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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-98-75

# THE CORE OF A CLASS OF NON-ATOMIC GAMES WHICH ARISE IN ECONOMIC APPLICATIONS*

Ezra Einy, Diego Moreno and Benyamin Shitovitz†

Abstract

We prove a representation theorem for the core of a non-atomic game of the form $v = f \circ \mu$, where $\mu$ is a finite dimensional vector of non-atomic measures and $f$ is a non-decreasing continuous concave function on the range of $\mu$. The theorem is stated in terms of the subgradients of the function $f$. As a consequence of this theorem we show that the game $v$ is balanced (i. e., has a non-empty core) iff the function $f$ is homogeneous of degree one along the diagonal of the range of $\mu$, and it is totally balanced (i.e., every subgame of $v$ has a non-empty core) iff the function $f$ is homogeneous of degree one in the entire range of $\mu$. We also apply our results to some non-atomic games which occur in economic applications.

Keywords: Non-atomic games, market games, core.

† Einy, Department of Economics, Ben-Grunion University of the Negev, Beer Sheva, Israel 84105; Moreno, Departamento de Economía, Universidad Carlos III de Madrid; Shitovitz, Department of Economics, University of Haifa, Israel 31905.

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## §1 - Introduction

One of the fundamental game theoretic concepts is the core of a coalitional game. It is the set of all feasible outcomes that no player or group of participants can improve upon by acting for themselves. The core of coalitional games with a finite or infinite set of players was investigated in many works (for a comprehensive survey see Kannai (1992)). In this work we study the core of the class of non-atomic games which can be represented in the form $v = f \circ \mu$ where $\mu$ is a finite dimensional vector of non-atomic measures and the function $f$ is non-decreasing, continuous, and concave on the range of $\mu$. Such games occur in several economic applications. For example, any non-atomic glove market game and every non-atomic linear production game of Billera and Raanan (1981) are of this form and so is any Aumann-Shapley-Shubik market game of an atomless economy with a finite number of types (see Section 4). We can also view these games as large production games where $\mu$ represents the distribution of production factors among the owners and $f$ is the production function.

Our main result is a representation theorem for the core of a non-atomic game of the above-mentioned form which is stated in terms of the subgradients of the function $f$ (see Theorem A). As a consequence of the representation theorem we show that a game of the above-mentioned form is balanced (i.e., it has a non-empty core) iff the function $f$ is homogeneous of degree one along the diagonal of the range of $\mu$. The game is totally balanced iff the function $f$ is homogeneous of degree one in the entire range of $\mu$.

In the last section of the paper (see Section 4) we apply our main results to some non-atomic games which occur in economic applications.

## §2 - Preliminaries

In this section we define some basic notions which are relevant to our work and prove a preliminary result which we use in the sequel.

Let $(T,\Sigma)$ be a measurable space, i.e., $T$ is a set and $\Sigma$ is a $\sigma$-field of subsets of $T$. We refer to the members of $T$ as *players* and to those of $\Sigma$ as *coalitions*. A *coalitional game*, or simply a *game* on $(T,\Sigma)$, is a function $v:\Sigma \to \Re$ with $v(\emptyset)=0$. A game $v$ on $(T,\Sigma)$ is *continuous at* $S \in \Sigma$ if for all sequences $\{S_n\}_{n=1}^{\infty}$ of coalitions such that $S_{n+1} \supseteq S_n$ and $\bigcup_{n=1}^{\infty} S_n = S$, and all sequences $\{S_n\}_{n=1}^{\infty}$ of coalitions such that $S_{n+1} \subseteq S_n$ and $\bigcap_{n=1}^{\infty} S_n = S$, we have $v(S_n) \to v(S)$.

A *payoff measure* in a game $v$ is a bounded finitely additive measure $\lambda:\Sigma \to \Re$ which satisfies $\lambda(T) \le v(T)$. The *core* of a game $v$, denoted by *Core(v)*, is the set of all payoff measures $\lambda$ such that $\lambda(S) \ge v(S)$ for all $S \in \Sigma$. As observed by Schmeidler (see the first part of the proof of Theorem 3.2 in Schmeilder (1972)), if $v$ is a continuous game at $T$, then every member of *Core(v)* is countably additive.

We denote by $ba = ba(T,\Sigma)$ the Banach space of all bounded finitely additive measures on $(T,\Sigma)$ with the variation norm. If $\mu$ is a countably additive measure on $(T,\Sigma)$ we denote by $ba(\mu) = ba(T,\Sigma,\mu)$ the subspace of $ba$ which consists of all bounded finitely additive measures on $(T,\Sigma)$ which vanish on the $\mu$-measure zero sets in $\Sigma$. The subspace of $ba$ which consists of all bounded countably additive measures on $(T,\Sigma)$ is denoted by $ca = ca(T,\Sigma)$. If $\mu$ is a measure in $ca$ then $ca(\mu) = ca(T,\Sigma,\mu)$ denotes the set of all members of $ca$ which are absolutely continuous with respect to $\mu$. If $A$ is a subset of an ordered vector space we denote by $A_+$ the set of all non-negative members of $A$.

Let $K$ be a convex subset of a Euclidean space and let $f: K \to \Re$ be a concave function. A vector $p$ is a *subgradient* of $f$ at $x \in K$ if $f(y) - f(x) \le p \cdot (y - x)$ for all $y \in K$. Note that the function $f$ is differentiable at a point $x$ in the relative interior of $K$ iff it has a unique subgradient at $x$ which, in this case, coincides with the gradient vector. The set of all subgradients of $f$ at $x$ will be denoted by $\partial f(x)$. It is well known that if $x$ is a point in the relative interior of $K$ then $\partial f(x) \ne \emptyset$ (see, for example, page 23 in Holmes (1975)). A function $f$ defined on a set $A \subseteq \Re^m$ is called non-decreasing if for every $x, y \in A$ we have $x \ge y$ implies $f(x) \ge f(y)$ (for two vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in $\Re^m$ the notation $x \ge y$ means that $x_i \ge y_i$ for all $1 \le i \le m$).

The following proposition will be useful in the sequel.

Proposition 2.1

*Let $K$ be a non-empty compact convex subset of $\Re^m_+$ such that $0 \in K$ and let $f: K \to \Re$ be a continuous, non-decreasing, and concave function. Then for every $x \in K \cap int\ \Re^m_+$ there exists $p \in \partial f(x)$ such that $p \ge 0$.*

Proof

For every $y \in \Re^m_+$ let $\bar{f}(y) = max\{f(x) \mid x \in K,\ x \le y\}$. Then it is easy to check that $\bar{f}$ is non-decreasing, concave, and continuous on $\Re^m_+$. Since $f$ is non-decreasing, we have $\bar{f}(x) = f(x)$ for every $x \in K$. Therefore for every $x \in K$ we have $\partial \bar{f}(x) \subseteq \partial f(x)$. Now $\bar{f}$ is concave on all $\Re^m_+$. Therefore $\partial \bar{f}(y) \ne \emptyset$ for every $y \in int\ \Re_+^m$. Since $\bar{f}$ is non-decreasing on $\Re_+^m$, for every $y \in \Re_+^m$ any subgradient of $\bar{f}$ at $y$ is non-negative (i.e., it has non-negative components). Let $x \in K \cap int\ \Re_+^m$. As $\partial \bar{f}(x) \subseteq \partial f(x)$, we obtain that $\partial f(x)$ contains a non-negative vector.

§3 - Characterization of the Core of a Class of Non-Atomic Games

In this section we state and prove a representation theorem for the core of a game $v$ of the form $v = f \circ \mu$ where $\mu$ is a finite dimensional vector of non-atomic measures in $ca_+$ and $f$ is a non-decreasing, continuous, and concave function on the range of $\mu$. We also use this theorem to characterize the balanced and totally balanced games of this form.

If $\mu = (\mu_1, \ldots, \mu_m)$ is a vector of a measure in $ca$ we denote by $R(\mu)$ the range of $\mu$.

We are now ready to state and prove the main result of our paper.

Theorem A

*Let $\mu = (\mu_1, \ldots \mu_m)$ be a vector of non-trivial non-atomic measures in $ca_+$. Assume that $f: R(\mu) \to \Re_+$ is a non-decreasing continuous concave function such that $f(0) = 0$. Then $\partial f(\mu(T)) \neq \emptyset$ and the core of the game $v = f \circ \mu$ is given by*

$$Core(v) = \{p \cdot \mu \mid p \in \partial f(\mu(T)) \text{ and } p \cdot \mu(T) = f(\mu(T))\}.$$

*In particular, $Core(v) \neq \emptyset$ iff there exists $p \in \partial f(\mu(T))$ such that $p \cdot \mu(T) = f(\mu(T))$.*

Proof

The fact that $\partial f(\mu(T)) \neq \varnothing$ follows from Proposition 2.1.

Let $M(v) = \{p \cdot \mu \mid p \in \partial f(\mu(T)) \text{ and } p \cdot \mu(T) = f(\mu(T))\}$.

We first show that $M(v) \subseteq Core(v)$. Let $\lambda \in M(v)$. Then there exists $p \in \partial f(\mu(T))$ such that $p \cdot \mu(T) = f(\mu(T))$ and $\lambda = p \cdot \mu$. Let $S \in \Sigma$. Then

$$\lambda(S) = \lambda(T) - \lambda(T \setminus S) = f(\mu(T)) - p \cdot \mu(T \setminus S) \geq f(\mu(S)) = v(S).$$

Thus, $\lambda \in Core(v)$.

We now show that $Core(v) \subseteq M(v)$. We split the proof into several steps.

Step 1: Let $\lambda \in Core(v)$. We show that $\lambda$ is a non-atomic measure in $ca_+$.

Since $f$ is continuous on $R(\mu)$, the game $v$ is continuous $(T, \Sigma)$. Therefore $Core(v) \subset ca_+$ and thus $\lambda \in ca_+$. We show that $\lambda$ is non-atomic. Assume, on the contrary, that there exists a coalition $A \in \Sigma$ which is an atom of $\lambda$. Then $\lambda(A) > 0$.

Since $f$ is continuous on $R(\mu)$, there exists a natural number $n$ such that

(2.1) $f(\mu(T)) - f(\mu(T) - \frac{1}{n}\mu(A)) < \lambda(A)$[1]

By Lyapunov's convexity theorem, there exists a partition $A_1, \dots, A_n$ of $A$ such that $\mu(A_i) = \frac{1}{n}\mu(A)$ for every $1 \leq i \leq n$. Since $A$ is an atom of $\lambda$, there exists $1 \leq i \leq n$ such that $\lambda(A_i) = \lambda(A)$. Now $\lambda \in Core(v)$. Therefore

$$\lambda(A) = \lambda(A_i) = \lambda(T) - \lambda(T \setminus A_i) \leq f(\mu(T)) - f(\mu(T \setminus A_i)) = f(\mu(T)) - f(\mu(T) - \frac{1}{n}\mu(A)).$$

But this contradicts (2.1).

[1] Note that $\mu(T) - \frac{1}{n}\mu(A) = (1 - \frac{1}{n})\mu(T) + \frac{1}{n}\mu(T \setminus A)$ is in $R(\mu)$ by Lyapunov's theorem.

Step 2: Let $\lambda \in Core(v)$. We will show that for each $S \in \Sigma$ there exists $p \in \partial f(\mu(T))$ such that $\lambda(S) \leq p \cdot \mu(S)$.

Let $S \in \Sigma$. Since $\mu_1, \ldots, \mu_m$ and $\lambda$ are non-atomic, for every natural number $n > 1$ there exists a coalition $S_n \in \Sigma$ such that $\mu(S_n) = \frac{1}{n}\mu(S)$ and $\lambda(S_n) = \frac{1}{n}\lambda(S)$. By Proposition 2.1, for every $n$ there exists $p_n \in \partial f(\mu(T \setminus S_n))$ such that $p_n \geq 0$. We first show that the sequence $\{p_n\}_{n=2}^{\infty}$ is bounded. For every $n$ we have

$$0 = f(0) \leq f(\mu(T \setminus S_n)) + p_n \cdot (\mu(S_n) - \mu(T)) \leq \lambda(T \setminus S_n) + \frac{1}{n} p_n \cdot \mu(S) - p_n \cdot \mu(T).$$

Therefore

$$p_n \cdot \mu(T) \leq \lambda(T) - \frac{1}{n}\lambda(S) + \frac{1}{n} p_n \cdot \mu(S).$$

Since $p_n \geq 0$, $p_n \cdot \mu(S) \leq p_n \cdot \mu(T)$. Therefore

$$(1 - \frac{1}{n}) p_n \cdot \mu(T) \leq \lambda(T) - \frac{1}{n}\lambda(S) \leq \lambda(T).$$

As $\mu(T) >> 0$ (i.e., every component of $\mu(T)$ is positive), we obtain that the sequence $\{p_n\}_{n=2}^{\infty}$ is bounded and therefore it has a convergent subsequence which converges to a vector $p \in \Re_+^m$. It is clear that $p \in \partial f(\mu(T))$. We will show that $\lambda(S) \leq p \cdot \mu(S)$. Indeed, for every $n$ we have

$$f(\mu(T)) \leq f(\mu(T \setminus S_n)) + p_n \cdot \mu(S_n) \leq \lambda(T \setminus S_n) + p_n \cdot \mu(S_n).$$

As $\lambda(T) = f(\mu(T))$, we obtain

$$\frac{1}{n}\lambda(S) = \lambda(S_n) \leq p_n \cdot \mu(S_n) \leq \frac{1}{n} p_n \cdot \mu(S).$$

Thus $\lambda(S) \leq p_n \cdot \mu(S)$ for every $n$. Therefore $\lambda(S) \leq p \cdot \mu(S)$.

Step 3: We show that the order of the quantifiers in Step 2 can be reversed, that is, if $\lambda \in Core(v)$ there exists $p \in \partial f(\mu(T))$ such that $\lambda(S) \leq p \cdot \mu(S)$ for all $S \in \Sigma$ .

Let $\sigma = \sum_{i=1}^{m} \mu_i$ . Then $\sigma$ is a non-atomic measure in $ca_+$. Let $B_+$ be the positive unit ball of $L_\infty(T,\Sigma,\sigma)$. Then $B_+$ is a weak*-compact convex subset of $L_\infty(T,\Sigma,\sigma)$. It is also easy to check that $\partial f(\mu(T))$ is a (non-empty) convex compact subset of $\Re^m$. Define a function $H$ on $\partial f(\mu(T)) \times B_+$ by

$$H(p,g) = p \cdot \int_T g\, d\mu - \int_T g\, d\lambda \ .$$

Since $\lambda$ and $\mu_1, \ldots \mu_m$ are absolutely continuous with respect to $\sigma$ ( $\lambda$ is absolutely continuous with respect to $\sigma$ by Step 2), by using the Radon-Nikodym Theorem and the fact that the weak*-topology on $B_+$ is metrizable, it is straightforward to check that the function $H$ is well defined and continuous on $\partial f(\mu(T)) \times B_+$. It is also easy to see that $H$ is affine in each of its variables separately. Thus the sets $\partial f(\mu(T))$, $B_+$, and the function $H$ satisfy the assumptions of Sion's minmax theorem (see Sion (1958)), and therefore

$$(2.2) \quad \min_{g \in B_+} \max_{p \in \partial f(\mu(T))} H(p,g) = \max_{p \in \partial f(\mu(T))} \min_{g \in B_+} H(p,g) .$$

Define now a function $F$ on $B_+$ by

$$F(g) = \max_{p \in \partial f(\mu(T))} H(p,g) .$$

Then $F$ is weak*-continuous on $B_+$ (see, for example, Lemma 2.2, page 89 in Rosenmuller (1981)). By Step 2, for every $S \in \Sigma$ we have $F(1_S) \geq 0$ (where $1_S$ denotes the characteristic function of S). Since $\sigma$ is non-atomic on $(T,\Sigma)$, the characteristic functions are weak*-dense in $B_+$ (see, for example, Lemma 3, p. 106 in

Holmes (1975) or Proposition 22.4 in Aumann and Shapley (1974)). Therefore by the continuity of $F$, we have $F(g) \geq 0$ for all $g \in \boldsymbol{B}_+$. Hence, $\min_{g \in \boldsymbol{B}_+} F(g) \geq 0$ and thus by (2.2), $\max_{p \in \partial f(\mu(T))} \min_{g \in \boldsymbol{B}_+} H(p,g) \geq 0$. Therefore there exists $p \in \partial f(\mu(T))$ such that $H(p,g) \geq 0$ for all $g \in \boldsymbol{B}_+$. In particular, $H(p,1_S) \geq 0$ for all $S \in \Sigma$. Thus, $\lambda(S) \leq p \cdot \mu(S)$ for all $S \in \Sigma$.

Now $\lambda$ and $p \cdot \mu$ are two measures in $ca$ such that $\lambda(S) \leq p \cdot \mu(S)$ for every $S \in \Sigma$ and $\lambda(T) = p \cdot \mu(T)$. Therefore we must have $\lambda = p \cdot \mu$. Thus $\lambda \in M(v)$.

Q.E.D.

The following remark is useful in applications of Theorem A.

Remark 3.1

Let $\bar{f}: \Re^m_+ \to \Re$ be an extension of the function $f$ of Theorem A (i.e., $\bar{f}(x) = f(x)$ for every $x \in R(\mu)$) which is non-decreasing, continuous, and concave on $\Re^m_+$ (such an extension always exists as shown in the proof of Proposition 2.1). Then since $\partial \bar{f}(x) \subseteq \partial f(x)$ for every $x \in R(\mu)$, exactly the same proof of that of Theorem A yields that the core of the game $v = f \circ \mu$ is given by

$$Core(v) = \left\{ p \cdot \mu \,\middle|\, p \in \partial \bar{f}(\mu(T)) \text{ and } p \cdot \mu(T) = f(\mu(T)) \right\}$$

If $f$ is a function which is defined on a neighborhood of point $x \in \Re^m$ and differentiable at $x$ we denote by $\nabla f(x)$ the gradient of $f$ at $x$.

Corollary 3.2

*Let $(\mu_1, \ldots, \mu_m)$ be a vector of non-trivial non-atomic measures in $ca_+$. Assume that $f: \Re^m_+ \to \Re$ is continuous non-decreasing concave function which is differentiable at*

*$\mu(T)$ and satisfies $f(0)=0$. Then the core of the game $v=f\circ\mu$ is non-empty iff*

$\nabla f(\mu(T))\cdot\mu(T)=f(\mu(T))$. Moreover, if $Core(v)\neq\emptyset$ then $Core(v)=\{\nabla f(\mu(T))\cdot\mu\}$.

A game $v$ on $(T,\Sigma)$ is called *balanced* if it has a non-empty core.

The following theorem shows that if $\mu$ is a finite dimensional vector of non-atomic measures in $ca_+$ and $f$ is a continuous non-decreasing and concave function on $R(\mu)$ with $f(0)=0$ then balancedness of the game $v=f\circ\mu$ is equivalent to homogeneity of degree one of $f$ along the diagonal of $R(\mu)$.

Theorem 3.3

*Let $\mu$ be a finite dimensional vector of non-trivial non-atomic measures in $ca_+$. Assume that $f:R(\mu)\to\Re$ is a continuous non-decreasing concave function which satisfies $f(0)=0$. Then the game $v=f\circ\mu$ is balanced iff for every $0\le\alpha\le 1$ we have $f(\alpha\mu(T))=\alpha f(\mu(T))$ (i.e., $f$ is homogeneous of degree one along the diagonal of $R(\mu)$).*

Proof

We first assume that the game $v=f\circ\mu$ is balanced and show that $f$ is homogeneous of degree one along the diagonal of $R(\mu)$. Let $0\le\alpha\le 1$. Since $f$ is concave on $R(\mu)$, $f(\alpha\mu(T))\ge\alpha f(\mu(T))$. We show that $f(\alpha\mu(T))\le\alpha f(\mu(T))$. Indeed, let $\lambda\in Core(v)$. By Lyapunov's convexity theorem, there exists $S\in\Sigma$ such that $\mu(S)=\alpha\,\mu(T)$ and $\lambda(S)=\alpha\,\lambda(T)$. As $\lambda\in Core(v)$,

$$\alpha f(\mu(T))=\alpha\,\lambda(T)=\lambda(S)\ge f(\mu(S))=f(\alpha\,\mu(T)).$$

Hence, $f(\alpha\,\mu(T))=\alpha f(\mu(T))$.

We now assume that $f$ is homogeneous of degree one along the diagonal of $R(\mu)$ and show that the game $v=f\circ\mu$ is balanced. By Theorem A, it is enough to show that there exists $p \in \partial f(\mu(T))$ such that $p \cdot \mu(T) = f(\mu(T))$. By Proposition 2.1, for every natural number $n > 1$ there exists $p_n \in \partial f((1-\frac{1}{n})\mu(T))$ such that $p_n \geq 0$. As $f(0) = 0$, for every $n$ we have

$$0 \leq f((1-\frac{1}{n})\mu(T)) - (1-\frac{1}{n})p_n \cdot \mu(T).$$

Hence,

$$(1-\frac{1}{n})p_n \cdot \mu(T) \leq f((1-\frac{1}{n})\mu(T)).$$

Since $f$ is continuous and $\mu(T) >> 0$, the sequence $\{p_n\}_{n=2}^{\infty}$ is bounded. Therefore it has a subsequence which converges to a vector $p \in \Re_+^m$. It is clear that $p \in \partial f(\mu(T))$ and $p \cdot \mu(T) \leq f(\mu(T))$. On the other hand, since $p_n \in \partial f((1-\frac{1}{n})\mu(T))$ and $f$ is homogeneous of degree one, for every $n$ we have

$$f(\mu(T)) \leq f((1-\frac{1}{n})\mu(T)) + \frac{1}{n}p_n \cdot \mu(T) = (1-\frac{1}{n})f(\mu(T)) + \frac{1}{n}p_n \cdot \mu(T).$$

Thus, $p_n \cdot \mu(T) \geq f(\mu(T))$ for every $n$. Therefore $p \cdot \mu(T) \geq f(\mu(T))$, and this completes the proof that $Core(v) \neq \emptyset$. Q.E.D.

Let $S \in \Sigma$. Denote $\Sigma_S = \{Q \in \Sigma \mid Q \subset S\}$. Then $\Sigma_S$ is a $\sigma$-field of subsets of $S$. Let $v$ be a game on $(T,\Sigma)$, and let $S \in \Sigma$. The *subgame* of $v$ which is determined by $S$ is the game $v_S$ on $(S,\Sigma_S)$ which is given by $v_S(Q) = v(Q)$ for every $Q \in \Sigma_S$. A game $v$ on $(T,\Sigma)$ is called *totally balanced* if for every $S \in \Sigma$ we have $Core(v_S) \neq \emptyset$.

The following theorem shows that if $\mu$ is a finite dimensional vector of non-atomic measures in $ca_+$ and $f$ is a continuous non-decreasing concave function on $R(\mu)$ with $f(0) = 0$, then total balancedness of the game $v = f \circ \mu$ is equivalent to homogeneity of degree one of $f$ on all $R(\mu)$.

Theorem 3.4

*Let $\mu$ be a finite dimensional vector of non-atomic measures in $ca_+$. Assume that $f : R(\mu) \to \Re$ is a non-decreasing continuous and concave function which satisfies $f(0) = 0$. Then the game $v = f \circ \mu$ is totally balanced iff f is homogeneous of degree one on $R(\mu)$ (i.e., $f\alpha x) = \alpha f(x)$ for every $x \in R(\mu)$ and $0 \le \alpha \le 1$).*

Proof

We first show that if the game $v = f \circ \mu$ is totally balanced then $f$ is homogeneous of degree one on $R(\mu)$. Let $0 \le \alpha \le 1$ and $S \in \Sigma$. Since $f$ is concave on $R(\mu)$, $f(\alpha\mu(S)) \ge \alpha f(\mu(S))$. Let $\lambda \in Core(v_S)$. By Lyapunov's convexity theorem, there exists $Q \in \Sigma_S$ such that $\lambda(Q) = \alpha\lambda(S)$ and $\mu(Q) = \alpha\mu(S)$. As $\lambda(S) = f(\mu(S))$, by a similar argument to that which was used in the proof of Theorem 4.3 we obtain that $f(\alpha\mu(S)) \le \alpha f(\mu(S))$. Therefore $f$ is homogeneous of degree one on $R(\mu)$.

We assume now that $f$ is homogeneous of degree one on $R(\mu)$ and show that the game $v = f \circ \mu$ is totally balanced. Let $S \in \Sigma$. We will show that $Core(v_S) \ne \emptyset$. Let $\hat{\mu}$ be the restriction of $\mu$ to $(S, \Sigma_S)$ and $\hat{f}$ be the restriction of $f$ to $R(\hat{\mu})$. Then $v_S = \hat{f} \circ \hat{\mu}$. Since $\hat{f}$ is continuous, non-decreasing, concave and homogeneous of degree one on $R(\hat{\mu})$, by Theorem 3.3, $Core(v_S) \ne \emptyset$. Q.E.D.

In the light of Theorems 3.3 and 3.4 it will be useful to give an example of a function $f$ which is defined on the range $R$ of a vector of non-atomic measures on a measurable space and such that $f$ is continuous, non-decreasing and concave on $R$, $f(0) = 0$, $f$ is homogeneous of degree one along the diagonal of $R$, but $f$ is not homogeneous of degree one in the entire range $R$. Indeed, let $R$ be the unit square in $\Re^2$ ($R$ is, for example, the range of the vector $(\lambda_1, \lambda_2)$ when the measureable space is [$0$, $2$] with its Borel subsets, $\lambda_1$ is the Lebesgue measure on [$0$, $1$] and $\lambda_2$ is the Lebesgue measure on [$1$, $2$] ). Define a function $f$ on $R$ by

$$f(x,y) = \sqrt{xy}\,(1 - \varepsilon(x-y)^2),$$

where $0 < \varepsilon < 10^{-7}$. It is clear that $f$ is continuous on $R$ and homogeneous of degree one along the diagonal of $R$ but not in all $R$. It is also easy to check (by computing the partial derivatives) that $f$ is non-decreasing. A direct computation gives that the Hessian of $f$ is negative semidefinite on $R$. Therefore $f$ is concave on $R$.

§4 - Applications

In this section we apply Theorem A to games which arise in economic applications. We start with the non-atomic glove market game whose core was studied in Billera and Raanan (1981) and Einy et al. (1996).

Let $\mu_1,\ldots,\mu_m$ be non-atomic measures in $ca_+$. The non-atomic glove market game is defined by

$$v(S) = min(\mu_1(S),\ldots,\mu_m(S)) \text{ for every } S \in \Sigma.$$

Billera and Raanan (see Billera and Raanan (1981), Corollary 2.7) proved that the core of $v$ coincides with the convex hull of the set $M = \{\mu_i \mid i = 1,\ldots,m \text{ and } \mu_i(T) = v(T)\}$. We now derive this result from Theorem A. It is clear that $M \subset Core(v)$. Since $Core(v)$ is convex, $co\,M \subseteq Core(v)$ ($co\,M$ denotes the convex hull of $M$). Define now $\bar{f}: \Re^m_+ \to R$ by $\bar{f}:(x_1,\ldots,x_m) = min(x_1,\ldots,x_m)$. Let $\lambda \in Core(v)$, then by Remark 3.1, there exists $p \in \partial\bar{f}(\mu(T))$ such that $p \cdot \mu(T) = v(T)$ and $\lambda = p \cdot \mu$. It is clear that $p \geq 0$

and $p_i = 0$ for every $i$ in which $\mu_i(T) > v(T)$. Therefore $v(T) = v(T)\sum_{i=1}^{m} p_i$. Now if $v(T) = 0$ the result is trivial. If $v(T) > 0$ then $\sum_{i=1}^{m} p_i = 1$ and thus $Core(v) \subseteq co\,M$.

We consider now a pure exchange economy $E$ in which the *commodity space* is $\Re^m_+$. The *traders' space* is represented by a measure space $(T, \Sigma, \mu)$, where $T$ is the set of traders and $\mu$ is a non-atomic probability measure on $\Sigma$. A *coalition* is a member of $\Sigma$. An *assignment* (of commodity bundles to traders) is an integrable function $x: T \to \Re^l_+$. There is a fixed *initial assignment* $\omega$. ($\omega(t)$ represents the *initial bundle density* of trader $t$.) We assume that $\int_T \omega\, d\mu >> 0$. An *allocation* is an assignment $x$ such that $\int_T x\, d\mu \leq \int_T \omega\, d\mu$. Each trader $t \in T$ has a *utility function* $u_t: \Re^m_+ \to \Re_+$.

We first study the case in which all the traders in the economy $E$ have the same utility function $u$ which is continuous, non-decreasing, concave and homogeneous of degree one on $\Re^m_+$. The *Aumann-Shapley-Shubik market game* which is associated with the economy $E$ (see Section 30 of Chapter VI in Aumann and Shapley (1974)) in this special case is defined by

(4.1) $v(S) = sup \left\{ \int_S u(x(t))\, d\mu \mid x \text{ is an allocation such that } \int_S x\, d\mu = \int_S \omega\, d\mu \right\}$.

Proposition 4.1

*Assume that every trader in the economy E has the same utility function* $u: \Re^m_+ \to \Re$ *which is continuous, non-decreasing, concave, homogeneous of degree one, and satisfies* $u(0) = 0$. *Let* $v$ *be the market game which is defined in (4.1). Then for every* $S \in \Sigma$ *we have* $v(S) = u(\int_S \omega\, d\mu)$ *and*

(4.2) $Core(v) = \left\{ p \cdot \int \omega \, d\mu \;\middle|\; p \in \partial u(\int_T \omega \, d\mu) \right\}$

Proof

From the definition of $v$ it is clear that for every $S \in \Sigma$ we have $v(S) \geq u(\int_S \omega \, d\mu)$. Let $S \in \Sigma$. Since $u$ is concave and homogeneous of degree one, by Jensen's inequality, for every allocation $x$ such that $\int_S x \, d\mu = \int_S \omega \, d\mu$ we have $\int_S u(x) d\mu \leq u(\int_S \omega \, d\mu)$. Therefore $v(S) \leq u(\int_S \omega \, d\mu)$ and thus $v(S) = u(\int_S \omega \, d\mu)$. Now (4.2) follows from Theorem 3.3 and Theorem A. Q.E.D.

Note that since the function $u$ of Proposition 4.1 is homogeneous of degree one on $\Re^m_+$, every $p \in \partial u(\int_T \omega \, d\mu)$ is a vector of competitive prices which corresponds to a transferable utility competitive equilibrium of the economy $E$ (see Section 32 on page 184 of Aumann and Shapley (1974)).

We now apply Theorem A to the case when the economy $E$ has a finite number of types.

Two traders in the economy $E$ are of the same *type* if they have identical initial bundles and identical utility functions . We assume that the number of different types of traders in $E$ is finite and it will be denoted by $n$. For every $1 \leq i \leq n$ we denote by $T_i$ the set of traders of type $i$. We assume that $T_i$ is measurable and $\mu(T_i) > 0$. The utility function of the traders of type $i$ will be denoted by $u_i$, and their initial bundle by $\omega_i$. We assume that for every $1 \leq i \leq n$, $u_i$ is non-decreasing, concave, and continuous on $\Re^m_+$.

The Aumann-Shapley-Shubik market game which is associated with the economy $E$ in the case of a finite number of types is

(4.2) $v(S) = sup\left\{\sum_{i=1}^{n}\int_{S\cap T_i} u_i(x(t))d\mu \,\middle|\, x \text{ is an allocation such that } \int_S x\, d\mu = \int_S \omega\, d\mu\right\}.$

Define now a function $f: \Re_+^n \times \Re_+^m \to \Re$ by

(4.3) $f(y,z) = max\left\{\sum_{i=1}^{n} y_i u_i(x_i) \,\middle|\, x_i \in \Re_+^m,\ \sum_{i=1}^{n} y_i x_i \le z\right\}.$

Then by Lemma 39.9 of Aumann and Shapley (1974), $f$ is concave, continuous, non-decreasing, and homogeneous of degree one on $\Re_+^n \times \Re_+^m$.

Proposition 4.2

*Let $v$ be the market games which is given by (4.2). Define an (n+m)-dimensional vector of non-atomic measures $\xi$ on $\Sigma$ by*

$$\xi(S) = (\mu(S\cap T_1),\ldots,\mu(S\cap T_n),\ \int_T \omega\, d\mu).$$

*Let $f$ be the function which is defined in (4.3).* Then $v = f \circ \xi$ and

(4.4) $Core(v) = \{p\cdot\xi \mid p \in \partial f(\xi(T))\}.$

Proof

By Lemma 39.16 of Aumann and Shapley (see also Lemma 4.6 in Dubey and Neyman (1981)), for every $S \in \Sigma$ we have $v(S) = f(\xi(S))$. Since $f$ is continuous, concave, non-decreasing, and homogeneous of degree one on $\Re_+^{n+m}$ (e.g., Lemma 39.9 of Aumann and Shapley (1974)), (4.4) follows from Theorem 3.3 and Theorem A. Q.E.D.

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