Pseudo-uniform Convexity in H^p and Some Extremal Problems on Sobolev Spaces

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We extend Newman and Keldysh theorems to the behavior of sequences of functions in $H^p(\mu)$ which explain geometric properties of discs in these spaces. Through Keldysh's theorem we obtain asymptotic results for extremal polynomials in Sobolev spaces.

Keywords: H^p space; Orthogonal polynomials

AMS (MOS) Subject Classifications: 32A35; 42C05

1. INTRODUCTION AND MAIN RESULTS

In this article, we extend Newman and Keldysh theorems to $H^p(\mu)$ with $0 < p < \infty$. These results are very useful for obtaining convergence in norm. It is known that $H^p, 0 < p \leq 1,$ are not locally uniformly convex spaces and they are used for checking that Hahn–Banach theorem fails in a nonlocally convex space with ''reasonable'' properties. That H^p would seem destined to be of further interest in the future can be guessed from the fact that the most common ''singularities'' in analysis, such as those given by rational functions, or carried on analytic subvarieties, or representable by Fourier integral (''Lagrangian'') distributions, are all of them locally in H^p, for some $p < 1$.

We will use these theorems for proving a result about asymptotics of extremal Sobolev polynomials. Sobolev orthogonal polynomials have been received considerable attention in the last two decades, as a natural consequence of the great importance of Sobolev spaces. Sobolev orthogonal polynomials are also connected with spectral

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theory for ordinary differential equations, matrix orthogonal polynomials, and higher order recurrence relations. Sobolev orthogonal polynomials also appear in a natural way in some problems of approximation theory where the derivatives are considered. Two updated surveys on Sobolev orthogonal polynomials are presented in [12,15] (look at the references therein). Asymptotics for Sobolev orthogonal polynomials have been described among others in [1,8,9,11,12].

We begin with the extensions of Newman and Keldysh theorems, that will be proved in Section 3. But first we set some notations. Let m be the normalized Lebesgue measure on $[0, 2\pi)$ and let μ be a positive Borel measure on $[0, 2\pi)$ satisfying Szegö's condition, i.e. $\mu \in \mathbf{S} \Leftrightarrow \log \mu'(\theta) \in L^1$, where $\mu'(\theta)$ is the Radon–Nikodym derivate of μ with respect to m. $H^p(\mu)$ is defined as the $L^p(\mu)$ closure of the polynomials in $e^{i\theta}$ and $\|f\|_{p,\mu} = \left(\int |f(e^{i\theta})|^p d\mu(\theta)\right)^{1/p}$ for $f \in H^p(\mu)$. For a sake of simplicity we will denote $H^p = H^p(m)$. Let $D_p(\mu, z)$ be the Szegö function

$$D_p(\mu, z) = \exp\left\{\frac{1}{p}\int_{\pi}^{\pi} \frac{\zeta + z}{\zeta - z} \log \mu'(\theta)\, dm(\theta)\right\}, \quad \zeta = e^{i\theta} \tag{1}$$

and

$$K_p(\mu, z) = \begin{cases} \dfrac{D_p(\mu, 0)}{D_p(\mu, z)}, & \text{if } z \in (S_a \cup \{z : |z| < 1\}), \\ 0, & \text{if } z \in S_s, \end{cases} \tag{2}$$

where S_a and S_s are a disjoint decomposition of the unit circle such that μ_a and μ_s live on these sets respectively, hereafter μ_a and μ_s denote the absolutely continuous and singular parts of μ with respect to m. $L^p_s(\mu) = \{f \in L^p(\mu) \colon f = 0,\ \mu_a - a.e.\}$ and $L^p_a(\mu) = \{f \in L^p(\mu) \colon f = 0,\ \mu_s - a.e.\}$. Similarly, we define $H^p_s(\mu)$ and $H^p_a(\mu)$. Set $\mathbb{D} = \{z \colon |z| < 1\}$ and $\mathbb{E} = \{z \colon |z| > 1\}$.

Theorem 1 *Assume that* $\mu \in \mathbf{S}$. *If* f_n *and* f *are in* $H^p_a(\mu)$, $0 < p < \infty$, *such that*

(i) $\lim_{n\to\infty} \|f_n\|_{p,\mu} = \|f\|_{p,\mu}$,
(ii) $\lim_{n\to\infty} f_n(z) = f(z)$, *holds uniformly on each compact subset of* $\mathbb{D}$,

then

$$\lim_{n\to\infty} \|f_n - f\|_{p,\mu} = 0. \tag{3}$$

In the next section, we are going to prove that if $f \in H^p(\mu)$, then there exist unique functions $\tilde{f}, f_s$ such that $f = K_p \tilde{f} + f_s$, $\tilde{f} \in H^p$, and $f_s \in L^p_s(\mu)$. With these notations we set

Theorem 2 *Let* $\{z_i\}_{i=1,\ldots,\Lambda}$ *be a set of points in* $\mathbb{D}$ *where* Λ *can be finite or infinite,* $\mu \in \mathbf{S}$, *and* $\{f_n\} \subset H^p(\mu)$, $0 < p < \infty$, *such that*

(i) $\lim_{n\to\infty} \tilde{f}_n(0) = 1$;
(ii) $\lim_{n\to\infty} \tilde{f}_n(z_i) = 0, \quad i = 1, 2, \ldots$;
(iii) $\sum_{i=1}^{\Lambda} (1 - |z_i|) < +\infty$;
(iv) $\lim_{n\to\infty} \|f_n\|_{p,\mu} = D_p(\mu, 0) / \prod_{i=1}^{\Lambda} |z_i|^p$.

Then

(a) $\lim_{n\to\infty}\tilde{f}_n(z) = \prod_{i=1}^{\Lambda}((z-z_i)/(\bar{z}_i z - 1))(z_i/|z_i|^2)$, *holds uniformly on each compact subset of* $\mathbb{D}$.

(b) $\lim_{n\to\infty}\|f_n - \prod_{i=1}^{\Lambda}((z-z_i)/(\bar{z}_i z - 1))(z_i/|z_i|^2))K_p(z)\|_{p,\mu} = 0$.

The other results that will be proved in Section 4, give us strong asymptotics for extremal polynomials, $\{P_n\}_{n=0,1,\ldots}$, solving the following extremal problem:

$$\tau_{n,\mu_0,\ldots,\mu_k,p} = \inf\left\{\sum_{j=0}^{k}\|Q^{(j)}\|_{p,\mu_j}\colon Q(z) = z^n + \cdots\right\}, \tag{4}$$

where $0 < p < \infty$ and $\mu_0, \ldots, \mu_k$ are positive Borel measures on $[0, 2\pi)$, with $\mu_0 \not\equiv 0$. In the case $p = 2$, the polynomials $\{P_n\}_{n=0,1,\ldots}$ are usually said to be Sobolev orthogonal polynomials. The special case $k = 1$ has been studied by many authors (see, for instance [10,12,18]). For $k = 0$ we have the classical orthogonality.

Theorem 3 *The following statements are equivalent.*

(i) $\mu_k \in \mathbf{S}$;

(ii) $\limsup_{n\to\infty}(\tau_{n,\mu_0,\ldots,\mu_k,p}/n^k) > 0$;

(iii) *There exists a function* $\Delta \in H_a^p(\mu_k)$ *with* $\Delta(0) = 1$ *such that*

$$\lim_{n\to\infty}\int\left|\frac{P_n^{(k)}(z)}{n^k z^{n\ k}} - \overline{\Delta(1/\bar{z})}\right|^p \mu_k'(\theta)\,dm(\theta) = 0, \quad z = e^{i\theta}.$$

Moreover, if (i) *holds then*

$$\lim_{n\to\infty}\frac{\tau_{n,\mu_0,\ldots,\mu_k,p}}{n^k} = D_p(\mu_k, 0),$$

$\Delta(z) = K_p(\mu_k, z)$, *and*

$$\lim_{n\to\infty}\frac{P_n^{(k)}(z)}{n^k z^{n\ k}} = \overline{K_p(\mu_k, 1/\bar{z})}, \tag{5}$$

holds uniformly on each compact subset of $\mathbb{E}$.

When $k = 0$ the extremal problem (4) was studied by Geronimus (see [4]) who stated more precisely that Theorem 3 holds for $k = 0$. Nevertheless, the case of $k \geq 1$ is interesting because it describes for the extremal polynomials how the norms of their derivatives balance. In the following theorem we observe that there exist polynomials asymptotically extremal for all j, $0 \leq j \leq k$.

Theorem 4 *If the measures* $\mu_l \in \mathbf{S}$, *for* $j \leq l \leq k$, *then for all* l, $j \leq l \leq k$,

$$\lim_{n\to\infty}\frac{P_n^{(l)}(z)}{n^l z^{n\ l}} = \overline{K_p(\mu_k, 1/\bar{z})}, \tag{6}$$

holds uniformly on each compact subset of $\mathbb{E}$.

The framework of this article is in Section 2 we describe some known properties of the H^p-extremal Szegö function and the necessary results for proving the three main results. Finally in Sections 3 and 4 we prove all the theorems presented in Section 1.

2. AUXILIARY RESULTS

Given $\mu \in \mathbf{S}$ the corresponding Szegö function satisfies the following properties:

1 $D_p(\mu, z)$ is analytic on $\mathbb{D}$, more precisely, $D_p(\mu, z) \in H^p$;
2 $D_p(\mu, z) \neq 0$ in $\mathbb{D}$, and $D_p(\mu, 0) > 0$;
3 $|D_p(\mu, e^{i\theta})|^p = \mu'(\theta)$ a.e. on $[0, 2\pi)$.

The function $D_p(\mu, .)$ is not uniquely determined by the conditions 1–3. To this aim it is also required that $D_p(\mu, .)$ must be an outer function (see page 277 in [19] or page 118 in [15]).

Notice that $H_a^p(\mu)$ denotes the space of analytic functions, f, on $\mathbb{D}$, such that $fD_p(\mu, .) \in H^p$. If $1 \leq p \leq \infty$, then $H_a^p(\mu)$ is a Banach space with the norm:

$$\|f\|_{\mu_a, p} = \lim_{R \uparrow 1} \left(\int |f(Re^{i\theta}) D_p(\mu, Re^{i\theta})|^p \, dm(\theta) \right)^{1/p}.$$

For $0 < p < 1$, $H_a^p(\mu)$ is not a normed space, but it is a complete metric space with the distance $d(f, g) = \|f - g\|_{\mu_a}^p$. If $f \in H_a^p(\mu)$, then there exists its radial limit a.e. on the unit circle and

$$\|f\|_{\mu_a, p}^p = \int |f(z)|^p \, \mu'(\theta) \, dm(\theta), \quad z = e^{i\theta},$$

where $f(e^{i\theta}) = \lim_{R \uparrow 1} f(Re^{i\theta})$ a.e. on $[0, 2\pi)$. We shall need the following well-known result (for clarity we include the proof).

LEMMA 1 *There is a unique solution for the extremal problem*

$$\inf\left\{ \|\Phi\|_{\mu_a, p} \colon \Phi \in H_a^p(\mu), \ \Phi(0) = 1 \right\} = D_p(\mu, 0).$$

In fact such a function is $K_p(\mu, z)$.

Proof First, let us assume that $p = 2$. Let $\Phi \in H_a^2(\mu)$ be such that $\Phi(0) = 1$, then $\left|\Phi(z) D_2(\mu, z)\right|^2$ is a subharmonic function on $\mathbb{D}$, so $D_2(\mu, 0)^2 = \left|\Phi(0) D_2(\mu, 0)\right|^2 \leq \|\Phi\|_{\mu_a, 2}^2$. Otherwise, we have that K_2 belongs to $H_a^2(\mu)$ (Krein–Kolmogorov–Szegö theorem), its value at $z = 0$ is 1, and $\|K_2\|_{\mu_a, 2}^2 = D_2(\mu, 0)^2$. The uniqueness follows immediately from the parallelogram law. If $p \neq 2$ we reduce these cases to $p = 2$ because of each function $f \in H_a^p(\mu)$ has a decomposition $f(z) = B(z)[h(z)]^{2/p}$, where B is the Blaschke product associated with the zeros of f and $h \in H_a^2(\mu)$ with $\|f\|_p^p = \|h\|_2^2$ (see [7], p. 96). ■

It is very well known that the density of the space of polynomials in $L^p(\mu)$ can also be characterized in terms of the Szegö condition for μ. $H^p(\mu) = L^p(\mu)$ if and only if $\mu \notin \mathbf{S}$ (it is an immediate consequence of Geronimus and Weierstrass theorems and the fact that continuous functions are dense in $L^p(\mu)$). For Sobolev spaces ($p = 2$) look at [18]. On the other side is the characterization of $H^p(\mu)$ for $\mu \in \mathbf{S}$. The following theorem is very well known but we only found a reference for $1 \leq p \leq \infty$ (see [5] and page 22 in [16]).

THEOREM 5 *If we assume* $\mu \in \mathbf{S}$, *then* $H^p(\mu) = K_p H^p \oplus L^p_s(\mu)$. *This means that there exist unique functions* $\tilde{f}, f_s$ *such that* $f = K_p \tilde{f} + f_s$, $\tilde{f} \in H^p$, *and* $f_s \in L^p_s(\mu)$.

Proof Set $g \in H^p(\mu)$. Then $g \in L^p(\mu)$ and $g = g_1 + g_2$ with $g_1 \in L^p_a(\mu)$ and $g_2 \in L^p_s(\mu)$. We must prove that either $g_1 \in K_p H^p$ or that $g_1/K_p \in H^p$. It is enough to prove that there exists $\{j_n\}$, $j_n \in H^p$ such that $\|j_n - g_1/K_p\|_p \longrightarrow 0$. Since $g \in H^p(\mu)$ there exists a sequence of polynomials $\{h_n\}$ such that, $\|h_n - g\|_{p,\mu} \longrightarrow 0$. Hence, with $z = e^{i\theta}$,

$$\int \left| \frac{h_n(z)}{K_p(z)} - \frac{g_1(z)}{K_p(z)} \right|^p dm(\theta) = \int |h_n(z) - g_1(z)|^p \frac{\mu'(\theta)}{|D_p(\mu,0)|^p} dm(\theta)$$

$$= \int |h_n(z) - g(z)|^p \frac{\mu'(\theta)}{|D_p(\mu,0)|^p} dm(\theta) \leq \left\| \frac{h_n - g}{D_p(\mu,0)} \right\|^p_{p,\mu} \longrightarrow 0,$$

and $j_n = h_n/K_p \in H^p$ for each n. The uniqueness of the representation follows immediately from the fact $H^p_a(\mu) \cap L^p_s(\mu) = 0$. Hence, we have proved one of the inclusions. We are going to see that $K_p H^p \subset H^p(\mu)$. Consider $f = K_p \tilde{f}$ with $\tilde{f} \in H^p$. Then there exist polynomials h_n such that

$$\|\tilde{f} - h_n\|_p \to 0 \Longrightarrow \|K_p \tilde{f} - K_p h_n\|_{p,\mu} \to 0,$$

and because of $K_p h_n \in H^p(\mu)$, we get $f \in H^p(\mu)$.

Now set $f \in L^p_s(\mu)$. As $\mu_s \notin \mathbf{S}$, there exist polynomials Q_n such that

$$\|f - Q_n\|_{\mu_s,p} \to 0. \tag{7}$$

Moreover, because of $Q_n/K_p \in H^p$, there exists a sequence of polynomials $\{h_n\}$ such that

$$\left\| \frac{Q_n}{K_p} - h_n \right\|_p \to 0 \Longleftrightarrow \|Q_n - K_p h_n\|_{\mu_a,p} \to 0. \tag{8}$$

Combining (7) and (8) with

$$\|f - Q_n + K_p h_n\|_{\mu,p} = \|f - Q_n\|_{\mu_s,p} + \|Q_n - K_p h_n\|_{\mu_a,p},$$

the proof is concluded. ■

The last auxiliary result that we need is

LEMMA 2 (see [3], p. 21) *Let* $\varphi_n, \varphi \in L^p$, $0 < p < \infty$. *If* $\varphi_n(x) \to \varphi(x)$ *a.e. and* $\|\varphi_n\|_p \to \|\varphi\|_p$, *then* $\|\varphi_n - \varphi\|_p \to 0$.

3. PROOF OF NEWMAN AND KELDYSH EXTENSIONS

Proof of Theorem 1

Proof First, we consider the case $\mu = m$, the Lebesgue measure.

(a) It is easy to see that the theorem holds for $1 < p < \infty$. From (i) and (ii) we obtain $\lim_{n\to\infty} \|(f_n + f)/2\|_p = \|f\|_p$, so, because of the uniform convexity of L^p, $1 < p < \infty$, (or Clarkson inequalities, see [2], p. 3) we get $\|f_n - f\|_p \to 0$.

(b) Let us consider $0 < p \leq 1$. We can suppose that f is not identically 0. Otherwise the proof of our theorem is trivial, since $\|f_n\|_p \longrightarrow \|f\|_p = 0$.

(c) From Lemma 2, if (i) holds and for any $\Gamma \subset \mathbb{N}$, there exists $\Gamma' \subset \Gamma$ such that $\lim_n f_n(z) = f(z)$, a.e., $n \in \Gamma'$, then we have (3).

(d) If $f_n(z) \neq 0$ holds for $n \geq N_0$ and $z \in \mathbb{D}$, then according to Hurwitz's theorem also $f(z) \neq 0$ holds in $\mathbb{D}$. So we can fix a branch for $w^{p/2}$ such that $h_n = f_n^{p/2}$ and $h = f^{p/2}$ with $h_n, h \in H^2$, $\|h_n\|_2^2 = \|f_n\|_p^p$, $\|h\|_2^2 = \|f\|_p^p$, $\lim h_n = h$ uniformly on each compact subset of $\mathbb{D}$, and $\lim \|h_n\|_2 = \|h\|_2$. This means (i) and (ii) hold for h_n and h in H^2. Therefore, from (a) we have (3). Then according to the Riesz theorem, there exists $\Gamma \subset \mathbb{N}$ such that $\lim_{n\in\Gamma} h_n(z) = h(z), a.e.$, hence $\lim_{n\in\Gamma} f_n(z) = f(z)$, a.e. and from (c) we have (3).

(e) If f_n can be 0 in $\mathbb{D}$, then $f_n = B_n h_n$ where h_n is a zero-free function in H^p, $\|h_n\|_p = \|f_n\|_p$, and B_n is a Blaschke product so $B_n \in H^\infty$. Then $\{B_n\}$ and $\{h_n\}$ are uniformly bounded in each compact subset of $\mathbb{D}$. Hence, from the Montel theorem there exists a subsequence $\Gamma_1 \subset \mathbb{N}$ such that $\lim_{n\in\Gamma_1} h_n(z) = h(z)$, $h \in H^p$, and $\lim_{n\in\Gamma_1} B_n(z) = B(z)$ hold uniformly on each compact subset of $\mathbb{D}$. Otherwise, $\|h\|_p \leq \limsup \|h_n\|_p = \limsup \|f_n\|_p = \|f\|_p$, while $\|f\|_p \leq \|h\|_p \|B\|_\infty$ and $\|B\|_\infty \leq \limsup \|B_n\|_\infty = 1$. Thus, $\|hB\|_p = \|h\|_p$, and as a consequence $|B(e^{i\theta})| = 1$, a.e. Then $\lim_n \|h_n\|_p = \|h\|_p$ and $\lim_n \|B_n\|_2 = \|B\|_2$. Then from (d), there exists $\Gamma_2 \subset \Gamma_1$ such that $\lim_{n\in\Gamma_2} h_n(z) = h(z), a.e.$ and from (a) $\lim_{n\in\Gamma_2} B_n(z) = B(z), a.e.$, hence $\lim_{n\in\Gamma_2} f_n(z) = f(z), a.e.$ and from (c) the theorem is proved.

Now, in order to complete the proof it is enough to see that the functions $\tilde{f}_n$ and $\tilde{f}$ hold the assumption of theorem with Lebesgue measure. ■

Remark 1 Theorem 1 was setting by Newman (see [14]) for the cases $p = 1$ and $\mu = m$, the Lebesgue measure. This theorem gives an alternative look for the uniform convexity of the $H^p(\mu)$ spaces.

Proof of Theorem 2

Proof The sketch of the proof is the following. First, we will prove the theorem for Lebesgue measure and $\Lambda = \emptyset$ in two steps: $p = 2$ and $p \neq 2$. Second, we consider a general $\mu \in \mathbf{S}$ and again $\Lambda = \emptyset$. Finally, we prove the general case.

(A) Set $\mu = m$, $p = 2$, and $\Lambda = \emptyset$. Notice that in this case $D_p(\mu, z) \equiv 1$.

From the monotonicity of the means and triangular inequality, we get $|f_n(0) + 1| \leq \|f_n + 1\|_2 \leq \|f_n\|_2 + \|1\|_2$. Hence $\lim_{n\to\infty} \|f_n + 1\|_2 = 2$. Now, using the parallelogram law, we obtain $\lim_{n\to\infty} \|f_n - 1\|_2 = 0$, this is (b).

The statement (a) follows immediately from Cauchy formula and Hölder inequality.

(B) Now let us consider $p \neq 2$ and again $\mu = m$, and $\Lambda = \emptyset$. Using again the factorization theorem for H^p, we get that there exists $B_n \in H^\infty$, more precisely, Blaschke products, and $h_n \in H^2$, such that

$$f_n(z) = B_n(z) h_n(z)^{2/p} = \frac{B_n(z)}{B_n(0)} (B_n(0)^{p/2} h_n(z))^{2/p} \quad \text{and} \quad \|f_n\|_p^p = \|h_n\|_2^2.$$

We are going to see that $\bar{h}_n(z) = B_n(0)^{p/2} h_n(z) \in H^2$ holds the conditions studied in (A).

$$\begin{aligned} 1 &= \lim_{n\to\infty} |f_n(0)|^{p/2} = \lim_{n\to\infty} |B_n(0)^{p/2} h_n(0)| \leq \lim_{n\to\infty} \|B_n(0)^{p/2} h_n\|_2 \\ &= \lim_{n\to\infty} |B_n(0)|^{p/2} \|h_n\|_2 = \lim_{n\to\infty} |B_n(0)|^{p/2} \leq 1, \end{aligned}$$

because $|B_n(z)| = 1$ if $|z| = 1$ and from the Maximun Principle the inequality follows.

Hence

$$\lim_{n\to\infty} |B_n(0)|^{p/2} = 1, \tag{9}$$

and we obtain $\lim_{n\to\infty} \|\bar{h}_n\|_2 = 1$. Then, from the previous case, we have (a) and (b) for $\bar{h}_n$. The same holds for $\{\bar{B}_n(z) = B_n(z)/B_n(0)\}$, $\bar{B}_n \in H^\infty \subset H^2$. Hence $\{\bar{B}_n\}$ holds (a) and (b). Then, we have (a) for f_n.
It remains to see that (b) holds. Since (b) holds for $\bar{h}_n$ and $\bar{B}_n$, there exists $\{n_j\} \subset \Gamma$ such that

$$\lim_j \bar{h}_{n_j}(z) = 1, \text{ a.e.} \quad \text{and} \quad \lim_j \bar{B}_{n_j}(z) = 1, \text{ a.e.}$$

Using Lemma 2 the proof of this case is completed.

(C) In this step we consider a general $\mu \in \mathbf{S}$ and again $\Lambda = \emptyset$. The main idea is to apply the previous argument to $\tilde{f}_n$. In fact,

$$\lim_{n\to\infty} \|f_n\|_{p,\mu}^p = \lim_{n\to\infty} \|K_p \tilde{f}_n\|_{p,\mu_a}^p + \|f_{n,s}\|_{p,\mu_s}^p = D_p(\mu, 0)^p,$$

and this yields

$$\limsup_{n\to\infty} \|K_p \tilde{f}_n\|_{p,\mu_a}^p \leq D_p(\mu, 0)^p.$$

Then

$$\begin{aligned} &\limsup_{n\to\infty} \int |K_p(e^{i\theta}) \tilde{f}_n(e^{i\theta})|^p \mu'(\theta)\, dm(\theta) \\ &\quad = D_p(\mu, 0)^p \limsup_{n\to\infty} \int |\tilde{f}_n(e^{i\theta})|^p dm(\theta) \leq D_p(\mu, 0)^p, \end{aligned}$$

hence $\limsup_{n\to\infty}\|\tilde{f}_n\|_p^p \le 1$. From the case analyzed above, we get $\lim_{n\to\infty}\tilde{f}_n(z)=1$ uniformly on compact subsets of $\mathbb{D}$ and $\lim_{n\to\infty}\|\tilde{f}_n-1\|_p=0$. Therefore $\lim_{n\to\infty}\|\tilde{f}_n\|_p=1$ and $\lim_{n\to\infty}\|K_p\tilde{f}_n\|_{p,\mu_a}=D_p(\mu,0)$. Thus $\lim_{n\to\infty}\|f_{n,s}\|_{p,\mu_s}=0$ and as a consequence

$$\begin{aligned}\lim_{n\to\infty}\|f_n-K_p\|_{p,\mu}^p &= \lim_{n\to\infty}\left(\|f_n-K_p\|_{p,\mu_a}^p+\|f_n-K_p\|_{p,\mu_s}^p\right)\\ &=\lim_{n\to\infty}\left(\|K_p\tilde{f}_n-K_p\|_{p,\mu_a}^p+\|f_{n,s}\|_{p,\mu_s}^p\right)\\ &=\lim_{n\to\infty}\left(D_p(\mu,0)^p\|\tilde{f}_n-1\|_p^p+\|f_{n,s}\|_{p,\mu_s}^p\right)=0.\end{aligned}$$

(D) Finally, we prove the general case.
Let $f_n=K_p\tilde{f}_n+f_{n,s}$ with $\tilde{f}_n\in H^p$ and $\|f_n\|_{\mu,p}\ge\|K_p\tilde{f}_n\|_{\mu,p}$. Then $\limsup_{n\to\infty}\|\tilde{f}_n\|_p\le 1/\prod|z_i|^p$ from (iv). We are going to see that all convergent subsequence of $\tilde{f}_n$, converges to the same limit, uniformly on each compact subset of $\mathbb{D}$.
Let $\tilde{f}$ be a limit function. From (ii), $\tilde{f}$ is zero on z_i, $\tilde{f}\in H^p$ and $\|\tilde{f}\|_p\le 1/\prod|z_i|^p$.

$$\tilde{f}(z)=\prod\frac{z-z_i}{z\bar{z}_i-1}\frac{z_i}{|z_i|^2}\prod\frac{z-w_i}{z\bar{w}_i-1}\frac{w_i}{|w_i|^2}h(z),$$

where $\{z_i\}$, $\{w_i\}$ are zeros of $\tilde{f}$, h is a zero-free function in H^p, $h(0)=1$, and $\|\tilde{f}\|_p=\prod(1/|z_i|^p)\prod(1/|w_i|^p)\|h\|_p$. So $\|h\|_p\le 1$ and, as a consequence, $h\equiv 1$. Therefore, the set $\{w_i\}$ is empty and $\tilde{f}(z)=\prod((z-z_i)/(z\bar{z}_i-1))(z_i/|z_i|^2)$. Moreover, we have $\lim_{n\to\infty}\tilde{f}_n(z)=\tilde{f}(z)$ uniformly on each compact subset of $\mathbb{D}$ and $\|\tilde{f}_n\|_p\le\|\tilde{f}\|_p$. Then from Theorem 1, we obtain $\lim_{n\to\infty}\|\tilde{f}_n-\tilde{f}\|_p=0$ and this is the same that $\lim_{n\to\infty}\|K_p\tilde{f}_n-K_p\tilde{f}\|_{p,\mu_a}=0$. In particular, $\lim_{n\to\infty}\|K_p\tilde{f}_n\|_{p,\mu_a}=\|K_p\tilde{f}\|_{p,\mu_a}=(1/\prod|z_i|^p)\times D_p(\mu,0)$, and then $\lim_{n\to\infty}\|f_n\|_{p,\mu_s}=0$. Therefore, we obtain (b). ■

Remark 2 For the cases $\mu=m$ and $\Lambda=\emptyset$, Theorem 2 was setting by Keldysh (see [6]).

Remark 3 Newman and Keldysh theorems do not hold in H^∞ as the following example shows. Set $f_n(z)=(nz+n-1)/(n+(n-1)z)$. It is easy to check that $f_n\in H^\infty$, $\|f_n\|_\infty=1$, and $\lim_{n\to\infty}f_n=1$, uniformly on each compact subset of $\mathbb{D}$, but $\|f_n-1\|_\infty\not\to 0$.

4. ASYMPTOTICS FOR EXTREMAL SOBOLEV POLYNOMIALS

Proof of Theorem 3

Proof (i) ⇒ (ii)
We are going to prove that $\lim_{n\to\infty}\tau_{n,\,\mu_0,\dots,\mu_k,\,p}/n^k=D_p(\mu_k,0)>0$. Obviously $\tau_{n,\,\mu_0,\dots,\mu_k,\,p}\ge\|P_n^{(k)}\|_{p,\,\mu_k}\ge n^{(k)}\tau_{n\ \,k,\,\mu_k,\,p}$, with $n^{(k)}=n(n-1)\cdots(n-k+1)$. As μ_k satisfies the Szegö's condition, from the Geronimus theorem we obtain $\lim_{n\to\infty}\tau_{n\ \,k,\,\mu_k,\,p}=D_p(\mu_k,0)$. Thus

$$\liminf_{n\to\infty} \frac{\tau_{n}, \mu_0, \ldots, \mu_k, p}{n^k} \geq \liminf_{n\to\infty} \frac{n^{(k)} \tau_{n \quad k}, \mu_k, p}{n^k}$$

$$= \liminf_{n\to\infty} \tau_{n \quad k}, \mu_k, p = D_p(\mu_k, 0).$$

Now we prove

$$\limsup_{n\to\infty} \frac{\tau_{n,\, \mu_0, \ldots, \mu_k,\, p}}{n^k} \leq D_p(\mu_k, 0). \tag{10}$$

First, we consider $p \geq 1$. Let Q_n be the monic polynomial of degree n minimizing the norm $\| \cdot \|_{p,\mu_k}$; since we are on $\mathbb{T}$ and using Minkowski's inequality, we get

$$\tau_{n+m,\{\mu_j\},p} \leq \sum_{j=0}^{k} \|(z^m Q_n)^{(j)}\|_{p,\mu_j} = \sum_{j=0}^{k} \left\| \sum_{i=0}^{j} \binom{j}{i} (z^m)^{(i)} (Q_n)^{(j \quad i)} \right\|_{p,\mu_j}$$

$$\leq (m)_k \|Q_n\|_{p,\mu_k} + f(n) o(m^k).$$

Dividing these inequalities through by m^k and taking limits (first, $m \to \infty$, and then $n \to \infty$) we obtain (10).

Second, if $0 < p < 1$, then relation (10) follows easily. Indeed, we can assume $\mu_k([0, 2\pi)) = 1$, because

$$\tau_{n,\, \mu_0, \ldots,\, \mu_k,\, p} = \|\mu_k\| \tau_{n,\, \mu_0/\|\mu_k\|, \ldots,\, \mu_k/\|\mu_k\|,\, p}.$$

Moreover, notice that if μ is a probability measure then from Jensen's inequality we have $\tau_{n,\, \mu_0, \ldots,\, \mu_k,\, p} \leq \tau_{n,\, \mu_0, \ldots,\, \mu_k,\, 1}$. Then we get (10) for $0 < p < 1$ from the corresponding relation for $p = 1$.

(ii) $\Rightarrow$ (i)

Set $k = 1$ and assume that μ_1 does not satisfy the Szegö's condition. Then from the Geronimus theorem $\lim_{n\to\infty} \tau_{n,\, \mu_1,\, p} = 0$. For a fixed $\epsilon > 0$, there exists $n_0(\epsilon)$ such that for $n \geq n_0(\epsilon)$ the set

$$\{Q\colon Q(z) = z^n + \cdots,\ \|Q\|_{p,\, \mu_1} \leq \epsilon\}$$

is non empty. For each $n \geq n_0$ we consider the extremal problem:

$$\alpha_{n,\, \mu_0,\, \mu_1,\, p}(\epsilon) = \inf\left\{ \|Q\|_{p,\, \mu_0} + \|Q'\|_{p,\, \mu_1} \colon Q(z) = z^n + \cdots,\ \|Q\|_{p,\, \mu_1} \leq \epsilon \right\}.$$

It is obvious $\tau_{n,\, \mu_0,\, \mu_1,\, p} \leq \alpha_{n,\, \mu_0,\, \mu_1,\, p}(\epsilon)$ and through the same argument as before, for n large enough we have $\alpha_{n+n_0,\, \mu_0,\, \mu_1,\, p}(\epsilon) \leq \alpha_{n_0,\, \mu_0,\, \mu_1,\, p}(\epsilon) + n\epsilon$. Hence, we have $\limsup_{n\to\infty}(\tau_{n,\, \mu_0,\, \mu_1,\, p}/n) \leq \limsup_{n\to\infty}(\alpha_{n,\, \mu_0,\, \mu_1,\, p}(\epsilon)/n) \leq \epsilon$, and this is a contradiction.

Now by induction we obtain the general case.

(i) $\Rightarrow$ (iii)

If Q is a polynomial of degree n, then $Q^*(z) = z^n\overline{Q(1/\bar{z})}$ and if $|z| = 1$, then $|Q(z)| = |Q^*(z)|$. So

$$\tau_{n,\ \mu_0,\ldots,\mu_k,\ p} \geq \|(P_n^{(k)})^*\|_{p,\ \mu_k} \geq n^{(k)} D_p(\mu_k, 0).$$

Hence

$$\lim_{n\to\infty}\left\|\frac{(P_n^{(k)})^*}{n^k}\right\|_{p,\ \mu_k} = D_p(\mu_k, 0).$$

Therefore, the sequence of functions $\{(P_n^{(k)}(z))^*/n^k\}$ holds the hypothesis of Theorem 2, and hence (iii) is proved.

(iii) $\Rightarrow$ (ii)

From (iii) we have

$$\lim_{n\to\infty}\int\left|\frac{P_n^{(k)}(z)}{n^k z^{n-k}}\right|^p \mu_k'(\theta)dm(\theta) = \int\left|\overline{\Delta\left(\frac{1}{\bar{z}}\right)}\right|^p \mu_k'(\theta)dm(\theta) > 0.$$

On the other hand

$$\tau_{n,\ \mu_0,\ldots,\mu_k,\ p} \geq \|(P_n^{(k)})^*\|_{p,\ \mu_k} \geq \left(\int\left|\frac{P_n^{(k)}(z)}{z^{n-k}}\right|^p \mu_k'(\theta)dm(\theta)\right)^{1/p}.$$

Hence

$$\lim_{n\to\infty}\frac{\tau_{n,\ \mu_0,\ldots,\mu_k,\ p}}{n^k} \geq \left(\int\left|\frac{P_n^{(k)}(z)}{n^k z^{n-k}}\right|^p \mu_k'(\theta)dm(\theta)\right)^{1/p} > 0.$$

■

Proof of Theorem 4

Proof Let us consider $l = k-1$ and assume $\mu_{k-1}, \mu_k \in \mathbf{S}$. By definition $\tau_{n,\ \mu_0,\ldots,\mu_k,\ p}/n^k \geq \|(P_n^{(k-1)})^*\|_{p,\ \mu_{k-1}}/n^k + \|(P_n^{(k)})^*\|_{p,\ \mu_k}/n^k$. Because of $\mu_k \in \mathbf{S}$, from (ii) in Theorem 3 we get $\lim_{n\to\infty}\|P_n^{(k-1)}/n^k z^{n-k+1}\|_{p,\ \mu_{k-1}} = 0$. Hence, using $\mu_{k-1} \in \boldsymbol{S}$, the Cauchy integral formula, and Hölder inequality, we obtain $\lim_{n\to\infty} P_n^{(k-1)}(z)/n^k z^{n-k+1} = 0$, uniformly on each compact subset of $\mathbb{E}$. Thus $\lim_{n\to\infty}(P_n^{(k-1)}(z)/n^k z^{n-k+1})' = 0$. Taking into account $(n-k+1)P_n^{(k-1)}(z)/n^k z^{n-k+2} = P_n^{(k)}(z)/n^k z^{n-k+1} - (P_n^{(k-1)}(z)/n^k z^{n-k+1})'$, for $l = k-1$ (6) follows if it holds for $l = k$,

$$\lim_{n\to\infty} \frac{P_n^{(k-1)}(z)}{n^{k-1} z^{n-k+1}} = \lim_{n\to\infty} \frac{(n-k+1) P_n^{(k-1)}(z)}{n^k z^{n-k+1}}$$

$$= \lim_{n\to\infty} \frac{P_n^{(k)}(z)}{n^k z^{n-k}} - z\left(\frac{P_n^{(k-1)}(z)}{n^k z^{n-k+1}}\right)' = \frac{D_p(\mu_k, 0)}{D_p(\mu_k, 1/\bar{z})}.$$

Repeating this reasoning, we obtain the corresponding results for all l, with $j \le l \le k$. ■

Other extremal problems can be considered. For example, let $0 < p_0, p_1, \ldots, p_k < \infty$ and $\mu_0, \mu_1, \ldots, \mu_k$ be positive Borel measures in $[0, 2\pi)$, set

$$\inf\left\{\sum_{j=0}^{k} \|Q^{(j)}\|_{p_j, \mu_j} \colon Q(z) = z^n + \cdots\right\}, \text{ or}$$

$$\inf\left\{\left(\sum_{j=0}^{k} \|Q^{(j)}\|_{p, \mu_j}\right)^{1/p} \colon Q(z) = z^n + \cdots\right\}.$$

Of course, similar asymptotic results for the corresponding extremal polynomials can be proved.

Acknowledgments

The work of Bello Hernández and Mínguez Ceniceros have been supported by Dirección General de Investigación (Ministerio de Ciencia y Tecnología) of Spain under grant BFM2000-0206-C04-03 de la D.G.I. The work of Marcellán has been supported by Dirección General de Investigación (Ministerio de Ciencia y Technología) of Spain under grant BMF 2000-0206-C04-01 and INTAS 2000-272. The authors thank the referee for helping them to improve the presentation of this paper and for bringing [18] to our attention.

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