

On two models of orthogonal polynomials and their applications

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Abstract

This contribution deals with some models of orthogonal polynomials as well as their applications in several areas of mathematics. Some new trends in the theory of orthogonal polynomials are summarized. In particular, we emphasize on two kinds of orthogonality, i.e., the standard orthogonality in the unit circle and a non standard one, which is called multi-orthogonality. Both have attracted the interest of researchers during the past ten years.

Key words: *Linear functionals, Orthogonal polynomials, Stieltjes and Markov functions, rational approximation, Hermite-Padé approximation, multiple orthogonal polynomials, trigonometric Moment problem, Quadratures on the unit circle, linear prediction, electrostatics of zeros.*

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1 Introduction

The theory of orthogonal polynomials has experienced a relevant growth and an increasing interest during the last twenty years as a consequence, among others, of a substantial reevaluation of our perception concerning their nature and applicability. The popularity of orthogonal polynomials is due, in particular, to Louis de Branges's solution of the Beiberbach conjecture which uses an inequality of Askey and Gasper on Jacobi polynomials. But the main reason lies in their wide applications in many areas as approximation theory (Padé approximations [50], continued fractions) numerical analysis, scattering theory, digital signal processing, electrical engineering, theoretical chemistry, solid

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state physics (Toda lattices) atomic and nuclear physics (eigenfunctions of Hamiltonians) and so forth.

The contemporary general theory of orthogonal polynomials was initiated by G. Szegő in a series of papers starting in 1915. It was Szegő who realized that real orthogonal polynomials can better be understood by first studying complex orthogonal polynomials on the unit circle. His monograph “Orthogonal polynomials” [55], whose first edition was published in 1939, has been one of the basic and most popular references on the subject. In the decade of the seventies the connection with special functions (mostly hypergeometric and basic hypergeometric functions) provided a creative approach to new ideas coming from mathematical physics (quantum oscillators, Schrödinger equations, and Klein-Gordon equations among others). The monographs by G. Gasper and M. Rahman “Basic hypergeometric series” [25] as well as the books by A. F. Nikiforov and V. B. Uvarov [48] and A. F. Nikiforov, S. K. Suslov and V. B. Uvarov [47] constitute a good sample of this approach. For more information the survey contribution [3] is a nice presentation for the non specialized reader.

On the other hand, the new trends in numerical quadrature [24], spectral methods for boundary value problems [11, 26] and the powerful tools derived of the recent progress in potential theory have consolidated the research not only from a theoretical point of view but by the interactions with other domains as harmonic analysis, operator theory and matrix analysis.

The starting point is the concept of orthogonality with respect to a measure μ supported on an infinite subset Δ of the real line.

If we assume that $\int_{\Delta} p(x)d\mu(x)$ converges for every polynomial p , then we can introduce an inner product

$$\langle p, q \rangle = \int_{\Delta} p(x)q(x)d\mu(x),$$

where p, q are polynomials. For such an inner product, which we will say to be standard, a sequence of polynomials $\{p_n\}_{n \in \mathbb{N}}$ is said to be orthogonal with respect to the above inner product if

- (i) $p_n(x) = a_n x^n + \text{lower degree terms}$, and $a_n > 0$.
- (ii) $\langle p_n, p_m \rangle = \delta_{n,m}$, $m, n \in \mathbb{N}$.

From the definition of the inner product it is straightforward to prove that the sequence $\{p_n\}_{n \in \mathbb{N}}$ satisfies a three-term recurrence relation [19]

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad p_{-1}(x) = 0, \quad p_0(x) = 1, \quad (1)$$

where $a_n > 0$ and $b_n \in \mathbb{R}$. This recurrence relation contains a basic information both about the behavior of the polynomials as well as the properties of the orthogonality measure.

Non standard examples of orthogonality have been studied intensively during the last decades. For instance:

1. If the measure μ is supported in some domain of \mathbb{R}^N ($N \in \mathbb{N}$), then the orthogonality in the space of polynomials $\mathbb{R}[x_1, \dots, x_N]$ in N variables constitute an attractive research areas because of their connections with group theory, numerical cubature, etc. Unfortunately, very few monographs have been devoted to this subject and it constitutes one of the most promising subjects for the research in the next future [21].
2. If μ is a matrix of measures supported on some subset of the real line, the orthogonality in the space of matrix polynomials $\mathbb{R}^{N \times N}[x]$ has been analyzed in a wide set of contributions both from the analytical point of view as from their applications in numerical quadrature (see [22, 23] among others).
3. If $(\mu_0, \mu_1, \dots, \mu_r)$ is a vector of measures supported on subsets $(\Delta_0, \Delta_1, \dots, \Delta_r)$ of the real line, then an inner product on the linear space of polynomials

$$\langle p, q \rangle = \sum_{i=0}^r \int_{\Delta_i} p^{(i)}(x)q^{(i)}(x)d\mu_i(x),$$

yields an interesting family of polynomials (which are called Sobolev orthogonal polynomials) with potential applications in boundary value problems, smooth least square approximation, etc. [42, 45, 46].

In this contribution we will focus our attention in two kind of non standard families of orthogonal polynomials.

First, we will consider a positive Borel measure μ supported on the unit circle \mathbb{T} and we will introduce an Hermitian inner product

$$\langle p(z), q(z) \rangle = \int_{\mathbb{T}} p(z)\overline{q(z)}d\mu(z).$$

In Section 2 we will discuss some properties of the corresponding sequences of orthogonal polynomials. More precisely, we will analyze quadrature formulas related to knots located on \mathbb{T} as well as some recent results about differential properties of those orthogonal polynomials. As an interesting application we will present the modellization of optimal linear predictors for stationary discrete-time stochastic processes.

Second, we will deal with multiple orthogonal polynomials, i.e., we distribute the orthogonality conditions over a fixed number of intervals. In Section 3 we define two kinds of multiple orthogonality and we explain how they are closely related to simultaneous rational approximation of a system of Markov functions. Special emphasis will be done when the orthogonality conditions are considered with respect to discrete measures (Hahn, Meixner, Kravchuk and Charlier).

2 Orthogonal polynomials on the unit circle

Let μ be a probability measure supported on the unit circle $\mathbb{T} := \{z : |z| = 1\}$. Associated with μ we can introduce in the linear space \mathbb{P} of polynomials with

complex coefficients an inner product

$$\langle p(z), q(z) \rangle = \int_{\mathbb{T}} p(z) \overline{q(z)} d\mu, \quad p, q \in \mathbb{P}. \quad (2)$$

Taking into account the linear operator $\mathcal{H} : \mathbb{P} \mapsto \mathbb{P}$, $(\mathcal{H}p)(z) = zp(z)$ is a unitary operator with respect to the inner product (2), the Gram matrix \mathcal{T} associated with (2) in terms of the canonical basis $\{z^n\}_{n \in \mathbb{N}}$ has a special structure. The corresponding (j, k) entry is given by $\langle z^j, z^k \rangle = \langle z^{j-k}, 1 \rangle$ for $j \geq k$, and $\langle z^j, z^k \rangle = \langle 1, z^{k-j} \rangle$ for $j \leq k$.

In other words, in the first row of the infinite matrix we can concentrate the information about it. Indeed, the entries of every subdiagonal are equal. In the literature such a kind of matrices are called Toeplitz matrices. Let $c_n = \langle z^n, 1 \rangle$. We will say c_n is the n -th moment of μ .

If we use the Gram-Schmidt orthogonalization method for the canonical basis, we obtain a unique sequence of monic polynomials $\{\phi_n\}_{n \in \mathbb{N}}$ such that

$$\langle \phi_n(z), z^j \rangle = 0, \quad j = 0, 1, \dots, n-1.$$

Notice that this orthogonality condition means that the coefficients of the polynomials $\phi_n(z) = z^n + a_{n,n-1}z^{n-1} + \dots + a_{n,1}z + a_{n,0}$ are the solutions of a system of linear equations

$$\begin{array}{cccccc} a_{n,0}c_0 & + & a_{n,1}c_1 & + & \dots & + & a_{n,n-1}c_{n-1} & = & -c_n, \\ a_{n,0}\overline{c_1} & + & a_{n,1}c_0 & + & \dots & + & a_{n,n-1}c_{n-2} & = & -c_{n-1}, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{n,0}\overline{c_{n-1}} & + & a_{n,1}\overline{c_{n-2}} & + & \dots & + & a_{n,n-1}c_0 & = & -c_1. \end{array}$$

In other words

$$\mathcal{T}_n(a_{n,0}, a_{n,1}, \dots, a_{n,n-1})^T = -(c_n, c_{n-1}, \dots, c_1)^T, \quad (3)$$

where \mathcal{T}_n is the principal submatrix of \mathcal{T} with dimension n . Thus, from the orthogonality condition we get a system of linear equations whose matrix has an underlying Toeplitz structure. The solution of it yields in a natural way to the orthogonal polynomials with respect to the measure μ whose moments are $\{c_n\}_{n \in \mathbb{N}}$. They can also be characterized as the solution of the following extremal problem: *Minimize* $\langle p, p \rangle$ *over all monic polynomials* $p \in \mathbb{P}$.

In the sequel, we will consider the norm induced by the inner product (2) and we will denote it $\|p\| = \sqrt{\langle p, p \rangle}$.

From the orthogonality conditions it follows that the zeros of ϕ_n lie in the unit disk. Indeed, if $\phi_n(\alpha) = 0$, then from $\phi_n(z) = (z - \alpha)q_{n-1}(z)$ we get

$$0 < \|\phi_n\|^2 = (1 - |\alpha|^2) \|q_{n-1}\|^2,$$

i.e. $|\alpha| < 1$.

If we introduce the reversed polynomial $\phi_n^*(z) = z^n \overline{\phi_n(z^{-1})}$ we get

$$\langle \phi_n^*(z), z^k \rangle = 0, \quad 1 \leq k \leq n.$$

On the other hand,

$$\langle \phi_{n+1}(z) - z\phi_n(z), z^k \rangle = 0, \quad 1 \leq k \leq n.$$

Consequently, one gets

$$\phi_{n+1}(z) - z\phi_n(z) = \lambda_n \phi_n^*(z),$$

where $\lambda_n = \phi_{n+1}(0)$. The above expression leads to

$$\phi_{n+1}(z) = z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z), \quad (4)$$

which is a forward recurrence relation for the sequence $\{\phi_n\}_{n \in \mathbb{N}}$. Apparently, in order to obtain ϕ_{n+1} we only need the value $\phi_{n+1}(0)$ but it can be deduced from

$$\langle z\phi_n(z), 1 \rangle = -\phi_{n+1}(0) \langle \phi_n^*(z), 1 \rangle.$$

Taking into account

$$\langle \phi_n^*(z), 1 \rangle = \|\phi_n\|^2,$$

we get

$$\phi_{n+1}(0) = -\frac{c_{n+1} + \sum_{j=1}^n a_{n,j-1}c_j}{\|\phi_n\|^2},$$

together with

$$\|\phi_n\|^2 = c_0 + \sum_{j=1}^n \bar{a}_{n,n-j}c_j.$$

The recurrence relation (4) was introduced by G. Szegő and constitutes the polynomial counterpart of the Levinson algorithm for the solution of (3) in a finite number of steps [16]. The values $\{\phi_n(0)\}_{n \in \mathbb{N}}$ are called reflection parameters. Notice that $|\phi_n(0)| < 1$ for every $n \in \mathbb{N}$, which is clearly perceptible taking into account the fact that the zeros of ϕ_n lie in the unit disk.

On the other hand, the polynomial $\phi_{n+1}(z) - \phi_{n+1}(0)\phi_{n+1}^*(z)$ satisfies

$$\langle \phi_{n+1}(z) - \phi_{n+1}(0)\phi_{n+1}^*(z), z^k \rangle = 0, \quad 1 \leq k \leq n+1,$$

as well as it vanishes for $z = 0$. This means

$$\phi_{n+1}(z) - \phi_{n+1}(0)\phi_{n+1}^*(z) = s_n z\phi_n(z),$$

with $s_n = 1 - |\phi_{n+1}(0)|^2$. Hence,

$$\phi_{n+1}(z) = (1 - |\phi_{n+1}(0)|^2)z\phi_n(z) + \phi_{n+1}(0)\phi_{n+1}^*(z), \quad (5)$$

holds. The above expression (5) is a backward recurrence relation for the sequence $\{\phi_n\}_{n \in \mathbb{N}}$. It was introduced by G. Szegő and it is intimately related with the Schur-Cohn algorithm, a standard way to characterize the polynomials whose zeros lie in the unit disk [16]. This algorithm is very well known in the

theory of discrete linear systems and provides a tool for the study of their stability.

Notice that both recurrence relations (4) and (5) are substantially different with respect to the three-term recurrence relation associated with orthogonal polynomials in the real case. Here we need only one parameter in order to generate the sequence of monic polynomials but the reversed polynomial appears as a counterpart.

As a first conclusion, given a probability measure μ we can associate a sequence of moments $\{c_n\}_{n \in \mathbb{N}}$. From them, we obtain the reflection parameters $\{a_n\}_{n \in \mathbb{N}}$ where $a_n = \phi_n(0)$, and thus we get the sequence of monic polynomials orthogonal with respect to μ . What about the converse problem, i.e., how get the measure μ from the moments $\{c_n\}_{n \in \mathbb{N}}$ or from the reflection parameters $\{a_n\}_{n \in \mathbb{N}}$?

2.1 Trigonometric moment problem and quadrature formulas

Given a sequence of complex numbers $\{c_n\}_{n \in \mathbb{N}}$ the trigonometric moment problem consists in finding necessary and sufficient conditions for the existence of a probability measure μ supported on the unit circle such that

$$c_n = \int_{\mathbb{T}} z^n d\mu, \quad n \in \mathbb{N}.$$

Theorem 1 [30] *A necessary and sufficient condition for the solution of the trigonometric moment problem is*

$$\sum_{j,k} c_{j-k} x_j \bar{x}_k \geq 0,$$

for every $\mathbf{x} = (x_j)_{j \in \mathbb{N}}$, or, equivalently, the infinite Toeplitz matrix \mathcal{T} associated with the moments $\{c_n\}_{n \in \mathbb{N}}$ is positive-definite. Under this assumption, the measure μ is unique.

The basic question is to describe a constructive method to get the measure μ . There are two approaches to this problem.

The first one is based on the fact that the absolutely continuous measure $d\mu = \frac{1}{|\phi_n(z)|^2} \frac{dz}{2\pi iz}$ supported on the unit circle induces in \mathbb{P}_n (the linear space of polynomials with complex coefficients and degree less than or equal to n) the same inner product as μ . It is not so complicated to prove that μ_n converges to μ in the \star -weak topology [20].

The second one is related to quadrature formulas. Unfortunately, the Gaussian quadrature formulas for the real case cannot be considered on the unit circle because the zeros of orthogonal polynomials lie in the unit disk. Thus, it is not natural to recover the measure from mass points which do not live in the support of the measure. We proceed as follows:

Let $a_n(z) = \frac{z\phi_n(z)}{\phi_n^*(z)}$ a Blaschke product with $n+1$ zeros in the unit disk. Given $w \in \mathbb{T}$, the equation $a_n(z) = a_n(w)$ has $n+1$ roots on \mathbb{T} . We will

denote them $(\zeta_{n,j})_{j=0}^n$. Notice that w is a solution. We will order the roots according to the increasing arguments.

$$\zeta_{n,0} = w, \quad \arg \zeta_{n,j+1} \geq \arg \zeta_{n,j}, \quad j = 0, 1, \dots, n.$$

On the other hand, if we consider the kernel polynomials $\{K_n(x, y)\}_{n \in \mathbb{N}}$ associated with μ

$$K_n(x, y) = \sum_{j=0}^n \frac{\phi_j(z)\phi_j(y)}{\|\phi_j\|^2},$$

then we get a Christoffel-Darboux formula

$$K_n(x, y) = \frac{1}{\|\phi_{n+1}\|^2} \frac{\phi_{n+1}^*(x)\overline{\phi_{n+1}^*(y)} - \phi_{n+1}(x)\overline{\phi_{n+1}(y)}}{1 - x\bar{y}}.$$

This formula is the polynomial counterpart of the Gohberg-Semencul algorithm for the inversion of Toeplitz matrices [16].

Notice that $K_n(\zeta_{n,j}, w) = 0$ for $j = 1, 2, \dots, n$. Furthermore, the polynomial

$$B_{n+1}(z; w) = \phi_{n+1}(z) - \overline{\left(\frac{\phi_{n+1}^*(w)}{\phi_{n+1}(w)}\right)} \phi_{n+1}^*(z),$$

has as the set of zeros $\{\zeta_{n,j}\}_{j=0}^{n+1}$ and satisfies the orthogonality condition

$$\langle B_{n+1}(z; w), z^k \rangle = 0, \quad 1 \leq k \leq n.$$

They are called para-orthogonal polynomials.

Theorem 2 *The discrete measure $d\tilde{\mu}_n = \sum_{j=0}^n (\delta(z - \zeta_{n,j})K_n(\zeta_{n,j}, \zeta_{n,j}))^{-1}$ induces in \mathbb{P}_n the same inner product as μ . Furthermore, $\tilde{\mu}_n$ converges to μ in the \star -weak topology.*

The quadrature formula associated to $d\tilde{\mu}_n$ is called a Szegő quadrature formula [34]. It has been extensively studied in the last decade both from the numerical point of view as well as from an analytical point of view. From this perspective, L. B. Golinskii [29] proved:

Theorem 3 (i) *The sets $(\zeta_{n-1,j})_{j=0}^{n-1}$ and $(\zeta_{n,j})_{j=1}^n$ interlace, i.e., between two consecutive points of one of them there is exactly one point of the other.*

(ii) *If $\ln \mu' \in L^1(\mathbb{T})$, then*

$$|\zeta_{n,k+1} - \zeta_{n,k}| \leq \frac{C(\mu)}{\sqrt{n}},$$

for $k = 0, 1, \dots, n$ and $\zeta_{n,n+1} = \zeta_{n,0}$ as a convention.

(iii) If $(\mu')^{-r} \in L^1(\mathbb{T})$ for some $r > 0$, then

$$|\zeta_{n,k+1} - \zeta_{n,k}| \leq C(\mu, r) \frac{\ln n}{n}, \quad k \leq n, \quad \zeta_{n,n+1} = \zeta_{n,0},$$

where $C(\mu)$ and $C(\mu, r)$ are universal constants.

(iv) If μ is absolutely continuous and $0 < A \leq \mu' < B$, a.e. then

$$\frac{4}{n} \left(\frac{A}{B} \right)^{\frac{1}{2}} \leq |\zeta_{n,k+1} - \zeta_{n,k}| \leq \frac{4\pi}{n+1} \frac{B}{A}, \quad 1 \leq k \leq n, \quad \zeta_{n,n+1} = \zeta_{n,0}.$$

Thus, in terms of the properties of the measures we have upper estimates for the distance between two consecutive zeros of para-orthogonal polynomials.

Finally, a necessary and sufficient condition for the uniform distribution of zeros of para-orthogonal polynomials is given in the following:

Theorem 4 For a fixed $w \in \mathbb{T}$, the sequence $(\zeta_{n,j})_{j=0}^n$ $n \in \mathbb{N}$ is uniformly distributed in \mathbb{T} if and only if

$$\frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta}) d\mu \xrightarrow{*} \frac{d\theta}{2\pi}.$$

2.2 Differential equations

From the perspective of differential operators, polynomials orthogonal on the unit circle behave very differently with respect to the real case. Here, classical orthogonal polynomials (Jacobi, Laguerre, Hermite and Bessel) can be analyzed as eigenfunctions of second order differential operators with polynomial coefficients. S. Bochner [14] described all the polynomial solutions of a second order differential equation

$$a_2 y''(x) + a_1(x) y'(x) + a_0 y(x) = \lambda_n y(x),$$

where $(a_k)_{k=0}^2$ are polynomials of degree at most k . H. L. Krall [40] obtained the polynomial solutions of a fourth-order differential equation

$$\sum_{k=0}^4 a_k(x) y^{(k)}(x) = \lambda_n y(x), \quad \deg(a_k)_{k=0}^4 \leq k.$$

Among them three-families of polynomials orthogonal with respect to non absolutely continuous measures appear. He called them Legendre-type, Laguerre-type and Jacobi-type orthogonal polynomials appear. On the other hand, there is a characterization of classical orthogonal polynomials due to Sonin (and independently obtained by W. Hahn) in the sense that they are the families of orthogonal polynomials such that the sequence of their first derivatives constitutes a sequence of orthogonal polynomials.

During the last decade an interesting work was done in the study of differential properties of orthogonal polynomials on the unit circle. Unfortunately, as it was proved in [43], the analog of classical orthogonal polynomials in the Sonin-Hahn sense is reduced to the canonical family of polynomials $\{z^n\}_{n \in \mathbb{N}}$ which is orthogonal with respect to the Lebesgue measure supported on the unit circle. More recently, some works are done about the analysis of second order differential equations associated with polynomials orthogonal with respect to measures supported on the unit circle [1]. The goal was to give an electrostatic interpretation of their zeros. In [33] an absolutely continuous measure $d\mu = w d\theta$ is considered. Let w be positive and differentiable function in a neighborhood of the unit circle, evenmore, a function v is associated with w as follows: $w = \exp(-v)$. The function v is said to be an external field. If

$$\int_{\mathbb{T}} \frac{v'(z) - v'(y)}{z - y} y^n \frac{w(y) dy}{iy},$$

exists for every $n \in \mathbb{Z}$, then the corresponding monic orthogonal polynomials satisfy a differential relation

$$\phi'_n(z) = n(1 - |\phi_n(0)|^2)\phi_{n-1}(z) + M(z; n)\phi_n(z) + N(z; n)\phi_n^*(z), \quad (6)$$

where

$$\begin{aligned} M(z; n) &= i \int_{\mathbb{T}} \frac{v'(z) - v'(y)}{z - y} \frac{\phi_n(y)\overline{\phi_n(y)}}{\|\phi_n\|^2} w(y) dy \\ N(z; n) &= -i \int_{\mathbb{T}} \frac{v'(z) - v'(y)}{z - y} \frac{\phi_n(y)\phi_n^*(y)}{\|\phi_n\|^2} w(y) dy. \end{aligned}$$

If we assume $\phi_n(0) \neq 0$, then taking into account the forward recurrence relation we can rewrite (6) in the form

$$\phi'_n(z) = -A(z; n)\phi_n(z) + B(z; n)\phi_{n-1}(z). \quad (7)$$

Next we can define first order linear differential operators $\mathcal{L}_{n,1}$ and $\mathcal{L}_{n,2}$ as follows

$$\begin{aligned} \mathcal{L}_{n,1} &= \frac{d}{dz} + A(z; n) \\ \mathcal{L}_{n,2} &= -\frac{d}{dz} + C(z; n), \end{aligned}$$

where

$$C(n; z) = -A(z; n) + \frac{\|\phi_{n-1}\|^2}{\|\phi_n\|^2} \left(\frac{\phi_n(0)}{\phi_{n+1}(0)} + \frac{1}{z} \right) B(z; n).$$

Thus, the operators $\mathcal{L}_{n,1}$ and $\mathcal{L}_{n,2}$ are lowering and raising operators, respectively, i.e.

$$\begin{aligned} \mathcal{L}_{n,1}\phi_n(z) &= B(z; n)\phi_{n-1}(z) \\ \mathcal{L}_{n,2}\phi_n(z) &= \frac{B(z; n)}{z} \frac{\phi_n(0)}{\phi_{n+1}(0)} \left(1 - |\phi_n(0)|^2\right)^{-1} \phi_{n+1}(z). \end{aligned}$$

From both relations we deduce a second order linear differential equation for the polynomials ϕ_n [18, 33]

$$D(z; n)\phi_n''(z) + E(z; n)\phi_n'(z) + F(z; n)\phi_n(z) = 0, \quad (8)$$

where $D(z; n)$, $E(z; n)$ and $F(z; n)$ can be explicitly given in terms of $A(z; n)$ and $B(z; n)$. In particular, we get

$$\phi_n''(z) - \left[v'(z) + (n-1)z^{-1} + \frac{B'(z; n)}{B(z; n)} \right] \phi_n'(z) + G(z; n)\phi_n(z) = 0.$$

Notice that if $z_{n,k}$ is a zero of ϕ_n , then

$$\phi_n''(z_{n,j}) = \left[v'(z_{n,j}) + (n-1)z_{n,j}^{-1} + \frac{B'(z_{n,j}; n)}{B(z_{n,j}; n)} \right] \phi_n'(z_{n,j}).$$

This yields

$$\frac{\phi_n''(z_{n,j})}{\phi_n'(z_{n,j})} = v'(z_{n,j}) + (n-1)z_{n,j}^{-1} + \frac{B'(z_{n,j}; n)}{B(z_{n,j}; n)}.$$

Taking into account that

$$\frac{\phi_n''(z_{n,j})}{\phi_n'(z_{n,j})} = 2 \sum_{\substack{1 \leq k \leq n \\ k \neq j}} \frac{1}{z_{n,j} - z_{n,k}},$$

we get

$$v'(z_{n,j}) + (n-1)z_{n,j}^{-1} + \frac{B'(z_{n,j}; n)}{B(z_{n,j}; n)} + 2 \sum_{\substack{1 \leq k \leq n \\ k \neq j}} \frac{1}{z_{n,k} - z_{n,j}} = 0.$$

The left hand side of the above expression is the derivative of the function

$$H(z) = v(z) + (n-1) \ln z + \ln B(z; n) + \ln \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (z - z_{n,k})^2,$$

evaluated at $z = z_{n,j}$.

One can construct a real function

$$T(y_1, y_2, \dots, y_n) = \left| \prod_{j=1}^n y_j^{-(n+1)} \left[\frac{\exp(-v(z_j))}{B(y_j, n)} \right] \prod_{1 \leq j < k \leq n} (y_j - y_k)^2 \right|,$$

such that the zeros of ϕ_n are stationary points of this function. One can interpret this function as the total energy function for n unit charges in the unit disk interacting with a one-body confining potential $v(z) + \ln B(z; n)$, an attractive logarithmic potential with a charge $(n-1)$ at the origin, and repulsive logarithmic two-body potentials between pairs of charges. However, all the stationary points are saddle-points.

When v is a rational function, the external field is said to be semiclassical. In such a case, the functions $A(z; n)$ and $B(z; n)$ are rational functions.

Semiclassical orthogonal polynomials on the unit circle have attracted the interest of researchers during the last decade because they provide a constructive method of sequences of orthogonal polynomials [1, 2, 17, 27, 28, 32, 41].

Very few examples were known until such a moment in despite the development of an analytic theory for some very general families of measures supported on the unit circle. Among these families, we have [39]

1. The Szegő class of measures μ such that $\ln \mu' \in L^1(\mathbb{T})$ or, equivalently, $\phi_n(0) \in l_2$.
2. The Nevai class of measures μ such that $\phi_n(0) \rightarrow 0$. An element of such a class is $\mu' > 0$ a.e.
3. The Césaro-Nevai class of measures μ such that $\frac{1}{n+1} \sum_{k=0}^n |\phi_k(0)| \rightarrow 0$. In particular, the Rakhmanov measures μ defined by the condition $|\phi_n(e^{i\theta})|^2 d\mu \xrightarrow{*} \frac{\theta}{2\pi}$ belongs to the Césaro-Nevai class.

As a first example of semiclassical external field consider the weight function [56]

$$w(z) = |z - 1|^{2\alpha} |z + 1|^{2\beta} \frac{dz}{iz}.$$

In such a case, the reflection parameters are

$$\phi_n(0) = \frac{\alpha + (-1)^n \beta}{n + \alpha + \beta},$$

and the corresponding polynomials orthogonal with respect to $w(z)$ satisfy a second order linear differential equation (8) with

$$\begin{aligned} D(z; n) &= z(z^2 - 1) [(\alpha + (-1)^{n+1} \beta)z + (\alpha + (-1)^n \beta)] \\ E(z; n) &= \left[(\alpha + (-1)^{n+1} \beta)z + (\alpha + (-1)^n \beta) \right] \\ &\quad \left[(\alpha + \beta + 3 - n)z^2 - 2(\beta - \alpha)z + (n - 1) + \alpha + \beta \right] \\ &\quad - z(z^2 - 1)(\alpha + (-1)^{n+1} \beta) \\ F(z; n) &= \left[(\alpha + (-1)^{n+1} \beta)z + (\alpha + (-1)^n \beta) \right] [-(\alpha + \beta + 2)nz + (n - 1)(\beta - \alpha)] \\ &\quad + (\alpha + (-1)^{n+1} \beta)(nz^2 + (\beta - \alpha)z - (n + \alpha + \beta)). \end{aligned}$$

These polynomials are related to the Jacobi polynomials via the projective mapping of the unit circle onto the interval $[-1, 1]$, $z \mapsto \frac{1}{2}(z + z^{-1})$.

As a particular case ($\beta = 0$) we get the circular Jacobi orthogonal polynomials. They arise in a class fo random unitary matrix ensembles where the parameter α is related to the charge of an impurity fixed at $z = 1$ in a system of unit charges located on the unit circle at the complex values given by the eigenvalues of this matrix ensemble. In such a situation the monic orthogonal polynomials are hypergeometric polynomials $\phi_n(z) = \left(\frac{\alpha}{\alpha+n} \right) {}_2F_1(-n, \alpha + 1; -n + 1 - \alpha; z)$.

Other example of semiclassical orthogonal polynomials is related to the weight function

$$w(z) = \frac{1}{2\pi I_0(t)} \exp\left(\frac{t}{2}(z + z^{-1})\right),$$

where I_ν is the modified Bessel function. The corresponding system of orthogonal polynomials arises from studies of the length of longest increasing subsequences of random permutations and unitary matrix models.

Theorem 5 [33, 60]

(i) *The reflection parameters $a_n(t)$ for the above system of orthogonal polynomials satisfy a discrete Painlevé II equation*

$$a_{n+1} + \frac{2na_n}{t(1-a_n^2)} + a_{n-1} = 0, \text{ for } n \geq 1, \quad a_0(t) = 1, \quad a_1(t) = -\frac{I_1(t)}{I_0(t)}.$$

(ii) *As an alternative to this algebraic equation we get*

$$\begin{aligned} a_n'' &= \frac{1}{2} \left[\frac{1}{a_n + 1} - \frac{1}{a_n - 1} \right] (a_n')^2 - \frac{1}{t} a_n' \\ &\quad - a_n(1 - a_n^2) + \frac{n^2}{t^2} \frac{a_n}{1 - a_n^2}, \end{aligned}$$

with the boundary conditions determined by the expansion

$$a_n(t) \sim \frac{\left(-\frac{1}{2}t\right)^n}{n!} \left[1 + \left(\frac{n}{n+1} - \delta_{n,1} \right) \frac{1}{4} t^2 + \mathcal{O}(t^4) \right], \quad t \rightarrow 0,$$

for $n \geq 1$.

Finally, the modified Bessel orthogonal polynomials (ϕ_n) satisfy the differential equation

$$\frac{d\phi_n}{dt} = \frac{1}{2} \left[\frac{I_1(t)}{I_0(t)} + \frac{\phi_{n+1}(0)}{\phi_n(0)} \right] \phi_n(z) - \frac{1}{2} \left[1 + \frac{\phi_{n+1}(0)}{\phi_n(0)} z \right] (1 - |\phi_n(0)|^2) \phi_{n-1}(z),$$

for $n \geq 1$, where we have considered the derivative with respect to the dynamical parameter t .

On the other hand, taking into account

$$\frac{v'(z) - v'(t)}{z - t} = -\frac{t}{2} \left(\frac{1}{zt^2} + \frac{1}{z^2t} \right),$$

the relation (7) becomes

$$\begin{aligned} \frac{d\phi_n(z)}{dz} &= (1 - |\phi_n(0)|^2) \left[\left(n + \frac{t}{2z} + \frac{t}{2} \frac{\phi_{n-1}(0)}{\phi_n(0)} - \right. \right. \\ &\quad \left. \left. - \frac{t}{2} (1 - |\phi_{n+1}(0)|^2) \overline{\phi_{n+1}(0)} \phi_n(0) \right) \phi_{n-1}(z) - \frac{t}{2z} \frac{\phi_{n-1}(0)}{\phi_n(0)} \phi_n(z) \right]. \end{aligned}$$

2.3 An application to the prediction of time series

Suppose $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of complex random variables. This sequence is a discrete-time stochastic process, usually referred to as a time series in various applications [16]. The simplest case of such a process contains independent and identically distributed random variables with mean $E(z_n) = 0$ and finite variance $E(|z_n|^2) = \sigma^2$. Such a time series is said to be a white noise and has minimal prediction value: the knowledge of the past z_{n-k} ($k \in \mathbb{N}$) does not help in predicting the value z_n . In most applications the time series $\{x_n\}_{n \in \mathbb{Z}}$ contains random variables which are dependent. The knowledge of their dependence structure will be useful in the prediction of values from past observations. A special interest are the stationary time series for which

$$E(x_n) = 0, \quad E(\overline{x_n}x_{n+k}) = \text{cov}(x_n, x_{n+k}) = \gamma(k).$$

This means that the mean and the covariance are independent of the time n . In particular, the autocovariance $\gamma(k)$ depends only on the time lag k . The autocovariance of white noise is

$$\gamma(k) = \begin{cases} \sigma^2, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In such a case, if we only deal with the second order behavior of the time series then white noise can be generalized to uncorrelated random variables with zero mean and variance σ^2 , respectively, hence we write $z_n \sim WN(0, \sigma^2)$.

For the second order behavior of time series it is quite useful to consider the generating function

$$G(z) = \sum_{k \in \mathbb{Z}} \gamma(k)z^k. \tag{9}$$

This formal Laurent series will contain most information of the time series. We will assume that this Laurent series converges in some annulus, i.e. there exists a nonnegative real number $r < 1$ such that the series converges absolutely for $r < |z| < r^{-1}$. In particular, the series converges on the unit circle and thus

$$G(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \gamma(k)e^{ik\theta},$$

defines a function $S(\theta) = \frac{G(e^{i\theta})}{2\pi}$. Notice that $\gamma(k)$ can be written as the n -th Fourier coefficient of $S(\theta)$:

$$\gamma(k) = \int_0^{2\pi} S(\theta)e^{-ik\theta}.$$

The function $S(\theta)$ is said to be the spectral density of the time series.

A general linear process is a stationary time series of the form

$$x_n = \sum_{k \in \mathbb{Z}} \alpha_k z_{n-k}, \quad z_n \sim WN(0, \sigma^2).$$

It is straightforward to prove that $E(x_n) = 0$ and the autocovariance is $E(\bar{x}_n x_{n+k}) = \sigma^2 \sum_{k \in \mathbb{Z}} \alpha_j \bar{\alpha}_{j+k}$.

General linear processes are composed by means of a white noise but they have a certain covariance structure which makes possible to predict x_n using information of the time series at times different from n . One speaks of coloured noise in such a case. In practice, when one wants to predict the time series x_n at the present n , one only has observations x_{n-k} from the past ($k > 0$).

A time series of the form $x_n = \sum_{k=0}^{\infty} \alpha_k z_{n-k}$ containing only the white noise sequence of the past observations is known as a causal linear process. They are the most relevant in practice.

If we denote $\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^k$, the generating function of the parameters for the general linear process, then the generating function $G(z)$ introduced in (9) becomes

$$G(z) = \sigma^2 \bar{\alpha}(z^{-1}) \alpha(z).$$

In particular, the spectral density of a general linear process is

$$S(\theta) = \frac{\sigma^2}{2\pi} |\alpha(e^{i\theta})|^2,$$

which shows that the spectral density is a positive function on the unit circle. When α is a polynomial of degree m with $\alpha(0) = 1$, then the general linear process is causal and it is known as a moving average process $MA(m)$.

If $\alpha(z) = \frac{1}{\beta(z)}$, where β is a polynomial of degree ν with $\beta(0) = 1$ and the zeros of β lie outside the unit disk, then the process is again causal and is called an autoregressive process $AR(\nu)$. Such a process is given by

$$x_n + \sum_{k=1}^{\nu} \beta_k x_{n-k} = z_n, \quad z_n \sim WN(0, \sigma^2).$$

One of the main purposes of studying time series is to predict as good as possible the future from past observations. This means that we want to predict x_n by means of x_{n-k} ($k > 0$). For practical purposes, we will use only the last N observations from the past and thus we would like to predict x_n by means of a function $f(x_{n-1}, x_{n-2}, \dots, x_{n-N})$. Furthermore, we will restrict our analysis to linear predictors, i.e. linear functions of the N variables $x_{n-1}, x_{n-2}, \dots, x_{n-N}$. Our predictor $\hat{x}_n(N)$ can be expressed by

$$\hat{x}_n(N) = - \sum_{j=1}^N a_{N,j} x_{n-j}.$$

The coefficients $a_{N,j}$ can be determined by the condition about the prediction error

$$E \left(|\hat{x}_n(N) - x_n|^2 \right),$$

be minimal. Thus, we get the extremal problem

$$\min_{b_{N,0}=1} E \left(\left| \sum_{j=0}^N b_{N,j} x_{n-j} \right|^2 \right). \quad (10)$$

This problem can be transformed into an extremal problem in $L^2(\mu)$ where $\mu' = S(\theta)$.

Consider the inner product in $L^2(\mu)$ such that

$$\int_0^{2\pi} e^{-ik\theta} S(\theta) d\theta = \gamma(k).$$

Thus, the mapping $x_n \mapsto z^n$ gives an isometry between the inner product space for $\text{span}\{x_n\}$, $n \in \mathbb{Z}$ and \mathbb{P} with the $L^2(\mu)$ -norm. The expression (10) is equivalent to minimize

$$\int_0^{2\pi} |q_N(e^{i\theta})|^2 S(\theta) d\theta,$$

with the constraint $q_N(0) = 1$. But taking into account some previous result, we get $q_N(z) = \phi_N^*(z)$ where $\phi_N(z)$ is the monic polynomial of degree N which is orthogonal on the unit circle with respect to the measure μ with $\mu'(\theta) = S(\theta)$. The minimum is $\|\phi_N\|^2$.

If μ belongs to the Szegő class we have a useful interpretation in terms of prediction of stationary time series. Indeed, if $\ln \mu' \in L^1(\mathbb{T})$, then $\|\phi_N\| \rightarrow \alpha < \infty$ and thus it is not possible to find a linear predictor with a variance smaller than $\alpha^{-1} > 0$ even by allowing the complete past.

If $\ln \mu' \notin L^1(\mathbb{T})$, then in this case the predictor $\hat{x}_n(N)$ will converge to the actual random variable x_n if the number of observations in the past N increases. Such processes are called deterministic since we can predict the values in the future exactly from the information of the past.

A more general linear prediction problem can be formulated as follows: Given an integer number ν , let M_i be an increasing sequence of nested finite sets of integers such that $\nu \notin M_i$ and consider the construction of the best linear predictors of x_ν by elements x_n where $n \in M_i$. Let $e(M_i)$ denote the prediction error

$$e(M_i) = \min E \left(\left| x_\nu - \sum_{n \in M_i} c_n x_n \right|^2 \right).$$

This problem was first considered by Kolmogorov who proved that if the stochastic process $\{x_n\}_{n \in \mathbb{Z}}$ is real, $\nu = 0$ and $M_i = \{n : -i \leq n \leq i\}$ then

$$\lim_{i \rightarrow \infty} e(M_i) = \left(\frac{1}{\pi} \int_0^\pi \frac{d\theta}{S(\theta)} \right)^{-1}.$$

It follows that if $S(\theta)^{-1}$ is not integrable then the prediction error converges to zero.

The study of best linear predictors which special emphasis on the case when the reciprocal of the absolutely continuous part of the spectral function is not integrable, as well as the rate of convergence of $e(k_i)$ remains an open problem.

On the other hand, if $\{x_n\}$ and $\{y_n\}$ are two stationary discrete time stochastic processes with spectral densities $S(\theta)$ and $T(\theta)$, respectively, and

we assume S and T are comparable in the sense that $S(\theta) = g(\theta)T(\theta)$, then a natural question is to analyze the ratio

$$\lim_{i \rightarrow \infty} \frac{e(x_n; M_i)}{e(y_n; M_i)},$$

and find it explicitly for several examples.

We have analyzed this problem and given partial answers in [44] when g is a trigonometric positive rational function.

3 Multiple orthogonal polynomials

In this section we present the basic notions, definitions, and notations related with Multiple Orthogonal Polynomials (MOP). Special attention will be paid to the type II multiple discrete orthogonal polynomials. Furthermore, we will sketch the recent progress of the subject for the past few years.

During the past fifteen years there has been increased interest in multiple orthogonal polynomials, particularly promoted by the Soviet mathematical school. There are several survey papers about this topic (see, e.g., [5, 6, 15, 57] and Chapter 4 in the Nikishin and Sorokin book [50]), where the connection with the Hermite-Padé approximants is shown.

Recently, in [7, 8, 9] multiple orthogonal polynomials with respect to discrete measures have been considered. Other efforts in this direction have been considered (see [35, 36, 37, 50, 51, 52, 53, 54]). A quite complete collection of them are surveyed in [58].

For multiple orthogonal polynomials we will need multi-indices consisting in a vector of dimension r of positive integers, for which we use the notation $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$.

When the orthogonality conditions are distributed over r real intervals $\Delta_1, \dots, \Delta_r$ with respect to r different measures $\mu_1, \mu_2, \dots, \mu_r$, in order to extend the standard orthogonal polynomials two different ways appear: The so-called type I and type II multiple orthogonal polynomials. Let us start with the last ones:

3.1 Type II multiple orthogonal polynomials

Definition 1 *A polynomial $q_{\vec{n}}(x)$ is said to be a multiple orthogonal polynomial of a multi-index \vec{n} with respect to positive Borel measures*

$$\mu_1, \mu_2, \dots, \mu_r \quad \text{such that} \quad \text{supp } \mu_i = \Delta_i \subset \mathbb{R}, \quad i = 1, 2, \dots, r,$$

if it satisfies the following conditions:

$$\begin{aligned} (i) \quad & \deg q_{\vec{n}} \leq |\vec{n}| := n_1 + n_2 + \dots + n_r, \\ (ii) \quad & \int_{\Delta_i} q_{\vec{n}}(x) x^k d\mu_i(x) = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r. \end{aligned} \quad (11)$$

For $r = 1$ multiple orthogonal polynomial becomes standard orthogonal polynomial.

The *existence* of $q_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}}x^k$ is always guaranteed, because for its $|\vec{n}| + 1$ unknown coefficients the orthogonality conditions (11) give a system of $|\vec{n}|$ linear algebraic homogeneous equations, which always has a nontrivial solution. However, the matrix of coefficients for such a linear system can be singular. Therefore the *uniqueness* is in general not guaranteed. A simple *counterexample* is when the measures $\mu_1, \mu_2, \dots, \mu_r$ are all them identical on the same interval. Hence, we need some extra conditions on the r vector of measures in order that the above multiple orthogonal polynomial is unique.

Remark 1 *If one deals with the non-Hermitian complex orthogonality with respect to a set of complex valued functions*

$$m_1(z) = \sum_{k=0}^{\infty} \frac{m_{1,k}}{z^{k+1}}, \dots, m_r(z) = \sum_{k=0}^{\infty} \frac{m_{r,k}}{z^{k+1}}, \tag{12}$$

over the contours $\Gamma_i \subset \mathbb{C}$ ($i = 1, 2, \dots, r$) then, we can generalize the notion of multiple orthogonal polynomials as follows.

A multiple orthogonal polynomial $q_{\vec{n}}(z)$ with respect to complex weights (12) verifies

- (i) $\deg q_{\vec{n}} \leq |\vec{n}|,$
- (ii) $\oint_{\Gamma_i} q_{\vec{n}}(z)z^k m_i(z) dz = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$

Two different sequences of indices are usually considered. The so-called diagonal and step-line sequences, respectively (see [5, 36, 57]).

3.1.1 Type I multiple orthogonal polynomials

Definition 2 *A vector of polynomials $(v_{\vec{n},1}v_{\vec{n},2}, \dots, v_{\vec{n},r})$ is said to be a multiple orthogonal polynomial vector of type I if each polynomial $v_{\vec{n},i}$, where $i = 1, 2, \dots, r$, satisfies the conditions*

$$\begin{aligned} &\deg v_{\vec{n},i} \leq n_i - 1, \\ &\sum_{i=1}^r \int_{\Delta_i} v_{\vec{n},i}(x)x^k d\mu_i(x) = 0, \quad k = 0, 1, \dots, |\vec{n}| - 2, \quad i = 1, 2, \dots, r. \end{aligned} \tag{13}$$

When $r = 1$ one recovers the standard orthogonal polynomials.

Again the *existence* of all the polynomials $v_{\vec{n},i}$ ($i = 1, 2, \dots, r$) determined by Definition 2 is guaranteed, because there are $|\vec{n}| - 1$ orthogonality conditions which give $|\vec{n}| - 1$ linear algebraic homogeneous equations for the $|\vec{n}|$ unknown coefficients. Therefore the type I multiple orthogonal polynomial vector is determined up to a constant factor.

3.1.2 Connection with Hermite-Padé simultaneous rational approximants

Multiple orthogonal polynomials are intimately related to simultaneous Padé approximation, which is often known as Hermite-Padé approximation.

Let $\Delta_i = (a_i, b_i)$, $i = 1, 2, \dots, r$, be intervals on the real line, and $\mu_1, \mu_2, \dots, \mu_r$ be Borel measures on \mathbb{R} with infinitely many points of increase such that $\text{supp } \mu_i \subset \Delta_i$, $i = 1, 2, \dots, r$. The Markov functions (or Stieltjes functions)

$$m_i(z) = \int_{\Delta_i} \frac{d\mu_i}{z-x}, \quad z \notin \Delta_i \quad i = 1, 2, \dots, r,$$

can be simultaneously approximated by rational functions with prescribed order near infinity. Two different ways are considered to study such a kind of problems.

The first one: For the multi-index $\vec{n} = (n_1, n_2, \dots, n_r)$ of nonnegative integers it is well known [50] how to find a polynomial $q_{\vec{n}}(z) \not\equiv 0$ of degree at most $|\vec{n}|$, such that the expressions

$$q_{\vec{n}}(z)m_i(z) = p_{\vec{n},i}(z) + \frac{\zeta_i}{z^{n_i+1}} + \dots, \quad i = 1, 2, \dots, r, \quad (14)$$

hold, being $p_{\vec{n},i}(z)$ complex polynomials. Notice that $q_{\vec{n}}(z)$ always exists, because the relations (14) lead to a system of $|\vec{n}| + 1$ homogeneous linear equations. This approximation procedure, where one needs to find the polynomials $q_{\vec{n}}(z)$ and $p_{\vec{n},i}(z)$, is called type II Hermite-Padé approximation.

Through the rational function

$$\pi_i(\vec{n}, z) = \frac{p_{\vec{n},i}(z)}{q_{\vec{n}}(z)}, \quad i = 1, 2, \dots, r. \quad (15)$$

we denote the simultaneous Hermite-Padé approximants of the r Markov (or Stieltjes) functions $m_1(z), m_2(z), \dots, m_r(z)$, being $q_{\vec{n}}(z)$ the common denominator of the simultaneous approximants. This polynomial is precisely the multiple orthogonal polynomial due to the connection between the relations (14) and (11).

Thus, type II Hermite-Padé approximation is a rational approximation of the functions $m_i(z)$ ($i = 1, 2, \dots, r$) with the same denominator.

In [49], are considered simultaneous Hermite-Padé approximants of several Markov (or Stieltjes) functions, as well as the connection of this kind of approximation with the construction of linear forms on Markov (or Stieltjes) functions with polynomial coefficients.

For a fixed index-vector \vec{n} , although the existence of $q_{\vec{n}}(z)$ is guaranteed, the uniqueness is not determined by equalities (11) up to a normalizing constant. Even the rational functions $\pi_1(\vec{n}, z), \pi_2(\vec{n}, z), \dots, \pi_r(\vec{n}, z)$ are not constructed in a unique way, using the vector-index \vec{n} and the measures $\mu_1, \mu_2, \dots, \mu_r$. There are two cases where $q_{\vec{n}}(z)$ is unique determined up to a constant factor: The so-called Angelesco and Nikishin systems [50].

Chebyshev system of functions.

Definition 3 A set of continuous real functions $\{f_k(x)\}_0^n$ in the interval Δ is said to be a Chebyshev system (or T-system) of order n , if the determinant

$$V(x_0, \dots, x_n) = \det \begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n) \end{pmatrix}, \quad (16)$$

does not vanish for arbitrary different values x_0, \dots, x_n of Δ .

$$\sum_{k=0}^n |\lambda_k| > 0, \quad \lambda_k \in \mathbb{R}. \quad (17)$$

Thus, the definition 3 is equivalent to the following statement [50]: A set of functions $\{f_k(x)\}_0^n$ constitutes a T-system of order n if every linear combination

$$F(x) = \sum_{k=0}^n \lambda_k f_k(x),$$

where $\{\lambda_k\}_{k=0}^n$ verify (17), has at most n zeros in Δ .

For type II multiple orthogonal polynomials there is an useful system of functions.

Definition 4 An AT-system consists of r weights $\{\rho_k(x)\}_1^r$ supported on the same interval Δ such that

$$\rho_1(x), x\rho_1(x), \dots, x^{n_1-1}\rho_1(x), \dots, \rho_r(x), x\rho_r(x), \dots, x^{n_r-1}\rho_r(x), \quad (18)$$

is a T-system on Δ for every multi-index $\vec{n} = (n_1, \dots, n_r)$.

In this contribution we will focus our attention on the set of weights (the extension to measures is straightforward) which is an AT-system. See [7] for more information about the normality of the system of weights).

3.2 Brief description on the classical discrete polynomials.

The classical discrete orthogonal polynomials are those named after Hahn, Meixner, Kravchuk and Charlier. There are several approaches to the study (or characterization) of these polynomials. The more standard one is based on the fact that these discrete orthogonal polynomials are special cases of the basic hypergeometric series [25]. Other usual approaches are the group-theoretical approach [59] and the difference-equation approach on a lattice with a constant mesh [47, 48]. In the present contribution all the classical discrete orthogonal polynomials are considered as special cases of the type-II Hermite-Padé polynomials.

The Hahn polynomials $h_n^{(\alpha_0, \alpha_1)}(x, N)$ are polynomials of degree n which are orthogonal to all lower degree polynomials with respect to the weight function $\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_1 + 1)/(\Gamma(x + 1)\Gamma(N - x))$ on the set of points $x \in [0, N - 1]$ (mass points), where $\alpha_0, \alpha_1 > -1$. This means that the orthogonality conditions

$$\sum_{x=0}^{N-1} h_n^{(\alpha_0, \alpha_1)}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_1 + 1)}{\Gamma(x + 1)\Gamma(N - x)} x^k = 0, \quad (19)$$

$$k = 0, \dots, n - 1,$$

holds.

The Meixner polynomials $m_n^{(\gamma, \nu)}(x)$ (being $\gamma > 0$ and $0 < \nu < 1$) are orthogonal on the set of points $x \in [0, \infty)$ with respect to the *Pascal distribution* $\nu^x(\gamma)_x/\Gamma(x + 1)$, where $(\gamma)_x := \Gamma(\gamma + x)/\Gamma(\gamma)$, so that,

$$\sum_{x=0}^{\infty} m_n^{(\gamma, \nu)}(x) \frac{\nu^x \Gamma(\gamma + x)}{\Gamma(x + 1)} x^k = 0, \quad k = 0, \dots, n - 1. \quad (20)$$

The Kravchuk polynomials $k_n^{(p)}(x, N)$ are orthogonal on the set of points $x \in [0, N]$ with respect to the *binomial distribution* $N!p^x(1 - p)^{N-x}/(\Gamma(x + 1)\Gamma(N + 1 - x))$, where $p, (1 - p) > 0$. Therefore, the orthogonality conditions are

$$\sum_{x=0}^N k_n^{(p)}(x, N) \frac{N!p^x(1 - p)^{N-x}}{\Gamma(x + 1)\Gamma(N + 1 - x)} x^k = 0, \quad k = 0, \dots, n - 1. \quad (21)$$

Finally, the Charlier polynomials $c_n^{(a)}(x)$ are polynomials of degree n which are orthogonal with respect to the *Poisson distribution* $a^x/\Gamma(x + 1)$ ($a > 0$) on the mass points $x \in [0, \infty)$. They satisfy the orthogonality conditions

$$\sum_{x=0}^{\infty} c_n^{(a)}(x) \frac{a^x}{\Gamma(x + 1)} x^k = 0, \quad k = 0, \dots, n - 1. \quad (22)$$

The above four families of discrete orthogonal polynomials satisfy several properties which also allow to characterize them in several ways. Before proceed to comment these properties of classical discrete orthogonal polynomials, let define the forward and backward difference operators

$$\Delta y(x(s)) := \frac{\Delta}{\Delta x(s)} y(x(s)) = \frac{y(s + h) - y(s)}{x(s + h) - x(s)}, \quad (23)$$

$$\nabla y(x(s)) := \frac{\nabla}{\nabla x(s)} y(x(s)) = \Delta y(s - h),$$

respectively, for a functions y in terms of an arbitrary partition $x(s)$ with mesh h . For more simplicity, let us choose $x(s) = s$ and $\Delta x(s) = h = 1$. Such a situation is said to be canonical in the sense that, by a canonical variable we will use x instead of s . Thus, the formula

$$\nabla^n y(x) = \sum_{i=0}^n \frac{(-1)^i n!}{i!(n - i)!} y(x - i) = \sum_{i=0}^n \frac{(-n)_i}{i!} y(x - i), \quad (24)$$

holds. It can be proved easily by induction.

Other important property of the operator ∇ (and equivalently of Δ) is the formula of summation by parts

$$\sum_{x=a}^b y(x) \nabla z(x) = y(x)z(x)|_{a-1}^b - \sum_{x=a}^b z(x-1) \nabla y(x), \tag{25}$$

whose proof is straightforward taking into account

$$\nabla[y(x)z(x)] = y(x) \nabla z(x) + z(x-1) \nabla y(x). \tag{26}$$

One of the most useful properties of the classical discrete orthogonal polynomials is that they verify the hypergeometric-type difference equation

$$\sigma(x) \Delta \nabla y(x) + \tau(x) \Delta y(x) + \lambda_n y(x) = 0, \tag{27}$$

where σ and τ are polynomials independent of the degree n , with, $\deg \sigma \leq 2$, $\deg \tau = 1$, and λ_n is a constant depending on n .

This equation can be written in the self-adjoint form

$$\Delta[\sigma(x)\rho(x) \nabla y(x)] + \lambda_n \rho(x)y(x) = 0,$$

when the function $\rho(x)$ satisfies the *Pearson-type* equation

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x). \tag{28}$$

As a simple consequence of the second order linear difference equation (27) we get their polynomial solutions satisfy a finite-difference analog of the Rodrigues formula, i.e.,

$$p_n(x) = \frac{c_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^n \sigma(x+k) \right], \quad \nabla^n := \underbrace{\nabla \cdots \nabla}_{n\text{-times}}, \tag{29}$$

where c_n is a normalizing factor depending on n .

From (29) we can deduce the relation between $\Delta p_n(x)$ and the polynomials themselves. Hence, the first finite differences of the discrete orthogonal polynomials are again orthogonal polynomials of the same family, but with different parameters, i.e.,

$$\Delta p_n(x) = -\lambda_n \frac{c_n}{c_{n-1}} p_{n-1}(x),$$

where λ_n and c_n are the corresponding eigenvalue of (27) and normalizing factor of (29), respectively. The coefficient \tilde{c}_{n-1} is the normalizing constant in the Rodrigues formula (29) for the polynomial $p_{n-1}(x)$ obtained by replacing $\rho(x)$

by $\sigma(x+1)\rho(x+1)$. Indeed

$$\left\{ \begin{array}{l} \Delta h_n^{(\alpha_0, \alpha_1)}(x, N) = \frac{h_{n-1}^{(\alpha_0+1, \alpha_1+1)}(x, N-1)}{(n + \alpha_0 + \alpha_1 + 1)^{-1}}, \quad c_n = \frac{(-1)^n}{n!}, \\ \Delta m^{(\gamma, v)}(x) = \frac{n(v-1)}{v} m_{n-1}^{(\gamma+1, v)}(x), \quad c_n = v^{-n}, \\ \Delta k_n^{(p)}(x, N) = k_{n-1}^{(p)}(x, N-1), \quad c_n = \frac{(p-1)^n}{n!}, \\ \Delta c_n^{(a)}(x) = -\frac{n}{a} c_{n-1}^{(a)}(x) \quad c_n = a^{-n}. \end{array} \right. \quad (30)$$

From the orthogonality conditions all these families verify the three-term recurrence relation

$$xp_n(x) = \frac{a_n}{a_{n+1}} p_{n+1}(x) + \left[\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \right] p_n(x) + \frac{a_{n-1} \|p_n\|^2}{a_n \|p_{n-1}\|^2} p_{n-1}(x). \quad (31)$$

From the hypergeometric property, we get

$$\sigma(x) \nabla p_n(x) = \frac{\lambda_n}{n\tau'_n} \left[\tau_n(x) p_n(x) - \frac{c_n}{c_{n+1}} p_{n+1}(x) \right], \quad (32)$$

being $p_n(x) = a_n x^n + b_n x^{n-1} + \text{lower terms}$, and $\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x)$.

For the classical discrete orthogonal polynomials starting from any of the properties (27)-(32), or from one of the orthogonality conditions (19)-(22), can be deduced the other properties.

3.3 Discrete polynomials with simultaneous orthogonality

Now we will present five families of multiple discrete orthogonal polynomials which constitute an AT system (see [7]). Here we give their Rodrigues-type formulas [9].

3.3.1 Examples

Multiple Hahn polynomials. The multiple Hahn polynomials are orthogonal polynomials associated with an AT system consisting of Hahn weights on $[0, N-1]$. These polynomials verify simultaneous orthogonality conditions with respect to r measures over the same mass points belonging to the interval $[0, N-1]$. This system has different singularities at 0 and the same singularity at 1, when the set of mass points tends to infinity and one substitutes the variable $x \in [0, N-1]$ by $(N-1)s$. Let us discuss this affirmation in more detail. It is natural to expect that the Hahn polynomials $h_n^{(\alpha_0, \alpha_1)}(x, N)$, after the linear change of variable $x = (N-1)s$, which transforms the interval $[0, N-1]$ into $[0, 1]$, will tend to the Jacobi polynomials $P_n^{(\alpha_0, \alpha_1)}(s)$ when N tends to infinity (i.e., when the mesh $h = \Delta s = 1/(N-1)$ in the new variable s tends to 0), and that the weight function $\rho(x)$ will tend, up to a constant factor, to the

weight function $x^{\alpha_0}(1-x)^{\alpha_1}$, where $\alpha_0, \alpha_1 > -1$ for the Jacobi polynomials, orthogonal on $[0, 1]$.

More precisely, replacing x by $(N-1)s$ in the Hahn weight one has

$$\rho(x) = \frac{\Gamma(N - Ns + \alpha_1)\Gamma(Ns + 1 + \alpha_0)}{\Gamma(N - Ns)\Gamma(Ns + 1)}.$$

Using the well known relation [48]

$$\frac{\Gamma(z+a)}{\Gamma(z)} = z^a \left[1 + \mathcal{O}\left(\frac{1}{z^2}\right) \right], \quad |\arg z| \leq \pi - \delta, \quad \delta > 0, \quad (33)$$

or, equivalently

$$\lim_{z \rightarrow \infty} \frac{\Gamma(z+a)}{z^a \Gamma(z)} = 1, \quad (34)$$

one gets that $\rho(x)$ behaves as $N^{\alpha_0} N^{\alpha_1} s^{\alpha_0} (1-s)^{\alpha_1}$ when $N \rightarrow \infty$.

Let $\alpha_0 > -1$ and $\alpha_1, \dots, \alpha_r$ be such that each $\alpha_i > -1$, $i = 1, 2, \dots, r$, and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. The function $\hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N)$ denotes the monic multiple Hahn polynomial of degree $|\vec{n}| < N - 1$ for the multi-index $\vec{n} \in \mathbb{N}^r$ and $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ that satisfies the orthogonality conditions

$$\sum_{x=0}^{N-1} \hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x + 1)\Gamma(N - x)} x^k = 0, \quad (35)$$

$$k = 0, \dots, n_i - 1, \quad i = 1, \dots, r.$$

Proposition 6 *The following finite-difference analog of the Rodrigues formula*

$$\hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) = \frac{(-1)^{|\vec{n}|}}{\prod_{i=1}^r (|\vec{n} + n_i \vec{e}_i| + \alpha_0 + \alpha_i)_{n_i}} \times \frac{\Gamma(N - x)}{\Gamma(N + \alpha_0 - x)} \mathcal{D} \frac{\Gamma(N + \alpha_0 - x)}{\Gamma(N - |\vec{n}| - x)}, \quad (36)$$

where

$$\mathcal{D} := \prod_{i=1}^r \mathcal{D}_{i, n_i}, \quad \mathcal{D}_{i, n_i} = \frac{\Gamma(x + 1)}{\Gamma(x + \alpha_i + 1)} \nabla^{n_i} \frac{\Gamma(x + \alpha_i + n_i + 1)}{\Gamma(x + 1)},$$

holds [9].

Remark 2 *The product of the r difference operators \mathcal{D}_{i, n_i} can be taken in any order since these operators commute.*

From the orthogonality relations (35) we can write

$$\sum_{x=0}^{N-1} \hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x + 1)\Gamma(N - x)} \nabla \pi_{k+1}(x + 1) = 0,$$

$$k = 0, 1, \dots, n_i, \quad i = 1, 2, \dots, r,$$

where

$$\pi_k(x) = x(x-1)\cdots(x-k+1) = \frac{x!}{(x-k)!} = \frac{\nabla\pi_{k+1}(x+1)}{k+1}. \quad (37)$$

Hence, using the summation by parts (25), as well as the fact that $\pi_{k+1}(0) = 0$ and $\Gamma^{-1}(0) = 0$, one obtains

$$\sum_{x=0}^N \nabla \left[\hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x + 1)\Gamma(N - x)} \right] \pi_{k+1}(x) = 0, \\ k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

Thus, we get the raising operators

$$\nabla \left[\hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x + 1)\Gamma(N - x)} \right] \\ = c_{n,i} \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i)}{\Gamma(x + 1)\Gamma(N - x + 1)} \hat{h}_{\vec{n} + \vec{e}_i}^{(\alpha_0 - 1, \vec{\alpha} - \vec{e}_i)}(x, N + 1), \quad i = 1, 2, \dots, r, \quad (38)$$

where the constant factors $c_{n,i}$ are found comparing the coefficients of the power $|\vec{n}| + 1$ of x on the two sides of (38), i.e.,

$$c_{i,n} = -(|\vec{n}| + \alpha_0 + \alpha_i).$$

If we apply the above operator $(l_i - 1)$ times, where $l_i \in \mathbb{N}$, $(l_i > 1)$ on the expression (38), we get

$$\nabla^{l_i} \left[\hat{h}_{\vec{n}}^{(\alpha_0, \vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x + 1)\Gamma(N - x)} \right] = (-1)^{l_i} \\ (|\vec{n}| + \alpha_0 + \alpha_i)_{l_i} \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i)}{\Gamma(x + 1)\Gamma(N - x + 1)} \hat{h}_{\vec{n} + \vec{e}_i}^{(\alpha_0 - l_i, \vec{\alpha} - \vec{e}_i)}(x, N + l_i), \quad (39) \\ i = 1, 2, \dots, r.$$

The multiplication by the ratios $\Gamma(x + 1)\Gamma(N - x + 1)/\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i)$ and $\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_j)/\Gamma(x + 1)\Gamma(N - x + 1)$ on both sides of expressions (39), being $(j = 1, \dots, i - 1, i + 1, \dots, r)$, and the successive application of ∇^{l_j} on both sides of the equalities leads to (36).

In an analogous way to the multiple Hahn polynomials we can consider the discrete orthogonal polynomials of simultaneous orthogonality over the mass points $x = 0, 1, 2, \dots$, with respect to r different *Pascal distributions*. Here can be distinguished two different cases that we will show in more detail below.

Multiple Meixner polynomials (first kind)

The multiple Meixner polynomials of the first kind $\hat{m}_{\vec{n}}^{(\vec{\gamma}, v)}(x)$ are given by the weights $v^x \Gamma(x + \gamma_i)/\Gamma(x + 1)\Gamma(\gamma_i)$, where $v \in (0, 1)$ and $\gamma_i > 0$, for $i = 1, 2, \dots, r$. The assumption $\gamma_i - \gamma_j \notin \mathbb{Z}$ guarantees the AT property of the system of Meixner weights (or Pascal distributions).

So, the orthogonality conditions which determine the polynomial $\hat{m}_{\vec{n}}^{(\vec{\gamma},v)}(x)$ are

$$\sum_{x=0}^{\infty} \hat{m}_{\vec{n}}^{(\vec{\gamma},v)}(x) x^k \frac{v^x \Gamma(\gamma_i + x)}{\Gamma(x + 1)} = 0, \quad k = 0, \dots, n_i - 1, \quad i = 1, \dots, r. \quad (40)$$

From (40) it is no so difficult to deduce the raising operators

$$\mathcal{D}_i \hat{m}_{\vec{n}}^{(\vec{\gamma},v)}(x) = \frac{v - 1}{v(\gamma_i - 1)} m_{\vec{n} + \vec{e}_i}^{(\vec{\gamma} - \vec{e}_i, v)}(x), \quad \mathcal{D}_i := \frac{\Gamma(x + 1)}{v^x (\gamma_i - 1)_x} \nabla \frac{v^x (\gamma_i)_x}{\Gamma(x + 1)}, \quad (41)$$

where $(a)_x := \Gamma(a + x)/\Gamma(a)$ is the Pochhammer symbol.

A repeated use of the raising operator (41) gives the Rodrigues-type formula

$$\hat{m}_{\vec{n}}^{(\vec{\gamma},v)}(x) = \left(\frac{v}{v - 1} \right)^{|\vec{n}|} \left[\prod_{i=1}^r \frac{(\gamma_i + n_i - 1)!}{(\gamma_i - 1)!} \right] \Gamma(x + 1) \mathcal{D} \Gamma^{-1}(x + 1), \quad (42)$$

where $\mathcal{D} := \prod_{i=1}^r \mathcal{D}_{i, n_i}$, and $\mathcal{D}_{i, n_i} := v^{-x} (\gamma_i)_x^{-1} \nabla^{n_i} v^x (\gamma_i + n_i)_x$.

Multiple Meixner polynomials (second kind)

The other family of multiple Meixner polynomials appears when the simultaneous orthogonality conditions are distributed over the same set of discrete points \mathbb{N}_0 with respect to the weights $v_i^x (\gamma)_x / x!$, ($i = 1, \dots, r$). To avoid the system of measures will be identical system we assume $v_i \neq v_j$ whenever $i \neq j$. The restrictions $\gamma > 0$, and $v_i \in (0, 1)$, ($i = 1, \dots, r$) are essentially inherited from the classical case (20). Thus, the AT property is again guaranteed. Hence, the polynomial $\hat{m}_{\vec{n}}^{(\gamma, \vec{v})}(x)$ determined by the following orthogonality conditions

$$\sum_{x=0}^{\infty} \hat{m}_{\vec{n}}^{(\gamma, \vec{v})}(x) x^k \frac{v_i^x \Gamma(\gamma + x)}{\Gamma(x + 1)} = 0, \quad k = 0, \dots, n_i - 1, \quad i = 1, \dots, r, \quad (43)$$

is called multiple Meixner polynomial of the second kind.

An analogous procedure as that carried out for the multiple Meixner polynomial of the first kind allows to obtain a finite-difference analog of the Rodrigues formula

$$\begin{aligned} \hat{m}_{\vec{n}}^{(\gamma, \vec{v})}(x) &= (\gamma + |\vec{n}| - 1)!^2 \left[\prod_{i=1}^r \left(\frac{v_i}{v_i - 1} \right)^{n_i} \frac{1}{(\gamma + n_i - 1)!} \right] \\ &\times \Gamma(x + 1) \mathcal{D} \Gamma^{-1}(x + 1), \end{aligned} \quad (44)$$

where $\mathcal{D} := \prod_{i=1}^r \mathcal{D}_{i, n_i}$, and $\mathcal{D}_{i, n_i} = v_i^{-x} (\gamma + |\vec{n}| - n_i)_x^{-1} \nabla^{n_i} v_i^x (\gamma + |\vec{n}|)_x$.

Multiple Kravchuk polynomials

The multiple Kravchuk polynomials are orthogonal polynomials of degree $|\vec{n}| < N$, associated with an AT system of Kravchuk weights (*binomial distributions*). The function $\hat{k}_{\vec{n}}^{(\vec{p})}(x, N)$ denotes the monic multiple Kravchuk polynomials for which the r orthogonality conditions are given on the same set of finite number of points $x = 1, 2, \dots, N$ with respect to different binomial distributions. Thus, the orthogonality conditions become

$$\sum_{x=0}^N \hat{k}_{\vec{n}}^{(\vec{p})}(x, N) x^k \frac{N! p_i^x (1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N+1-x)} = 0, \quad 0 < p_i < 1, \quad (45)$$

$$k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

From (45), using (37) and (25) one obtains the raising operators

$$\mathcal{D}_i \hat{k}_{\vec{n}}^{(\vec{p})}(x, N) = -\frac{1}{p_i(1-p_i)} \hat{k}_{\vec{n}+\vec{e}_i}^{(\vec{p})}(x, N+1), \quad i = 1, 2, \dots, r \quad (46)$$

$$\mathcal{D}_i := \frac{N! p_i^x (1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N+1-x)} \nabla \frac{(N+1)! p_i^x (1-p_i)^{N+1-x}}{\Gamma(x+1)\Gamma(N+2-x)}.$$

An appropriate combination of the raising operators (46) leads to the Rodrigues-type formula

$$\hat{k}_{\vec{n}}^{(\vec{p})}(x, N) = (-1)^{|\vec{n}|} \left[\prod_{i=1}^r p_i^{n_i} \right] \Gamma(x+1) \mathcal{D} \Gamma^{-1}(x+1), \quad (47)$$

where $\mathcal{D} := \prod_{i=1}^r \mathcal{D}_{i, n_i}$, and $\mathcal{D}_{i, n_i} = \frac{\Gamma^{-1}(N+1) p_i^{-x} (1-p_i)^x}{\Gamma^{-1}(N+1-x)} \nabla^{n_i} \frac{\Gamma(N-n_i+1) p_i^x (1-p_i)^{-x}}{\Gamma(N-n_i+1-x)}$.

Multiple Charlier polynomials

Finally, the Poisson discrete measures

$$\mu_i = \sum_{x=0}^{\infty} \frac{a_i^x}{\Gamma(x+1)}, \quad a_i > 0, \quad i = 1, 2, \dots, r,$$

determine the associated system of simultaneous orthogonal polynomials $\hat{c}_{\vec{n}}^{(\vec{a})}(x)$, with $\vec{a} = (a_1, a_2, \dots, a_r)$ such that $a_i \neq a_j$, under the orthogonality conditions

$$\sum_{x=0}^{\infty} \hat{c}_{\vec{n}}^{(\vec{a})}(x) \frac{a_i}{\Gamma(x+1)} x^k = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r. \quad (48)$$

Thus, $\hat{c}_{\vec{n}}^{(\vec{a})}(x)$ is said to be the multiple Charlier polynomial (see also [10]).

In the same sense as we have proceeded in the above cases, the raising operators

$$\nabla \left[\hat{c}_{\vec{n}}^{(\vec{a})}(x) \frac{a_i^x}{\Gamma(x+1)} \right] = -\frac{1}{a_i} \frac{a_i^x}{\Gamma(x+1)} \hat{c}_{\vec{n}+\vec{e}_i}^{(\vec{a})}(x), \quad i = 1, 2, \dots, r, \quad (49)$$

can be obtained from (48).

A repeated use of the raising operators (49) gives the Rodrigues-type formula

$$\hat{c}_{\vec{n}}^{(\vec{a})}(x) = (-1)^{|\vec{n}|} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r} \Gamma(x+1) \underbrace{\left[\prod_{i=1}^r a_i^{-x} \nabla^{n_i} a_i^x \right]}_{\mathcal{D}} \Gamma^{-1}(x+1). \quad (50)$$

3.4 Recurrence relation for multiple discrete orthogonal polynomials

Let \mathcal{D}_{n_i} , where n_i is the i -th coordinate of the vector index \vec{n} , be a difference operator defined by

$$\mathcal{D}_{n_i} := g_{i,1}(x; \alpha_1, \dots, \alpha_n) \nabla^{n_i} g_{i,2}(x; \beta_1, \dots, \beta_n), \quad \alpha_k, \beta_k \in \mathbb{R}, \quad k = 1, \dots, n,$$

being $g_{i,1}$ and $g_{i,2}$ certain functions depending, in general, on the i -th orthogonality measure.

The Rodrigues-type formulas allow us to introduce the following notation for the MDOP

$$q_{\vec{n}}(x) = c_{\vec{n},r} f_1(x) \mathcal{D}_{\vec{n}} f_2(x), \quad \mathcal{D}_{\vec{n}} := \prod_{i=1}^r \mathcal{D}_{n_i}, \quad (51)$$

where the coefficient $c_{\vec{n},r}$ depends on the parameters of the measures μ_1, \dots, μ_r , and $f_1(x)$, $f_2(x)$ can also depend, in general, on $|\vec{n}|$ and μ_1, \dots, μ_r .

Lemma 7 *Let $n_k \in \mathbb{N}$. Then, the product of n_k backward difference operators over the function $x f(x)$ can be written as*

$$\nabla^{n_k} x f(x) = n_k \nabla^{n_k-1} f(x) + (x - n_k) \nabla^{n_k} f(x), \quad (52)$$

where $\nabla^{n_k} := \underbrace{\nabla \cdots \nabla}_{n_k \text{ times}}$.

From (24) and (26), the proof is straightforward using induction, i.e.,

$$\begin{aligned} \nabla^{n_k} x f(x) &= \nabla^{n_k-1} [\nabla x f(x)] = \nabla^{n_k-1} f(x) + \nabla^{n_k-1} [(x-1) \nabla f(x)] \\ &= 2 \nabla^{n_k-1} f(x) + \nabla^{n_k-2} [(x-2) \nabla^2 f(x)] = \cdots = \\ &= n_k \nabla^{n_k-1} f(x) + (x - n_k) \nabla^{n_k} f(x). \end{aligned}$$

Corollary 8 *The relation*

$$\mathcal{D}_{n_i} x f(x) = n_i \mathcal{D}_{n_i-1} f(x) + (x - n_i) \mathcal{D}_{n_i} f(x),$$

holds.

Lemma 9 *Let \mathcal{D}_{n_i} , where n_i is the i -th coordinate of the vector-index \vec{n} , be a difference operator defined in (51). Then,*

$$\mathcal{D}_{\vec{n}-\vec{e}} x f(x) = \left[\sum_{i=1}^r (n_i - 1) \prod_{j=1}^r \mathcal{D}_{n_j - \delta_{j,i} - 1} + (x - |\vec{n}| + r) \mathcal{D}_{\vec{n}-\vec{e}} \right] f(x), \quad (53)$$

where $\mathcal{D}_{\vec{n}-\vec{e}} := \mathcal{D}_{n_1-1} \cdots \mathcal{D}_{n_r-1} = \prod_{j=1}^r \mathcal{D}_{n_j-1}$ and $\delta_{i,j}$ Kronecker's delta.

Applying r times Corollary 8 we get (53).

If the condition

$$\nabla [g_{n_i,2}(x) f_2(x)] = g_{n_i,2}(x) [ax + b] f_2(x),$$

is verified by the set of functions $g_{n_i,2}(x)$ ($i = 1, \dots, r$), then the MDOP $q_{\vec{n}}(x)$ satisfies a $r + 2$ recurrence relation.

Thus, we have proved the following theorem, which generalizes the results presented in [36] for any vector index \vec{n} (see theorem 13 below).

Theorem 10 *The multiple discrete orthogonal polynomials satisfy a $(r + 1)$ -order recurrence relation (for every row and column in the Padé Table), where r is the number of orthogonality conditions.*

3.5 Application of MOP and open problems

This section deals with the application and open problems in which are involved the MOP and various fields of mathematics. Among them we can emphasize on number theory, special functions, and the spectral analysis of non-symmetric banded Hessenberg operators.

Number theory

Number theory is perhaps the more natural field of application of MOP. Indeed, the roots of these mathematical objects go back to the nineteenth century. More precisely, in 1878 Hermite used the Hermite-Padé approximants to prove the transcendence of the number “ e ” [31]. Traditionally, the Hermite method has been considered the main tool in order to investigate the arithmetic properties of real numbers. Related with the transcendence of π we remind the solution of the famous old problem about the quadrature of the circle given by Lendemann in 1882.

The multiple orthogonal polynomials seem to be an useful tool to prove the irrationality and transcendence of certain numbers.

Let us comment the problem about the arithmetic nature of the values of Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

at the odd positive integer numbers, because the first result in this direction has been obtained just in 1979 by Apéry.

The proof of Apéry is based on the following elementary lemma:

Lemma 11 *Let x be a real number, and p_n, q_n two sequences of integers ($n \in \mathbb{N}$). If p_n and q_n are such that*

- i) $q_n x - p_n \neq 0$, for all $n \in \mathbb{N}$,*
- ii) $\lim_{n \rightarrow \infty} (q_n x - p_n) = 0$,*

then x is irrational.

Theorem 12 (Apéry [4]) $\zeta(3)$ *is irrational.*

Apéry found the sequence of numbers

$$\begin{aligned} p_n &= \sum_{j=0}^n \binom{n}{j}^2 \binom{n+j}{n}^2 \left[\sum_{m=1}^n \frac{1}{m^3} + \sum_{m=1}^j \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right] \\ q_n &= \sum_{j=0}^n \binom{n}{j}^2 \binom{n+j}{n}^2, \end{aligned} \tag{54}$$

which after some normalization give a sequence of integers \bar{p}_n and \bar{q}_n such that

$$0 \neq r_n = \bar{p}_n - \bar{q}_n \zeta(3) \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, using the previous Lemma 11 the statement holds.

The Beukers' contributions [12, 13] helped to understand where the sequences of integers (54) come from. On the other hand, the proofs given by Sorokin [53] and Van Assche [57] based on multiple orthogonal polynomials about the irrationality of $\zeta(3)$ are very constructive and interesting by themselves.

Concerning to the arithmetic nature of the values of Riemann zeta-function there is a challenging open problem which consists in proving the irrationality of $\zeta(5), \zeta(7), \dots$, as well as the transcendency of $\zeta(3), \dots$, etc.

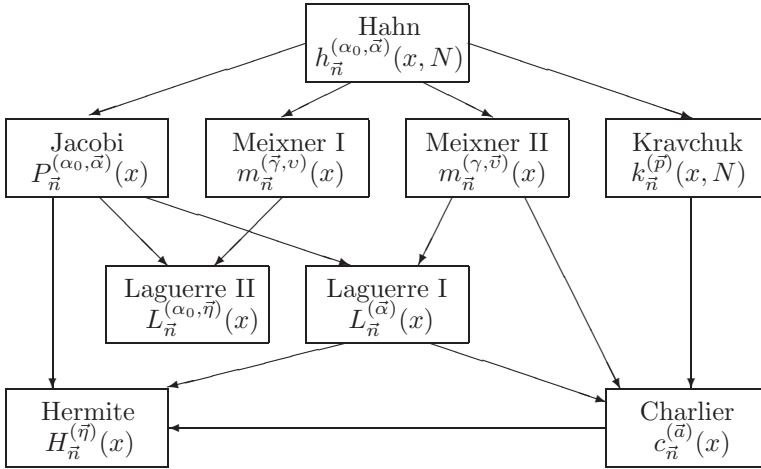
Special functions and limit relations

The Rodrigues-type formula for the MOP suggests that the MOP could be expressed in terms of hypergeometric series. So, it would be a very good contribution from the point of view of special functions to clarify this question.

Other interesting problem is to classify all the classical multiple orthogonal polynomials and to establish the limit relations between them. It would be a nice approach to do this starting from the multiple Askey-Wilson polynomials

(see [38] for classical cases) because all the other cases like multiple q -Hahn, q -Meixner, q -Kravchuk and q -Charlier can be obtained by limit transitions (see [8] for the q examples of MOP on the linear q lattice).

In [58] some families of classical MOP are deduced via the connection by limit transitions. In [9] the limit relations between discrete MOP and continuous MOP for AT systems are obtained (see the table below).



Non-symmetric band operators

Here we will show the relationship between MOP and the spectral theory of non-symmetric operators. Let us start mentioning few classical results. Assuming that the higher order difference operator is represented by a band matrix, i.e.,

$$H = (a_{i,j})_{i,j=0}^{\infty}, \quad a_{i,j} = 0, \quad (i > j + k, j > i + m), \quad \text{and } a_{n,n-k} \neq 0, a_{n,n+m} \neq 0, \tag{55}$$

in the standard basis of the Hilbert space $l_2(\mathbb{N}_0)$

$$e_k = (\underbrace{0, \dots, 0}_{k\text{-times}}, 1, 0, \dots), \quad k \in \mathbb{N}_0,$$

one concludes that if H is a Jacobi matrix (i.e., symmetric tridiagonal matrix with real coefficients and positive extreme diagonals) then, the moment problem associated with H is determined. Hence by the Stone theorem, the class of closed operators

$$He_n = a_n e_{n-1} + b_n e_n + a_{n+1} e_{n+1}, \quad \text{where } \begin{cases} a_n = a_{n,n-1} \\ b_n = a_{n,n} \end{cases},$$

coincides with the class of simple spectrum self-adjoint operators (also known as Lebesgue operators). Therefore, by the Von Neumann spectral theorem

there exists a unique operator valued measure \mathcal{F}_t , for which H admits the representation

$$H = \int_{\mathbb{R}} t \mathcal{F}_t.$$

The spectral measure for the operator H is the positive Borel measure

$$\mu(t) = \langle \mathcal{F}_t e_0, e_0 \rangle.$$

Thus, the Weyl function is

$$\mathcal{S}(z) = \langle \mathcal{R}_z e_0, e_0 \rangle, \tag{56}$$

where \mathcal{R}_z is the resolvent operator defined as

$$\mathcal{R}_t = (zI - H)^{-1} = \int_{\mathbb{R}} \frac{d\mathcal{F}_t}{z - t}.$$

For the self-adjoint operator the Weyl function becomes the Markov (or Stieltjes) function

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}. \tag{57}$$

The theory of orthogonal polynomial enjoys a very important result, known as Favard’s theorem, which connects the spectral theory of self-adjoint operators and the theory of orthogonal polynomials. This theorem says that a sequence of polynomials $q_n(t)$ which verifies the recurrence relation (1) is always the orthogonal polynomial sequence with respect to the spectral measure

$$\int_{\mathbb{R}} q_n(t) t^k d\mu(t) = 0, \quad k = 0, \dots, n - 1. \tag{58}$$

Consequently, the orthogonality (58) and the recurrence relation (1) are equivalent ways to describe orthogonal polynomials.

Remark 3 *The three-term recurrence relation (1) can be written as*

$$\underbrace{\begin{pmatrix} b_0 & a_1 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} \end{pmatrix}}_{H_n} \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ \vdots \\ q_{n-1}(t) \end{pmatrix} = t \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ \vdots \\ q_{n-1}(t) \end{pmatrix} - a_n q_n(t) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Since the zeros of orthogonal polynomials are simple, one concludes that the eigenvalues of the Jacobi matrix H_n are the zeros of $q_n(t)$. Hence, the connection with the spectral theory of self-adjoint operators is clearly established when one considers the infinite matrix H (instead of H_n) acting as an operator $H : l_2 \rightarrow l_2$ on appropriate domains.

On the other hand, the rational function $\pi_n(z) = \frac{p_n(z)}{q_n(z)}$ (see (15) for the multiple case) being $p_n(z)$ the other linearly independent solution of the difference equation (1), i.e.,

$$\begin{aligned} ty_n(t) &= a_{n+1}y_{n+1}(t) + b_n y_n(t) + a_n y_{n-1}(t), \quad n = 0, 1, \dots, \\ p_1(t) &= \frac{1}{a_1}, \quad p_{-1}(t) = 0, \end{aligned} \quad (59)$$

is the diagonal Padé approximant for the Markov (or Stieltjes) function

$$m(z) - \pi_n(z) = \frac{\zeta_n}{z^{2n+1}} + \dots$$

Despite the non-symmetric character of certain operators H , a proper choice of the rational approximants for the Weyl functions (56) (see also (57)) of the operator guarantees the connection with the entries of the matrix H .

In [36] an example of an operator H associated to non-symmetric $p + 2$ diagonal matrix (55) is analyzed. This example shows how the three-term recurrence relation and the Jacobi matrix have a natural extension for multiple orthogonal polynomials.

Let $\vec{k}(n)$ ($n \in \mathbb{N}$) be a sequence of multi-indices such that $n = kr + j$, where $0 \leq j < r$, and then set

$$\vec{k}(n) = \underbrace{(k + 1, \dots, k + 1, k, \dots, k)}_{j\text{-times}}. \quad (60)$$

If all these indices are normal, then we have a weakly complete system. This condition is guaranteed if we consider the spectral problem for H i.e., if $q_n(z)$ and $p_n^{(j)}(z)$ ($j = 1, \dots, r$) are the $r + 1$ linearly independent solutions of $(r + 1)$ -order difference equation

$$\begin{aligned} zy_n &= a_{n,n-r}y_{n-r} + \dots + a_{n,n}y_n + a_{n,n+1}y_{n+1}, \\ p_j^{(j)} &= \frac{1}{a_{j-1,j}}, \quad p_n^{(j)} = 0, \quad n < j, \quad j = 1, \dots, r. \end{aligned}$$

Then, the connection between the Hermite-Padé approximants for the system of functions

$$m_j(z) = \langle \mathcal{R}_z e_{j-1}, e_0 \rangle, \quad j = 1, \dots, r, \quad (61)$$

and the spectral problem is given in the following

Theorem 13 (Kalyagin [36]) *For $n = kr + j$ the vector of rational functions*

$$(\pi_1(\vec{n}, z), \dots, (\pi_r(\vec{n}, z)),$$

(see (15)) *is the Hermite-Padé approximant of index (60) for the system (61)*

Notice that in general for non-symmetric operators the notion of spectral positive measure has no sense. However, for non-symmetric operators the multiple orthogonal polynomials (Hermite-Padé polynomials) can be used, instead of the notion of standard orthogonal polynomials with respect to the positive spectral measure supported on the real line.

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