From (31) it follows that

$$\begin{aligned} & \left([2] \frac{P_2(x)}{p_2} c_1(q^{-1}x) - \frac{P_1(x)}{p_1} c_2(q^{-1}x) \right) D_q \mathbf{u}_1 \\ & + \left([2] \frac{P_2(x)}{p_2} (D_q c_1)(q^{-1}x) - \frac{P_1(x)}{p_1} (D_q c_2)(q^{-1}x) \right) \mathbf{u}_1 = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} & D_q \left[\left([2] \frac{P_2(qx)}{p_2} c_1(x) - \frac{P_1(qx)}{p_1} c_2(x) \right) \mathbf{u}_1 \right] \\ & = \left(\frac{P_1(x)}{p_1} (D_q c_2)(q^{-1}x) - [2] \frac{P_2(x)}{p_2} (D_q c_1)(q^{-1}x) + (D_q g)(x) \right) \mathbf{u}_1, \end{aligned}$$

where

$$g(x) = [2] \frac{P_2(x)}{p_2} c_1(q^{-1}x) - \frac{P_1(x)}{p_1} c_2(q^{-1}x).$$

Hence, we obtain

$$D_q[\phi_1(x)\mathbf{u}_1] = \psi_1(x)\mathbf{u}_1,$$

which coincides with (20).

(ii) Eliminating $D_q\mathbf{u}_1$ in the system (31) we obtain $A_3(x)\mathbf{u}_0 = B_2(x)\mathbf{u}_1$, where polynomials A_3 and B_2 are given in (25) and (26), respectively.

(iii) Finally, by using (9) appropriately we have

$$\begin{aligned} D_q[\phi_1(x)\phi_1(q^{-1}x)B_2(x)\mathbf{u}_0] &= D_q[\phi_1(x)B_2(x)\phi_1(q^{-1}x)\mathbf{u}_0] \\ &= D_q[\phi_1(x)B_2(x)A_3(x)\mathbf{u}_0] \\ &= D_q[\phi_1(x)B_2(x)B_2(x)\mathbf{u}_1] \\ &= D_q[B_2^2(x)\phi_1(x)\mathbf{u}_1] \\ &= B_2^2(q^{-1}x)D_q[\phi_1(x)\mathbf{u}_1] + (D_qB_2^2)(q^{-1}x)\phi_1(x)\mathbf{u}_1 \\ &= B_2^2(q^{-1}x)\psi_1(x)\mathbf{u}_1 + (B_2(q^{-1}x)(D_qB_2)(q^{-1}x) \\ &\quad + B_2(x)(D_qB_2)(q^{-1}x))\phi_1(x)\mathbf{u}_1 \\ &= B_2(q^{-1}x)\psi_1(x)A_3(q^{-1}x)\mathbf{u}_0 + (D_qB_2)(q^{-1}x)\phi_1(x)B_2(q^{-1}x)\mathbf{u}_1 \\ &\quad + (D_qB_2)(q^{-1}x)\phi_1(x)B_2(x)\mathbf{u}_1, \end{aligned}$$

where by using again (24) we get the result. \square

5. General problem of q -coherence

In Theorem 1 we have proved that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair of linear functionals, then both $\mathbf{u}_0, \mathbf{u}_1$ are q -semiclassical functionals of class at most 6 and 1, respectively. The main goal of this section is to prove that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair of linear functionals, then at least one of the functionals $\mathbf{u}_0, \mathbf{u}_1$ has to be a q -classical functional. In order to give a scheme of the proof let us denote by ξ and η the zeros of the polynomial B_2 defined in (26). The proof of this statement will consist in three steps. In the first one, we prove that if $\eta = q\xi$, \mathbf{u}_0 must be a q -classical linear functional (Theorem 2). In the second step we prove that if $\xi \neq \eta$ and $\eta \neq q\xi$, then \mathbf{u}_1 must be a q -classical linear functional (Theorem 3). Finally, as a remark, we study the case $\xi = \eta$.

Proposition 5. Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of linear functionals, being $\{P_n\}_n$ and $\{T_n\}_n$ the corresponding MOPS associated with \mathbf{u}_0 and \mathbf{u}_1 , respectively. Let c_n be the polynomials defined in (23). For each $n \geq 1$ we have

$$[n] \frac{P_n(x)}{p_n} B_2(x) = A_3(x)(D_q c_n)(q^{-1}x) + c_n(q^{-1}x)\pi(x), \quad (32)$$

where polynomials A_3 and B_2 are defined in (25) and (26), respectively, and

$$\pi(x) = \psi_1(x) - (D_q A_3)(x) \quad (33)$$

with ψ_1 given in (22).

Proof. Using (24), (30) and (9) we obtain

$$\begin{aligned} [n] \frac{P_n(x)}{p_n} B_2(x)\mathbf{u}_1 &= [n] \frac{P_n(x)}{p_n} A_3(x)\mathbf{u}_0 = A_3(x)D_q[c_n(x)\mathbf{u}_1] \\ &= A_3(x)\{c_n(q^{-1}x)D_q\mathbf{u}_1 + (D_q c_n)(q^{-1}x)\mathbf{u}_1\}. \end{aligned}$$

From Theorem 1 the functional \mathbf{u}_1 verifies Eq. (20). Using (9) and (25) it follows that

$$A_3(x)D_q\mathbf{u}_1 = \phi_1(q^{-1}x)D_q\mathbf{u}_1 = \pi(x)\mathbf{u}_1, \quad (34)$$

and then (32) holds. \square

Theorem 2. Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of linear functionals. Let ξ and η be the zeros of the polynomial B_2 defined in (26). Suppose that $\eta = q\xi$, i.e.,

$$B_2(x) = \frac{\sigma_1\sigma_2}{t_1t_2}(x - \xi)(x - q\xi). \quad (35)$$

Then,

(i) The functional $\tilde{\mathbf{u}} = (x - \xi)\mathbf{u}_1$ is a q -classical linear functional and

$$D_q[\tilde{\phi}(x)\tilde{\mathbf{u}}] = \tilde{\psi}(x)\tilde{\mathbf{u}}, \quad (36)$$

for some polynomials $\tilde{\phi}$ and $\tilde{\psi}$ with $\deg(\tilde{\phi}) \leq 2$ and $\deg(\tilde{\psi}) = 1$. Moreover,

$$\tilde{\phi}(q^{-1}x)\mathbf{u}_0 = q \frac{\sigma_1\sigma_2}{t_1t_2} \tilde{\mathbf{u}}. \quad (37)$$

(ii) The functional \mathbf{u}_0 is a q -classical linear functional and

$$D_q[\tilde{\phi}(q^{-1}x)\mathbf{u}_0] = \tilde{\psi}_0(x)\mathbf{u}_0, \quad (38)$$

for some polynomial $\tilde{\psi}_0$ with $\deg(\tilde{\psi}_0) = 1$.

Proof. We have proved in Theorem 1 that \mathbf{u}_1 is a q -semiclassical linear functional satisfying the distributional equation $D_q[\phi_1(x)\mathbf{u}_1] = \psi_1(x)\mathbf{u}_1$, where polynomials ϕ_1 and ψ_1 are defined in (21) and (22), respectively.

(i) From the definition of B_2 in (35) it is easy to check that

$$(D_q B_2)(x) = \frac{\sigma_1 \sigma_2}{t_1 t_2} (q+1)(x-\xi),$$

and also

$$(D_q B_2)(x) = [2] \frac{\sigma_2}{t_2} c_1(x), \quad (39)$$

using (26). Thus we obtain that $c_1(\xi) = 0$. From (26) doing $x = q\xi$ it follows that

$$0 = B_2(q\xi) = -c_2(\xi) \frac{\sigma_1}{t_1}.$$

Thus $c_2(\xi) = 0$ and from (21) we obtain $\phi_1(\xi) = 0$ as well as $\psi_1(\xi) = 0$ using (21) and (22).

Hence, we can write

$$\phi_1(x) = (x-\xi)\tilde{\phi}(x), \quad \psi_1(x) = (x-\xi)\tilde{\psi}(x).$$

Let us define $\tilde{\mathbf{u}} = (x-\xi)\mathbf{u}_1$. From (20) and the definition of polynomials $\tilde{\phi}$ and $\tilde{\psi}$, it follows that $\tilde{\mathbf{u}}$ satisfies Eq. (36). In Theorem 1 we have proved that $\deg(\phi_1) \leq 3$, so $\deg(\tilde{\phi}) \leq 2$. Since $\deg(\psi_1) \leq 2$, we obtain that $\deg(\tilde{\psi}) \leq 1$. If we prove that $\tilde{\psi}$ cannot be a constant polynomial, then we deduce part (i) of the Theorem. In order to do it we distinguish two situations:

1. If $\tilde{\psi}$ is a non-zero constant v ($\tilde{\psi}(x) \equiv v$), then

$$\langle \mathbf{u}_1, v(x-\xi) \rangle = \langle v(x-\xi)\mathbf{u}_1, 1 \rangle = \langle v\tilde{\mathbf{u}}, 1 \rangle = \langle D_q[\tilde{\phi}(x)\tilde{\mathbf{u}}], 1 \rangle = 0.$$

Hence $T_1(x) = x - \xi$. Since c_1 is defined by

$$c_1(x) = \sigma_1 \frac{T_1(x)}{t_1} - \frac{1}{t_0},$$

then $c_1(\xi) \neq 0$ and this contradicts that $c_1(\xi) = 0$.

2. Suppose that $\tilde{\psi} \equiv 0$. From (6) we get

$$\begin{aligned} \langle \phi_1(x)\mathbf{u}_1, x^n \rangle &= - \left\langle D_q[\phi_1(x)\mathbf{u}_1], \frac{x^{n+1}}{[n+1]} \right\rangle = - \left\langle D_q[\tilde{\phi}(x)\tilde{\mathbf{u}}], \frac{x^{n+1}}{[n+1]} \right\rangle \\ &= \langle \tilde{\psi}(x)\tilde{\mathbf{u}}, x^n \rangle = 0. \end{aligned}$$

So, $\langle \phi_1(x)\mathbf{u}_1, p(x) \rangle = 0$ for every $p \in \mathbb{P}$, and then \mathbf{u}_1 should not be a quasi-definite linear functional. Hence $\tilde{\psi} \neq 0$.

From the above situations we conclude that $\deg(\tilde{\psi}) = 1$.

Moreover, since $c_1(\xi) = c_2(\xi) = 0$ then c_1 divides c_2 . From (24) and (25) we can write

$$(q^{-1}x - \xi)\tilde{\phi}(q^{-1}x)\mathbf{u}_0 = B_2(x)\mathbf{u}_1,$$

or

$$(x - q\xi)\tilde{\phi}(q^{-1}x)\mathbf{u}_0 = qB_2(x)\mathbf{u}_1.$$

Multiplying both sides of the above expression by $(x - q\xi)^{-1}$, and using (4) we obtain

$$\tilde{\phi}(q^{-1}x)\mathbf{u}_0 = q(x - q\xi)^{-1}B_2(x)\mathbf{u}_1 + \tilde{\phi}(q^{-1}x)\mathbf{u}_0\delta_{q\xi}.$$

Applying again (4) we get

$$\tilde{\phi}(q^{-1}x)\mathbf{u}_0 = q\frac{\sigma_1\sigma_2}{t_1t_2}(x - \xi)\mathbf{u}_1,$$

if we normalize in a convenient way the linear functionals \mathbf{u}_0 and \mathbf{u}_1 .

(ii) By applying the operator D_q to both sides of the above equality we obtain

$$D_q\left[\tilde{\phi}(q^{-1}x)\mathbf{u}_0\right] = D_q\left[q\frac{\sigma_1\sigma_2}{t_1t_2}(x - \xi)\mathbf{u}_1\right] = q\frac{\sigma_2}{t_2}D_q[c_1(x)\mathbf{u}_1] = q\frac{\sigma_2}{t_2}\frac{P_1(x)}{p_1}\mathbf{u}_0,$$

using (30). If we define

$$\tilde{\psi}_0(x) = q\frac{\sigma_2}{t_2}\frac{P_1(x)}{p_1},$$

then $\deg(\tilde{\psi}_0) = 1$ and (38) holds. \square

In order to prove the behaviour in the other situations we need some previous lemmas.

Lemma 1. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of linear functionals and A_3 and B_2 the polynomials defined in (25) and (26), respectively. Suppose that ξ is a zero of B_2 such that $A_3(\xi) \neq 0$. Then, there exists a non-zero parameter k independent of n such that*

$$c_n(q^{-1}\xi) + k(D_q c_n)(q^{-1}\xi) = 0 \quad \text{for all } n \geq 1. \quad (40)$$

Proof. Since $B_2(\xi) = 0$, Eq. (32) for $n = 1$ and $x = \xi$ reads as

$$0 = A_3(\xi)(D_q c_1)(q^{-1}\xi) + c_1(q^{-1}\xi)\pi(\xi).$$

Since $(D_q c_1)(q^{-1}\xi) = \sigma_1/t_1$ is a non-zero constant and from the hypothesis $A_3(\xi) \neq 0$ then $\pi(\xi) \neq 0$ and

$$c_1(q^{-1}\xi) = -\frac{A_3(\xi)\sigma_1}{\pi(\xi)t_1} \neq 0.$$

If we define

$$k = \frac{A_3(\xi)}{\pi(\xi)},$$

then from Eq. (32) in $x = \xi$ our result holds for every $n \geq 1$. \square

Lemma 2. *Suppose that there exist parameters $\xi_1, \xi_2, k_1 \neq 0$, and $k_2 \neq 0$ such that*

$$c_n(\xi_1) + k_1(D_q c_n)(\xi_1) = 0 \quad \text{for all } n \geq 1, \quad (41)$$

$$c_n(\xi_2) + k_2(D_q c_n)(\xi_2) = 0 \quad \text{for all } n \geq 1. \quad (42)$$

If $\xi_i \neq q\xi_j$, then $\xi_1 = \xi_2$ and $k_1 = k_2$.

Proof. From the definition of c_n in (23), Eqs. (41) and (42) can be written

$$\sigma_n \left\{ \frac{T_n(\xi_j)}{t_n} + k_j \frac{(D_q T_n)(\xi_j)}{t_n} \right\} = \frac{T_{n-1}(\xi_j)}{t_{n-1}} + k_j \frac{(D_q T_{n-1})(\xi_j)}{t_{n-1}}, \quad j = 1, 2.$$

Let us denote

$$h_n^{(j)}(\xi_j) = \frac{T_n(\xi_j)}{t_n} + k_j \frac{(D_q T_n)(\xi_j)}{t_n}, \quad n \geq 0, \quad j = 1, 2.$$

Then for each $n \geq 1$ and for $j = 1, 2$ we can write

$$\sigma_n h_n^{(j)}(\xi_j) = h_{n-1}^{(j)}(\xi_j). \quad (43)$$

Observe that

$$h_0^{(j)}(\xi_j) = \frac{1}{t_0}, \quad j = 1, 2.$$

From (43) it follows that $h_n^{(j)}(\xi_j) \neq 0$ for all $n \geq 0$ and we obtain

$$\frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)} = \frac{h_{n-1}^{(1)}(\xi_1)}{h_{n-1}^{(2)}(\xi_2)}, \quad n \geq 1.$$

Repeating this process we get

$$\frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)} = \frac{h_0^{(1)}(\xi_1)}{h_0^{(2)}(\xi_2)} = \frac{1/t_0}{1/t_0} = 1, \quad n \geq 1,$$

i.e., $h_n^{(1)}(\xi_1) = h_n^{(2)}(\xi_2)$ for $n \geq 1$ or, equivalently,

$$T_n(\xi_1) + k_1(D_q T_n)(\xi_1) = T_n(\xi_2) + k_2(D_q T_n)(\xi_2), \quad n \geq 1. \quad (44)$$

So from the initial problem of characterizing ξ_1 , ξ_2 , k_1 , and k_2 such that (41) and (42) hold simultaneously, we arrive at a new problem: determine ξ_1 , ξ_2 , k_1 and k_2 such that (44) holds.

In order to find the solutions of problem (44), we study a more general one: find all μ , ν , δ and η such that

$$\mu T_n(\xi_1) + \nu(D_q T_n)(\xi_1) = \delta T_n(\xi_2) + \eta(D_q T_n)(\xi_2), \quad n \geq 1. \quad (45)$$

Assume that $\{T_n\}_n$ satisfies the three-term recurrence relation

$$T_{n+1}(x) = (x - \beta_n^T)T_n(x) - \gamma_n^T T_{n-1}(x), \quad n \geq 0. \quad (46)$$

Then, applying the D_q operator we obtain

$$(D_q T_{n+1})(x) = T_n(x) + (qx - \beta_n^T)(D_q T_n)(x) - \gamma_n^T (D_q T_{n-1})(x). \quad (47)$$

Using (46) and (47) a new equation is obtained:

$$\begin{aligned} & (\nu + \mu\xi_1)T_n(\xi_1) + q\nu\xi_1(D_q T_n)(\xi_1) \\ & = (\eta + \delta\xi_2)T_n(\xi_2) + q\eta\xi_2(D_q T_n)(\xi_2). \end{aligned} \quad (48)$$

Let us repeat the process from (45) (48), but starting with Eq. (48) instead of (45). If we do so, we obtain a new equation which should be verified. Finally, mimicking the process starting with this new last equation we find an homogeneous system of four linear equations with variables $T_n(\xi_1)$, $T_n(\xi_2)$, $(D_q T_n)(\xi_1)$ and $(D_q T_n)(\xi_2)$. The determinant of the matrix of coefficients is

$$-k_1 k_2 (\xi_1 - \xi_2)^2 (q\xi_2 - \xi_1)(q\xi_2 - \xi_1)(k_1 + \xi_1(1 - q))(k_2 + \xi_2(1 - q)), \quad (49)$$

after replacing $\mu = 1$, $\nu = k_1$, $\delta = 1$ and $\eta = k_2$. Then, we need to study the solutions of this linear system, depending on the value of (49).

If the determinant (49) is different of zero, then the solution of the linear system is

$$T_n(\xi_i) = (D_q T_n)(\xi_i) = 0, \quad n \geq 3, \quad i = 1, 2.$$

Hence it should be

$$T_n(\xi_1) = T_n(\xi_2) = T_n(q\xi_1) = T_n(q\xi_2) = 0, \quad n \geq 3,$$

but this contradicts that $\{T_n\}_n$ is an MOPS.

Now we discuss what happens when the determinant (49) is equal to zero. Because of the hypothesis of this lemma only two situations can appear:

1. If $\xi_1 = \xi_2$, it is trivial to check that $k_1 = k_2$ and then the result holds.
2. If $k_i = (q - 1)\xi_i$, then

$$T_n(\xi_i) + (q - 1)\xi_i(D_q T_n)(\xi_i) = 0,$$

and from the definition of the q -difference operator we get

$$T_n(q\xi_i) = 0 \quad \text{for every } n \geq 3,$$

which is not possible since $\{T_n\}_n$ is an MOPS.

Lemma 3. *Let A_3 , B_2 and c_n be the polynomials defined in (25), (26) and (23), respectively. Suppose that B_2 has not a double zero and that no zero of B_2 is a root of $(D_q B_2)(x) = 0$. Then, there exists a parameter ξ such that $B_2(\xi) = A_3(\xi) = 0$. Furthermore, we have $c_1(\xi) \neq 0$, $c_1(q^{-1}\xi) \neq 0$ and $\pi(\xi) = 0$.*

Proof. Let us denote ξ_1 and ξ_2 the zeros of B_2 . If both ξ_i ($i = 1, 2$) are not zeros of A_3 , we can apply Lemma 1 to obtain two constants $k_1 \neq 0$ and $k_2 \neq 0$ such that

$$c_n(q^{-1}\xi_1) + k_1(D_q c_n)(q^{-1}\xi_1) = 0, \quad c_n(q^{-1}\xi_2) + k_2(D_q c_n)(q^{-1}\xi_2) = 0,$$

for all $n \geq 1$. Using Lemma 2 we get $k_1 = k_2$ as well as $\xi_1 = \xi_2$ in contradiction with the hypothesis of Lemma.

Let us denote ξ the common zero of B_2 and A_3 . If $B_2(\xi) = 0$, we have $(D_q B_2)(\xi) \neq 0$, and hence $c_1(\xi) \neq 0$. But we also obtain that $c_1(q^{-1}\xi) \neq 0$, because if $c_1(q^{-1}\xi) = 0$, then it should be $B_2(q^{-1}\xi) = (D_q B_2)(q^{-1}\xi) = 0$ which is not possible.

From (32) putting $n = 1$ and $x = \xi$ we get $\pi(\xi) = 0$.

Theorem 3. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of linear functionals. Suppose that B_2 defined in (26) has not a double zero and also that no zero of B_2 is a root of $(D_q B_2)(x) = 0$. Then,*

(i) *there exist a parameter ξ and polynomials \tilde{A} , π_1 with $\deg(\tilde{A}) \leq 2$ and $\deg(\pi_1) \leq 1$ such that*

$$\tilde{A}(x)\mathbf{u}_0 = \frac{\sigma_1\sigma_2}{t_1t_2}(x - \xi)\mathbf{u}_1, \quad (50)$$

$$\pi_1(x)\mathbf{u}_0 = \frac{\sigma_1\sigma_2}{t_1t_2}(x - \xi)D_q\mathbf{u}_1. \quad (51)$$

(ii) *if $\tilde{A}(\xi) = 0$, then $\pi_1(\xi) = 0$.*

(iii) \mathbf{u}_1 *is a q -classical linear functional verifying*

$$D_q[\tilde{A}(qx)\mathbf{u}_1] = \tilde{\psi}_1(x)\mathbf{u}_1, \quad (52)$$

where $\deg(\tilde{\psi}_1) = 1$.

Proof. (i) Let us denote ξ_1 and ξ_2 the zeros of B_2 . Using Lemma 3, at least one of them is also a zero of A_3 . Suppose that $A_3(\xi_1) = 0$. Using again Lemma 3 we obtain that $\pi(\xi_1) = 0$. Define

$$B_2(x) = (x - \xi_1)\tilde{\mathbf{B}}(x), \quad (53)$$

$$A_3(x) = (x - \xi_1)\tilde{\mathbf{A}}(x), \quad (54)$$

$$\pi(x) = (x - \xi_1)\pi_1(x). \quad (55)$$

Then we can divide both members of (32) by $x - \xi_1$ and we obtain

$$[n] \frac{P_n(x)}{p_n} \tilde{\mathbf{B}}(x) = \tilde{\mathbf{A}}(x)(D_q c_n)(q^{-1}x) + c_n(q^{-1}x)\pi_1(x) \quad (56)$$

for every $n \geq 1$.

From (31), we get

$$\pi(x)\mathbf{u}_0 = B_2(x)D_q\mathbf{u}_1, \quad (57)$$

and thus, using (53), (54) and (24)

$$\tilde{\mathbf{A}}(x)\mathbf{u}_0 = \tilde{\mathbf{B}}(x)\mathbf{u}_1, \quad (58)$$

if we normalize properly the linear functionals \mathbf{u}_0 and \mathbf{u}_1 , which was to be proved. Furthermore, from (55) and (57) it yields

$$\pi_1(x)\mathbf{u}_0 = \tilde{\mathbf{B}}(x)D_q\mathbf{u}_1 + M\delta_{\xi_1}, \quad (59)$$

$$\tilde{\mathbf{A}}(x)D_q\mathbf{u}_1 = \pi_1(x)\mathbf{u}_1 + K\delta_{\xi_1}. \quad (60)$$

Hence from (56) and (30) we deduce for $n \geq 1$

$$\begin{aligned} & (\tilde{\mathbf{A}}(x)(D_q c_n)(q^{-1}x) + c_n(q^{-1}x)\pi_1(x))\mathbf{u}_0 \\ &= \left(n\tilde{\mathbf{B}}(x) \frac{P_n(x)}{p_n} \right) \mathbf{u}_0 = \tilde{\mathbf{B}}(x)((D_q c_n)(q^{-1}x)\mathbf{u}_1 + c_n(q^{-1}x)D_q\mathbf{u}_1), \end{aligned}$$

i.e.,

$$(D_q c_n)(q^{-1}x)(\tilde{\mathbf{A}}(x)\mathbf{u}_0 - \tilde{\mathbf{B}}(x)\mathbf{u}_1) = c_n(q^{-1}x)(\tilde{\mathbf{B}}(x)D_q\mathbf{u}_1 - \pi_1(x)\mathbf{u}_0).$$

From (58) and (59) we have

$$M c_n(q^{-1}x) = 0 \quad \text{for every } n \geq 1.$$

Since $c_1(q^{-1}\xi_1) \neq 0$ we obtain $M = 0$ and this proves (51).

(ii) From the definition of $\tilde{\mathbf{A}}$ we have $\tilde{\mathbf{A}}(\xi_2) = 0$ and then, using Lemma 3 it follows that $\pi_1(\xi_2) = 0$, so part (ii) of the theorem is proved.

(iii) Finally, using (56) with $n = 1$

$$\tilde{\mathbf{A}}(x)(D_q c_1)(q^{-1}x) = \frac{P_1(x)}{p_1} \tilde{\mathbf{B}}(x) - c_1(q^{-1}x)\pi_1(x)$$

holds. Multiplying the first equation of (31) by \tilde{A} and using the above equation we get

$$\begin{aligned}\tilde{A}(x)\frac{P_1(x)}{p_1}\mathbf{u}_0 &= \tilde{A}(x)(D_q c_1)(q^{-1}x)\mathbf{u}_1 + \tilde{A}(x)c_1(q^{-1}x)D_q\mathbf{u}_1 \\ &= \left(\frac{P_1(x)}{p_1}\tilde{B}(x) - c_1(q^{-1}x)\pi_1(x)\right)\mathbf{u}_1 + \tilde{A}(x)c_1(q^{-1}x)D_q\mathbf{u}_1,\end{aligned}$$

whence

$$\frac{P_1(x)}{p_1}(\tilde{A}(x)\mathbf{u}_0 - \tilde{B}(x)\mathbf{u}_1) = c_1(q^{-1}x)(\tilde{A}(x)D_q\mathbf{u}_1 - \pi_1(x)\mathbf{u}_1).$$

From (50) and (60) we obtain $Kc_1(q^{-1}\xi_1) = 0$ and since $c_1(q^{-1}\xi_1) \neq 0$, then $K = 0$. Thus (60) reads as

$$\tilde{A}(x)D_q\mathbf{u}_1 = \pi_1(x)\mathbf{u}_1.$$

Now $D_q[\tilde{A}(qx)\mathbf{u}_1] = (D_q\tilde{A})(x)\mathbf{u}_1 + \tilde{A}(x)D_q\mathbf{u}_1 = ((D_q\tilde{A})(x) + \pi_1(x))\mathbf{u}_1 = \tilde{\psi}_1(x)\mathbf{u}_1$, being $\deg(\tilde{\psi}_1) \leq 1$. As in Theorem 2, we use that \mathbf{u}_1 is quasi-definite to conclude $\deg(\tilde{\psi}_1) = 1$, i.e., \mathbf{u}_1 is a q -classical linear functional. \square

Remark 1. If B_2 has a double zero ξ , from (39) we have

$$c_1(x) = \frac{\sigma_1}{t_1} \left(x - \frac{2\xi}{q+1} \right). \quad (61)$$

Since the sequence $\{Q_n\}_n$ defined in (11) is orthogonal with respect to the linear functional $\phi_0\mathbf{u}_0$ (see Proposition 2), it is convenient to write (18) for $n = 0$ using (17) and the definition of c_1 in (23) as

$$\frac{\phi_0(x)}{\langle \mathbf{u}_0, \phi_0(x) \rangle} \mathbf{u}_0 = -c_1(x)\mathbf{u}_1.$$

We can now use the definition of ϕ_0 given in (28) as well as (24) in order to obtain

$$\frac{\phi_1(x)B_2^2(x)}{\langle \mathbf{u}_0, \phi_0(x) \rangle} \mathbf{u}_1 = -c_1(x)\mathbf{u}_1.$$

Since \mathbf{u}_1 is a quasi-definite linear functional it follows that

$$\frac{\phi_1(x)B_2^2(x)}{\langle \mathbf{u}_0, \phi_0(x) \rangle} + c_1(x) = 0. \quad (62)$$

When $x = \xi$, the above expression can be written

$$\frac{(q-1)\xi\sigma_1}{(q+1)t_1} = 0, \quad (63)$$

according to (61) and $B_2(\zeta) = 0$, so ζ must be 0 and Theorem 2 can be applied in order to obtain that \mathbf{u}_0 is a q -classical linear functional.

6. Examples

In Section 5 we have proved that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a q -coherent pair of linear functionals, at least one of them has to be a q -classical linear functional. In this section we give the coherent pairs of positive-definite linear functionals when one of the functionals is the little q -Jacobi linear functional $\mathbf{u}^{(a,b)}$ defined as [23, Eq. (3.12.2)]

$$\langle \mathbf{u}^{(a,b)}, p \rangle = \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq)^k p(q^k),$$

$$0 < aq < 1, \quad b < q^{-1} \quad \text{for every } p \in \mathbb{P}, \quad (64)$$

where the q -shifted factorials $(c; q)_k$ are defined in (15) and when one of the linear forms is the little q -Laguerre/Wall linear functional $\mathbf{u}^{(a)}$ given in (14).

Concerning notations, let $(\mathbf{u}_0, \mathbf{u}_1)$ be a coherent pair of linear functionals and let B_2 be the polynomial defined in (26) with zeros ζ and η . These zeros can be complex, but in the computations below we shall assume they are real for the sake of simplicity. From the study done in the previous section, at least one of the functionals has to be a q -classical linear functional. More concretely, if $\eta = q\zeta$, then \mathbf{u}_0 is a q -classical linear functional and if $\eta \neq q\zeta$ and $\eta \neq \zeta$, then \mathbf{u}_1 is a q -classical linear functional.

6.1. Little q -Jacobi linear functional

Let $\mathbf{u}^{(a,b)}$ be the little q -Jacobi linear functional defined in (64). Let us consider $a = q^\alpha$, $b = q^\beta$ with $\alpha, \beta > -1$. In this situation the little q -Jacobi linear functional will be denoted by $\mathbf{u}^{(q^\alpha, q^\beta)} \equiv \mathbf{v}^{(\alpha, \beta)}$.

6.1.1. Case $\zeta = q\eta$

If $\eta = q\zeta$, then \mathbf{u}_0 is a q -classical linear functional. Suppose that $\mathbf{u}_0 = \mathbf{v}^{(\alpha, \beta)}$ is the little q -Jacobi linear functional. Then $\phi_0(x) = q^{\alpha+\beta+1}x(x - q^{-(\beta+1)})$ and from (37) and (38) we obtain $(x - \zeta)\mathbf{u}_1 = x(1 - q^{\beta+1}x)\mathbf{u}_0 = x(1 - q^{\beta+1}x)\mathbf{v}^{(\alpha, \beta)}$. Thus, $\mathbf{u}_1 = (x - \zeta)^{-1}x(1 - q^{\beta+1}x)\mathbf{v}^{(\alpha, \beta)} + M\delta_\zeta$. Since $x(1 - q^{\beta+1}x)\mathbf{v}^{(\alpha, \beta)} = \mathbf{v}^{(\alpha+1, \beta+1)}$ we get

$$\mathbf{u}_1 = (x - \zeta)^{-1}\mathbf{v}^{(\alpha+1, \beta+1)} + M\delta_\zeta, \quad M \geq 0, \quad \zeta \leq 0. \quad (65)$$

Let us prove that $(\mathbf{v}^{(\alpha, \beta)}, \mathbf{u}_1)$ is a q -coherent pair of linear functionals. If we denote $\{p_n(x; q^\alpha, q^\beta | q)\}_n$ the MOPS of little q -Jacobi associated with $\mathbf{v}^{(\alpha, \beta)}$ and $\{T_n\}_n$ the MOPS with respect to \mathbf{u}_1 , we have

$$\begin{aligned}
& \langle \mathbf{v}^{(\alpha+1, \beta+1)}, T_n(x) p_k(x; q^{\alpha+1}, q^{\beta+1} | q) \rangle \\
&= \langle x \mathbf{v}^{(\alpha)}, T_n(x) p_k(x; q^{\alpha+1}, q^{\beta+1} | q) \rangle \\
&= \langle (x - \xi) \mathbf{u}_1, T_n(x) p_k(x; q^{\alpha+1}, q^{\beta+1} | q) \rangle \\
&= \langle \mathbf{u}_1, T_n(x) p_k(x; q^{\alpha+1}, q^{\beta+1} | q) (x - \xi) \rangle = 0,
\end{aligned} \tag{66}$$

if $0 \leq k \leq n - 2$. Hence, for $n \geq 1$ with the notation $p_n(x; q^\alpha, q^\beta | q) \equiv p_n(x)$ we get

$$\begin{aligned}
T_n(x) &= p_n(x; q^{\alpha+1}, q^{\beta+1} | q) - \sigma_n p_{n-1}(x; q^{\alpha+1}, q^{\beta+1} | q) \\
&= \frac{(D_q p_{n+1})(x)}{[n+1]} - \sigma_n \frac{(D_q p_n)(x)}{[n]},
\end{aligned}$$

where the last equality is a consequence of

$$\frac{(D_q p_{n+1})(x)}{[n+1]} = p_n(x; q^{\alpha+1}, q^{\beta+1} | q), \quad n \geq 0. \tag{67}$$

6.1.2. Case $\eta \neq \xi$ and $\eta \neq q\xi$

From Theorem 3, \mathbf{u}_1 is a q -classical linear functional. Suppose that $\mathbf{u}_1 \equiv \mathbf{v}^{(\alpha, \beta)}$ is the little q -Jacobi linear functional. We shall denote $\{P_n\}_n$ the MOPS with respect to \mathbf{u}_0 . Since $\phi_1(x) = q^{\alpha+\beta+1} x(x - q^{-(\beta+1)})$ from (50) and (52) we have $\phi_1(q^{-1}x) \mathbf{u}_0 = (x - \xi) \mathbf{v}^{(\alpha, \beta)}$ and we can consider the following cases:

(1) If $\alpha, \beta > 0$, since $\phi_1(q^{-1}x) \mathbf{v}^{(\alpha-1, \beta-1)} = \mathbf{v}^{(\alpha, \beta)}$ then $(x - \xi) \mathbf{v}^{(\alpha, \beta)} = (x - \xi) x \mathbf{v}^{(\alpha-1, \beta-1)}$, so

$$\mathbf{u}_0 = (x - \xi) \mathbf{v}^{(\alpha-1, \beta-1)} + M \delta_0, \quad M \geq 0. \tag{68}$$

From (51) we have

$$\phi_1(x) \pi_1(x) \mathbf{u}_0 = (x - \xi) \pi_1(x) \mathbf{v}^{(\alpha, \beta)} = (x - \xi) \pi_1(x) \phi_1(x) \mathbf{v}^{(\alpha-1, \beta-1)}.$$

Using (68) and the above equation we get

$$\mathbf{u}_0 = (x - \xi) \mathbf{v}^{(\alpha-1, \beta-1)}, \quad \xi \leq 0. \tag{69}$$

To check that $(\mathbf{u}_0, \mathbf{v}^{(\alpha, \beta)})$ defines a q -coherent pair we compute

$$\begin{aligned}
\langle \mathbf{u}_0, P_k(x) p_{n+1}(x; q^{\alpha-1}, q^{\beta-1} | q) \rangle &= \langle \mathbf{v}^{(\alpha-1)}, P_k(x) p_{n+1}(x; q^{\alpha-1}, q^{\beta-1} | q) (x - \xi) \rangle \\
&= 0 \quad \text{for } 0 \leq k \leq n - 1.
\end{aligned}$$

Thus, we have

$$p_{n+1}(x; q^{\alpha-1}, q^{\beta-1} | q) = P_{n+1}(x) - \sigma_n P_n(x), \quad n \geq 1.$$

Applying the D_q -operator to the above relation and using (67) we obtain

$$p_n(x; q^\alpha, q^\beta | q) = \frac{(D_q p_{n+1})(x)}{[n+1]} - \frac{q^n - 1}{q^{n+1} - 1} \sigma_n \frac{(D_q p_n)(x)}{[n]}, \quad n \geq 1.$$

(2) If $\alpha = 0$ and $\beta > 0$ since $\pi_1(x) = (q^\beta - 1)x/(q(1 - q))$ then, from Lemma 3, $\zeta = 0$ and we obtain

$$\mathbf{u}_0 = \mathbf{v}^{(0, \beta-1)} + M\delta_0, \quad M \geq 0. \quad (70)$$

Using the relation

$$p_{n+1}(0; 1, q^{\beta-1} | q) + A(n)p_n(0; 1, q^{\beta-1} | q) = 0,$$

$$A(n) = \frac{q^n(q^{n+1} - 1)(q^{\beta+n} - 1)}{(q^{\beta+2n} - 1)(q^{\beta+2n+1} - 1)},$$

and since

$$\begin{aligned} & \langle \mathbf{u}_0, (p_{n+1}(x; 1, q^{\beta-1} | q) + A(n)p_n(x; 1, q^{\beta-1} | q))P_k(x) \rangle \\ &= \langle \mathbf{v}^{(0, \beta-1)}, (p_{n+1}(x; 1, q^{\beta-1} | q) + A(n)p_n(x; 1, q^{\beta-1} | q))P_k(x) \rangle \\ & \quad + M(p_{n+1}(0; 1, q^{\beta-1} | q) + A(n)p_n(0; 1, q^{\beta-1} | q)) \\ &= \langle \mathbf{v}^{(0, \beta)}, (p_{n+1}(x; 1, q^{\beta-1} | q) + A(n)p_n(x; 1, q^{\beta-1} | q))P_k(x) \rangle \\ &= 0 \quad \text{for } 0 \leq k \leq n-1, \end{aligned}$$

then we can write

$$p_{n+1}(x; 1, q^{\beta-1} | q) + A(n)p_n(x; 1, q^{\beta-1} | q) = P_{n+1}(x) - \sigma_n P_n(x).$$

Applying the D_q -operator to the above equation and taking into account

$$p_n(x; 1, q^\beta | q) = \frac{(D_q \tilde{p}_{n+1})(x)}{[n+1]} + \frac{A(n)}{[n+1]} (D_q \tilde{p}_n)(x), \quad (71)$$

where $\tilde{p}_n(x) = p_n(x; 1, q^{\beta-1} | q)$, we obtain the q -coherence of the pairs $(\mathbf{v}^{(0, \beta-1)} + M\delta_0, \mathbf{v}^{(0, \beta)})$, $M \geq 0$.

(3) If $\alpha > 0$ and $\beta = 0$, then $\pi_1(x) = (q^\alpha - 1)(1 - x)/(q(1 - q))$. So from Lemma 3 we obtain $\zeta = 1$ and

$$\mathbf{u}_0 = \mathbf{v}^{(\alpha-1, 0)} + M\delta_1, \quad M \geq 0. \quad (72)$$

Let us prove that $(\mathbf{u}_0, \mathbf{v}^{(\alpha, \beta)})$ is a q -coherent pair of linear functionals. Since

$$\begin{aligned} & \langle \mathbf{u}_0, (p_{n+1}(x; q^{\alpha-1}, 1 | q) + B(n)p_n(x; q^{\alpha-1}, 1 | q))P_k(x) \rangle \\ &= \langle \mathbf{v}^{(\alpha-1, 0)}, (p_{n+1}(x; q^{\alpha-1}, 1 | q) + B(n)p_n(x; q^{\alpha-1}, 1 | q))P_k(x) \rangle \\ & \quad + M(p_{n+1}(1; q^{\alpha-1}, 1 | q) + B(n)p_n(1; q^{\alpha-1}, 1 | q)) \\ &= \langle \mathbf{v}^{(\alpha-1, 0)}, (p_{n+1}(x; q^{\alpha-1}, 1 | q) + B(n)p_n(x; q^{\alpha-1}, 1 | q))P_k(x) \rangle \\ &= 0 \quad \text{for } 0 \leq k \leq n-1, \end{aligned}$$

using the relation

$$p_{n+1}(1; q^{\alpha-1}, 1 | q) + B(n)p_n(1; q^{\alpha-1}, 1 | q) = 0,$$

$$B(n) = \frac{q^{2n+\alpha}(q^{n+1} - 1)(q^{n+\alpha} - 1)}{(q^{2n+\alpha} - 1)(q^{2n+\alpha+1} - 1)},$$

then we can write

$$p_{n+1}(x; q^{x-1}, 1|q) + B(n)p_n(x; q^{x-1}, 1|q) = P_{n+1}(x) - \sigma_n P_n(x).$$

Applying the D_q -operator and taking into account

$$p_n(x; q^x, 1|q) = \frac{(D_q \hat{p}_{n+1})(x)}{[n+1]} + \frac{B(n)}{[n+1]} (D_q \hat{p}_n)(x), \quad (73)$$

where $\hat{p}_n(x) \equiv p_n(x; q^{x-1}, 1|q)$, we obtain the q -coherence of the pairs $(\mathbf{v}^{(\alpha-1,0)} + M\delta_1, \mathbf{v}^{(\alpha,0)})$, $M \geq 0$.

6.2. Little q -Laguerre/Wall linear functional

The little q -Laguerre/Wall linear functional defined in (14) is a particular case of little q -Jacobi linear functional defined in (64) when $b = 0$ [8, p. 117]. Let $\mathbf{w}^{(\alpha)}$ be the little q -Laguerre/Wall linear functional with parameter $a = q^\alpha$, $\alpha > -1$. By using similar arguments as in Section 6.1 we obtain:

Theorem 4. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a q -coherent pair of positive-definite linear functionals.*

1. *If $\mathbf{u}_0 = \mathbf{w}^{(\alpha)}$ is the little q -Laguerre/Wall linear functional, then*

$$\mathbf{u}_1 = \frac{1}{(x - \xi)} \mathbf{w}^{(\alpha+1)} + M\delta_\xi \quad \text{with } \xi \leq 0, M \geq 0.$$

2. *If $\mathbf{u}_1 = \mathbf{w}^{(\alpha)}$ is the little q -Laguerre/Wall linear functional, and*

(a) *if $\alpha > 0$, then $\mathbf{u}_0 = (x - \xi)\mathbf{w}^{(\alpha-1)}$ with $\xi \leq 0$,*

(b) *if $\alpha = 0$, then $\mathbf{u}_0 = \mathbf{w}^{(0)} + M\delta_0$ with $M \geq 0$,*

(c) *if $-1 < \alpha < 0$, then $\mathbf{u}_0 = \mathbf{u}_1$.*

7. The limit transitions

In [10], Meijer proved that if $(\mathbf{u}_0, \mathbf{u}_1)$ is a coherent pair of linear functionals (2), then at least one of the functionals has to be a classical continuous one, i.e., Hermite, Laguerre or Jacobi linear functional. He gave the classification of coherent pairs of linear functionals which can be represented by distribution functions. In this section, limit transitions from little q -Jacobi linear functional to Jacobi linear functional and from little q -Laguerre linear functional to Laguerre linear functional are obtained. From Sections 5 and 6 and using these limit relations we recover the classification of all coherent pairs of positive definite linear functionals in case of Jacobi and Laguerre linear functionals. Therefore, as stated in [1, p. 13] and [10, p. 333], there exists no coherent pair of linear functionals if one of them is the Hermite linear functional. Thus, we recover the classification of all positive definite linear functionals given in [10].

Let us consider the little q -Jacobi MOPS $\{p_n(x, q^\alpha, q^\beta | q)\}_n$. The polynomials

$$2^n p_n\left(\frac{1-x}{2}; q^\alpha, q^\beta | q\right)$$

are orthogonal with respect to a linear functional \mathbf{u}_q which satisfies an equation as (10) with

$$\begin{aligned}\phi(x) &= q^\alpha(1-x)(2+q^{1+\beta}(x-1)), \\ \psi(x) &= \frac{1-2q^{1+\alpha}-q^{2+\alpha+\beta}(x-1)+x}{(q-1)q}.\end{aligned}\tag{74}$$

If we take the limit when $q \uparrow 1$ in (10) we obtain

$$\mathcal{D}[(1-x^2)\mathbf{v}] = (\beta - \alpha - x(2 + \alpha + \beta))\mathbf{v},\tag{75}$$

which is the distributional equation of the Jacobi linear functional $\mathbf{v} = \mathbf{v}_J^{(\alpha, \beta)}$ given by

$$\langle \mathbf{v}_J^{(\alpha, \beta)}, p \rangle = \int_{-1}^1 p(x) \frac{(1-x)^\alpha (1+x)^\beta \Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} dx \quad \forall p \in \mathbb{P}.\tag{76}$$

Therefore, $\mathbf{u}_q \rightarrow \mathbf{v}_J^{(\alpha, \beta)}$ when $q \uparrow 1$.

By applying the limit when $q \uparrow 1$ in definition (26) of the polynomial B_2 , it converges to a new polynomial B of degree 2 which has two zeros, ζ and ϑ . If we take the limit when $q \uparrow 1$ and using the above limit transition we obtain (compare with [10]):

Theorem 5. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a coherent pair of positive-definite linear functionals.*

1. *If $\mathbf{u}_0 = \mathbf{v}_J^{(\alpha, \beta)}$ is the Jacobi linear functional, then*

$$\mathbf{u}_1 = \frac{1}{|x - \zeta|} \mathbf{v}_J^{(\alpha+1, \beta+1)} + M \delta_\zeta \quad \text{with } \alpha > -1, \beta > -1, |\zeta| \geq 1, M \geq 0.$$

2. *If $\mathbf{u}_1 = \mathbf{v}_J^{(\alpha, \beta)}$ is the Jacobi linear functional, and*

(a) *if $\alpha > 0$ and $\beta > 0$, then $\mathbf{u}_0 = |x - \zeta| \mathbf{v}_J^{(\alpha-1, \beta-1)}$ with $|\zeta| > 1$,*

(b) *if $\alpha = 0$ and $\beta > 0$, then $\mathbf{u}_0 = \mathbf{v}_J^{(0, \beta-1)} + M \delta_1$ with $M \geq 0$,*

(c) *if $\alpha > 0$ and $\beta = 0$, then $\mathbf{u}_0 = \mathbf{v}_J^{(\alpha-1, 0)} + M \delta_{-1}$ with $M \geq 0$.*

Let us consider $p_n(x; q^\alpha | q)$ the little q -Laguerre/Wall MOPS with parameter q^α . The polynomials $p_n((1-q)x; q^\alpha | q)$ are orthogonal with respect to a linear functional \mathbf{u}_q which satisfies an equation of type (10) with

$$\phi(x) = q^\alpha x, \quad \psi(x) = \frac{-1 + q^{1+\alpha} + (1-q)x}{(q-1)q}.\tag{77}$$

If we take the limit when $q \uparrow 1$ in (10) we obtain

$$\mathcal{D}[x\mathbf{v}] = (\alpha + 1 - x)\mathbf{v}, \quad (78)$$

which is the distributional equation of the Laguerre linear functional $\mathbf{v} = \mathbf{v}_L^{(\alpha)}$ given by

$$\left\langle \mathbf{v}_L^{(\alpha)}, p \right\rangle = \int_0^\infty p(x) \frac{x^\alpha e^{-x}}{\Gamma(\alpha + 1)} dx \quad \forall p \in \mathbb{P}. \quad (79)$$

Therefore, $\mathbf{u}_q \rightarrow \mathbf{v}_L^{(\alpha)}$ when $q \uparrow 1$. From this limit transition and Theorem 4 we obtain:

Theorem 6. *Let $(\mathbf{u}_0, \mathbf{u}_1)$ be a coherent pair of positive-definite linear functionals.*

1. *If $\mathbf{u}_0 = \mathbf{v}_L^{(\alpha)}$ is the Laguerre linear functional, then*

$$\mathbf{u}_1 = \frac{1}{x - \zeta} \mathbf{v}_L^{(\alpha+1)} + M\delta_\zeta \quad \text{with } \alpha > -1, \zeta \leq 0, M \geq 0.$$

2. *If $\mathbf{u}_1 = \mathbf{v}_L^{(\alpha)}$ is the Laguerre linear functional, and*

(a) *if $\alpha > 0$, then $\mathbf{u}_0 = (x - \zeta)\mathbf{v}_L^{(\alpha-1)}$ with $\zeta < 0$,*

(b) *if $\alpha = 0$, then $\mathbf{u}_0 = \mathbf{v}_L^{(0)} + M\delta_0$ with $M \geq 0$,*

(c) *if $-1 < \alpha < 0$, then $\mathbf{u}_0 = \mathbf{u}_1$.*

Remark. Finally, let us mention that if we change the q -difference operator by a difference operator on non-uniform lattices [24], we guess that an extension of this theory could be done.

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