

## GENERALIZED $\Delta$ -COHERENT PAIRS

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ABSTRACT. A pair of quasi-definite linear functionals  $\{u_0, u_1\}$  is a generalized  $\Delta$ -coherent pair if monic orthogonal polynomials

$$\{P_n(x)\}_{n=0}^{\infty}$$

and

$$\{R_n(x)\}_{n=0}^{\infty}$$

relative to  $u_0$  and  $u_1$ , respectively, satisfy a relation

$$R_n(x) = \frac{1}{n+1} \Delta P_{n+1}(x) - \frac{\sigma_n}{n} \Delta P_n(x) - \frac{\tau_{n-1}}{n-1} \Delta P_{n-1}(x), \quad n \geq 2,$$

where  $\sigma_n$  and  $\tau_n$  are arbitrary constants and  $\Delta p = p(x+1) - p(x)$  is the difference operator.

We show that if  $\{u_0, u_1\}$  is a generalized  $\Delta$ -coherent pair, then  $u_0$  and  $u_1$  must be discrete-semiclassical linear functionals. We also find conditions under which either  $u_0$  or  $u_1$  is discrete-classical.

### 1. Introduction

Concerning the problem of evaluating the Fourier coefficients in the Fourier expansion of functions by polynomials orthogonal with respect to a Sobolev inner product

$$(1.1) \quad \phi_\lambda(f, g) := \int_{-\infty}^{\infty} f(x)g(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} f'(x)g'(x)d\mu_1(x),$$

where  $d\mu_0$  and  $d\mu_1$  are positive Borel measures with finite moments and  $\lambda \in \mathbb{R}^+$ , Iserles et al. [8] introduced the concept of coherency and symmetric coherency for the measures  $d\mu_0$  and  $d\mu_1$ .

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After the work by Iserles et al. [8], there have been many works [4, 12, 13, 14, 15, 17, 18, 20] on coherency from different points of view even allowing  $d\mu_0$  and  $d\mu_1$  to be signed or even complex valued measures. In particular, in [10], we introduced generalized coherency which unifies both coherency and symmetric coherency.

In [2, 3], they introduced a discrete version of coherency, that is,  $\Delta$ -coherency. Here  $\Delta$  is the difference operator defined as  $\Delta f(x) = f(x+1) - f(x)$ .

In this work, we will study the generalized  $\Delta$ -coherency in a more general setting by using the formal approach to orthogonality via linear functionals as was done in [10]. See also [2, 3, 14, 15].

In Section 2, we collect basic definitions, notations, and lemmas that we will use later. In Section 3, we define (see Definition 4.1) and analyze the generalized  $\Delta$ -coherency.

## 2. Preliminaries

Let  $\mathbb{P}$  be the linear space of all polynomials in one variable with complex coefficients. We denote the degree of a polynomial  $P(x)$  by  $\deg(P)$  with the convention that  $\deg(0) = -1$ . A polynomial system(PS) is a sequence of polynomials  $\{P_n(x)\}_{n=0}^{\infty}$  with  $\deg(P_n) = n$ ,  $n \geq 0$ .

A linear functional  $u$  on  $\mathbb{P}$  is called a moment functional and we denote its action on a polynomial  $\phi(x)$  by  $\langle u, \phi \rangle$ . We say that a moment functional  $u$  is quasi-definite(positive-definite, respectively) if its moments  $a_n := \langle u, x^n \rangle$ ,  $n \geq 0$ , satisfy the Hamburger condition

$$\Delta_n(u) := \det[a_{i+j}]_{i,j=0}^n \neq 0, \quad (\Delta_n(u) > 0, \text{ respectively}), \quad n \geq 0.$$

DEFINITION 2.1. A PS  $\{P_n(x)\}_{n=0}^{\infty}$  is said to be an orthogonal polynomial system(OPS) if there is a linear functional  $u$  on  $\mathbb{P}$  such that

$$\langle u, P_m P_n \rangle = p_n \delta_{mn}, \quad m, n \geq 0,$$

where  $p_n$  are non-zero constants. In this case, we call  $\{P_n(x)\}_{n=0}^{\infty}$  an OPS relative to  $u$  and  $u$  is said to be an orthogonalizing moment functional of  $\{P_n(x)\}_{n=0}^{\infty}$ . A linear functional  $u$  is quasi-definite if and only if there is an OPS  $\{P_n(x)\}_{n=0}^{\infty}$  relative to  $u$  (see [6]). Moreover, in this case, each  $P_n(x)$  is uniquely determined up to a non-zero constant factor.

For a moment functional  $u$ , a polynomial  $\phi(x)$ , and a constant  $c$ , we define moment functionals  $\Delta u$ ,  $\phi u$ , and  $(x-c)^{-1}u$  by

$$\langle \Delta u, p(x) \rangle := -\langle u, \Delta p(x-1) \rangle;$$

$$\begin{aligned} \langle \phi u, p \rangle &:= \langle u, \phi p \rangle; \\ \langle (x - c)^{-1} u, p \rangle &:= \langle u, \frac{p(x) - p(c)}{x - c} \rangle, \quad p \in \mathbb{P}. \end{aligned}$$

Then we have for polynomials  $p(x)$  and  $q(x)$

$$\begin{aligned} \Delta(p(x)q(x)) &= q(x)\Delta p(x) + p(x+1)\Delta q(x), \\ \Delta(p(x)u) &= p(x+1)\Delta u + \Delta p(x)u. \end{aligned}$$

For a constant  $c$ , let  $\delta(x - c)$  be the moment functional defined by

$$\langle \delta(x - c), p(x) \rangle = p(c), \quad p(x) \in \mathbb{P}.$$

For a PS  $\{P_n(x)\}_{n=0}^\infty$ , the dual basis of  $\{P_n(x)\}_{n=0}^\infty$  is the sequence  $\{u_n\}_{n=0}^\infty$  of moment functionals defined by the relation

$$\langle u_n, P_m \rangle = \delta_{mn}, \quad m, n \geq 0.$$

In particular,  $u_0$  is said to be the canonical moment functional of  $\{P_n(x)\}_{n=0}^\infty$ . If  $\{P_n(x)\}_{n=0}^\infty$  is a monic OPS(MOPS), then  $\{P_n(x)\}_{n=0}^\infty$  must be orthogonal with respect to  $u_0$  and

$$u_n = \frac{P_n(x)}{p_n} u_0, \quad n \geq 0.$$

DEFINITION 2.2. ([16]) A quasi-definite moment functional  $u$  is said to be discrete-semiclassical if  $u$  satisfies

$$(2.1) \quad \Delta(\varphi u) = \psi u,$$

for some polynomials  $\varphi(x)$  and  $\psi(x)$  with  $(\varphi, \psi) \neq (0, 0)$ . We then have  $\deg(\varphi) \geq 0$  and  $\deg(\psi) \geq 1$ . The corresponding OPS is called a discrete-semiclassical OPS.

For a discrete-semiclassical moment functional  $u$ ,

$$s := \min \max(\deg(\varphi) - 2, \deg(\psi) - 1)$$

the class number of  $u$ , where the minimum is taken over all pairs  $(\varphi, \psi) \neq (0, 0)$  of polynomials satisfying (2.1). In particular, a discrete-semiclassical moment functional of class 0 is called a discrete-classical moment functional.

Discrete-classical moment functionals can be characterized in many other ways. For an MOPS  $\{P_n(x)\}_{n=0}^\infty$  relative to  $u$ , the following statements are all equivalent ([1]):

- (i)  $\{P_n(x)\}_{n=0}^\infty$  is a discrete-classical OPS, that is,  $\Delta(\varphi u) = \psi u$  for some polynomial  $\varphi$  and  $\psi$  with  $0 \leq \deg(\varphi) \leq 2$  and  $\deg(\psi) = 1$ ;

(ii) ([7])

$$\{Q_n(x) := \frac{1}{n+1} \Delta P_{n+1}\}_{n=0}^\infty$$

is also an MOPS. Then  $\{Q_n(x)\}_{n=0}^\infty$  is orthogonal relative to  $\tilde{u} = \varphi u$  satisfying

$$(2.2) \quad \Delta(\varphi(x)\tilde{u}) = (\psi(x) + \Delta\varphi(x-1))\tilde{u};$$

(iii) There are polynomials  $\varphi$  and  $\psi$  with  $0 \leq \deg(\varphi) \leq 2$  and  $\deg(\psi) = 1$  such that

$$(2.3) \quad \begin{aligned} & \varphi(x)\Delta^2 P_n(x) + \psi(x)\Delta P_n(x) \\ &= \left(\frac{1}{2}n(n-1)\Delta^2\varphi(x) + n\Delta\psi(x)\right)P_n(x+1), \quad n \geq 0 \text{ ([11]).} \end{aligned}$$

It is well-known that there are essentially four distinct discrete-classical OPS's, up to a linear change of variable ([7, 19]):

- (i) Charlier polynomials  $\{c_n^{(\mu)}(x)\}_{n=0}^\infty$ :  $\varphi(x) = \mu$ ,  $\psi(x) = \mu - x$  ( $\mu > 0$ );
- (ii) Meixner polynomials  $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^\infty$ :  $\varphi(x) = \mu(\gamma + x)$ ,  $\psi(x) = \mu\gamma - x(1 - \mu)$  ( $\gamma > 0$ ,  $\mu \in (0, 1)$ );
- (iii) Kravchuk polynomials  $\{k_n^{(p)}(x; N)\}_{n=0}^\infty$ :  $\varphi(x) = N - x$ ,  $\psi(x) = \frac{Np-x}{p}$  ( $p \in (0, 1)$ ,  $N \in \mathbb{Z}^+$ );
- (iv) Hahn polynomials  $\{h_n^{(\alpha,\beta)}(x, N)\}_{n=0}^\infty$ :  $\varphi(x) = (N-x-1)(x+\beta+1)$ ,  $\psi(x) = (N-1)(\beta+1) - x(\alpha+\beta+2)$  ( $\alpha, \beta > -1$ ,  $N \in \mathbb{Z}^+$ ).

We denote by  $u_c^{(\mu)}$ ,  $u_m^{(\gamma,\mu)}$ ,  $u_k^{(p,N)}$ , and  $u_h^{(\alpha,\beta,N)}$  the orthogonalizing moment functionals for Charlier, Meixner, Kravchuk, and Hahn polynomials, respectively. Notice that the moment functionals for Kravchuk and Hahn polynomials are not quasi-definite.

For an MOPS  $\{P_n(x)\}_{n=0}^\infty$  relative to  $u$  and complex numbers  $\xi$  and  $c$ , let  $\{P_n^*(\xi; x)\}_{n=0}^\infty$ ,  $\{P_n^{(1)}(x)\}_{n=0}^\infty$ , and  $\{P_n(c; x)\}_{n=0}^\infty$  be the monic kernel polynomials, the monic numerator polynomials(also called the associated polynomials of first kind (see [6])), and the monic co-recursive polynomials of  $\{P_n(x)\}_{n=0}^\infty$ , respectively:

$$P_n^*(\xi; x) = \frac{\langle u, P_n^2 \rangle}{P_n(\xi)} \sum_{k=0}^n \frac{P_k(x)P_k(\xi)}{\langle u, P_k^2 \rangle}, \quad n \geq 0 \text{ ([6]);}$$

$$(2.4) \quad P_n(x) = P_n^*(\xi; x) - \frac{P_{n-1}(\xi)}{P_n(\xi)} \frac{\langle u, P_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle} P_{n-1}^*(\xi; x), \quad n \geq 1 \text{ ([9]);}$$

$$(2.5) \quad P_n(c; x) = P_n(x) - cP_{n-1}^{(1)}(x), \quad n \geq 1 \text{ ([5]).}$$

It is well-known (see Theorem 7.1 on p. 36 in [6]) that for a quasi-definite moment functional  $u$  with MOPS  $\{P_n(x)\}_{n=0}^\infty$  and a complex number  $\xi$ ,  $(x - \xi)u$  is also quasi-definite if and only if  $P_n(\xi) \neq 0$ ,  $n \geq 1$ . Then the MOPS relative to  $(x - \xi)u$  is  $\{P_n^*(\xi; x)\}_{n=0}^\infty$ . Moreover (see Theorem 3.6 in [9]), if  $u$  is discrete-semiclassical of class  $s$  satisfying (2.1), then  $(x - \xi)u$  is also discrete-semiclassical of class

$$\begin{cases} s - 1 & \text{if } \varphi(\xi) = \psi(\xi) = 0 \\ s & \text{if } \varphi(\xi) = 0 \text{ and } \psi(\xi) \neq 0 \\ s + 1 & \text{if } \varphi(\xi) \neq 0. \end{cases}$$

Conversely if  $(x - \xi)u$  is discrete-semiclassical of class  $s$ , then  $u$  is discrete-semiclassical of class either  $s - 1$ ,  $s$ , or  $s + 1$ .

**PROPOSITION 2.1.** *Let  $\{P_n(x)\}_{n=0}^\infty$  and  $\{Q_n(x)\}_{n=0}^\infty$  be the MOPS's relative to  $u$  and  $v$  respectively. Then,  $\{Q_n(x)\}_{n=0}^\infty = \{P_n^*(\xi; x)\}_{n=0}^\infty$  for some complex number  $\xi$  if and only if there are complex numbers  $\alpha_n (n \geq 1)$  such that  $\alpha_1 \neq 0$  and*

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-1}(x), \quad n \geq 0 \quad (Q_{-1}(x) = 0, \alpha_0 \text{ arbitrary}).$$

In this case  $\alpha_n \neq 0$ ,  $n \geq 1$  (cf. (2.4)),  $P_n(\xi) \neq 0$ ,  $n \geq 1$  and  $(x - \xi)u = v$ .

*Proof.* See Theorem 3.2, Theorem 3.3, and Theorem 3.4 in [9]. □

### 3. Generalized $\Delta_w$ -coherency

Consider the inner product on  $\mathbb{P}$

$$(3.1) \quad \phi_\lambda(f, g) = \int_{\mathbb{R}} f(x)g(x)d\rho_0(x) + \lambda \sum_{k=1}^\infty \Delta_{w_1} f(y_k)\Delta_{w_1} g(y_k)\rho_1(y_k),$$

where  $\rho_1$  is a discrete measure supported on a uniform lattice  $\{y_k\}_{k=0}^\infty$  with step  $w_1$ .

We let  $\Delta_{w_1}$  be the difference operator defined by

$$\Delta_{w_1} f(x) = \frac{f(x + w_1) - f(x)}{w_1}.$$

Notice that  $\lim_{w_1 \rightarrow 0} \Delta_{w_1}$  is the standard derivative operator.

We will consider the basis  $x^{[0]} = 1$ ,  $x^{[n]} = x(x - w_1) \cdots (x - (n - 1)w_1)$ ,  $n = 1, 2, \dots$  in the linear space  $\mathbb{P}$ . Notice that  $\Delta_{w_1} x^{[n]} = nx^{[n-1]}$ . This basis will play in our work the same role as the canonical basis for the derivative operator.

We introduce the generalized moments for the inner product (3.1) as follows

$$\mu_{m,n} = \phi_\lambda(x^{[m]}, x^{[n]}) = \mu_{m,n}^{(0)} + \lambda mn \mu_{m-1,n-1}^{(1)}.$$

Here  $\mu_{m,n}^{(0)}$  and  $\mu_{m,n}^{(1)}$  will denote the moments associated with the basis  $(x^{[n]})_{n \in \mathbb{N}}$  for the inner products

$$\begin{aligned} \langle f, g \rangle_0 &= \int_{\mathbb{R}} f(x)g(x)d\rho_0(x), \\ \langle f, g \rangle_1 &= \sum_{k=1}^{\infty} f(y_k)g(y_k)\rho_1(y_k). \end{aligned}$$

Using the standard Gram-Schmidt orthogonalization process, we can obtain a sequence  $\{Q_n(x; \lambda)\}_{n=0}^{\infty}$  of monic polynomials orthogonal with respect to the inner product (3.1). Notice that when  $w_1 \rightarrow 0$ , (3.1) becomes a Sobolev inner product in the standard sense.

Thus, the monic polynomial  $\{Q_n(x; \lambda)\}_{n=0}^{\infty}$  can be explicitly given by a determinantal expression

$$\begin{aligned} &Q_n(x; \lambda) \\ &= \frac{\begin{vmatrix} \mu_{0,0}^{(0)} & \mu_{1,0}^{(0)} & \cdots & \mu_{n,0}^{(0)} \\ \mu_{0,1}^{(0)} & \mu_{1,1}^{(0)} + \lambda \mu_{0,0}^{(1)} & \cdots & \mu_{n,1}^{(0)} + \lambda n \mu_{n-1,0}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{0,n-1}^{(0)} & \mu_{1,n-1}^{(0)} + \lambda(n-1)\mu_{0,n-2}^{(1)} & \cdots & \mu_{n,n-1}^{(0)} + \lambda n(n-1)\mu_{n-1,n-2}^{(1)} \\ 1 & x^{[1]} & \cdots & x^{[n]} \end{vmatrix}}{\det \left[ \mu_{k,j}^{(0)} + \lambda k j \mu_{k-1,j-1}^{(1)} \right]_{k,j=0}^{n-1}}. \end{aligned}$$

Dividing the numerator and the denominator by  $\lambda^{n-2}$  and taking limit in the resulting expression when  $\lambda \rightarrow \infty$ , we get

$$\begin{aligned} S_n(x) &= \lim_{\lambda \rightarrow \infty} Q_n(x; \lambda) \\ &= \frac{\begin{vmatrix} \mu_{0,0}^{(0)} & \mu_{1,0}^{(0)} & \cdots & \mu_{n,0}^{(0)} \\ 0 & \mu_{0,0}^{(1)} & \cdots & n \mu_{n-1,0}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (n-1)\mu_{0,n-2}^{(1)} & \cdots & n(n-1)\mu_{n-1,n-2}^{(1)} \\ 1 & x^{[1]} & \cdots & x^{[n]} \end{vmatrix}}{\det \left[ k j \mu_{k-1,j-1}^{(1)} \right]_{k,j=0}^{n-1}} \end{aligned}$$

with the convention  $\mu_{0,0}^{(0)} = 1$ , i.e., we assume that  $\rho_0$  is a probability measure.

THEOREM 3.1. *The following statements hold.*

- (i)  $\langle S_n(x), 1 \rangle_0 = 0$ ;
- (ii)  $\langle \Delta_{w_1} S_n(x), x^{[k]} \rangle_1 = 0, k = 0, 1, \dots, n - 2$ .

*Proof.* Both results are direct consequences of the determinantal representation of  $S_n(x)$ . □

If  $\{P_n(x)\}_{n=0}^\infty$  and  $\{R_n(x)\}_{n=0}^\infty$  denote, respectively, the MOPS relative to  $\rho_0$  and  $\rho_1$ , then we get from Theorem 3.1

$$\Delta_{w_1} S_n(x) = nR_{n-1}(x).$$

On the other hand

$$nR_{n-1}(x) = \Delta_{w_1} P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} \Delta_{w_1} P_k(x).$$

Thus

$$S_n(x) = P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} P_k(x) + \alpha_{n,0} P_0(x).$$

But by taking into account of (i) in Theorem 3.1,  $\alpha_{n,0} = 0$  and, as a consequence,

$$S_n(x) = P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} P_k(x).$$

DEFINITION 3.1. The pair of measures  $\{\rho_0, \rho_1\}$  is said to be a generalized  $\Delta_w$ -coherent pair if there is a non-negative integer  $N$  such that

$$(3.2) \quad nR_{n-1}(x) = \Delta_w P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} \Delta_w P_k(x)$$

with  $\alpha_{n,n-N} \neq 0$ .

In particular, if  $N = 1$  we get the usual  $\Delta_w$ -coherent pairs considered by I. Area, E. Godoy, and F. Marcellán [2, 3] for  $w = 1$ .

On the other hand, if we expand the polynomial  $S_n(x)$  in terms of the MOPS  $\{Q_n(x; \lambda)\}_{n=0}^\infty$ , then we get

$$S_n(x) = Q_n(x; \lambda) + \sum_{k=0}^{n-1} \beta_{n,k} Q_k(x; \lambda),$$

where

$$\beta_{n,k} = \frac{\phi_\lambda(S_n(x), Q_k(x; \lambda))}{\phi_\lambda(Q_k(x; \lambda), Q_k(x; \lambda))}.$$

Notice that according to (3.1), the numerator is

$$\begin{aligned} & \langle S_n(x), Q_k \rangle_0 + \lambda \langle \Delta_{w_1} S_n(x), \Delta_{w_1} Q_k(x; \lambda) \rangle_1 \\ &= \langle S_n(x), Q_k(x) \rangle_0 + \lambda \langle nR_{n-1}(x), \Delta_{w_1} Q_k(x; \lambda) \rangle_1. \end{aligned}$$

From (3.2), the first term vanishes when  $k < n - N$ , while the second one vanishes for  $k \leq n - 1$ .

Thus  $\beta_{n,k} = 0$  for  $k < n - N$ . For  $k = n - N$ , we get

$$\beta_{n,n-N} = \frac{\alpha_{n,n-N} \langle P_{n-N}(x), P_{n-N}(x) \rangle_0}{\phi_\lambda(Q_{n-N}(x; \lambda), Q_{n-N}(x; \lambda))} \neq 0.$$

Thus, generalized  $\Delta_{w_1}$ -coherency yields

$$(3.3) \quad Q_n(x; \lambda) + \sum_{k=n-N}^{n-1} \beta_{n,k} Q_k(x; \lambda) = P_n(x) + \sum_{k=0}^{n-1} \alpha_{n,k} P_k(x),$$

where  $\beta_{n,n-N} \neq 0$  and  $\alpha_{n,n-N} \neq 0$ . Here

$$\beta_{n,n-N} = \alpha_{n,n-N} \frac{\langle P_{n-N}(x), P_{n-N}(x) \rangle_0}{\phi_\lambda(Q_{n-N}(x; \lambda), Q_{n-N}(x; \lambda))}.$$

Notice that if (3.3) holds, then taking into account of (3.1) for  $j = 0, 1, \dots, n - N - 1$ ,

$$\begin{aligned} (3.4) \quad 0 &= \phi_\lambda(Q_n(x; \lambda) + \sum_{k=n-N}^{n-1} \beta_{n,k} Q_k(x; \lambda), x^{[j]}) \\ &= \langle P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x), x^{[j]} \rangle_0 + \lambda \langle \Delta_{w_1}(P_n(x) \\ &+ \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)), \Delta_{w_1} x^{[j]} \rangle_1 \\ &= \lambda \langle \Delta_{w_1}(P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)), jx^{[j-1]} \rangle_1, \end{aligned}$$

i.e.,

$$\Delta_{w_1}(P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)) = nR_{n-1}(x) + \sum_{k=n-N-1}^{n-2} \gamma_{n,k} R_k(x),$$

according to the orthogonality condition (3.4).



In this work we are interested in the case of generalized  $\Delta_{w_1}$ -coherent pairs when  $N = 2$ , i.e, the MOPS's relative to  $\rho_0$  and  $\rho_1$  satisfy

$$(3.5) \quad nR_{n-1}(x) = \Delta_{w_1}(P_n(x) + \alpha_{n,n-1}P_{n-1}(x) + \alpha_{n,n-2}P_{n-2}(x))$$

with  $\alpha_{n,n-2} \neq 0$ . For a sake of simplicity we will assume  $w_1 = 1$ .

We now give an example of generalized  $\Delta$ -coherent pair for  $N = 2$ .

Let  $\rho_0(x) = \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(x + 1)\Gamma(\gamma)}$  ( $0 < \mu < 1, \gamma > 0$ ) be the Meixner weight function supported in the set  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ . It is well known [19] that the sequence of monic Meixner polynomials  $\{M_n^{(\gamma, \mu)}(x)\}_{n=0}^\infty$  satisfies

$$(3.6) \quad \begin{aligned} M_n^{(\gamma, \mu)}(x) &= \frac{1}{n+1} \Delta M_{n+1}^{(\gamma-1, \mu)}(x), \\ M_n^{(\gamma, \mu)}(x) &= \frac{1}{n+1} \Delta M_{n+1}^{(\gamma, \mu)}(x) + \frac{\mu}{1-\mu} \Delta M_n^{(\gamma, \mu)}(x). \end{aligned}$$

Thus, if the sequence of monic polynomials  $\{R_n(x)\}_{n=0}^\infty$  orthogonal relative to a discrete measure  $\rho_1$  satisfies

$$R_{n-1}(x) = M_{n-1}^{(\gamma, \mu)}(x) + \beta_{n-1}M_{n-2}^{(\gamma, \mu)}(x)$$

with  $\beta_{n-1} \neq 0$ , i.e., the pair  $\{\rho_0, \rho_1\}$  is a  $\Delta$ -coherent pair with  $N = 1$ , then from (3.6) we get

$$\begin{aligned} nR_{n-1}(x) &= \Delta(M_n^{(\gamma, \mu)}(x) + \frac{\mu}{1-\mu}nM_{n-1}^{(\gamma, \mu)}(x)) \\ &\quad + \beta_{n-1}(\frac{n}{n-1}\Delta M_{n-1}^{(\gamma, \mu)}(x) + \frac{\mu}{1-\mu}M_{n-2}^{(\gamma, \mu)}(x)), \end{aligned}$$

i.e., (3.5) holds.

Thus every  $\Delta$ -coherent pair with  $\rho_0$ (Meixner weight) and  $N = 1$  is a generalized  $\Delta$ -coherent pair with  $N = 2$ .

#### 4. Generalized $\Delta$ -coherent pairs

Let  $u_0$  and  $u_1$  be quasi-definite moment functionals with corresponding MOPS's  $\{P_n(x)\}_{n=0}^\infty$  and  $\{R_n(x)\}_{n=0}^\infty$ , respectively, satisfying three-term recurrence relations

$$(4.1) \quad \begin{aligned} P_{n+1}(x) &= (x - b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geq 0 \\ &\text{and } \langle u_0, P_n^2 \rangle = p_n, \quad n \geq 0; \end{aligned}$$

$$(4.2) \quad \begin{aligned} R_{n+1}(x) &= (x - \beta_n)R_n(x) - \gamma_nR_{n-1}(x), \quad n \geq 0 \\ &\text{and } \langle u_1, R_n^2 \rangle = r_n, \quad n \geq 0. \end{aligned}$$

DEFINITION 4.1.  $\{u_0, u_1\}$  is a generalized  $\Delta$ -coherent pair if there exist complex numbers  $\{\sigma_n\}_{n=1}^\infty$  and  $\{\tau_n\}_{n=1}^\infty$  such that

$$(4.3) \quad R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x) - \tau_{n-1} Q_{n-2}(x), \quad n \geq 0,$$

where  $Q_{-1}(x) = Q_{-2}(x) = 0$ ,  $Q_n(x) = \frac{1}{n+1} \Delta P_{n+1}(x)$ ,  $n \geq 0$ , and  $\sigma_0 = \tau_{-1} = \tau_0 = 0$ .

In particular, if  $\sigma_n \neq 0$  for some  $n \geq 1$  and  $\tau_n = 0$ ,  $n \geq 1$  (resp.  $\tau_n \neq 0$  for some  $n \geq 1$ ), then we call  $\{u_0, u_1\}$  a 2-term (resp. 3-term)  $\Delta$ -coherent pair.

In these cases, we call  $u_1$  (resp.  $u_0$ ) a ‘‘companion’’ of  $u_0$  (resp.  $u_1$ ).

In the following, we always assume that  $\{u_0, u_1\}$  is a generalized  $\Delta$ -coherent pair unless stated otherwise.

PROPOSITION 4.1. *We have*

$$(4.4) \quad n \frac{P_n(x)}{p_n} u_0 = \Delta(G_n(x)u_1), \quad n \geq 1,$$

where

$$(4.5) \quad G_n(x) = \frac{\tau_n}{r_{n+1}} R_{n+1}(x) + \frac{\sigma_n}{r_n} R_n(x) - \frac{1}{r_{n-1}} R_{n-1}(x), \quad n \geq 1,$$

so that  $n - 1 \leq \deg(G_n) \leq n + 1$ .

*Proof.* Let  $u_n^{(0)}$ ,  $\tilde{u}_n^{(0)}$ , and  $u_n^{(1)}$ ,  $n \geq 0$  be the dual bases of  $\{P_n(x)\}_{n=0}^\infty$ ,  $\{Q_n(x)\}_{n=0}^\infty$ , and  $\{R_n(x)\}_{n=0}^\infty$ , respectively. Then, it is easy to see that

$$\tilde{u}_n^{(0)} = u_n^{(1)} - \sigma_{n+1} u_{n+1}^{(1)} - \tau_{n+1} u_{n+2}^{(1)} = -G_{n+1} u_1 \quad (n \geq 0).$$

Hence,

$$\Delta(\tilde{u}_n^{(0)}) = -(n+1)u_{n+1}^{(0)} = -(n+1)\frac{1}{p_{n+1}} P_{n+1} u_0 = -\Delta(G_{n+1} u_1), \quad n \geq 0.$$

Therefore, we have the result. □

THEOREM 4.2. *Both  $u_0$  and  $u_1$  are discrete-semiclassical (of class  $\leq 6$  for  $u_0$  and of class  $\leq 2$  for  $u_1$ ) satisfying*

$$(4.6) \quad \Delta(\rho_i u_i) = \eta_i u_i, \quad i = 0, 1,$$

as well as

$$(4.7) \quad \rho_1(x+1)\Delta u_1 = \nu(x)u_1, \quad \rho_1(x)u_0 = H(x)u_1, \quad \nu(x)u_0 = H(x)\Delta u_1,$$

where

$$(4.8) \quad \begin{aligned} \rho_1(x) &:= 2 \frac{P_2(x-1)}{p_2} G_1(x) - \frac{P_1(x-1)}{p_1} G_2(x), \\ \eta_1(x) &:= 2 \frac{\Delta P_2(x-1)}{p_2} G_1(x) - \frac{\Delta P_1(x)}{p_1} G_2(x), \end{aligned}$$

$$(4.9) \quad \begin{aligned} \rho_0(x) &:= \rho_1(x)H(x), \\ \eta_0(x) &:= H(x+1)\nu(x) + \rho_1(x+1)(\Delta H(x) + \Delta H(x-1)), \end{aligned}$$

$$(4.10) \quad \begin{aligned} H(x) &:= G_1(x+1)\Delta G_2(x) - G_2(x+1)\Delta G_1(x), \\ \nu &:= \frac{P_1(x)}{p_1} \Delta G_2(x) - 2 \frac{P_2(x)}{p_1} \Delta G_1(x). \end{aligned}$$

Moreover,

$$(4.11) \quad n \frac{P_n(x)}{p_n} H(x) = \rho_1(x+1)\Delta G_n(x) + \nu(x)G_n(x), \quad n \geq 1.$$

*Proof.* Set  $n = 1$  and  $2$  in (4.4). Then

$$(4.12) \quad \frac{P_1(x)}{p_1} u_0 = \Delta G_1(x)u_1 + G_1(x+1)\Delta u_1,$$

$$(4.13) \quad 2 \frac{P_2(x)}{p_2} u_0 = \Delta G_2(x)u_1 + G_2(x+1)\Delta u_1.$$

Eliminating  $u_0$ ,  $u_1$ , and  $\Delta u_1$  from (4.12) and (4.13) gives (4.6) for  $i = 1$  and (4.7).

We also have  $\Delta(\rho_0(x)u_0) = \Delta(\rho_1(x)H(x)u_0) = \Delta(H(x-1)H(x)u_1) = \eta_0 u_0$  by (4.7) and (4.9), which gives (4.6) for  $i = 0$ .

By (4.4) and (4.7), we have

$$\begin{aligned} &n \frac{P_n(x)}{p_n} H(x)u_1 \\ &= n \frac{P_n(x)}{p_n} \rho_1(x+1)u_0 = (\rho_1(x+1)\Delta G_n(x) + \nu(x)G_n(x+1))u_1 \end{aligned}$$

since  $\rho_1(x + 1)\Delta u_1 = \nu(x)u_1$  so that (4.11) holds. It is now easy to see that  $H = \frac{\tau_1\tau_2}{r_2r_3}x^4 + \text{lower degree terms}$  so that  $\deg(H) \leq 4$  and

$$(4.14) \quad \deg(H) = \begin{cases} 4 & \text{if } \tau_1\tau_2 \neq 0 \\ 3 & \text{if } \tau_1 = 0, \sigma_1\tau_2 \neq 0 \\ 2 & \text{if (i) } \sigma_1 = \tau_1 = 0, \tau_2 \neq 0 \text{ or} \\ & \text{(ii) } \tau_1 \neq 0, \tau_2 = 0, \sigma_1\sigma_2 + \tau_1 \neq 0 \text{ or} \\ & \text{(iii) } \tau_1 = \tau_2 = 0, \sigma_1\sigma_2 \neq 0 \\ 1 & \text{if (i) } \tau_1 \neq 0, \tau_2 = \sigma_1\sigma_2 + \tau_1 = 0 \text{ or} \\ & \text{(ii) } \sigma_1 = \tau_1 = \tau_2 = 0, \sigma_2 \neq 0 \\ 0 & \text{if } \sigma_2 = \tau_1 = \tau_2 = 0. \end{cases}$$

Hence  $H \neq 0$  so that  $0 \leq \deg(H) \leq 4, 0 \leq \deg(\rho_1) \leq 4, 0 \leq \deg(\rho_0) \leq 8,$  and  $0 \leq \deg(\nu) \leq 3,$  by (4.7) and (4.9). Hence  $u_0$  and  $u_1$  are discrete-semiclassical of class  $\leq 6$  and  $\leq 2,$  respectively, and so  $1 \leq \deg(\eta_1) \leq 3, 1 \leq \deg(\eta_0) \leq 7.$  □

Marcellán et al. ([2]) proved: if  $\{u_0, u_1\}$  is a 2-term  $\Delta$ -coherent pair, then either  $u_0$  or  $u_1$  must be classical under some extra relations between  $u_0$  and  $u_1.$

We say that a quasi-definite moment functional  $u$  with MOPS

$$\{P_n(x)\}_{n=0}^\infty$$

is strongly discrete-classical if there is another MOPS  $\{S_n(x)\}_{n=0}^\infty$  relative to  $w$  such that  $P_n(x) = \frac{1}{n+1}\Delta S_{n+1}(x), n \geq 0.$  Then  $u$  and  $w$  must be discrete-classical moment functionals of the same type satisfying

$$\Delta(\varphi(x)u) = \psi(x)u, \Delta(\varphi(x)w) = (\psi(x+1) - \Delta\varphi(x))w, \text{ and } \varphi(x)w = u.$$

Discrete-classical moment functionals  $u_c^{(\mu)}$  and  $u_m^{(\gamma,\mu)}$  ( $\gamma > 1$ ) are strongly discrete-classical.

In our more general case, both  $u_0$  and  $u_1$  may not be discrete-classical but we have:

**THEOREM 4.3.** ([10]) *Assume that either  $u_0$  is discrete-classical or  $u_1$  is strongly discrete-classical.*

- (i) *If  $\tau_k = 0$  for some  $k \geq 1,$  then  $\tau_n = 0$  for all  $n \geq 1.$*
- (ii) *If  $\sigma_j = 0$  for some  $j \geq 1$  and  $\tau_k = 0$  for some  $k \geq 1,$  then  $\sigma_n = \tau_n = 0$  for all  $n \geq 1$  so that  $u_0$  and  $u_1$  must be discrete-classical of the same family.*

*Proof.* See the proof of Theorem 3.4 in [10]. □

PROPOSITION 4.4. *If  $u_0$  is discrete-classical, then  $G_1u_1$  is also discrete-classical of the same type as  $u_0$ . Moreover if  $\deg(G_1) = 0$ , then  $\sigma_n = \tau_n = 0$ ,  $n \geq 1$ . Moreover if  $\deg(G_1) = 1$ , i.e.,  $G_1(x) = g_1(x - \xi)$  ( $g_1 \neq 0$ ), then  $Q_n(\xi) \neq 0$ ,  $\sigma_n = \frac{R_{n-1}(\xi)}{R_n(\xi)}\gamma_n$ ,  $\tau_n = 0$ ,  $n \geq 1$ , and  $\{Q_n(x)\}_{n=0}^\infty = \{R_n^*(\xi; x)\}_{n=0}^\infty$ .*

*Proof.* Assume  $u_0$  is a discrete-classical moment functional satisfying  $\Delta(\varphi u_0) = \psi u_0$  with  $0 \leq \deg(\varphi) \leq 2$  and  $\deg(\psi) = 1$ . Then (cf. (2.3))

$$\varphi(x)\Delta^2P_n(x) + \psi(x)\Delta P_n(x) = \lambda_n P_n(x+1), n \geq 0,$$

where  $\lambda_n = \frac{1}{2}n(n-1)\Delta^2\varphi(x) + n\Delta\psi(x)$  and  $\lambda_n \neq 0$ ,  $n \geq 1$ . Hence  $\psi(x) = \lambda_1 P_1(x)$  so that by (3.4) for  $n = 1$

$$\Delta(\varphi u_0) = \lambda_1 P_1 u_0 = \lambda_1 p_1 \Delta(G_1 u_1).$$

Therefore  $G_1u_1 = (\lambda_1 p_1)^{-1}\varphi u_0$  is also a discrete-classical moment functional of the same type as  $u_0$ . If  $\deg(G_1) = 0$ , then  $\sigma_1 = \tau_1 = 0$  (cf. (4.5)) so that  $\sigma_n = \tau_n = 0$ ,  $n \geq 1$  by Theorem 4.3. If  $\deg(G_1) = 1$ , then  $\sigma_1 \neq 0$  and  $\tau_1 = 0$  so that  $\sigma_n \neq 0$ ,  $\tau_n = 0$ ,  $n \geq 1$ , and  $(x - \xi)u_1 = (\lambda_1 p_1 g_1)^{-1}\varphi u_0$ . Hence,  $(x - \xi)u_1$  is quasi-definite so that  $Q_n(\xi) \neq 0$ ,  $n \geq 1$ ,  $\{Q_n(x)\}_{n=0}^\infty = \{R_n^*(\xi; x)\}_{n=0}^\infty$ , and  $\sigma_n = \frac{R_{n-1}(\xi)}{R_n(\xi)}\gamma_n$ ,  $n \geq 1$  (see (2.4)). □

The discrete-semiclassical character of  $u_0$  and  $u_1$  depends on  $\deg(H)$ . It is the same as for generalized coherent pair [10]. In this paper, we see only the cases when  $\{u_0, u_1\}$  has discrete-classical character, which occurs when  $\deg(H) = 2$  (iii) and  $\deg(H) = 4$  in (4.14).

Consider the case  $\deg(H) = 2$  (iii), that is,  $\tau_1 = \tau_2 = 0$  and  $\sigma_1\sigma_2 \neq 0$ . In this case, there are three cases:  $H(x) = h(x - \xi)(x - \xi - 1)$  or  $H(x) = h(x - \xi)(x - \zeta)$  ( $\zeta \neq \xi, \xi \pm 1$ ) and  $\tau_n = 0$ ,  $n \geq 1$  or  $H(x) = h(x - \xi)(x - \zeta)$  ( $\zeta \neq \xi, \xi \pm 1$ ) and  $\tau_n \neq 0$  for some  $n \geq 3$ .

THEOREM 4.5. (cf. [3, 10]) *Assume  $\tau_1 = \tau_2 = 0$  and  $\sigma_1\sigma_2 \neq 0$  so that  $\deg(H) = 2$ .*

- (i) *If  $H(x) = h(x - \xi)(x - \xi - 1)$ , then  $u_0$  and  $G_1u_1$  are discrete-classical of the same type,  $\deg(\eta_1) = 2$ , and  $\sigma_n \neq 0$ ,  $\tau_n = 0$ ,  $n \geq 1$ .*
- (ii) *If  $H(x) = h(x - \xi)(x - \zeta)$  ( $\zeta \neq \xi, \xi \pm 1$ ),  $\tau_n = 0$ ,  $n \geq 1$ , then  $u_1$  is discrete-classical. Moreover, if  $u_1$  is strongly discrete-classical, then  $\sigma_n \neq 0$ ,  $n \geq 1$ .*
- (iii) *If  $\tau_n \neq 0$  for some  $n \geq 3$ , then  $H(x) = h(x - \xi)(x - \zeta)$  ( $\zeta \neq \xi, \xi \pm 1$ ) and  $1 \leq s_0 \leq 3$ ,  $0 \leq s_1 \leq 1$ , and  $u_1$  cannot be strongly discrete-classical.*

*Proof.* Note that  $\deg(G_1) = 1$  and  $\deg(G_2) = 2$  when  $\tau_1 = \tau_2 = 0$  and  $\sigma_1\sigma_2 \neq 0$ .

- (i) The following proof is essentially the same as that of Theorem 4.2 in [3], where it is assumed that  $\sigma_n \neq 0$  and  $\tau_n = 0, n \geq 1$ .

Assume  $H(x) = h(x - \xi)(x - \xi - 1)$  ( $h = g_1g_2$ ). Then

$$H(\xi) = G_1(\xi+1)\Delta G_2(\xi) - g_1G_2(\xi+1) = 0 \text{ and } \Delta H(\xi) = 2g_2G_1(\xi+1) = 0$$

so that  $G_1(\xi + 1) = G_2(\xi + 1) = 0$ . Hence  $G_1(x) = g_1(x - \xi - 1)$  and  $G_2(x) = G_1(x)\tilde{G}_2(x)$ ,  $\deg(\tilde{G}_2) = 1$ . Then

$$\begin{aligned} \rho_1(x) &= G_1(x)\tilde{\rho}_1(x), \quad 0 \leq \deg(\tilde{\rho}_1) \leq 2 \\ \eta_1(x) &= G_1(x)\tilde{\eta}_1(x), \quad 0 \leq \deg(\tilde{\eta}_1) \leq 1. \end{aligned}$$

Multiplying (4.12) by  $\tilde{G}_2$  and then subtracting (4.13), we have

$$(4.15) \quad \tilde{\rho}_1(x)u_0 = G_1(x)\Delta\tilde{G}_2u_1 = g_2G_1(x)u_1$$

so that by (4.12)  $\Delta(\tilde{\rho}_1u_0) = g_2\Delta(G_1(x)u_1) = g_2\frac{P_1(x)}{p_1}u_0$ . Therefore,  $u_0$  is discrete-classical and  $G_1u_1$  is also discrete-classical of the same type as  $u_0$  by Proposition 4.4 satisfying

$$\Delta(\tilde{\rho}_1(x)G_1(x)u_1) = \tilde{\eta}_1(x)G_1(x)u_1.$$

Hence  $\deg(\tilde{\eta}_1) = 1$  and so  $\deg(\eta_1) = 2$ . Finally  $\sigma_n \neq 0$  and  $\tau_n = 0, n \geq 1$ , by Theorem 4.3.

- (ii) It is also proved in Theorem 4.6 in [3] assuming  $\sigma_n \neq 0$  and  $\tau_n = 0, n \geq 1$ . But, the inspection of the proof of Theorem 4.6 in [3] reveals that we only need  $\sigma_1\sigma_2 \neq 0$  and  $\tau_n = 0, n \geq 1$ . Then, by Theorem 4.3,  $\sigma_n \neq 0, n \geq 1$ , if  $u_1$  is strongly discrete-classical.
- (iii) Assume  $\tau_n \neq 0$  for some  $n \geq 3$ . Then,  $H(x)$  cannot have a repeated zero by (i) so that  $H(x) = h(x - \xi_1)(x - \zeta)$  ( $\zeta \neq \xi, \xi \pm 1$ ) and the conclusion follows from Theorem 4.3. □

The relation (4.15) between  $u_0$  and  $u_1$  also holds in case Theorem 4.5 (ii) (see Theorem 4.6 in [3]) for  $\xi = \xi_1$  or  $\xi_2$ . Hence we have in case (i) or (ii) in Theorem 4.5

$$hu_1 = (x - \xi - 1)^{-1}\tilde{\rho}_1u_0 + r_0\delta(x - \xi).$$

Now consider the case  $\deg(H) = 4$ , that is,  $\tau_1\tau_2 \neq 0$ .

**THEOREM 4.6.** *If  $G_1$  divides  $G_2$ , then  $u_0$  and  $G_1u_1$  are discrete-classical of the same type,  $\deg(\eta_1) = 3$ , and  $\tau_n \neq 0, n \geq 1$ . More precisely, we have:*

(i) If  $H(x) = h(x - \xi)(x - \xi - 1)(x - \xi - 2)(x - \xi - 3)$  and  $G_1(\xi + 1) = 0$ , then  $G_1$  divides  $G_2$

$$u_1 = (g_2)^{-1}(x - \xi)^{-2}\tilde{\rho}_1 u_0 + r_0\delta(x - \xi) + (R_1(0) + \xi)r_0\delta'(x - \xi).$$

(ii) If  $H(x) = h(x - \xi)(x - \xi - 1)(x - \zeta)(x - \zeta - 1)$  ( $\zeta \neq \xi, \xi + 1, \xi + 2$ ) and  $G_1(\xi + 1) \neq 0$ , then  $G_1$  divides  $G_2$  and

$$u_1 = (p_1 p_2 g_2)^{-1}(x - \xi_1)^{-1}(x - \xi_2)^{-1}\tilde{\rho}_1 u_0 + \frac{r_0}{\xi_1 - \xi_2} [(R_1(0) + \xi_1)\delta(x - \xi_2) - (R_1(0) + \xi_2)\delta(x - \xi_1)].$$

Here,  $\rho_1(x) = G_1(x)\tilde{\rho}_1(x)$ .

*Proof.* Since  $\tau_1\tau_2 \neq 0$ ,  $\deg(G_1) = 2$  and  $\deg(G_2) = 3$ . Assume that  $G_1$  divides  $G_2$ . Set  $G_2 = G_1\tilde{G}_2$ ,  $\deg(\tilde{G}_2) = 1$ . Then

$$\begin{aligned} \rho_1(x) &= G_1(x)\tilde{\rho}_1 \quad (0 \leq \deg(\tilde{\rho}_1) \leq 2) \\ \eta_1 &= G_1(x)\tilde{\eta}_1 \quad (0 \leq \deg(\tilde{\eta}_1) \leq 1). \end{aligned}$$

As in the proof of Theorem 4.2, by eliminating  $\Delta u_1$  from (4.12) and (4.13), we obtain

$$(4.16) \quad \tilde{\rho}_1(x + 1)u_0 = G_1(x)\Delta\tilde{G}_2 u_1$$

so that by (4.12),

$$\Delta(\tilde{\rho}_1(x + 1)u_0) = \Delta\tilde{G}_2(x)\Delta(G_1(x)u_1) = \Delta\tilde{G}_2(x)\frac{P_1(x)}{p_1}u_0.$$

Hence  $u_0$  and  $G_1 u_1$  are discrete-classical of the same type by Proposition 4.4 and  $\Delta(\tilde{\rho}_1(x)G_1(x)u_1) = \tilde{\eta}_1(x)G_1(x)u_1$ . Hence,  $\deg(\tilde{\eta}_1(x)) = 1$  and so  $\deg(\eta_1(x)) = 3$ . Finally,  $\tau_n \neq 0$ ,  $n \geq 1$  by Theorem 4.3.

(i)  $H(x) = h(x - \xi)(x - \xi - 1)(x - \xi - 2)(x - \xi - 3)$ . Then  $\Delta^3 H(x) = 24h(x - \xi)$ . By (4.10), we have

$$(4.17) \quad \begin{aligned} H(x) &= G_1(x + 1)\Delta G_2(x) - G_2(x + 1)\Delta G_1(x) \\ \Delta H(x) &= G_1(x + 1)\Delta^2 G_2(x) - G_2(x + 1)\Delta^2 G_1 \\ \Delta^2 H(x) &= G_1(x + 2)\Delta^3 G_2 + \Delta G_1(x + 1)\Delta^2 G_2(x + 1) \\ &\quad - \Delta G_2(x + 1)\Delta^2 G_1 \end{aligned}$$

$$(4.18) \quad \Delta^3 H(x) = \Delta^3 G_2(\Delta G_1(x + 1) + \Delta G_1(x + 2)).$$

Since  $G_1(\xi + 1) = 0$  i.e.,  $G_1(x) = g_1(x - \xi - 1)(x - m)$  so we have

$$(4.19) \quad \Delta G_1(x) = g_1(2x - \xi - m).$$

Since  $\Delta^3 H(\xi) = 0$ , we have  $\Delta G_1(\xi + 1) + \Delta G_1(\xi + 2) = 0$  so that  $m = \xi + 3$ . Hence  $G_1(\xi + 3) = 0$ . Since  $H(\xi) = H(\xi + 2) = 0$  but

$\Delta G_1(\xi) \neq 0$  and  $\Delta G_1(\xi+2) \neq 0$ , we have  $G_2(\xi+1) = G_2(\xi+3) = 0$  by (4.18). Hence  $G_1(x)$  divides  $G_2(x)$ .

- (ii) Assume  $H(x) = h(x - \xi)(x - \xi - 1)(x - \zeta)(x - \zeta - 1)$  ( $\zeta \neq \xi, \xi + 1, \xi + 2$ ). Then since  $\Delta^3 H(x) = 24h(x - \frac{\xi+\zeta+2}{2})$ , by using (4.19), we have  $m = \zeta + 1$ . Hence  $G_1(\zeta + 1) = 0$ . Since  $H(\xi) = H(\zeta) = 0$ , but  $\Delta G_1(\xi) \neq 0$  and  $\Delta G_1(\zeta) \neq 0$ , by (4.18) we have  $G_2(\xi + 1) = G_2(\zeta + 1) = 0$ . Hence  $G_1(x)$  divides  $G_2(x)$ .  $\square$

By the essentially same methods used in [10] for ordinary coherent pairs, we now have:

**THEOREM 4.7.** *Let  $u_0$  be a discrete-classical moment functional satisfying  $\Delta(\varphi u_0) = \psi u_0$  ( $0 \leq \deg(\varphi) \leq 2, \deg(\psi) = 1$ ). Assume  $\langle u_0, \varphi \rangle = 1$ . Then  $u_1$  is a 3-term companion of  $u_0$  if and only if either*

$$(4.20) \quad u_1 = (x - \xi_1)^{-1}(x - \xi_2)^{-1}\varphi u_0 + a\delta(x - \xi_1) + b\delta(x - \xi_2)$$

or

$$(4.21) \quad u_1 = (x - \xi_1)^{-2}\varphi u_0 + a\delta(x - \xi_1) + b\delta'(x - \xi_1)$$

for some complex numbers  $\xi_1 \neq \xi_2, a$ , and  $b$  satisfying

- (i) in case of (4.20)

$$\begin{cases} a + b \neq 0, \\ a(\xi_1 - a\xi_1 - b\xi_2)^2 + b(\xi_2 - a\xi_1 - b\xi_2)^2 \neq 1, \\ \left| \begin{array}{cc} \langle u_1, P_{n+1} \rangle & \langle u_1, P_n \rangle \\ \langle (x - \xi_1)u_1, P_{n+1} \rangle & \langle (x - \xi_1)u_1, P_n \rangle \end{array} \right| \neq 0, \quad n \geq 0; \end{cases}$$

- (ii) in case of (4.21)

$$\begin{cases} a \neq 0, \quad a - b^2 \neq 0, \\ \left| \begin{array}{cc} \langle u_1, P_{n+1} \rangle & \langle u_1, P_n \rangle \\ \langle (x - \xi_1)u_1, P_{n+1} \rangle & \langle (x - \xi_1)u_1, P_n \rangle \end{array} \right| \neq 0, \quad n \geq 0. \end{cases}$$

*Proof.* See Theorem 4.5 in [10].  $\square$

Conversely we have:

**THEOREM 4.8.** *Let  $u_1$  be a strongly discrete-classical moment functional satisfying  $\Delta(\varphi u_1) = \psi u_1$  ( $0 \leq \deg(\varphi) \leq 2, \deg(\psi) = 1$ ) and  $\{T_n(x)\}_{n=0}^\infty$  the discrete-classical MOPS relative to  $w$  with  $\frac{1}{n+1}\Delta T_{n+1}(x) = R_n(x), n \geq 0$ . Then  $u_0$  is a 3-term companion of  $u_1$  if and only if either*

$$u_0 = (x - \xi_1)(x - \xi_2)w$$



for some complex numbers  $\xi_1 \neq \xi_2$  satisfying

$$\begin{vmatrix} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T_n(\xi_2) & T_{n+1}(\xi_2) \end{vmatrix} \neq 0, \quad n \geq 1$$

or

$$u_0 = (x - \xi_1)^2 w$$

for some complex number  $\xi_1$  satisfying

$$\begin{vmatrix} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T'_n(\xi_1) & T'_{n+1}(\xi_1) \end{vmatrix} \neq 0, \quad n \geq 1.$$

*Proof.* See Theorem 4.6 in [10]. □

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