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GENERALIZED Δ-COHERENT PAIRS

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Abstract. A pair of quasi-definite linear functionals $\{u_0, u_1\}$ is a generalized Δ-coherent pair if monic orthogonal polynomials

$$\{P_n(x)\}_{n=0}^{\infty}$$

and

$$\{R_n(x)\}_{n=0}^{\infty}$$

relative to u_0 and u_1, respectively, satisfy a relation

$$R_n(x) = \frac{1}{n+1}\Delta P_{n+1}(x) - \frac{\sigma_n}{n}\Delta P_n(x) - \frac{\tau_{n-1}}{n-1}\Delta P_{n-1}(x),\ n \geq 2,$$

where σ_n and τ_n are arbitrary constants and $\Delta p = p(x+1) - p(x)$ is the difference operator.

We show that if $\{u_0, u_1\}$ is a generalized Δ-coherent pair, then u_0 and u_1 must be discrete-semiclassical linear functionals. We also find conditions under which either u_0 or u_1 is discrete-classical.

1. Introduction

Concerning the problem of evaluating the Fourier coefficients in the Fourier expansion of functions by polynomials orthogonal with respect to a Sobolev inner product

$$\phi_\lambda(f,g) := \int_{-\infty}^{\infty} f(x)g(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} f'(x)g'(x)d\mu_1(x), \tag{1.1}$$

where $d\mu_0$ and $d\mu_1$ are positive Borel measures with finite moments and $\lambda \in \mathbb{R}^+$, Iserles et al. [8] introduced the concept of coherency and symmetric coherency for the measures $d\mu_0$ and $d\mu_1$.

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After the work by Iserles et al. [8], there have been many works [4, 12, 13, 14, 15, 17, 18, 20] on coherency from different points of view even allowing $d\mu_0$ and $d\mu_1$ to be signed or even complex valued measures. In particular, in [10], we introduced generalized coherency which unifies both coherency and symmetric coherency.

In [2, 3], they introduced a discrete version of coherency, that is, Δ-coherency. Here Δ is the difference operator defined as $\Delta f(x) = f(x+1) - f(x)$.

In this work, we will study the generalized Δ-coherency in a more general setting by using the formal approach to orthogonality via linear functionals as was done in [10]. See also [2, 3, 14, 15].

In Section 2, we collect basic definitions, notations, and lemmas that we will use later. In Section 3, we define (see Definition 4.1) and analyze the generalized Δ-coherency.

2. Preliminaries

Let $\mathbb{P}$ be the linear space of all polynomials in one variable with complex coefficients. We denote the degree of a polynomial $P(x)$ by $\deg(P)$ with the convention that $\deg(0) = -1$. A polynomial system(PS) is a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n,\ n \geq 0$.

A linear functional u on $\mathbb{P}$ is called a moment functional and we denote its action on a polynomial $\phi(x)$ by $\langle u, \phi \rangle$. We say that a moment functional u is quasi-definite(positive-definite, respectively) if its moments $a_n := \langle u, x^n \rangle,\ n \geq 0$, satisfy the Hamburger condition

$$\Delta_n(u) := det[a_{i+j}]_{i,j=0}^{n} \neq 0, \quad (\Delta_n(u) > 0, \text{ respectively}), \quad n \geq 0.$$

Definition 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is said to be an orthogonal polynomial system(OPS) if there is a linear functional u on $\mathbb{P}$ such that

$$\langle u, P_m P_n \rangle = p_n \delta_{mn},\ m, n \geq 0,$$

where p_n are non-zero constants. In this case, we call $\{P_n(x)\}_{n=0}^{\infty}$ an OPS relative to u and u is said to be an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. A linear functional u is quasi-definite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to u (see [6]). Moreover, in this case, each $P_n(x)$ is uniquely determined up to a non-zero constant factor.

For a moment functional u, a polynomial $\phi(x)$, and a constant c, we define moment functionals $\Delta u,\ \phi u$, and $(x-c)^{-1}u$ by

$$\langle \Delta u, p(x) \rangle := -\langle u, \Delta p(x-1) \rangle;$$

$$\langle \phi u, p\rangle := \langle u, \phi p\rangle;$$

$$\langle (x-c)^{-1}u, p\rangle := \langle u, \frac{p(x)-p(c)}{x-c}\rangle, \quad p \in \mathbb{P}.$$

Then we have for polynomials $p(x)$ and $q(x)$

$$\Delta(p(x)q(x)) = q(x)\Delta p(x) + p(x+1)\Delta q(x),$$

$$\Delta(p(x)u) = p(x+1)\Delta u + \Delta p(x)u.$$

For a constant c, let $\delta(x-c)$ be the moment functional defined by

$$\langle \delta(x-c), p(x)\rangle = p(c), \ p(x) \in \mathbb{P}.$$

For a PS $\{P_n(x)\}_{n=0}^{\infty}$, the dual basis of $\{P_n(x)\}_{n=0}^{\infty}$ is the sequence $\{u_n\}_{n=0}^{\infty}$ of moment functionals defined by the relation

$$\langle u_n, P_m\rangle = \delta_{mn}, \quad m, n \geq 0.$$

In particular, u_0 is said to be the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. If $\{P_n(x)\}_{n=0}^{\infty}$ is a monic OPS(MOPS), then $\{P_n(x)\}_{n=0}^{\infty}$ must be orthogonal with respect to u_0 and

$$u_n = \frac{P_n(x)}{p_n}u_0, \quad n \geq 0.$$

Definition 2.2. ([16]) A quasi-definite moment functional u is said to be discrete-semiclassical if u satisfies

$$\Delta(\varphi u) = \psi u, \tag{2.1}$$

for some polynomials $\varphi(x)$ and $\psi(x)$ with $(\varphi, \psi) \neq (0, 0)$. We then have $\deg(\varphi) \geq 0$ and $\deg(\psi) \geq 1$. The corresponding OPS is called a discrete-semiclassical OPS.

For a discrete-semiclassical moment functional u,

$$s := \min \max(\deg(\varphi) - 2, \deg(\psi) - 1)$$

the class number of u, where the minimum is taken over all pairs $(\varphi, \psi) \neq (0, 0)$ of polynomials satisfying (2.1). In particular, a discrete-semiclassical moment functional of class 0 is called a discrete-classical moment functional.

Discrete-classical moment functionals can be characterized in many other ways. For an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to u, the following statements are all equivalent ([1]):

(i) $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete-classical OPS, that is, $\Delta(\varphi u) = \psi u$ for some polynomial φ and ψ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$;

(ii) ([7])

$$\{Q_n(x) := \frac{1}{n+1}\Delta P_{n+1}\}_{n=0}^{\infty}$$

is also an MOPS. Then $\{Q_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to $\tilde{u} = \varphi u$ satisfying

$$\Delta(\varphi(x)\tilde{u}) = (\psi(x) + \Delta\varphi(x-1))\tilde{u}; \tag{2.2}$$

(iii) There are polynomials φ and ψ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$ such that

$$\begin{aligned} &\varphi(x)\Delta^2 P_n(x) + \psi(x)\Delta P_n(x) \\ &= (\frac{1}{2}n(n-1)\Delta^2\varphi(x) + n\Delta\psi(x))P_n(x+1),\ n \geq 0 \text{ ([11]).} \end{aligned} \tag{2.3}$$

It is well-known that there are essentially four distinct discrete-classical OPS's, up to a linear change of variable ([7, 19]):

(i) Charlier polynomials $\{c_n^{(\mu)}(x)\}_{n=0}^{\infty}$: $\varphi(x) = \mu$, $\psi(x) = \mu - x$ ($\mu > 0$);

(ii) Meixner polynomials $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$: $\varphi(x) = \mu(\gamma + x)$, $\psi(x) = \mu\gamma - x(1-\mu)$ ($\gamma > 0$, $\mu \in (0,1)$);

(iii) Kravchuk polynomials $\{k_n^{(p)}(x;N)\}_{n=0}^{\infty}$: $\varphi(x) = N - x$, $\psi(x) = \frac{Np-x}{p}$ ($p \in (0,1)$, $N \in \mathbb{Z}^+$);

(iv) Hahn polynomials $\{h_n^{(\alpha,\beta)}(x,N)\}_{n=0}^{\infty}$: $\varphi(x) = (N-x-1)(x+\beta+1)$, $\psi(x) = (N-1)(\beta+1) - x(\alpha+\beta+2)$ ($\alpha, \beta > -1$, $N \in \mathbb{Z}^+$).

We denote by $u_c^{(\mu)}, u_m^{(\gamma,\mu)}, u_k^{(p,N)}$, and $u_h^{(\alpha,\beta,N)}$ the orthogonalizing moment functionals for Charlier, Meixner, Kravchuk, and Hahn polynomials, respectively. Notice that the moment functionals for Kravchuk and Hahn polynomials are not quasi-definite.

For an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to u and complex numbers ξ and c, let $\{P_n^*(\xi;x)\}_{n=0}^{\infty}$, $\{P_n^{(1)}(x)\}_{n=0}^{\infty}$, and $\{P_n(c;x)\}_{n=0}^{\infty}$ be the monic kernel polynomials, the monic numerator polynomials(also called the associated polynomials of first kind (see [6])), and the monic co-recursive polynomials of $\{P_n(x)\}_{n=0}^{\infty}$, respectively:

$$P_n^*(\xi;x) = \frac{\langle u, P_n^2\rangle}{P_n(\xi)}\sum_{k=0}^{n}\frac{P_k(x)P_k(\xi)}{\langle u, P_k^2\rangle},\ n \geq 0\text{([6])};$$

$$P_n(x) = P_n^*(\xi;x) - \frac{P_{n-1}(\xi)}{P_n(\xi)}\frac{\langle u, P_n^2\rangle}{\langle u, P_{n-1}^2\rangle}P_{n-1}^*(\xi;x),\ n \geq 1\text{([9])}; \tag{2.4}$$

$$P_n(c;x) = P_n(x) - cP_{n-1}^{(1)}(x),\ n \geq 1\text{([5])}. \tag{2.5}$$

It is well-known (see Theorem 7.1 on p. 36 in [6]) that for a quasi-definite moment functional u with MOPS $\{P_n(x)\}_{n=0}^{\infty}$ and a complex number ξ, $(x-\xi)u$ is also quasi-definite if and only if $P_n(\xi) \neq 0,\ n \geq 1$. Then the MOPS relative to $(x-\xi)u$ is $\{P_n^*(\xi;x)\}_{n=0}^{\infty}$. Moreover (see Theorem 3.6 in [9]), if u is discrete-semiclassical of class s satisfying (2.1), then $(x-\xi)u$ is also discrete-semiclassical of class

$$\begin{cases} s-1 & \text{if } \varphi(\xi)=\psi(\xi)=0 \\ s & \text{if } \varphi(\xi)=0 \text{ and } \psi(\xi)\neq 0 \\ s+1 & \text{if } \varphi(\xi)\neq 0. \end{cases}$$

Conversely if $(x-\xi)u$ is discrete-semiclassical of class s, then u is discrete-semiclassical of class either $s-1$, s, or $s+1$.

Proposition 2.1. *Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the MOPS's relative to u and v respectively. Then, $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\xi;x)\}_{n=0}^{\infty}$ for some complex number ξ if and only if there are complex numbers $\alpha_n (n\geq 1)$ such that $\alpha_1 \neq 0$ and*

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-1}(x),\ n\geq 0\ (Q_{-1}(x)=0, \alpha_0 \text{ arbitrary}).$$

In this case $\alpha_n \neq 0,\ n\geq 1$ (cf. (2.4)), $P_n(\xi)\neq 0,\ n\geq 1$ and $(x-\xi)u = v$.

Proof. See Theorem 3.2, Theorem 3.3, and Theorem 3.4 in [9]. □

3. Generalized Δ_w-coherency

Consider the inner product on $\mathbb{P}$

$$(3.1)\quad \phi_\lambda(f,g) = \int_{\mathbb{R}} f(x)g(x)d\rho_0(x) + \lambda \sum_{k=1}^{\infty} \Delta_{w_1} f(y_k)\Delta_{w_1} g(y_k)\rho_1(y_k),$$

where ρ_1 is a discrete measure supported on a uniform lattice $\{y_k\}_{k=0}^{\infty}$ with step w_1.

We let Δ_{w_1} be the difference operator defined by

$$\Delta_{w_1} f(x) = \frac{f(x+w_1)-f(x)}{w_1}.$$

Notice that $\lim\limits_{w_1\to 0} \Delta_{w_1}$ is the standard derivative operator.

We will consider the basis $x^{[0]}=1$, $x^{[n]} = x(x-w_1)\cdots(x-(n-1)w_1)$, $n=1,2,\dots$ in the linear space $\mathbb{P}$. Notice that $\Delta_{w_1} x^{[n]} = nx^{[n-1]}$. This basis will play in our work the same role as the canonical basis for the derivative operator.

We introduce the generalized moments for the inner product (3.1) as follows

$$\mu_{m,n} = \phi_\lambda(x^{[m]}, x^{[n]}) = \mu^{(0)}_{m,n} + \lambda mn\mu^{(1)}_{m-1,n-1}.$$

Here $\mu^{(0)}_{m,n}$ and $\mu^{(1)}_{m,n}$ will denote the moments associated with the basis $(x^{[n]})_{n\in\mathbb{N}}$ for the inner products

$$\langle f, g\rangle_0 = \int_{\mathbb{R}} f(x)g(x)d\rho_0(x),$$

$$\langle f, g\rangle_1 = \sum_{k=1}^{\infty} f(y_k)g(y_k)\rho_1(y_k).$$

Using the standard Gram-Schmidt orthogonalization process, we can obtain a sequence $\{Q_n(x;\lambda)\}_{n=0}^{\infty}$ of monic polynomials orthogonal with respect to the inner product (3.1). Notice that when $w_1 \to 0$, (3.1) becomes a Sobolev inner product in the standard sense.

Thus, the monic polynomial $\{Q_n(x;\lambda)\}_{n=0}^{\infty}$ can be explicitly given by a determinantal expression

$$Q_n(x;\lambda) = \frac{\begin{vmatrix} \mu^{(0)}_{0,0} & \mu^{(0)}_{1,0} & \cdots & \mu^{(0)}_{n,0} \\ \mu^{(0)}_{0,1} & \mu^{(0)}_{1,1} + \lambda\mu^{(1)}_{0,0} & \cdots & \mu^{(0)}_{n,1} + \lambda n\mu^{(1)}_{n-1,0} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^{(0)}_{0,n-1} & \mu^{(0)}_{1,n-1} + \lambda(n-1)\mu^{(1)}_{0,n-2} & \cdots & \mu^{(0)}_{n,n-1} + \lambda n(n-1)\mu^{(1)}_{n-1,n-2} \\ 1 & x^{[1]} & \cdots & x^{[n]} \end{vmatrix}}{\det\left[\mu^{(0)}_{k,j} + \lambda kj\mu^{(1)}_{k-1,j-1}\right]_{k,j=0}^{n-1}}.$$

Dividing the numerator and the denominator by λ^{n-2} and taking limit in the resulting expression when $\lambda \to \infty$, we get

$$S_n(x) = \lim_{\lambda\to\infty} Q_n(x;\lambda) = \frac{\begin{vmatrix} \mu^{(0)}_{0,0} & \mu^{(0)}_{1,0} & \cdots & \mu^{(0)}_{n,0} \\ 0 & \mu^{(1)}_{0,0} & \cdots & n\mu^{(1)}_{n-1,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (n-1)\mu^{(1)}_{0,n-2} & \cdots & n(n-1)\mu^{(1)}_{n-1,n-2} \\ 1 & x^{[1]} & \cdots & x^{[n]} \end{vmatrix}}{\det\left[kj\mu^{(1)}_{k-1,j-1}\right]_{k,j=0}^{n-1}}$$

with the convention $\mu^{(0)}_{0,0} = 1$, i.e., we assume that ρ_0 is a probability measure.

THEOREM 3.1. *The following statements hold.*

(i) $\langle S_n(x), 1\rangle_0 = 0$;

(ii) $\langle \Delta_{w_1} S_n(x), x^{[k]}\rangle_1 = 0$, $k = 0, 1, \ldots, n-2$.

Proof. Both results are direct consequences of the determinantal representation of $S_n(x)$. □

If $\{P_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ denote, respectively, the MOPS relative to ρ_0 and ρ_1, then we get from Theorem 3.1

$$\Delta_{w_1} S_n(x) = nR_{n-1}(x).$$

On the other hand

$$nR_{n-1}(x) = \Delta_{w_1} P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} \Delta_{w_1} P_k(x).$$

Thus

$$S_n(x) = P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} P_k(x) + \alpha_{n,0} P_0(x).$$

But by taking into account of (i) in Theorem 3.1, $\alpha_{n,0} = 0$ and, as a consequence,

$$S_n(x) = P_n(x) + \sum_{k=1}^{n-1} \alpha_{n,k} P_k(x).$$

DEFINITION 3.1. The pair of measures $\{\rho_0,\ \rho_1\}$ is said to be a generalized Δ_w-coherent pair if there is a non-negative integer N such that

$$nR_{n-1}(x) = \Delta_w P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} \Delta_w P_k(x) \tag{3.2}$$

with $\alpha_{n,n-N} \neq 0$.

In particular, if $N = 1$ we get the usual Δ_w-coherent pairs considered by I. Area, E. Godoy, and F. Marcellán [2, 3] for $w = 1$.

On the other hand, if we expand the polynomial $S_n(x)$ in terms of the MOPS $\{Q_n(x;\lambda)\}_{n=0}^{\infty}$, then we get

$$S_n(x) = Q_n(x;\lambda) + \sum_{k=0}^{n-1} \beta_{n,k} Q_k(x;\lambda),$$

where

$$\beta_{n,k} = \frac{\phi_\lambda(S_n(x), Q_k(x;\lambda))}{\phi_\lambda(Q_k(x;\lambda), Q_k(x;\lambda))}.$$

Notice that according to (3.1), the numerator is

$$\begin{aligned}&\langle S_n(x), Q_k\rangle_0 + \lambda\langle \Delta_{w_1} S_n(x), \Delta_{w_1} Q_k(x;\lambda)\rangle_1\\ &= \langle S_n(x), Q_k(x)\rangle_0 + \lambda\langle nR_{n-1}(x), \Delta_{w_1} Q_k(x;\lambda)\rangle_1.\end{aligned}$$

From (3.2), the first term vanishes when $k < n-N$, while the second one vanishes for $k \le n-1$.

Thus $\beta_{n,k} = 0$ for $k < n-N$. For $k = n-N$, we get

$$\beta_{n,n-N} = \frac{\alpha_{n,n-N}\langle P_{n-N}(x), P_{n-N}(x)\rangle_0}{\phi_\lambda(Q_{n-N}(x;\lambda), Q_{n-N}(x;\lambda))} \neq 0.$$

Thus, generalized Δ_{w_1}-coherency yields

$$Q_n(x;\lambda) + \sum_{k=n-N}^{n-1} \beta_{n,k} Q_k(x;\lambda) = P_n(x) + \sum_{k=0}^{n-1} \alpha_{n,k} P_k(x), \tag{3.3}$$

where $\beta_{n,n-N} \neq 0$ and $\alpha_{n,n-N} \neq 0$. Here

$$\beta_{n,n-N} = \alpha_{n,n-N} \frac{\langle P_{n-N}(x), P_{n-N}(x)\rangle_0}{\phi_\lambda(Q_{n-N}(x;\lambda), Q_{n-N}(x;\lambda))}.$$

Notice that if (3.3) holds, then taking into account of (3.1) for $j = 0, 1, \ldots, n-N-1$,

$$\begin{aligned}0 &= \phi_\lambda(Q_n(x;\lambda) + \sum_{k=n-N}^{n-1} \beta_{n,k} Q_k(x;\lambda), x^{[j]})\\ &= \langle P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x), x^{[j]}\rangle_0 + \lambda\langle \Delta_{w_1}(P_n(x)\\ &\qquad + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)), \Delta_{w_1} x^{[j]}\rangle_1 \qquad (3.4)\\ &= \lambda\langle \Delta_{w_1}(P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)), j x^{[j-1]}\rangle_1,\end{aligned}$$

i.e.,

$$\Delta_{w_1}(P_n(x) + \sum_{k=n-N}^{n-1} \alpha_{n,k} P_k(x)) = nR_{n-1}(x) + \sum_{k=n-N-1}^{n-2} \gamma_{n,k} R_k(x),$$

according to the orthogonality condition (3.4).

In this work we are interested in the case of generalized Δ_{w_1}-coherent pairs when $N=2$, i.e, the MOPS's relative to ρ_0 and ρ_1 satisfy

$$nR_{n-1}(x) = \Delta_{w_1}(P_n(x) + \alpha_{n,n-1}P_{n-1}(x) + \alpha_{n,n-2}P_{n-2}(x)) \tag{3.5}$$

with $\alpha_{n,n-2} \neq 0$. For a sake of simplicity we will assume $w_1 = 1$.

We now give an example of generalized Δ-coherent pair for $N=2$.

Let $\rho_0(x) = \dfrac{\mu^x\Gamma(\gamma+x)}{\Gamma(x+1)\Gamma(\gamma)}$ $(0<\mu<1,\ \gamma>0)$ be the Meixner weight function supported in the set $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$. It is well known [19] that the sequence of monic Meixner polynomials $\{M_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$ satisfies

$$\begin{aligned} M_n^{(\gamma,\mu)}(x) &= \frac{1}{n+1}\Delta M_{n+1}^{(\gamma-1,\mu)}(x),\\ M_n^{(\gamma,\mu)}(x) &= \frac{1}{n+1}\Delta M_{n+1}^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}\Delta M_n^{(\gamma,\mu)}(x). \end{aligned} \tag{3.6}$$

Thus, if the sequence of monic polynomials $\{R_n(x)\}_{n=0}^{\infty}$ orthogonal relative to a discrete measure ρ_1 satisfies

$$R_{n-1}(x) = M_{n-1}^{(\gamma,\mu)}(x) + \beta_{n-1}M_{n-2}^{(\gamma,\mu)}(x)$$

with $\beta_{n-1}\neq 0$, i.e., the pair $\{\rho_0,\rho_1\}$ is a Δ-coherent pair with $N=1$, then from (3.6) we get

$$\begin{aligned} nR_{n-1}(x) = \Delta(&M_n^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}nM_{n-1}^{(\gamma,\mu)}(x))\\ &+ \beta_{n-1}(\frac{n}{n-1}\Delta M_{n-1}^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}M_{n-2}^{(\gamma,\mu)}(x)), \end{aligned}$$

i.e., (3.5) holds.

Thus every Δ-coherent pair with ρ_0(Meixner weight) and $N=1$ is a generalized Δ-coherent pair with $N=2$.

4. Generalized Δ-coherent pairs

Let u_0 and u_1 be quasi-definite moment functionals with corresponding MOPS's $\{P_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$, respectively, satisfying three-term recurrence relations

$$\begin{aligned} P_{n+1}(x) = &\quad (x-b_n)P_n(x) - c_nP_{n-1}(x),\ n\geq 0\\ &\quad\text{and } \langle u_0, P_n^2\rangle = p_n,\ n\geq 0; \end{aligned} \tag{4.1}$$

$$\begin{aligned} R_{n+1}(x) = &\quad (x-\beta_n)R_n(x) - \gamma_nR_{n-1}(x),\ n\geq 0\\ &\quad\text{and } \langle u_1, R_n^2\rangle = r_n,\ n\geq 0. \end{aligned} \tag{4.2}$$

Definition 4.1. $\{u_0, u_1\}$ is a generalized Δ-coherent pair if there exist complex numbers $\{\sigma_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that

$$R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x) - \tau_{n-1} Q_{n-2}(x),\ n \geq 0, \tag{4.3}$$

where $Q_{-1}(x) = Q_{-2}(x) = 0$, $Q_n(x) = \frac{1}{n+1}\Delta P_{n+1}(x)$, $n \geq 0$, and $\sigma_0 = \tau_{-1} = \tau_0 = 0$.

In particular, if $\sigma_n \neq 0$ for some $n \geq 1$ and $\tau_n = 0$, $n \geq 1$(resp. $\tau_n \neq 0$ for some $n \geq 1$), then we call $\{u_0, u_1\}$ a 2-term(resp. 3-term) Δ-coherent pair.

In these cases, we call u_1(resp. u_0) a "companion" of u_0(resp. u_1).

In the following, we always assume that $\{u_0, u_1\}$ is a generalized Δ-coherent pair unless stated otherwise.

Proposition 4.1. *We have*

$$n\frac{P_n(x)}{p_n}u_0 = \Delta(G_n(x)u_1),\ n \geq 1, \tag{4.4}$$

where

$$G_n(x) = \frac{\tau_n}{r_{n+1}}R_{n+1}(x) + \frac{\sigma_n}{r_n}R_n(x) - \frac{1}{r_{n-1}}R_{n-1}(x),\ n \geq 1, \tag{4.5}$$

so that $n - 1 \leq \deg(G_n) \leq n + 1$.

Proof. Let $u_n^{(0)}$, $\tilde{u}_n^{(0)}$, and $u_n^{(1)}$, $n \geq 0$ be the dual bases of $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{R_n(x)\}_{n=0}^{\infty}$, respectively. Then, it is easy to see that

$$\tilde{u}_n^{(0)} = u_n^{(1)} - \sigma_{n+1}u_{n+1}^{(1)} - \tau_{n+1}u_{n+2}^{(1)} = -G_{n+1}u_1\ (n \geq 0).$$

Hence,

$$\Delta(\tilde{u}_n^{(0)}) = -(n+1)u_{n+1}^{(0)} = -(n+1)\frac{1}{p_{n+1}}P_{n+1}u_0 = -\Delta(G_{n+1}u_1),\ n \geq 0.$$

Therefore, we have the result. □

Theorem 4.2. *Both u_0 and u_1 are discrete-semiclassical* (*of class ≤ 6 for u_0 and of class ≤ 2 for u_1*) *satisfying*

$$\Delta(\rho_i u_i) = \eta_i u_i,\ i = 0, 1, \tag{4.6}$$

as well as

$$\rho_1(x+1)\Delta u_1 = \nu(x)u_1,\ \rho_1(x)u_0 = H(x)u_1,\ \nu(x)u_0 = H(x)\Delta u_1, \tag{4.7}$$

where

$$(4.8)\quad \rho_1(x) := 2\frac{P_2(x-1)}{p_2}G_1(x) - \frac{P_1(x-1)}{p_1}G_2(x),$$
$$\eta_1(x) := 2\frac{\Delta P_2(x-1)}{p_2}G_1(x) - \frac{\Delta P_1(x)}{p_1}G_2(x),$$

$$(4.9)\quad \rho_0(x) := \rho_1(x)H(x),$$
$$\eta_0(x) := H(x+1)\nu(x) + \rho_1(x+1)(\Delta H(x) + \Delta H(x-1)),$$

$$(4.10)\quad H(x) := G_1(x+1)\Delta G_2(x) - G_2(x+1)\Delta G_1(x),$$
$$\nu := \frac{P_1(x)}{p_1}\Delta G_2(x) - 2\frac{P_2(x)}{p_1}\Delta G_1(x).$$

Moreover,

$$(4.11)\qquad n\frac{P_n(x)}{p_n}H(x) = \rho_1(x+1)\Delta G_n(x) + \nu(x)G_n(x),\ n \geq 1.$$

Proof. Set $n = 1$ and 2 in (4.4). Then

$$(4.12)\qquad \frac{P_1(x)}{p_1}u_0 = \Delta G_1(x)u_1 + G_1(x+1)\Delta u_1,$$
$$(4.13)\qquad 2\frac{P_2(x)}{p_2}u_0 = \Delta G_2(x)u_1 + G_2(x+1)\Delta u_1.$$

Eliminating u_0, u_1, and Δu_1 from (4.12) and (4.13) gives (4.6) for $i = 1$ and (4.7).

We also have $\Delta(\rho_0(x)u_0) = \Delta(\rho_1(x)H(x)u_0) = \Delta(H(x-1)H(x)u_1) = \eta_0 u_0$ by (4.7) and (4.9), which gives (4.6) for $i = 0$.

By (4.4) and (4.7), we have

$$n\frac{P_n(x)}{p_n}H(x)u_1$$
$$= n\frac{P_n(x)}{p_n}\rho_1(x+1)u_0 = (\rho_1(x+1)\Delta G_n(x) + \nu(x)G_n(x+1))u_1$$

since $\rho_1(x+1)\Delta u_1 = \nu(x)u_1$ so that (4.11) holds. It is now easy to see that $H = \frac{\tau_1\tau_2}{r_2r_3}x^4$+lower degree terms so that $\deg(H) \leq 4$ and

$$\deg(H) = \begin{cases} 4 \text{ if} & \tau_1\tau_2 \neq 0 \\ 3 \text{ if} & \tau_1 = 0,\ \sigma_1\tau_2 \neq 0 \\ 2 \text{ if} & \text{(i) } \sigma_1 = \tau_1 = 0, \tau_2 \neq 0 \text{ or} \\ & \text{(ii) } \tau_1 \neq 0, \tau_2 = 0, \sigma_1\sigma_2 + \tau_1 \neq 0 \text{ or} \\ & \text{(iii) } \tau_1 = \tau_2 = 0, \sigma_1\sigma_2 \neq 0 \\ 1 \text{ if} & \text{(i) } \tau_1 \neq 0, \tau_2 = \sigma_1\sigma_2 + \tau_1 = 0 \text{ or} \\ & \text{(ii) } \sigma_1 = \tau_1 = \tau_2 = 0, \sigma_2 \neq 0 \\ 0 \text{ if} & \sigma_2 = \tau_1 = \tau_2 = 0. \end{cases} \tag{4.14}$$

Hence $H \not\equiv 0$ so that $0 \leq \deg(H) \leq 4$, $0 \leq \deg(\rho_1) \leq 4$, $0 \leq \deg(\rho_0) \leq 8$, and $0 \leq \deg(\nu) \leq 3$, by (4.7) and (4.9). Hence u_0 and u_1 are discrete-semiclassical of class ≤ 6 and ≤ 2, respectively, and so $1 \leq \deg(\eta_1) \leq 3$, $1 \leq \deg(\eta_0) \leq 7$. □

Marcellán et al. ([2]) proved: if $\{u_0, u_1\}$ is a 2-term Δ-coherent pair, then either u_0 or u_1 must be classical under some extra relations between u_0 and u_1.

We say that a quasi-definite moment functional u with MOPS

$$\{P_n(x)\}_{n=0}^{\infty}$$

is strongly discrete-classical if there is another MOPS $\{S_n(x)\}_{n=0}^{\infty}$ relative to w such that $P_n(x) = \frac{1}{n+1}\Delta S_{n+1}(x),\ n \geq 0$. Then u and w must be discrete-classical moment functionals of the same type satisfying

$$\Delta(\varphi(x)u) = \psi(x)u,\ \Delta(\varphi(x)w) = (\psi(x+1) - \Delta\varphi(x))w, \text{ and } \varphi(x)w = u.$$

Discrete-classical moment functionals $u_c^{(\mu)}$ and $u_m^{(\gamma,\mu)}$ $(\gamma > 1)$ are strongly discrete-classical.

In our more general case, both u_0 and u_1 may not be discrete-classical but we have:

Theorem 4.3. ([10]) *Assume that either u_0 is discrete-classical or u_1 is strongly discrete-classical.*

(i) *If $\tau_k = 0$ for some $k \geq 1$, then $\tau_n = 0$ for all $n \geq 1$.*
(ii) *If $\sigma_j = 0$ for some $j \geq 1$ and $\tau_k = 0$ for some $k \geq 1$, then $\sigma_n = \tau_n = 0$ for all $n \geq 1$ so that u_0 and u_1 must be discrete-classical of the same family.*

Proof. See the proof of Theorem 3.4 in [10]. □

Proposition 4.4. *If u_0 is discrete-classical, then G_1u_1 is also discrete-classical of the same type as u_0. Moreover if* $\deg(G_1) = 0$, *then* $\sigma_n = \tau_n = 0$, $n \geq 1$. *Moreover if* $\deg(G_1) = 1$, *i.e.,* $G_1(x) = g_1(x - \xi)$ $(g_1 \neq 0)$, *then* $Q_n(\xi) \neq 0$, $\sigma_n = \frac{R_{n-1}(\xi)}{R_n(\xi)}\gamma_n$, $\tau_n = 0$, $n \geq 1$, *and* $\{Q_n(x)\}_{n=0}^{\infty} = \{R_n^*(\xi; x)\}_{n=0}^{\infty}$.

Proof. Assume u_0 is a discrete-classical moment functional satisfying $\Delta(\varphi u_0) = \psi u_0$ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$. Then (cf. (2.3))

$$\varphi(x)\Delta^2 P_n(x) + \psi(x)\Delta P_n(x) = \lambda_n P_n(x+1), n \geq 0,$$

where $\lambda_n = \frac{1}{2}n(n-1)\Delta^2\varphi(x) + n\Delta\psi(x)$ and $\lambda_n \neq 0$, $n \geq 1$. Hence $\psi(x) = \lambda_1 P_1(x)$ so that by (3.4) for $n = 1$

$$\Delta(\varphi u_0) = \lambda_1 P_1 u_0 = \lambda_1 p_1 \Delta(G_1 u_1).$$

Therefore $G_1u_1 = (\lambda_1 p_1)^{-1}\varphi u_0$ is also a discrete-classical moment functional of the same type as u_0. If $\deg(G_1) = 0$, then $\sigma_1 = \tau_1 = 0$(cf. (4.5)) so that $\sigma_n = \tau_n = 0$, $n \geq 1$ by Theorem 4.3. If $\deg(G_1) = 1$, then $\sigma_1 \neq 0$ and $\tau_1 = 0$ so that $\sigma_n \neq 0$, $\tau_n = 0$, $n \geq 1$, and $(x - \xi)u_1 = (\lambda_1 p_1 g_1)^{-1}\varphi u_0$. Hence, $(x-\xi)u_1$ is quasi-definite so that $Q_n(\xi) \neq 0$, $n \geq 1$, $\{Q_n(x)\}_{n=0}^{\infty} = \{R_n^*(\xi; x)\}_{n=0}^{\infty}$, and $\sigma_n = \frac{R_{n-1}(\xi)}{R_n(\xi)}\gamma_n$, $n \geq 1$ (see (2.4)). □

The discrete-semiclassical character of u_0 and u_1 depends on $\deg(H)$. It is the same as for generalized coherent pair [10]. In this paper, we see only the cases when $\{u_0, u_1\}$ has discrete-classical character, which occurs when $\deg(H) = 2$ (iii) and $\deg(H) = 4$ in (4.14).

Consider the case $\deg(H) = 2$ (iii), that is, $\tau_1 = \tau_2 = 0$ and $\sigma_1\sigma_2 \neq 0$. In this case, there are three cases: $H(x) = h(x-\xi)(x-\xi-1)$ or $H(x) = h(x-\xi)(x-\zeta)$ $(\zeta \neq \xi, \xi \pm 1)$ and $\tau_n = 0$, $n \geq 1$ or $H(x) = h(x-\xi)(x-\zeta)$ $(\zeta \neq \xi, \xi \pm 1)$ and $\tau_n \neq 0$ for some $n \geq 3$.

Theorem 4.5. (cf. [3, 10]) *Assume* $\tau_1 = \tau_2 = 0$ *and* $\sigma_1\sigma_2 \neq 0$ *so that* $\deg(H) = 2$.

(i) *If* $H(x) = h(x-\xi)(x-\xi-1)$, *then* u_0 *and* G_1u_1 *are discrete-classical of the same type,* $\deg(\eta_1) = 2$, *and* $\sigma_n \neq 0$, $\tau_n = 0$, $n \geq 1$.
(ii) *If* $H(x) = h(x-\xi)(x-\zeta)$ $(\zeta \neq \xi, \xi \pm 1)$, $\tau_n = 0$, $n \geq 1$, *then* u_1 *is discrete-classical. Moreover, if* u_1 *is strongly discrete-classical, then* $\sigma_n \neq 0$, $n \geq 1$.
(iii) *If* $\tau_n \neq 0$ *for some* $n \geq 3$, *then* $H(x) = h(x-\xi)(x-\zeta)$ $(\zeta \neq \xi, \xi \pm 1)$ *and* $1 \leq s_0 \leq 3$, $0 \leq s_1 \leq 1$, *and* u_1 *cannot be strongly discrete-classical.*

Proof. Note that $\deg(G_1) = 1$ and $\deg(G_2) = 2$ when $\tau_1 = \tau_2 = 0$ and $\sigma_1\sigma_2 \neq 0$.

(i) The following proof is essentially the same as that of Theorem 4.2 in [3], where it is assumed that $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$.
Assume $H(x) = h(x-\xi)(x-\xi-1)$ $(h = g_1g_2)$. Then

$$H(\xi) = G_1(\xi+1)\Delta G_2(\xi) - g_1G_2(\xi+1) = 0 \text{ and } \Delta H(\xi) = 2g_2G_1(\xi+1) = 0$$

so that $G_1(\xi+1) = G_2(\xi+1) = 0$. Hence $G_1(x) = g_1(x-\xi-1)$ and $G_2(x) = G_1(x)\tilde{G}_2(x)$, $\deg(\tilde{G}_2) = 1$. Then

$$\rho_1(x) = G_1(x)\tilde{\rho}_1(x),\ 0 \leq \deg(\tilde{\rho}_1) \leq 2$$
$$\eta_1(x) = G_1(x)\tilde{\eta}_1(x),\ 0 \leq \deg(\tilde{\eta}_1) \leq 1.$$

Multiplying (4.12) by $\tilde{G}_2$ and then subtracting (4.13), we have

$$\tilde{\rho}_1(x)u_0 = G_1(x)\Delta\tilde{G}_2u_1 = g_2G_1(x)u_1 \tag{4.15}$$

so that by (4.12) $\Delta(\tilde{\rho}_1u_0) = g_2\Delta(G_1(x)u_1) = g_2\frac{P_1(x)}{p_1}u_0$. Therefore, u_0 is discrete-classical and G_1u_1 is also discrete-classical of the same type as u_0 by Proposition 4.4 satisfying

$$\Delta(\tilde{\rho}_1(x)G_1(x)u_1) = \tilde{\eta}_1(x)G_1(x)u_1.$$

Hence $\deg(\tilde{\eta}_1) = 1$ and so $\deg(\eta_1) = 2$. Finally $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$, by Theorem 4.3.

(ii) It is also proved in Theorem 4.6 in [3] assuming $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$. But, the inspection of the proof of Theorem 4.6 in [3] reveals that we only need $\sigma_1\sigma_2 \neq 0$ and $\tau_n = 0$, $n \geq 1$. Then, by Theorem 4.3, $\sigma_n \neq 0$, $n \geq 1$, if u_1 is strongly discrete-classical.

(iii) Assume $\tau_n \neq 0$ for some $n \geq 3$. Then, $H(x)$ cannot have a repeated zero by (i) so that $H(x) = h(x-\xi_1)(x-\zeta)$ $(\zeta \neq \xi, \xi \pm 1)$ and the conclusion follows from Theorem 4.3. □

The relation (4.15) between u_0 and u_1 also holds in case Theorem 4.5 (ii) (see Theorem 4.6 in [3]) for $\xi = \xi_1$ or ξ_2. Hence we have in case (i) or (ii) in Theorem 4.5

$$hu_1 = (x-\xi-1)^{-1}\tilde{\rho}_1u_0 + r_0\delta(x-\xi).$$

Now consider the case $\deg(H) = 4$, that is, $\tau_1\tau_2 \neq 0$.

THEOREM 4.6. *If G_1 divides G_2, then u_0 and G_1u_1 are discrete-classical of the same type, $\deg(\eta_1) = 3$, and $\tau_n \neq 0$, $n \geq 1$. More precisely, we have:*

(i) *If* $H(x) = h(x-\xi)(x-\xi-1)(x-\xi-2)(x-\xi-3)$ *and* $G_1(\xi+1) = 0$, *then* G_1 *divides* G_2

$$u_1 = (g_2)^{-1}(x-\xi)^{-2}\tilde{\rho}_1 u_0 + r_0\delta(x-\xi) + (R_1(0)+\xi)r_0\delta'(x-\xi).$$

(ii) *If* $H(x) = h(x-\xi)(x-\xi-1)(x-\zeta)(x-\zeta-1)$ $(\zeta \neq \xi, \xi+1, \xi+2)$ *and* $G_1(\xi+1)$, *then* G_1 *divides* G_2 *and*

$$\begin{aligned} u_1 =& (p_1p_2g_2)^{-1}(x-\xi_1)^{-1}(x-\xi_2)^{-1}\tilde{\rho}_1 u_0 \\ &+ \frac{r_0}{\xi_1-\xi_2}[(R_1(0)+\xi_1)\delta(x-\xi_2) - (R_1(0)+\xi_2)\delta(x-\xi_1)]. \end{aligned}$$

Here, $\rho_1(x) = G_1(x)\tilde{\rho}_1(x)$.

Proof. Since $\tau_1\tau_2 \neq 0$, $\deg(G_1) = 2$ and $\deg(G_2) = 3$. Assume that G_1 divides G_2. Set $G_2 = G_1\tilde{G}_2$, $\deg(\tilde{G}_2) = 1$. Then

$$\begin{aligned} \rho_1(x) &= G_1(x)\tilde{\rho}_1 \ (0 \leq \deg(\tilde{\rho}_1) \leq 2) \\ \eta_1 &= G_1(x)\tilde{\eta}_1 \ (0 \leq \deg(\tilde{\eta}_1) \leq 1). \end{aligned}$$

As in the proof of Theorem 4.2, by eliminating Δu_1 from (4.12) and (4.13), we obtain

$$\tilde{\rho}_1(x+1)u_0 = G_1(x)\Delta\tilde{G}_2 u_1 \tag{4.16}$$

so that by (4.12),

$$\Delta(\tilde{\rho}_1(x+1)u_0) = \Delta\tilde{G}_2(x)\Delta(G_1(x)u_1) = \Delta\tilde{G}_2(x)\frac{P_1(x)}{p_1}u_0.$$

Hence u_0 and G_1u_1 are discrete-classical of the same type by Proposition 4.4 and $\Delta(\tilde{\rho}_1(x)G_1(x)u_1) = \tilde{\eta}_1(x)G_1(x)u_1$. Hence, $\deg(\tilde{\eta}_1(x)) = 1$ and so $\deg(\eta_1(x)) = 3$. Finally, $\tau_n \neq 0,\ n \geq 1$ by Theorem 4.3.

(i) $H(x) = h(x-\xi)(x-\xi-1)(x-\xi-2)(x-\xi-3)$. Then $\Delta^3 H(x) = 24h(x-\xi)$. By (4.10), we have

$$\begin{aligned} H(x) &= G_1(x+1)\Delta G_2(x) - G_2(x+1)\Delta G_1(x) \\ \Delta H(x) &= G_1(x+1)\Delta^2 G_2(x) - G_2(x+1)\Delta^2 G_1 \end{aligned}$$

$$\begin{aligned} \Delta^2 H(x) =& G_1(x+2)\Delta^3 G_2 + \Delta G_1(x+1)\Delta^2 G_2(x+1) \\ &- \Delta G_2(x+1)\Delta^2 G_1 \end{aligned} \tag{4.17}$$

$$\Delta^3 H(x) = \Delta^3 G_2(\Delta G_1(x+1) + \Delta G_1(x+2)). \tag{4.18}$$

Since $G_1(\xi+1) = 0$ i.e., $G_1(x) = g_1(x-\xi-1)(x-m)$ so we have

$$\Delta G_1(x) = g_1(2x-\xi-m). \tag{4.19}$$

Since $\Delta^3 H(\xi) = 0$, we have $\Delta G_1(\xi+1) + \Delta G_1(\xi+2) = 0$ so that $m = \xi+3$. Hence $G_1(\xi+3) = 0$. Since $H(\xi) = H(\xi+2) = 0$ but

$\Delta G_1(\xi) \neq 0$ and $\Delta G_1(\xi+2) \neq 0$, we have $G_2(\xi+1) = G_2(\xi+3) = 0$ by (4.18). Hence $G_1(x)$ divides $G_2(x)$.

(ii) Assume $H(x) = h(x-\xi)(x-\xi-1)(x-\zeta)(x-\zeta-1)$ $(\zeta \neq \xi, \xi+1, \xi+2)$. Then since $\Delta^3 H(x) = 24h(x - \frac{\xi+\zeta+2}{2})$, by using (4.19), we have $m = \zeta+1$. Hence $G_1(\zeta+1) = 0$. Since $H(\xi) = H(\zeta) = 0$, but $\Delta G_1(\xi) \neq 0$ and $\Delta G_1(\zeta) \neq 0$, by (4.18) we have $G_2(\xi+1) = G_2(\zeta+1) = 0$. Hence $G_1(x)$ divides $G_2(x)$. □

By the essentially same methods used in [10] for ordinary coherent pairs, we now have:

THEOREM 4.7. *Let u_0 be a discrete-classical moment functional satisfying $\Delta(\varphi u_0) = \psi u_0$ $(0 \leq deg(\varphi) \leq 2,\ deg(\psi) = 1)$. Assume $\langle u_0, \varphi \rangle = 1$. Then u_1 is a 3-term companion of u_0 if and only if either*

$$u_1 = (x-\xi_1)^{-1}(x-\xi_2)^{-1}\varphi u_0 + a\delta(x-\xi_1) + b\delta(x-\xi_2) \tag{4.20}$$

or

$$u_1 = (x-\xi_1)^{-2}\varphi u_0 + a\delta(x-\xi_1) + b\delta'(x-\xi_1) \tag{4.21}$$

for some complex numbers $\xi_1 \neq \xi_2$, a, and b satisfying

(i) *in case of* (4.20)

$$\begin{cases} a+b \neq 0, \\ a(\xi_1 - a\xi_1 - b\xi_2)^2 + b(\xi_2 - a\xi_1 - b\xi_2)^2 \neq 1, \\ \begin{vmatrix} \langle u_1, P_{n+1}\rangle & \langle u_1, P_n\rangle \\ \langle (x-\xi_1)u_1, P_{n+1}\rangle & \langle (x-\xi_1)u_1, P_n\rangle \end{vmatrix} \neq 0, \ n \geq 0; \end{cases}$$

(ii) *in case of* (4.21)

$$\begin{cases} a \neq 0, \ a - b^2 \neq 0, \\ \begin{vmatrix} \langle u_1, P_{n+1}\rangle & \langle u_1, P_n\rangle \\ \langle (x-\xi_1)u_1, P_{n+1}\rangle & \langle (x-\xi_1)u_1, P_n\rangle \end{vmatrix} \neq 0, \ n \geq 0. \end{cases}$$

Proof. See Theorem 4.5 in [10]. □

Conversely we have:

THEOREM 4.8. *Let u_1 be a strongly discrete-classical moment functional satisfying $\Delta(\varphi u_1) = \psi u_1$ $(0 \leq deg(\varphi) \leq 2,\ deg(\psi) = 1)$ and $\{T_n(x)\}_{n=0}^{\infty}$ the discrete-classical MOPS relative to w with $\frac{1}{n+1}\Delta T_{n+1}(x) = R_n(x)$, $n \geq 0$. Then u_0 is a 3-term companion of u_1 if and only if either*

$$u_0 = (x-\xi_1)(x-\xi_2)w$$

for some complex numbers $\xi_1 \neq \xi_2$ *satisfying*

$$\begin{vmatrix} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T_n(\xi_2) & T_{n+1}(\xi_2) \end{vmatrix} \neq 0, \ n \geq 1$$

or

$$u_0 = (x - \xi_1)^2 w$$

for some complex number ξ_1 *satisfying*

$$\begin{vmatrix} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T'_n(\xi_1) & T'_{n+1}(\xi_1) \end{vmatrix} \neq 0, \ n \geq 1.$$

Proof. See Theorem 4.6 in [10]. □

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