On Fourier Series of a Discrete Jacobi–Sobolev Inner Product

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Communicated by Walter van Assche
Received March 2, 2001; Accepted December 31, 2001

Let $\mu$ be the Jacobi measure supported on the interval $[-1, 1]$ and introduce the discrete Sobolev type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k),$$

where $a_k, 1 < k < K$, are real numbers such that $|a_k| > 1$ and $M_{k,i} > 0$ for all $k, i$. This paper is a continuation of Marcellán et al. (On Fourier series of Jacobi Sobolev orthogonal polynomials, J. Inequal. Appl., to appear) and our main purpose is to study the behaviour of the Fourier series associated with such a Sobolev inner product. For an appropriate function $f$, we prove here that the Fourier Sobolev series converges to $f$ on $(-1, 1)$ and the derivatives of the series converge to $f^{(i)}(a_k)$ for all $i$ and $k$. Roughly speaking, the term appropriate means here the same as we need for a function $f$ in order to have convergence for its Fourier series associated with the standard inner product given by the measure $\mu$. No additional conditions are needed.

MSC: 42C05.

Key Words: orthogonal polynomials; Sobolev inner product; Fourier series.

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1. INTRODUCTION

Let $\mu$ be a finite positive Borel measure on the interval $[-1, 1]$ such that $\text{supp } \mu$ is an infinite set and let $a_k$, for $k = 1, \ldots, K$, be real numbers such that $|a_k| > 1$. For $f$ and $g$ in $L^2(\mu)$ such that there exist the derivatives in $a_k$, we can introduce the Sobolev-type inner product

$$\langle f, g \rangle = \int_1^1 f(x)g(x) \, d\mu(x) + \sum_{k=1}^K \sum_{i=0}^{N_k} M_k f^{(i)}(a_k)g^{(i)}(a_k),$$

(1)

where $M_k > 0$ for $i = 0, \ldots, N_k$ and $k = 1, \ldots, K$. Let $(\mathcal{B}_n(x))_{n=0}^\infty$ be the sequence of orthonormal polynomials with respect to this inner product,

$$\langle \mathcal{B}_n, \mathcal{B}_k \rangle = \delta_{n,k}, \quad k, n = 0, 1, \ldots.$$ 

For every function $f$ such that $\langle f, \mathcal{B}_k \rangle$ exists for $k = 0, 1, \ldots$, we introduce the formal associated Fourier–Sobolev series

$$\sum_{k=0}^\infty \langle f, \mathcal{B}_k \rangle \mathcal{B}_k(x).$$

In this paper, we continue the work presented in [3] and its main purpose is to prove the relations

$$\sum_{k=0}^\infty \langle f, \mathcal{B}_k \rangle \mathcal{B}_k(x) = f(x), \quad x \in (-1, 1),$$

$$\sum_{k=0}^\infty \langle f, \mathcal{B}_k \rangle \mathcal{B}_k^{(i)}(a_k) = f^{(i)}(a_k), \quad 0 \leq i \leq N_k, \quad 1 \leq k \leq K,$$

under standard sufficient conditions for $f$ when the Jacobi measure, $d\mu(x) = (1-x)^\alpha(1+x)^\beta \, dx$, $\alpha > -1$, $\beta > -1$, is considered. The precise terms of this result are given in Section 4.

In order to obtain it, we previously need some estimates for the polynomials $\mathcal{B}_n(x)$ in $[-1, 1]\bigcup_{k=1}^K \{a_k\}$ as well as for the involved derivatives $\mathcal{B}_k^{(i)}(a_k)$. These estimates are studied in Section 3 not only for the Jacobi measure but also for every measure $\mu$ belonging to the Szegö class. We start with a representation of $\mathcal{B}_n(x)$ in terms of the polynomials $(q_n(x))_{n=0}^\infty$ which are orthonormal with respect to the measure $w_N(x) \, d\mu(x)$, where $w_N(x)$ is a polynomial with zeros of multiplicity $N_k + 1$ at the points $a_k$ and $N = \sum_{k=1}^K (N_k + 1)$ is the degree of $w_N(x)$. In Section 2, we prove that

$$\mathcal{B}_n(x) = \sum_{i=0}^N A_{n,q_n,i}(x).$$
and the sequences \(\{A_{ik}\}_{k=0}^N\), \(i = 0, \ldots, N\), are convergent when the measure \(\mu\) is such that \(\mu' > 0\) a.e. One consequence of this result is the strong and ratio asymptotics for the polynomials \(\tilde{B}_n(x)\). The relative asymptotics are well known from papers by López, et al. [2] and Marcellán and Van Assche [5], where they solved this problem using a different representation of the polynomials \(\tilde{B}_n(x)\).

In order to obtain some results, we will use the auxiliary space

\[
S = \{ f : f \in L^2(\mu), \ f^{(i)}(a_k) \text{ exists, } i = 0, \ldots, N_k, \ k = 1, \ldots, K \}
\]

with the inner product \((1)\), where \(f\) is assumed to be defined in a neighbourhood of \(a_k\) and its derivatives are considered in the ordinary sense. The space \(S\) behaves like a vector space with one component in \(L^2(\mu)\) and a finite number of real components.

The fact that the points \(a_k\) are outside the interval \([-1, 1]\) plays an important role in the whole paper because, in this case, \(w_N(x)\) is a continuous function in that interval. Note that some estimates of the polynomials \(\tilde{B}_n\), when a mass point at \(a = 1\) is considered, have been obtained in [1]. The problem of the estimates and the behaviour of the Fourier series when the mass points \(a_k\) lie on the interval \([-1, 1]\) remains open.

### 2. AUXILIARY RESULTS

Let \(N = \sum_{k=1}^K (N_k + 1)\) and let \(w_N(x)\) be the polynomial

\[
w_N(x) = \prod_{k=1}^K (x - a_k)^{N_k+1}.
\]

In order to have positivity for \(w_N(x)\) and also to make the notation more comfortable, without loss of generality, we will assume that all points \(a_k\) belong to the interval \((-\infty, -1)\); otherwise, we only have to change the corresponding factor \((x - a_k)\) by \((a_k - x)\) in the definition of \(w_N(x)\).

Let us consider the polynomials

\[
\{1, x - a_1, (x - a_1)^2, \ldots, (x - a_1)^{N_1+1}, (x - a_1)^{N_1+1}(x - a_2), \ldots, (x - a_1)^{N_1+1}(x - a_2)^{N_2+1}, \ldots, (x - a_K)^{N_K+1} \}
\]

and denote them as \(w_{k,1}(x)\) for \(k = 0, \ldots, N - 1\). It is clear that they constitute a basis of the vector space \(P_{N-1}\) of the polynomials of degree less than \(N\). Let \(w_{N,1}(x)w_N(x) = w_N(x)\).

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Let \( (q_n(x))_{n=0}^{\infty} \) be the sequence of orthonormal polynomials with respect to the measure \( w_N(x) \, d\mu(x) \) and let

\[
\Pi_N(x) = 1 + \sum_{k=1}^{N} b_k T_k(x),
\]

where \( T_k(x) = \cos k\theta, \; x = \cos \theta \) are the Tchebichef polynomials of the first kind, the \( N \)th polynomial orthogonal with respect to \( \frac{1}{\sqrt{w_N(x)}} \). We will also denote \( \kappa(x) \) the leading coefficient of any polynomial \( \pi(x) \).

**Lemma 2.1.** For \( n \gg N \), there exist constants \( A_{n,i} \) such that

\[
\mathbf{B}_n(x) = \sum_{i=0}^{N} A_{n,i} q_n(x), \quad A_{n,N} \neq 0.
\]

If the measure \( \mu \) is such that \( \mu'(x) > 0 \) a.e. in \([-1, 1] \), then \( \lim_{n \to \infty} A_{n,i} = A_i \), where

\[
A_0 = \frac{1}{\sqrt{2^N b_N}}, \quad A_i = \frac{h_i}{\sqrt{2^N b_N}}, \quad 1 \leq i \leq N.
\]

**Proof.** Since \( \mathbf{B}_n(x) = \sum_{i=0}^{n} A_{n,i} q_n(x) \) and

\[
A_{n,j} = \int_{-1}^{1} \mathbf{B}_n(x) q_n(x) w_N(x) \, d\mu(x) = \langle \mathbf{B}_n, q_n \rangle w_N = 0, \quad N < j \leq n,
\]

taking into account that \( A_{n,N} = \langle \mathbf{B}_n, q_N \rangle w_N \neq 0 \), the first assertion holds.

On the other hand, from the orthonormality of \( \mathbf{B}_n(x) \), we get

\[
\sum_{i=0}^{N} A_{n,i}^2 = \int_{-1}^{1} \mathbf{B}_n^2(x) w_N(x) \, d\mu(x) \leq \max_{x \in [-1,1]} |w_N(x)|
\]

and, as a consequence, \( |A_{n,i}| \) are bounded. Moreover,

\[
A_{n,0} = \frac{k(B_n)}{k(q_n)} \quad A_{n,N} = \langle \mathbf{B}_n(x), w_N(x) q_n(x) \rangle = \frac{k(q_n) x}{k(B_n)} = \frac{k(q_n) x}{k(B_n)} \frac{1}{A_{n,0}}.
\]

(2)

Also from the orthonormality of \( \mathbf{B}_n(x) \), the sequences \( (B_n^0(a_k))_{n=0}^{\infty} \) for \( 0 \leq i \leq N \); \( k = 1, \ldots, K \) are bounded.

Let \( A \) be a family of non-negative integers such that \( (A_{n,i})_{n=0}^{\infty} \) is convergent for each \( i = 0, 1, \ldots, N \) and let \( A_i = \lim_{n \to \infty} A_{n,i} \). As it is well known (see
[6, 7]), the condition \( \mu' > 0 \) a.e. gives ratio asymptotics and the equalities

\[
\lim_{n \to \infty} \frac{\kappa(q_n, x)}{\kappa(q_0)} = \frac{1}{2^n},
\]

\[
\lim_{n \to \infty} \int_1^1 f(x)q_n(x)w_N(x) \, d\mu(x) = \frac{1}{\pi} \int_1^1 f(x)T_k(x) \frac{dx}{\sqrt{1 - x^2}}
\]

for any continuous function \( f(x) \) also hold. As a consequence, taking into account that \( \frac{1}{w_N(x)} \) is a continuous function on \([-1, 1],\)

\[
\lim_{n \to \infty} \int_1^1 \mathcal{B}_n(x)q_n(x)w_{k,1}(x) \, d\mu(x) = \lim_{n \to \infty} \sum_{i=0}^N A_{n,i} \int_1^1 q_n(x)w_{k,1}(x) \, d\mu(x)
\]

\[
= \sum_{i=0}^N A_{n,i} \int_1^1 T_i(x) \frac{dx}{w_N(x) \sqrt{1 - x^2}}
\]

\[
= \frac{1}{\pi} \int_1^1 \sum_{i=0}^N A_i T_i(x)w_{k,1}(x) \frac{dx}{w_N(x) \sqrt{1 - x^2}}.
\]

Since \( A_N = \frac{1}{\pi}, \) if we prove that \( \lim_{n \to \infty} \int_1^1 \mathcal{B}_n(x)q_n(x)w_{k,1}(x) \, d\mu(x) = 0 \) for \( k = 0, \ldots, N - 1, \) the statement of the lemma follows because this means that \( \sum_{i=0}^N A_i T_i(x) \) is an orthogonal polynomial of degree \( N \) with respect to \( \frac{dx}{w_N(x) \sqrt{1 - x^2}}, \) and, since \( A_0 > 0 \) because \( A_N < \infty, 1 + \sum_{i=1}^N A_i T_i(x) = \Pi_N(x). \) Let us prove the previous assertion.

Consider the basis of \( \mathbb{P} \)

\[\{1, w_{1,2}(x), w_{2,2}(x), \ldots, w_{N-1,2}(x), w_{N,2}(x) = w_N(x), xw_N(x), \ldots\}.\]

If we write \( \mathcal{B}_n(x) \) in terms of this basis, we have

\[
\mathcal{B}_n(x) = \sum_{i=0}^N z_{n,i} w_{i,2}(x) + w_N(x) \sum_{i=1}^N \beta_{n,i} x^i,
\]

where \( w_{0,2}(x) = 1. \) There exists a constant \( C, \) independent of \( n, \) such that \( |z_{n,i}| < C \) for \( i = 0, 1, \ldots, N - 1, \) because \( z_{n,0} = \mathcal{B}_n(a_k) \) which is bounded as we already know, and if we assume that \( z_{n,0}, z_{n,1}, \ldots, z_{n,i} \) are proved bounded, since

\[
w_{i+1,2}(x) = (x - \xi)w_{i,2}(x) \quad \text{for} \quad \xi \in \{a_1, \ldots, a_K\},
\]
we have one of the following two possibilities:

First, writing \( w_{1+2}(x) = (x - z)^i \pi(x) \) for a polynomial \( \pi(x) \) such that \( \pi(x) \neq 0 \), when \( n \) is less than the multiplicity of \( z \) as a zero of \( w_N(x) \),

\[
\begin{align*}
\mathfrak{a}_{n+1} &= \lim_{x \to z} \frac{\mathcal{B}_n(x) - \sum_{i=0}^{n} a_{n,i} w_{1,2}(x)}{w_{1+2}(x)} = \frac{\mathcal{B}_n^{(i)}(z) - \sum_{i=0}^{n} a_{n,i} w_{1,2}^{(i)}(z)}{i! \pi(z)}.
\end{align*}
\]

Second, when \( n \) is equal to such a multiplicity, denoting \( z^* \) the consecutive zero in the construction of the \( w_{1,2}(x) \),

\[
\begin{align*}
\mathfrak{a}_{n+1}^* &= \frac{\mathcal{B}_n(z^*) - \sum_{i=0}^{n} a_{n,i} w_{1,2}(z^*)}{w_{1+2}(z^*)}.
\end{align*}
\]

In both cases \( \mathfrak{a}_{n+1} \) is bounded because \( \mathcal{B}_n^{(i)}(z) \) and \( \mathcal{B}_n(z^*) \) are also bounded sequences.

Now, we get

\[
\begin{align*}
\int_1^1 \mathcal{B}_n(x) q_n(x) w_{2,1}(x) \, d\mu(x) \\
= \sum_{i=0}^{N} \mathfrak{a}_{n,i} \int_1^1 w_{1,2}(x) q_n(x) w_{2,1}(x) \, d\mu(x) + \sum_{i=0}^{N} \mathfrak{a}_{n,i} \int_1^1 w_N(x) x^i q_n(x) w_{2,1}(x) \, d\mu(x) \\
= \sum_{i=0}^{N} \mathfrak{a}_{n,i} \int_1^1 w_{1,2}(x) w_{2,1}(x) q_n(x) \, d\mu(x)
\end{align*}
\]

from the orthogonality of \( q_n(x) \) and because, for \( N - k \leq i \leq N \), \( w_{1,2}(x) w_{2,1}(x) = w_N(x) \pi(x) \), where \( \pi(x) \) is a polynomial of degree less than or equal to \( k \).

Taking limits,

\[
\begin{align*}
\lim_{n \to \infty} \int_1^1 \mathcal{B}_n(x) q_n(x) w_{2,1}(x) \, d\mu(x) &= \lim_{n \to \infty} \sum_{i=0}^{N} \mathfrak{a}_{n,i} \int_1^1 w_{1,2}(x) w_{2,1}(x) q_n(x) \, d\mu(x) \\
&= \lim_{n \to \infty} \sum_{i=0}^{N} \mathfrak{a}_{n,i} \int_1^1 w_{1,2}(x) w_{2,1}(x) \frac{w_N(x)}{w_N(x)} \times q_n(x) w_N(x) \, d\mu(x) = 0,
\end{align*}
\]

because \( \mathfrak{a}_{n,i} \) are bounded sequences and \( \int_1^1 \frac{w_N(z) w_{1,2}(x)}{w_N(x)} q_n(x) w_N(x) \, d\mu(x) \) are the Fourier coefficients of the continuous function \( \frac{w_N(z) w_{1,2}(x)}{w_N(x)} \) which tend to zero. Hence, Lemma 2.1 is proved. \( \blacksquare \)
The computation of the coefficients $b_k$, $k = 1, \ldots, N$, is straightforward. Let us consider the function

$$F(z) = \frac{1}{\pi} \int_1^1 \frac{\Pi_N(x)}{x - z} \frac{dx}{\sqrt{1 - x^2}}.$$  

From the orthogonality of $\Pi_N(x)$, we have

$$0 = \frac{1}{\pi} \int_1^1 \frac{\Pi_N(x) w_N(x)}{(x - a_k)^j w_N(x)} \frac{dx}{\sqrt{1 - x^2}}, \quad i = 1, \ldots, N_k + 1, \quad k = 1, \ldots, K.$$  

Then,

$$F^{(i)}(a_k) = 0, \quad i = 0, \ldots, N_k, \quad k = 1, \ldots, K. \quad (3)$$  

Since $F(z) = \frac{1}{\sqrt{z^2 - 1}} (1 + \sum_{k=1}^N b_k(\phi(z))^k)$, where $\phi(z) = z - \sqrt{z^2 - 1}$ with the square root taken in such a way that $|\phi(z)| < 1$ for $z \notin [-1, 1]$, (3) means that $1 + \sum_{k=1}^K b_k(\phi(z))^k$ vanishes at $a_k$ with multiplicity $N_k + 1$ for $k = 1, \ldots, K$. As a consequence, taking into account that the function $w = \phi(z)$ is a conformal mapping from $\mathbb{C} \setminus [-1, 1]$ to $|w| < 1$,

$$1 + \sum_{k=1}^N b_k(\phi(z))^k = \prod_{k=1}^K (1 - \phi(a_k))^{N_k + 1} \prod_{k=1}^K (\phi(z) - \phi(a_k))^{N_k + 1}$$

and from the relations between $A_k$ and $b_k$ given in Lemma 2.1,

$$\sum_{k=0}^N A_k(\phi(z))^k = \frac{1}{\sqrt{2N} \prod_{k=1}^K (1 - \phi(a_k))^{N_k + 1} \prod_{k=1}^K (\phi(z) - \phi(a_k))^{N_k + 1}}.$$  

Note that if some points $a_k$ belong to $(1, \infty)$ in such a way that $\kappa(w_N) = -1$, in (2) we would have $A_{N+1} = -\frac{a_{N+1}}{x(\mu)}$ which gives $A_i = \frac{1}{\sqrt{2N}}$ in Lemma 2.1. It yields

$$\sum_{k=0}^N A_k(\phi(z))^k = \frac{1}{\sqrt{2N} \prod_{k=1}^K |\phi(a_k)|^{N_k + 1} \prod_{k=1}^K (\phi(z) - \phi(a_k))^{N_k + 1}}$$

for the general case.

As a straightforward consequence, one obtains the strong (resp. ratio) asymptotics for the polynomials $B_n(x)$ provided that $\mu$ belongs to Szegö (resp. Nevai) class. As it was previously mentioned, these results were also obtained by López, et al. [2] and Marcellán and Van Assche [5].
Corollary 2.1. If $\mu'(x) > 0$ a.e. $x \in [-1, 1]$, then

(i) $$\lim_{n \to \infty} \frac{\tilde{B}_n(x)}{q_n(x)} = \frac{1}{\sqrt{2N} \prod_{k=1}^{K} |\varphi(a_k)|^{N+1}} \prod_{k=1}^{K} (\varphi(x) - \varphi(a_k))^{N+1}$$

uniformly on compact sets of $\mathbb{C}([-1, 1])$, where $\varphi(x) = x - \sqrt{x^2 - 1}$.

(ii) $n - N$ zeros of $\tilde{B}_n(x)$ are in $[-1, 1]$ and, for $k = 1, \ldots, K$, $N_k + 1$ zeros tend to $a_k$.

(iii) $$\lim_{n \to \infty} \frac{\tilde{B}_{n+1}(x)}{\tilde{B}_n(x)} = x + \sqrt{x^2 - 1}$$

uniformly on compact sets of $\mathbb{C}([-1, 1] \cup \bigcup_{k=1}^{K} \{a_k\})$.

(iv) If $\int_{-1}^{1} \log \mu'(x) \frac{dx}{\sqrt{1 - x^2}} > -\infty$, then

$$\lim_{n \to \infty} \frac{\tilde{B}_n(x)}{(x + \sqrt{x^2 - 1})^n} = \frac{1}{\sqrt{2N} \prod_{k=1}^{K} |\varphi(a_k)|^{N+1}} \prod_{k=1}^{K} (\varphi(x) - \varphi(a_k))^{N+1} S(x)$$

uniformly on compact sets of $\mathbb{C}([-1, 1])$, where $S(x)$ is the Szegő function of $w_N(x)\mu'(x)$ (see [9, Theorem 12.1.2] as well as the definition in p. 276).

Item (ii) is a consequence of the fact that $\int_{-1}^{1} \tilde{B}_n(x) w_N(x) x^k \, d\mu(x)$ is equal to zero for $k = 0, 1, \ldots, n - N - 1$ as well as from the asymptotic formula (i).

If we write the polynomials $w_N(x)\tilde{B}_n(x)$ in terms of $\tilde{B}_j(x)$ for $j = 0, \ldots, n + N$, taking into account that

$$\langle w_N(x)\tilde{B}_n(x), \tilde{B}_j(x) \rangle = \langle \tilde{B}_n(x), w_N(x)\tilde{B}_j(x) \rangle = \int_{-1}^{1} \tilde{B}_n(x)\tilde{B}_j(x) w_N(x) \, d\mu(x),$$

which, in turn, is zero for $j = 0, \ldots, n - N - 1$, we have $w_N(x)\tilde{B}_n(x) = \sum_{j=0}^{N} x_{n,j} \tilde{B}_{n+j}(x)$ and, consequently, they satisfy a $2N + 1$ recurrence relation. Since $x_{n,j} = x_{n-j,j}$, the recurrence relation can be written as

$$w_N(x)\tilde{B}_n(x) = \sum_{j=0}^{N} x_{n,j} \tilde{B}_{n+j}(x) + \sum_{j=1}^{N} x_{n,j} \tilde{B}_{n-j}(x).$$
Besides, for \( 0 \leq j \leq N \),

\[
    z_{n,j} = \int_1^1 \hat{B}_n(x) \hat{B}_{n+j}(x) w_N(x) \, d\mu(x)
    = \sum_{k=0}^N \sum_{k'=0}^N A_{n,k} A_{n+j,k'} \int_1^1 q_{n,k} (x) q_{n+j,k'} (x) w_N(x) \, d\mu(x),
\]

and, if \( \mu'(x) > 0 \) a.e., Lemma 2.1 gives

\[
    \lim_{n \to \infty} z_{n,j} = \sum_{k=0}^N \sum_{k'=0}^N A_{n,k} A_{n+j,k'} \int_1^1 T_{\mu,k+j}(x) \frac{dx}{\sqrt{1-x^2}} = z_j.
\]

**Corollary 2.2.** There are constants \( z_{n,k} \) such that

\[
    w_N(x) \hat{B}_n(x) = \sum_{k=0}^N z_{n,k} \hat{B}_{n+k}(x) + \sum_{k=1}^N z_{n,k} \hat{B}_{n+k}(x).
\]

Moreover, if \( \mu'(x) > 0 \) a.e., there exist real numbers \( z_{n,k} \) such that \( \lim_{n \to \infty} z_{n,k} = z_k \) for \( k = 0, \ldots, N \).

In the case of only one point \( a_k \) and \( N_k = 1 \), explicit values of \( z_k \) are given in \([3]\) and, in the general case, the values \( z_k \) can be seen in \([2, 5]\). For our purpose in this paper, we only need to know that the sequences \( (z_{n,k})_{j=0}^\infty \) are convergent. From Lemma 2.1, it is possible to obtain the weak asymptotic formula

\[
    \lim_{n \to \infty} \int_1^1 f(x) \hat{B}_n(x) \hat{B}_{n+k}(x) \, d\mu(x) = \frac{1}{\pi} \int_1^1 f(x) T_k(x) \frac{dx}{\sqrt{1-x^2}}
\]

for any continuous function \( f(x) \). This formula was proved in \([2, 5]\).

In order to study the behaviour of the Fourier–Sobolev series, we need one more result. Let us consider the already defined space \( S \) and let \( \Phi \) be the family of polynomials

\[
    \Phi = \left\{ \frac{w_N(x)}{x-a_1}, \ldots, \frac{w_N(x)}{x-a_1}^{N_1+1}, \ldots, \frac{w_N(x)}{x-a_2}, \ldots, \frac{w_N(x)}{x-a_2}^{N_2+1}, \ldots, \frac{w_N(x)}{x-a_K}, \ldots \right\}.
\]

**Lemma 2.2.** \( S \) is a Hilbert space and the family of polynomials \( \Phi \) is maximal in \( S \).

**Proof.** Since \( \|f(x)\|_S^2 = \langle f(x), f(x) \rangle = \|f(x)\|^{2}_{L^2} + \sum_{k=1}^K \sum_{l=0}^{N_k} M_{k,l} |f^{(l)}(a_k)|^2 \), a Cauchy sequence in \( S, (f_n)_{n=0}^\infty \), is a Cauchy sequence in \( L^2(\mu) \) and
the same happens for the sequences \((f_{n}^{0}(a_{k}))_{n=0}^{\infty}\) in \(\mathbb{R}\). Then, any function \(f\), defined in a neighbourhood of \(a_{k}\) such that \(f^{0}(a_{k}) = \lim_{n \to \infty} f_{n}^{0}(a_{k})\) and that in \([-1, 1]\) is the \(L^{2}(\mu)\) limit of \(f_{n}\), is a limit of \(f_{n}\) in \(\mathcal{S}\). Therefore, \(\mathcal{S}\) is a Hilbert space.

On the other hand, if \(\langle f(x), x^{k}w_{N}(x) \rangle = 0\) for \(k = 0, 1, \ldots\), then

\[
\int_{1}^{1} f(x)x^{k}w_{N}(x) \, d\mu(x) = 0, \quad k = 0, 1, \ldots
\]

and thus \(w_{N}(x)f(x) = 0\) \(\mu\text{-a.e.}\). But \(w_{N}(x) > 0\) for \(x \in [-1, 1]\), hence \(f(x) = 0\) \(\mu\text{-a.e.}\). In this case, \(\langle f(x), g(x) \rangle = \sum_{i=1}^{N} \sum_{i=1}^{N} M_{ij} f^{0}(a_{n})d^{0}(a_{k})\) and, from \(\langle f(x), w_{N}(x) \rangle = 0\), \(f^{0}(a_{n}) = 0\) for \(i = 1, \ldots, N_{k} + 1\) and \(k = 1, \ldots, K\). As a consequence, \(f = 0\) in \(\mathcal{S}\) and the lemma is proved. □

3. ESTIMATES FOR SOBOLEV POLYNOMIALS

In order to obtain estimates for \(\hat{B}_{n}(x)\) when \(x \in [-1, 1]\), the measure \(\mu\) is considered to be in the Nevai class.

\textbf{Lemma 3.1.} Let \(\mu\) be a measure such that \(\mu'(x) > 0\) \(a.e.\) \(x \in [-1, 1]\). Let \((p_{n}(x))_{n=0}^{\infty}\) be the sequence of orthonormal polynomials with respect to \(\mu\). Let \(a \in \mathbb{R}[-1, 1]\) and let \((t_{n}(x))_{n=0}^{\infty}\) be the sequence of orthonormal polynomials with respect to \(|x - a|\) \(d\mu(x)\). There exists a positive constant \(C\) such that

\[
|x - a|t_{n}(x) \leq C(|p_{n+1}(x)| + |p_{n}(x)|)
\]

for every \(x\) and for all \(n\).

\textbf{Proof.} For the polynomials \(t_{n}(x)\) we have \(t_{n}(x) = \sum_{j=0}^{n} \lambda_{n,j} p_{j}(x)\), where

\[
\lambda_{n,j} = \int_{1}^{1} t_{n}(s) p_{j}(s) \, d\mu(s) = p_{j}(a) \int_{1}^{1} t_{n}(s) \, d\mu(s)
\]

\[
+ \int_{1}^{1} t_{n}(s)(s - a) \sum_{k=1}^{n} \frac{p_{k}^{(j)}(a)}{k!} (s - a)^{k} \, d\mu(s) = p_{j}(a) \int_{1}^{1} t_{n}(s) \, d\mu(s).
\]

Hence,

\[
t_{n}(x) = \int_{1}^{1} t_{n}(s) \, d\mu(s) \sum_{j=0}^{n} p_{j}(a) p_{j}(x)
\]

\[
= \int_{1}^{1} t_{n}(s) \, d\mu(s) \frac{K(p_{n})}{K(p_{n+1})} \frac{p_{n+1}(x)p_{n}(a) - p_{n+1}(a)p_{n}(x)}{x - a},
\]

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and, as a consequence,

$$
|t_a(x)| \leq \frac{k(p_n)}{k(p_{n+1})} \left| \frac{1}{|x-a|} \left( |p_{n+1}(x)| + \frac{|p_{n+1}(a)|}{p_n(a)} \right) \right|.
$$

But $\frac{k(p_n)}{k(p_{n+1})}$ and $\frac{p_{n+1}(a)}{p_n(a)}$ are bounded because these polynomials have ratio asymptotics. Thus,

$$
\int_1^1 \left| t_a(s) p_n(a) d\mu(s) \right| = \left| \int_1^1 t_a(s) p_n(s) d\mu(s) \right| \leq \left( \int_1^1 \left| t_a(s) \right|^2 d\mu(s) \right)^{1/2} \frac{1}{\sqrt{\text{dist}(a, [-1, 1])}}.
$$

Then, $|x-a||t_a(x)| \leq C(|p_{n+1}(x)| + |p_n(x)|)$ for every $x$ and for some constant $C$.

By iteration of this lemma, for the polynomials $(q_n(x))_{n=0}^\infty$, orthonormal with respect to $w_N(x) d\mu(x)$, we get

**Corollary 3.1.** If $\mu'(x) > 0$ a.e. and $p_n(x)$ are orthonormal with respect to $\mu$, there exists a positive constant $C$ such that

$$
|w_N(x)||q_n(x)| \leq C(|p_{n+1}(x)| + \cdots + |p_n(x)|)
$$

for every $x$ and for all $n$.

For the Sobolev orthonormal polynomials $(\hat{B}_n(x))_{n=0}^\infty$, this inequality and Lemma 2.1 give

**Corollary 3.2.** If $\mu'(x) > 0$ a.e. and $p_n(x)$ are orthonormal with respect to $\mu$, there exists a positive constant $C$ such that

$$
|w_N(x)||\hat{B}_n(x)| \leq C(|p_{n+1}(x)| + \cdots + |p_n(x)|)
$$

for every $x$ and for all $n$.

**Corollary 3.3.** If $\mu'(x) > 0$ a.e. and there is a function $h(x)$ such that the polynomials $p_n(x)$, orthonormal with respect to $\mu$, satisfy the condition $|p_n(x)| \leq h(x)$, $x \in [-1, 1]$, then there exists a constant $C$ such that

$$
|\hat{B}_n(x)| \leq Ch(x)
$$

for $x \in [-1, 1]$ and for all $n$.

It is clear that the constants $C$ in the previous corollaries may be different despite the fact that we use the same symbol.
The last corollary will be very useful for the study of Fourier–Sobolev series when \( \mu \) is the Jacobi measure because in this case the function \( h(x) \) is very well known.

In order to study the Fourier series, we also need estimates of \( \# \langle B_j \rangle n(x)_{ak} \), \( j = 0, \ldots, N_0 \) and \( k = 1, \ldots, K \). This problem will be considered now. The condition \( \mu'(x) > 0 \) a.e. is not sufficient for our purposes in what follows. Thus, from now on, we will consider the measure \( \mu \) in the Szegö class, i.e. \( \int_1^1 \log \mu'(x) \frac{dx}{\sqrt{1 - x^2}} > -\infty \).

**Lemma 3.2.** Let \( \mu \) be a measure in the Szegö class and let \( q_n(x) \) be the orthonormal polynomials with respect to \( w_N \langle x \rangle \, d\mu(x) \). There is a constant \( C \) such that

\[
\left| \int_1^1 q_n(x) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x) \right| \leq C \frac{n!}{|a_k + \sqrt{a_k^2 - 1}|^n}
\]

for \( i = 1, \ldots, N_0 + 1, \ k = 1, \ldots, K \) and \( n \) large enough.

**Proof.** We proceed by induction. By orthogonality,

\[
\int_1^1 q_n(x) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x) = \frac{1}{q_n(a_k)} \int_1^1 q_n(x)q_n(a_k) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x)
\]

\[
= \frac{1}{q_n(a_k)} \int_1^1 q_n(x)q_n(a_k)
\]

\[
+ (x - a_k) w_N \langle x \rangle \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x)
\]

for any polynomial \( \pi \langle x \rangle \) of degree less than \( n \). Then,

\[
\left| \int_1^1 q_n(x) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x) \right| \leq \frac{1}{|q_n(a_k)|} \left| \int_1^1 q_n'(x) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x) \right| \leq \frac{C_1}{|q_n(a_k)|}.
\]

Suppose

\[
\left| \int_1^1 q_n(x) \frac{w_N \langle x \rangle}{x - a_k} \, d\mu(x) \right| \leq C_2 \frac{n!}{|q_n(a_k)|}
\]

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for some constant \( C_j \) and for \( 1 \leq j \leq i \leq N_k \). Then,

\[
\int_1^1 g_\alpha(x) \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x) = \frac{1}{q_\alpha(a_k)} \int_1^1 g_\alpha(x) \left\{ \left( q_\alpha(a_k) + \sum_{i=1}^{i} \frac{q^{(i)}_\alpha(a_k)}{i!} (x - a_k)^i \right) \frac{w_N(x)}{(x - a_k)^{i+1}} \right\} d\mu(x) \\
- \frac{1}{q_\alpha(a_k)} \int_1^1 g_\alpha(x) \sum_{i=1}^{i} \frac{q^{(i)}_\alpha(a_k)}{i!} (x - a_k)^i \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x) \\
= \frac{1}{q_\alpha(a_k)} \int_1^1 g_\alpha(x) \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x) - \frac{1}{q_\alpha(a_k)} \sum_{i=1}^{i} \frac{q^{(i)}_\alpha(a_k)}{i!} \left( \frac{w_N(x)}{(x - a_k)^{i+1}} \right) d\mu(x) \\
\times \int_1^1 g_\alpha(x) \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x).
\]

By induction,

\[
\left| \int_1^1 g_\alpha(x) \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x) \right| \leq \frac{C^*}{|q_\alpha(a_k)|} + \sum_{i=1}^{i} \frac{|q^{(i)}_\alpha(a_k)|}{|q_\alpha(a_k)|} \frac{C_{i+1} n^i}{|q_\alpha(a_k)|}
\]

but, since \( \mu \) belongs to the Szegő class, \( \frac{|q^{(i)}_\alpha(a_k)|}{|q_\alpha(a_k)|} \leq C n^i \), and

\[
\left| \int_1^1 g_\alpha(x) \frac{w_N(x)}{(x - a_k)^{i+1}} d\mu(x) \right| \leq \frac{C^*}{|q_\alpha(a_k)|} + \frac{C n^i}{|q_\alpha(a_k)|} \sum_{i=1}^{i} \frac{C_{i+1} n^i}{|q_\alpha(a_k)|} \leq \frac{C_{i+1} n^i}{|q_\alpha(a_k)|}
\]

for some constant \( C_{i+1} \) and \( n \) large enough.

**Corollary 3.4.** If \( \mu \) belongs to the Szegő class, there is a constant \( C \) such that

\[
|\hat{B}_n^{(i)}(a_k)| \leq C \frac{n^i}{|a_k + \sqrt{a_k^2 - 1}|}
\]

for \( i = 0, \ldots, N_k, \; k = 1, \ldots, K \) and \( n \) large enough.

**Proof.** We use induction again. Since

\[
0 = \left( \frac{B_n(x)}{x - a_k} \frac{w_N(x)}{x - a_k} \right)
\]

\[
= \int_1^1 B_n(x) \frac{w_N(x)}{x - a_k} d\mu(x) + M_k N_k \hat{B}_n^{(N_k+1)}(a_k) \frac{w_N^{(N_k+1)}(a_k)}{N_k + 1},
\]

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we get
\[ |\hat{B}_n^{(N)}(a_k)| = \frac{N_k + 1}{M_{k,k_1}w_N^{(N_1+1)}(a_k)} \left| \int_1^{N} \hat{B}_n(x) \frac{w_N(x)}{x - a_k} d\mu(x) \right|. \]

Hence, Lemmas 2.1 and 3.2 give \( \hat{B}_n^{(N)}(a_k) = O\left( \frac{1}{(a_k + \sqrt{a_k^2 - 1})^n} \right) \).

We assume \( \hat{B}_n^{(N)}(a_k) = O\left( \frac{a_k^n}{(a_k + \sqrt{a_k^2 - 1})^n} \right) \) for \( 0 \leq j \leq i < N_k \). Then, we have
\[
0 = \left\langle \hat{B}_n(x), \frac{w_N(x)}{(x - a_k)^{N_1}} \right\rangle = \int_1^{N} \hat{B}_n(x) \frac{w_N(x)}{(x - a_k)^{N_1}} d\mu(x)
+ M_{k,k+1} \hat{B}_n^{(i+1)}(a_k) \frac{(i + 1)w_N^{(N_1+1)}(a_k)}{(N_k + 1)!} + O\left( \frac{n^{N_1} (i+2)}{(a_k + \sqrt{a_k^2 - 1})^n} \right),
\]
whence
\[
\hat{B}_n^{(i+1)}(a_k) = \frac{-(N_k + 1)!}{M_{k,k+1}(i + 1)!w_N^{(N_1+1)}(a_k)} \int_1^{N} \hat{B}_n(x) \frac{w_N(x)}{(x - a_k)^{N_1}} d\mu(x)
+ O\left( \frac{n^{N_1} (i+2)}{(a_k + \sqrt{a_k^2 - 1})^n} \right).
\]

But, from Lemmas 2.1 and 3.2,
\[
\int_1^{N} \hat{B}_n(x) \frac{w_N(x)}{(x - a_k)^{N_1}} d\mu(x) = O\left( \frac{n^{N_1} (i+1)}{(a_k + \sqrt{a_k^2 - 1})^n} \right).
\]

Then \( \hat{B}_n^{(i+1)}(a_k) = O\left( \frac{n^{N_1} (i+1)}{(a_k + \sqrt{a_k^2 - 1})^n} \right) \). This completes the proof. 

4. FOURIER SERIES

In Lemma 2.2 we proved that, with the inner product (1),
\[
\mathcal{S} = \left\{ f(x) : \int_1^{N} |f(x)|^2 d\mu(x) < \infty, f^{(i)}(a_k) \text{ exists for } i = 0, \ldots, N_k, \ k = 1, \ldots, K \right\}
\]
is a Hilbert space and the polynomials constitute a maximal family. Then, 
\[ S_n(f) \to f \text{ in } \mathcal{S} \text{ for any function } f \in \mathcal{S}, \]
where 
\[ S_n(x; f) = \sum_{k=0}^{n} \langle f, \mathcal{B}_k \rangle \mathcal{B}_k(x) \]
is the \( n \)th partial sum of the Fourier–Sobolev series of \( f \). Write \( \|f\|_{\mathcal{S}}^2 = \|f\|_{L^2(\mu)}^2 + \|f\|_{H}^2 \). Convergence in \( \mathcal{S} \) induces convergence in \( L^2(\mu) \) as well as convergence for the derivatives at the points \( a_k \) because \( \|f\|_{H}^2 \leq \|f\|_{\mathcal{S}}^2 \) and \( \|f\|_{L^2(\mu)}^2 \leq \|f\|_{\mathcal{S}}^2 \). So, for any function \( f \) in \( \mathcal{S} \), we have
\[ S_n(x; f) \to f(x), \quad S_n^0(a_k; f) \to f^0(a_k), \quad 0 \leq i \leq N_k, \quad k = 1, \ldots, K. \]

For \( i = 0, \ldots, N_k \) and \( k = 1, \ldots, K \), let us consider the functions \( f_{k,i} \) such that \( f_{k,i}(x) = 0, \ x \in [-1, 1] \), \( f_{k,j}(a_i) = 1 \) when \( t = k, \ j = i \), and 0 otherwise. Since \( S_n(f_{k,i}) \) converges to \( f_{k,i} \) in \( \mathcal{S} \) and \( \langle f_{k,i}, \mathcal{B}_k \rangle = M_{k,i} \mathcal{B}_k(a_k) \), we get
\[
\sum_{n=0}^{\infty} \mathcal{B}_n(a_k) \mathcal{B}_n(x) = 0, \quad \sum_{n=0}^{\infty} \mathcal{B}_n(a_k) \mathcal{B}_n(a_i) = 0, \quad t \neq k \text{ or } j \neq i,
\]
\[
\sum_{n=0}^{\infty} (\mathcal{B}_n(a_k))^2 = \frac{1}{M_{k,i}}.
\]
Let \( \mu \) be the Jacobi measure, \( d \mu(x) = (1-x)^\alpha (1+x)^\beta \, dx, \ x > -1, \ \beta > -1, \) and let \( p_\alpha(x) = p_\alpha^{(x,\beta)}(x) \) the corresponding orthonormal polynomials (from now on, the orthonormal Jacobi–Sobolev polynomials). As it is well known (see [8, Theorem 3.14, p. 101]) that
\[
(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4} |p_\alpha(x)| \leq C, \quad x \in [-1, 1].
\]
Let \( \mathcal{B}_n(x) = B_n^{(x,\beta)}(x) \) be the orthonormal polynomials with respect to the inner product (1) when \( \mu \) is the Jacobi measure. Corollary 3.3 yields the uniform bound
\[
|\mathcal{B}_n(x)| \leq \frac{C}{(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}} = h(x), \quad x \in (-1, 1).
\]
From inequality (4) and Corollary 3.4, the series \( \sum_{n=0}^{\infty} \mathcal{B}_n(a_k) \mathcal{B}_n(x), \ 0 \leq i \leq N_k \), has the majorant \( \sum_{n=0}^{\infty} C n^{\alpha/2} |(a_k - \sqrt{a_k^2 - 1})^n| \) in compact sets of \( (-1, 1) \) for some constant \( C \). Then, the series is a continuous function in \( (-1, 1) \). But we have convergence to 0 in \( L^2(\mu) \) for the series. Hence, it has a subsequence which converges pointwise to 0 a.e. As a consequence, \( \sum_{n=0}^{\infty} \mathcal{B}_n(a_k) \mathcal{B}_n(x) = 0 \) for all \( x \in (-1, 1) \). We summarize the above as follows.
**Theorem 4.1.** Let \( \hat{B}_n(x) \) be the orthonormal Jacobi Sobolev polynomials. Then,

1. \( \sum_{n=0}^{\infty} \hat{B}_n^{(j)}(a_k) \hat{B}_n(x) = 0 \) for every \( x \in (-1, 1) \), \( i = 0, \ldots, N_0 \) and \( k = 1, \ldots, K \).
2. \( \sum_{n=0}^{\infty} \hat{B}_n^{(j)}(a_k) \hat{B}_n^{(i)}(a_t) = 0 \) for \( t \neq k \) or \( j \neq i \).
3. \( \sum_{n=0}^{\infty} (\hat{B}_n^{(j)}(a_k))^2 \leq \frac{1}{\Delta_{j^2}} \) for \( i = 0, \ldots, N_0 \) and \( k = 1, \ldots, K \).

From now on, we will study the pointwise convergence of \( S_n(f) \) to \( f \) on the interval \([-1, 1]\) when there are standard sufficient conditions for the function \( f \). First of all, we need the analogous of the Christoffel–Darboux formula for the Sobolev polynomials but, if \( x_0 \in [-1, 1] \), the polynomial \( w_N(x) - w_N(x_0) \) can have two zeros in the interval \([-1, 1]\) when there are points \( a_k \) in \((-\infty, -1)\) and in \((1, \infty)\) simultaneously. Then, this polynomial is not convenient for representing the Dirichlet kernel. Instead of \( w_N(x) \), we will consider a different polynomial which also allows a Christoffel–Darboux-type formula and which has better properties. Let \( w_{N+1}(x) = \int_0^1 w_N(t) \, dt \) and let \( c = \min\{w_{N+1}(x) : x \in [-1, 1]\} \). Let \( w_{N+1}(x) \) be the polynomial \( w_{N+1}(x) + |c| + 1 \). It is clear that \( w_{N+1}(x) \) does not have zeros in \([-1, 1]\) and, when \( x_0 \in [-1, 1] \), \( w_{N+1}(x) - w_{N+1}(x_0) \) has the only zero \( x_0 \) in \([-1, 1]\) because its derivative \( w_N(x) \) does not vanish at this interval. The important facts are that \( \frac{w_{N+1}}{w_{N+1}(x)} \) in \([-1, 1]\) and that we can obtain an expression for the Dirichlet kernel in terms of \( w_{N+1}(x) \). Since the derivatives of \( w_{N+1}(x) \) are equal to zero at the points \( a_k \),

\[
\langle w_{N+1}(x)f(x), g(x) \rangle = \langle f(x), w_{N+1}(x)g(x) \rangle,
\]

and, as a consequence, we have the following recurrence relations for the polynomials \( \hat{B}_n(x) \),

\[
w_{N+1}(x)\hat{B}_n(x) = \sum_{k=0}^{N+1} \alpha_{n,k} \hat{B}_{n+k}(x) + \sum_{k=1}^{N+1} \alpha_{n,k} \hat{B}_{n-k}(x).
\]

Moreover, the coefficients \( \alpha_{n,k} \) are bounded because

\[|\alpha_{n,k}| = |\langle w_{N+1} \hat{B}_n, \hat{B}_{n+k} \rangle| \leq \int_1^1 \hat{B}_n(x) \hat{B}_{n+k}(x) w_{N+1}(x) \, d\mu(x) + \sum_{i=1}^K \int_1^1 M_{i,0} w_{N+1}(a_i) \hat{B}_n(a_i) \hat{B}_{n+k}(a_i) \, d\mu(x) \]

and the first term is bounded by \( \max_{x \in [-1, 1]} |w_{N+1}(x)| \) and the other one is also bounded from Corollary 3.4.

The Christoffel–Darboux formula takes now the following form (see [4]).
Lemma 4.1. Orthonormal polynomials with respect to the inner product (1) satisfy the following Christoffel Darboux-type formula:

\[
\{w_{N+1}(x) - w_{N+1}(y)\} \sum_{n=0}^{v} \hat{B}_n(x) \hat{B}_n(y) = z_{v+1}(\hat{B}_{v+1}(x) \hat{B}_n(y) - \hat{B}_{v+1}(y) \hat{B}_n(x))
\]

\[
+ z_{v,2}(\hat{B}_{v+2}(x) \hat{B}_n(y) - \hat{B}_{v+2}(y) \hat{B}_n(x)) + \cdots + z_{v,N+1}(\hat{B}_{v+N+1}(x) \hat{B}_n(y) - \hat{B}_{v+N+1}(y) \hat{B}_n(x))
\]

Furthermore, if the measure belongs to the Szegö class, the coefficients are bounded.

Proof. As usual, from (5) we have

\[
w_{N+1}(x) \hat{B}_n(x) \hat{B}_n(y) = \sum_{k=0}^{N+1} z_{n,k} \hat{B}_{n+k}(x) \hat{B}_n(y) + \sum_{k=1}^{N+1} z_{n,k} \hat{B}_{n+k}(x) \hat{B}_n(y),
\]

\[
w_{N+1}(y) \hat{B}_n(x) \hat{B}_n(y) = \sum_{k=0}^{N+1} z_{n,k} \hat{B}_{n+k}(x) \hat{B}_n(y) + \sum_{k=1}^{N+1} z_{n,k} \hat{B}_{n+k}(x) \hat{B}_n(y).
\]

Then

\[
\{w_{N+1}(x) - w_{N+1}(y)\} \hat{B}_n(x) \hat{B}_n(y) = \sum_{k=N+1}^{1} z_{n,k} (\hat{B}_{n+k}(x) \hat{B}_n(y) - \hat{B}_{n+k}(y) \hat{B}_n(x))
\]

\[
- \sum_{k=1}^{N+1} z_{n,k} (\hat{B}_{n+k}(x) \hat{B}_n(y) - \hat{B}_{n+k}(y) \hat{B}_n(x)).
\]

Writing \(F_n^a(x, y) = x^n (\hat{B}_{n+k}(x) \hat{B}_n(y) - \hat{B}_{n+k}(y) \hat{B}_n(x))\) and taking into account that \(F_n^a(x, y) = 0\) for negative integer values of \(n\), we get

\[
\{w_{N+1}(x) - w_{N+1}(y)\} \sum_{n=0}^{v} \hat{B}_n(x) \hat{B}_n(y)
\]

\[
= \sum_{n=0}^{v} \{F_n^1(x, y) - F_n^1(y, x)\} + \sum_{n=0}^{v} \{F_n^2(x, y) - F_n^2(y, x)\} + \cdots + \{F_n^{N+1}(x, y) - F_n^{N+1}(y, x)\}
\]

\[
= F_1^1(x, y) + F_1^2(x, y) + F_1^3(x, y) + \cdots + F_v^{N+1}(x, y)
\]
then

\[
\begin{align*}
= & x_{v,3}(\hat{B}_{v+1}(x)\hat{B}_t(y) - \hat{B}_{v+1}(y)\hat{B}_t(x)) + x_{v,2}(\hat{B}_{v+2}(x)\hat{B}_t(y) - \hat{B}_{v+1}(y)\hat{B}_t(x)) \\
& + x_{v,1}(\hat{B}_{v+1}(x)\hat{B}_t(y) - \hat{B}_{v+1}(y)\hat{B}_t(x)) + \cdots \\
& + x_{v,N+1}(\hat{B}_{v+N+1}(x)\hat{B}_t(y) - \hat{B}_{v+N+1}(y)\hat{B}_t(x)) + \cdots \\
& + x_{v,N,N+1}(\hat{B}_{v+N+1}(x)\hat{B}_t(x) - \hat{B}_{v+N+1}(y)\hat{B}_t(y)).
\end{align*}
\]

**Theorem 4.2.** Let \(x_0 \in (-1,1)\) and let \(f\) be a function with derivatives at the points \(a_k\) such that \(\frac{f(x_0)}{x_0}\) belongs to \(L^2(\mu)\) where \(\mu\) is the Jacobi measure. Then,

(i) \[
\sum_{n=0}^{\infty} \langle f, \hat{B}_n \rangle \hat{B}_n(x_0) = f(x_0).
\]

(ii) \[
\sum_{n=0}^{\infty} \langle f, \hat{B}_n \rangle \hat{B}_n^{(i)}(a_k) = f^{(i)}(a_k), \quad i = 0, \ldots, N_k, \quad k = 1, \ldots, K.
\]

**Proof.** Since \(f \in L^2(\mu)\) provided that \(\frac{f(x_0)}{x_0}\) \(\in L^2(\mu)\), (ii) is proved. Thus, we only need to prove (i). Let us denote \(D_n(x, t) = \sum_{j=0}^{\infty} \hat{B}_j(x)\hat{B}_j(t)\). We have

\[
\begin{align*}
\langle f(x_0) - S_n(x_0; f) \rangle &= \langle f(x_0) - f(t), D_n(x_0, t) \rangle \\
&= \int_1^1 (f(x_0) - f(t)) D_n(x_0, t) \, d\mu(t) \\
&+ \sum_{k=1}^K M_{k,0}(f(x_0) - f(a_k)) D_n(x_0, a_k) \\
&+ \sum_{k=1}^K \sum_{i=1}^{N_k} M_{k,i} f^{(i)}(a_k) \frac{\partial^i D_n}{\partial x^i}(x_0, a_k).
\end{align*}
\]

But Theorem 4.1 yields \(\lim_{n \to \infty} D_n(x_0, a_k) = \lim_{n \to \infty} \frac{\partial D_n}{\partial x}(x_0, a_k) = 0\) for \(i = 1, \ldots, N_k, \quad k = 1, \ldots, K\). Then,

\[
\lim_{n \to \infty} \langle f(x_0) - S_n(x_0; f) \rangle = \lim_{n \to \infty} \int_1^1 (f(x_0) - f(t)) D_n(x_0, t) \, d\mu(t).
\]

From the Christoffel–Darboux formula (Lemma 4.1), \(D_n(x_0, t)\) is a sum of a finite number of terms depending on \(N\) of the following type:

\[
x_{v,\ell, \frac{j}{wN+1}(x_0)} \hat{B}_{v+i}(x_0) \hat{B}_{v+j}(t), \quad 0 \leq i \leq N, \quad 1 \leq j \leq N + 1.
\]
Taking into account that

\[
\left| \int_1^1 (f(x) - f(t))z_{a_i} w_{N+1}(x_0) w_{N+1}(t) d\mu(t) \right|
\]

as well as \( |\hat{B}_{a_i}(\cdot(x_0))| \leq h(x_0) \) from (4), and that \( z_{a_i} \) are bounded from Lemma 4.1, since \( \frac{f(x)}{x_0} \frac{f(t)}{x_0} \frac{z_{a_i}}{w_{N+1}(x_0)} \) belongs to \( L^2(\mu) \) because \( \frac{f(x)}{x_0} \frac{f(t)}{x_0} \frac{z_{a_i}}{w_{N+1}(x_0)} \) is a continuous function at \([-1, 1] \) and, by hypothesis, \( \frac{f(x)}{x_0} \frac{f(t)}{x_0} \frac{z_{a_i}}{w_{N+1}(x_0)} \) belongs to \( L^2(\mu) \), Lemma 2.1 gives

\[
\lim_{n \to \infty} z_{a_i} \hat{B}_{a_i}(\cdot(x_0)) \int_1^1 \frac{f(x_0) - f(t)}{x_0 - t} \frac{x_0 - t}{w_{N+1}(x_0) - w_{N+1}(t)} d\mu(t) = 0.
\]

Hence, \( \lim_{n \to \infty} (f(x_0) - S_n(x_0; f)) = 0 \) and the proof is complete.  \\[\]

**Theorem 4.3.** Let \( f(x) \) be a function with derivatives at the points \( a_k \) satisfying a Lipschitz condition of order \( \eta < 1 \) uniformly in \([-1, 1] \), i.e., \( |f(x + h) - f(x)| \leq M|h|^{\eta} \) for \( |h| < \delta \) and for some \( \delta > 0 \). If \( c_n = \langle f, \hat{B}_{a_k} \rangle \), then

\[
\sum_{n=0}^{\infty} c_n \hat{B}_{a_k}(x) = f(x), \quad x \in (-1, 1),
\]

and the convergence is uniform in \([-1 + \varepsilon, 1 - \varepsilon] \) for every \( \varepsilon \) such that \( 0 < 2 \varepsilon < 1 \). Moreover, \( \sum_{n=0}^{\infty} c_n \hat{B}_{a_k}^2(a_k) = f^{(0)}(a_k) \) for \( i = 0, \ldots, N_k \) and \( k = 1, \ldots, K \).

**Proof.** In the same way as before, we only need to prove that \( \int_1^1 f(t) D_n(x, t) d\mu(t) \) converges to \( f(x) \) for \( x \in (-1, 1) \). Besides,

\[
\left| \int_1^1 (f(x) - f(t)) D_n(x, t) d\mu(t) \right| = \int_{|x - t| \geq \delta} (f(x) - f(t)) D_n(x, t) d\mu(t) + \int_{|x - t| < \delta} (f(x) - f(t)) D_n(x, t) d\mu(t) = f^{(1)}(x) + f^{(2)}(x).
\]

Since \( \frac{f(x)}{w_{N+1}(x_0)} \frac{f(t)}{w_{N+1}(t)} (1 - \chi_{[\delta x \in \delta \mu \in \delta]}(t)) \), where \( \chi_{[\delta x \in \delta \mu \in \delta]}(t) \) is the characteristic function of the interval, belongs to \( L^2(\mu) \), using Christoffel-Darboux formula and the same procedure as in the previous Theorem, the term \( f^{(1)}(x) \) tends to zero.
On the other hand, $I_{n}(x)$ is a sum of a finite number of terms

$$z_n \sum_{s \in S} \int_{[0,1]} \frac{f(x) - f(t)}{x - t} \frac{x - t}{w_n(x) - w_n(t)} \hat{B}_n(t) d\mu(t),$$

where the coefficients $z_n \sum_{s \in S} \hat{B}_n(t)$ are uniformly bounded in closed subsets of $(-1, 1)$ from Lemma 4.1 and (4). Furthermore, when $x$ belongs to $(-1, 1)$, the Lipschitz condition gives

$$\left| \int_{[0,1]} \frac{f(x) - f(t)}{x - t} \frac{x - t}{w_n(x) - w_n(t)} \hat{B}_n(t) d\mu(t) \right| \leq C \int_{[0,1]} \frac{d\mu(t)}{|x - t|^\gamma}$$

where the constant $C$ depends on $\max\{\frac{\|f\|_{L^2}}{|w_n(t)|} : t \in [-1, 1]\}$, on the constant of the Lipschitz condition and on $h(x)$, where $h(x)$ is the function such that $|\hat{B}_n(x)| \leq h(x)$ on the interval $(-1, 1)$. Hence, since $\mu$ is the Jacobi measure, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|I_{n}(x)| < \varepsilon$ and the pointwise convergence is proved. The uniform convergence in a compact subset $F$ of $(-1, 1)$ is an easy consequence of the uniform continuity of $\frac{f(t)}{w_n(t)}$ when $(y, t)$ belong to $\{(y, t) : |y - t| \leq \delta, |t - x| \leq \delta, x, y \in F\}$ for a fixed $x \in F$ and for a fixed $\delta$ such that $\int_{\mathbb{R}} \frac{d\mu(t)}{|x - t|^\gamma} < \varepsilon$, and of the compactness of $F$.

As usual, denote

$$w(\delta) = w(\delta, f) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [-1, 1], |x_1 - x_2| \leq \delta\},$$

the modulus of continuity of a function $f(x)$ in $[-1, 1]$.

**Theorem 4.4.** Let $f(x)$ be a function such that its modulus of continuity $w(\delta)$ satisfies the condition

$$w(\delta) = O\left(\log \left(1 + \frac{1}{\delta}\right)^\frac{1}{2}\right)$$

for $\varepsilon > 0$ with derivatives at the points $a_k$. If $\ell_2 = \langle f, \hat{B}_n(t) \rangle$, then $\sum_{n=0}^{\infty} c_n \hat{B}_n(x) = f(x)$ a.e. in $[-1, 1]$. Moreover, $\sum_{n=0}^{\infty} c_n B_n^{(i)}(a_k) = f^{(i)}(a_k)$ for $i = 1, \ldots, N_k$ and $k = 1, \ldots, K$.

**Proof.** Note that $\sum_{n=0}^{\infty} c_n B_n^{(i)}(a_k) = f^{(i)}(a_k)$ holds because $f(x)$ belongs to $S$ and the only thing to prove is the a.e. convergence in $[-1, 1]$.

We consider again the polynomial $w_n(x)$ and the orthonormal polynomials $q_n(x)$ with respect to $w_n(x) d\mu(x)$. Since $w_n(x)$ has no zeros in $[-1, 1]$, the modulus of continuity of $\frac{f(t)}{w_n(t)}$ satisfies the condition $w(\delta, \frac{f(t)}{w_n(t)}) = O(\log \left(1 + \frac{1}{\delta}\right)^\frac{1}{2})$.

Let $d_n = \int_{[-1, 1]} f(x) q_n(x) d\mu(x)$ be the Fourier coefficients of $\frac{f(t)}{w_n(t)}$ in terms of $q_n(x)$. By Jackson’s Approximation Theorem (see [8, Chapt. I], there is a
polynomial $p_n(x)$ such that $|\frac{f(x)}{w_N(x)} - p_n(x)| = O\left(\frac{1}{\log^2 n}\right)$. Hence,

$$
\sum_{k=n}^{\infty} c_k^2 = \int_1^1 \left( \frac{f(x)}{w_N(x)} - p_n(x) \right)^2 w_N(x) \, d\mu(x) = O\left(\frac{1}{\log^2 \gamma n}\right). \tag{6}
$$

From Lemma 2.1,

$$
c_n = \langle f, \tilde{B}_n \rangle = \sum_{i=0}^{N} A_{n,i} d_{n,i} + \sum_{i=1}^{K} \sum_{j=0}^{N_i} M_i, f^{(i)}(a_k) \tilde{B}_n^{(j)}(a_k).
$$

From the bounds of Corollary 3.4 and taking into account the Cauchy–Schwarz inequality, i.e. $|\sum_{k=n}^{\infty} d_k d_{k,i}| \leq (\sum_{k=n}^{\infty} d_k^2)^{1/2} (\sum_{k=n}^{\infty} d_{k,i}^2)^{1/2}$, Eq. (6) gives

$$
\sum_{k=n}^{\infty} c_k^2 = O\left(\frac{1}{\log^2 \gamma n}\right).
$$

As a consequence (see [8, Theorem 3.3, p. 137]), $\sum_{n=0}^{\infty} c_n^2 \log^2 n < \infty$ and thus (see [8, Theorem 2.5, p. 126]), $\sum_{n=0}^{\infty} c_n \tilde{B}_n(x)$ converges a.e. $x \in [-1, 1]$ to some function $g(x)$ (taking into account that $\|f\|^2_2 \leq \|f\|^2_1$ for any $f \in \mathbb{S}$). But $f(x)$ belongs to $\mathbb{S}$ by continuity, so convergence in $\mathbb{S}$ of $\sum_{n=0}^{\infty} c_n \tilde{B}_n(x)$ to $f(x)$ gives $g(x) = f(x)$ a.e. \[\blacksquare\]

ACKNOWLEDGMENTS

The work of F. Marcellán was supported by a grant of Dirección General de Investigación (Ministerio de Ciencia y Tecnología) of Spain BFM 2000 0206 C04 01 and by an INTAS Grant 2000/272.

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