Verblunsky parameters and linear spectral transformations.

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Abstract

In this paper we analyze the behavior of Verblunsky parameters for hermitian linear functionals deduced from canonical linear spectral transformations of a quasi-definite hermitian linear functional. Some illustrative examples are studied.

Key words: Quasi-definite hermitian linear functionals, orthogonal polynomials, Christoffel transformation, Uvarov transformation, Geronimus transformation, Verblunsky parameters.

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1 Introduction and preliminary results.

Spectral transformations appear in the literature related to bispectral problems, self-similar reductions, and factorization of matrices (see [1], [12], [18], [21]). They are connected with perturbations of linear functionals in the linear space of polynomials with complex coefficients, Jacobi matrices as a representation of the multiplication operator in terms of orthogonal polynomial bases, and LU/QR factorizations of such matrices.

The extension to other contexts has been started in [2] where polynomial perturbations of bilinear functionals have been considered. In such a situation, the represen-

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tation of the multiplication operator with respect to an orthogonal polynomial basis is a Hessenberg matrix.

In the case of bilinear functionals with respect to probability measures supported on the unit circle some linear spectral transforms have been introduced in the literature. In particular, polynomial and rational perturbations have been considered in [6], [7], [9], and [11] where explicit expressions for polynomials orthogonal with respect to the perturbed measure have been obtained in terms of the orthogonal polynomials with respect to the initial probability measure.

Later on, following the ideas of the spectral transformations associated with the spectral measures of Jacobi matrices (see [1], [18], [21]), in [3], [4], [13] some analog problems have been considered, mainly from the point of view of the Hessenberg matrices and their LU and QR factorization, respectively.

Suppose $\mathcal{L}$ is a linear functional in the linear space Λ of the Laurent polynomials ($\Lambda = span\{z^n\}_{n\in\mathbb{Z}}$) such that $\mathcal{L}$ is Hermitian, i. e. $c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$. Then, in the linear space $\mathbb{P}$ of polynomials with complex coefficients, a bilinear functional associated with $\mathcal{L}$ can be introduced as follows (see [5], [10])

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \right\rangle \tag{1}$$

where $p, q \in \mathbb{P}$.

In terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$, the Gram matrix associated with this bilinear functional is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{2}$$

i.e., a Toeplitz matrix [8].

The linear functional is said to be quasi-definite if the principal leading submatrices of $\mathbf{T}$ are non-singular. If such submatrices have a positive determinant, then the linear functional is said to be positive definite. Every positive definite linear functional has an integral representation

$$\langle \mathcal{L}, p(z) \rangle = \int_{\mathbb{T}} p(z) d\sigma(z), \tag{3}$$

where σ is a nontrivial probability Borel measure supported on the unit circle (see [5], [8], [10], [17]), assuming $c_0 = 1$.

If $\mathcal{L}$ is a quasi-definite linear functional then a unique sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ such that

$$\langle \Phi_n, \Phi_m \rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}, \tag{4}$$

can be introduced, where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with $\mathcal{L}$.

Let σ be a non trivial probability measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then there exists a sequence $\{\varphi_n\}_{n\geqslant 0}$ of orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \quad \kappa_n > 0,$$

such that

$$\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})} d\sigma(\theta) = \delta_{m,n}, \quad m, n \geqslant 0. \tag{5}$$

The corresponding monic polynomials are then defined by

$$\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n}.$$

These polynomials satisfy the following recurrence relations (see [5], [8], [17], [19])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n = 0, 1, 2, ... \tag{6}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)} z\Phi_n(z), \quad n = 0, 1, 2, ..., \tag{7}$$

Here $\Phi_n^*(z) = z^n \overline{\Phi}_n(1/z)$ is the reversed polynomial associated with $\Phi_n(z)$ (see [17]), and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are called reflection (or Verblunsky) parameters. Notice that $|\Phi_n(0)| < 1$ for every $n \geqslant 1$.

It is well known that given a probability measure σ supported on the unit circle, there exists a unique sequence of Verblunsky parameters $\{\Phi_n(0)\}_{n\geqslant 1}$ associated with σ. The converse is also true, i.e., given a sequence of complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$, with $\Phi_n(0) \in \mathbb{D}$, there exists a nontrivial probability measure on the unit circle such that those numbers are the associated Verblunsky parameters.

The family of Verblunsky parameters provides a quantitative information about the measure and the corresponding sequence of orthogonal polynomials.

The measure σ can be decomposed into a part that is purely absolutely continuous with respect to the Lebesgue measure $\frac{d\theta}{2\pi}$ and a singular measure. We denote $\omega = \sigma'$, and thus

$$d\sigma(\theta) = \omega(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta).$$

Definition 1 *[17],[19] Suppose the Szegő condition,*

$$\int_{\mathbb{T}} \log(\omega(\theta))\frac{d\theta}{2\pi} > -\infty, \tag{8}$$

holds. Then, the Szegő function, D(z), is defined by

$$D(z) = \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(\omega(\theta))d\theta\right). \tag{9}$$

The Szegő condition (8) is equivalent to $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$.

On the other hand, the measure σ is said to be of bounded variation if

$$\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$$

holds.

The Christoffel function associated with a nontrivial probability measure σ, supported on the unit circle, is defined as follows. Let introduce

$$\lambda_n(\zeta) = \min\left\{\int |\pi(e^{i\theta})|^2 d\sigma(\theta), \quad \deg\pi \leqslant n, \text{such that} \quad \pi(\zeta) = 1\right\}.$$

Notice that λ_n is a decreasing function in n, and thus we can define

$$\begin{aligned}\lambda_\infty(\zeta) &= \lim_{n\to\infty}\lambda_n(\zeta) = \inf_n \lambda_n(\zeta)\\ &= \inf\left\{\int |\pi(e^{i\theta})|^2 d\sigma(\theta), \quad \pi \in \mathbb{P}, \text{such that} \quad \pi(\zeta) = 1\right\}.\end{aligned}$$

There is a relation between the Christoffel function and the family of Verblunsky parameters associated with a given probability measure σ.

Theorem 2 *[17] Let σ be a nontrivial probability measure supported on the unit circle, and let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the corresponding family of Verblunsky parameters. Then,*

(i) If $|\zeta| > 1, \lambda_\infty(\zeta) = 0.$

(ii) If $|\zeta| = 1, \lambda_\infty(\zeta) = \sigma(\{\zeta\}).$

(iii) *If* $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = \infty$, *then* $\lambda_\infty(\zeta) = 0$ *for all* ζ *with* $|\zeta| < 1$.

(iv) *If* $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$, *then* $\lambda_\infty(\zeta) > 0$ *for all* ζ *with* $|\zeta| < 1$.

We also have
$$\lambda_n(\zeta) = \frac{1}{K_n(\zeta,\zeta)},$$
where $K_n(z,y)$ is the *n-th reproducing kernel polynomial* associated with $\{\varphi_n\}_{n\geqslant 0}$, defined by
$$K_n(z,y) = \sum_{j=0}^{n} \overline{\varphi_j(y)}\varphi_j(z) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j},$$
with $\mathbf{k}_j = ||\Phi_j||^2 = (\kappa_j(\sigma))^{-2}$. There is a direct formula to compute $K_n(z,y)$,

Theorem 3 *(Christoffel-Darboux Formula) For any* $n \geqslant 0$ *and* $z, y \in \mathbb{C}$ *with* $\bar{y}z \neq 1$,
$$K_n(z,y) = \sum_{j=0}^{n} \overline{\varphi_j(y)}\varphi_j(z) = \frac{\overline{\varphi_{n+1}^*(y)}\varphi_{n+1}^*(z) - \overline{\varphi_{n+1}(y)}\varphi_{n+1}(z)}{1-\bar{y}z}.$$

The functions
$$q_j(t) = \int_{\mathbb{T}} \frac{\overline{\varphi_j(z)}}{t-z} d\sigma(z), \quad t \notin \mathbb{T}, \quad j \geqslant 0, \tag{10}$$
are called *functions of second kind* associated with σ. We also denote
$$Q_j(t) = \int_{\mathbb{T}} \frac{\overline{\Phi_j(z)}}{t-z} d\sigma(z) = (\kappa_j(\sigma))^{-1} q_j(t).$$

For a class of perturbations of the measure σ, some properties of the perturbed measure $\tilde{\sigma}$ have been studied ([3], [6], [12], [14]), such as the corresponding families of orthogonal polynomials, and necessary and sufficient conditions for the definite (quasi-definite) positiveness of the new measure $\tilde{\sigma}$, assuming the definite (quasi-definite) positiveness of σ. Three canonical cases have been studied.

(i) If $d\tilde{\sigma} = |z-\alpha|^2 d\sigma$, $|z| = 1$, then the so-called canonical Christoffel transformation appears.

(ii) If $d\tilde{\sigma} = d\sigma + \boldsymbol{m}\delta(z-z_0)$, $|z_0| = 1$, $\boldsymbol{m} \in \mathbb{R}_+$, then the so-called canonical Uvarov transformation appears.

(iii) If $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2} d\sigma$, $|z| = 1$, and $|\alpha| > 1$, then a special case of the Geronimus transform appears.

In this work, we analyze these transformations from the point of view of the families of Verblunsky parameters. We get explicit expressions for the Verblunsky parameters associated with $\tilde{\sigma}$ in terms of the parameters associated with σ. We also study if the measure $\tilde{\sigma}$ is of bounded variation, provided that σ is.

The structure of the manuscript is as follows. In section 2 we analyze the behavior of the Verblunsky parameters when a Christoffel canonical transform of a probability measure supported on the unit circle is considered. Section 3 is focussed on the Uvarov transformation in a more general framework than one analyzed in [20]. A particular example of the Geronimus transform for the Lebesgue probability measure is studied.

2 The Christoffel transformation.

Let α be a complex number. Consider the Hermitian bilinear functional

$$\langle p,q\rangle_{\mathcal{L}_C} := \langle (z-\alpha)p,(z-\alpha)q\rangle_{\mathcal{L}}, \quad p,q\in\mathbb{P}. \tag{11}$$

If $\mathcal{L}$ is quasi-definite, then necessary and sufficient conditions for $\mathcal{L}_C$ to be quasi-definite have been studied in [14].

Proposition 4 *[14]*

(i) *$\mathcal{L}_C$ is quasi-definite if and only if $K_n(\alpha,\alpha)\neq 0$ for every $n\in\mathbb{N}$.*

(ii) *If $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ denotes the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_C$, then*

$$\widetilde{\Phi}_n(z) = \frac{1}{z-\alpha}\left(\Phi_{n+1}(z) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}K_n(z,\alpha)\right). \tag{12}$$

$\mathcal{L}_C$ is said to be the canonical Christoffel transformation of the linear functional $\mathcal{L}$.

Proposition 5 *Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the Verblunsky parameters associated with $\{\Phi_n(z)\}_{n\geqslant 0}$, the monic orthogonal polynomial sequence with respect to $\mathcal{L}$. Then, the Verblunsky parameters associated with $\{\widetilde{\Phi}(z)\}_{n\geqslant 0}$, are given by*

$$\widetilde{\Phi}_n(0) = \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_{n+1}(0)}{\alpha}, \quad n\geqslant 1. \tag{13}$$

Proof. From (12), the evaluation in $z=0$ yields

$$\widetilde{\Phi}_n(0) = -\alpha^{-1}\left(\Phi_{n+1}(0) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}\sum_{j=0}^{n}\varphi_j(0)\overline{\varphi_j(\alpha)}\right).$$

Applying the Christoffel-Darboux formula, we get

$$\widetilde{\Phi}_n(0) = -\alpha^{-1}\left(\Phi_{n+1}(0) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}\varphi_n^*(0)\overline{\varphi_n^*(\alpha)}\right) \tag{14}$$
$$= \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_{n+1}(0)}{\alpha} \tag{15}$$

since $\Phi_n^*(0) = 1$. ■

Another way to express (15) is

$$\begin{aligned}\widetilde{\Phi}_n(0) &= \frac{[\alpha\Phi_n(\alpha) + \Phi_{n+1}(0)\Phi_n^*(\alpha)]\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_n(0)}{\alpha},\\ &= \left[\frac{|\Phi_n^*(\alpha)|^2}{\mathbf{k}_n K_n(\alpha,\alpha)} - 1\right]\frac{\Phi_{n+1}(0)}{\alpha} + \frac{\Phi_n(\alpha)\Phi_n^*(\alpha)}{\mathbf{k}_n K_n(\alpha,\alpha)},\end{aligned}$$

i.e., there is a linear relation between both families of Verblunsky parameters.

Notice that, if $|\alpha| \neq 1$, from the Christoffel-Darboux formula, we deduce

$$\begin{aligned}K_n(\alpha,\alpha) &= \frac{\overline{\varphi_n^*(\alpha)}\varphi_n^*(\alpha) - |\alpha|^2\overline{\varphi_n(\alpha)}\varphi_n(\alpha)}{1-|\alpha|^2},\\ &= \frac{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}{\mathbf{k}_n(1-|\alpha|^2)},\end{aligned}$$

and the expression for the Verblunsky parameters $\{\widetilde{\Phi_n}(0)\}_{n\geqslant 1}$ in terms of $\Phi_n(\alpha)$ and $\Phi_n^*(\alpha)$ is therefore given by

$$\begin{aligned}\widetilde{\Phi}_n(0) &= \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{\alpha[|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2]} - \frac{\Phi_{n+1}(0)}{\alpha},\\ &= \frac{[\alpha\Phi_n(\alpha) + \Phi_{n+1}(0)\Phi_n^*(\alpha)]\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{\alpha[|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2]} - \frac{\Phi_{n+1}(0)}{\alpha},\\ &= \frac{1}{\alpha}\frac{(\alpha\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)} + \Phi_{n+1}(0)|\Phi_n^*(\alpha)|^2)(1-|\alpha|^2)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}\\ &- \frac{1}{\alpha}\frac{\Phi_{n+1}(0)|\Phi_n^*(\alpha)|^2 + |\alpha|^2\Phi_{n+1}(0)|\Phi_n(\alpha)|^2}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}.\end{aligned}$$

Thus,

$$\widetilde{\Phi}_n(0) = \frac{\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2) + \bar{\alpha}\Phi_{n+1}(0)[|\Phi_n(\alpha)|^2 - |\Phi_n^*(\alpha)|^2]}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2},$$

As a conclusion, $\widetilde{\Phi}_n(0)$ can be expressed as

$$\widetilde{\Phi}_n(0) = A(\alpha;n)\Phi_{n+1}(0) + B(\alpha;n),$$

with

$$\begin{aligned} A(\alpha;n) &= \frac{\bar{\alpha}[|\Phi_n(\alpha)|^2 - |\Phi_n^*(\alpha)|^2]}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}, \\ B(\alpha;n) &= \frac{\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}. \end{aligned}$$

On the other hand, if $|\alpha| = 1$, we have

$$\begin{aligned} K_n(z,\alpha) &= \frac{\overline{\varphi_{n+1}^*(\alpha)}\varphi_{n+1}^*(z) - \overline{\varphi_{n+1}(\alpha)}\varphi_{n+1}(z)}{1-\bar{\alpha}z} \\ &= \frac{\alpha\overline{\varphi_{n+1}(\alpha)}\varphi_{n+1}(z) - \bar{\alpha}^n\varphi_{n+1}(\alpha)\varphi_{n+1}^*(z)}{z-\alpha}, \end{aligned}$$

and applying L'Hospital's rule, we obtain

$$\begin{aligned} K_n(\alpha,\alpha) = \lim_{z\to\alpha} K_n(z,\alpha) &= \alpha\overline{\varphi_{n+1}(\alpha)}\varphi_{n+1}^{'}(z) - \bar{\alpha}^n\varphi_{n+1}(\alpha)\varphi_{n+1}^{*'}(z), \\ &= \alpha\overline{\varphi_{n+1}(\alpha)}\varphi_{n+1}^{'}(\alpha) - \bar{\alpha}^n\varphi_{n+1}(\alpha)\varphi_{n+1}^{*'}(\alpha). \end{aligned}$$

Therefore

$$\begin{aligned} \widetilde{\Phi_n}(0) &= \frac{\mathbf{k}_{n+1}\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi_{n+1}'(\alpha) - \bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi_{n+1}^{*'}(\alpha)]} - \frac{\Phi_{n+1}(0)}{\alpha}, \\ &= \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\Phi_{n+1}(0)|^2)}{\alpha[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi_{n+1}'(\alpha) - \bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi_{n+1}^{*'}(\alpha)]} - \frac{\Phi_{n+1}(0)}{\alpha}, \end{aligned}$$

with $\Phi_{n+1}^{*'}(\alpha) = \alpha^{-1}[(n+1)\Phi_{n+1}^*(\alpha) - (\Phi_{n+1}^{'})^*(\alpha)]$.

Theorem 6 ([15], [16]) *Suppose* $\sum_{n=0}^{\infty}|\Phi_n(0)|^2 < \infty$ *and* $\sum_{n=0}^{\infty}|\Phi_{n+1}(0) - \Phi_n(0)| < \infty$*. Then, for any* $\delta > 0$,

$$\sup_{n;\delta<\arg(z)<2\pi-\delta} |\Phi_n^*(z)| < \infty$$

and away from $z = 1$, we have that $\lim_{n\to\infty} \Phi_n^*(z)$ exists, is continuous, and equal to $D(0)D(z)^{-1}$. Furthermore, $d\mu_s = 0$ or else a pure mass point at $z = 1$.

Proposition 7 *Suppose $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ and $\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$. Then, for $|\alpha| \leqslant 1$, $\alpha \neq 1$,*

(i) $\sum_{n=0}^{\infty} |\widetilde{\Phi}_n(0)|^2 < \infty$.

(ii) $\sum_{n=0}^{\infty} |\widetilde{\Phi}_{n+1}(0) - \widetilde{\Phi}_n(0)| < \infty$.

Proof.

(i) We denote

$$t_{n+1} = \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha \mathbf{k}_n K_n(\alpha, \alpha)}.$$

Let assume $|\alpha| = 1$. Notice that $\overline{\Phi_{n+1}(\alpha)} = \overline{\alpha^{n+1}}\Phi_{n+1}^*(\alpha)$ and, from Theorem 6, $\lim_{n\to\infty} \Phi_n^*(\alpha) = D(0)D(\alpha)^{-1}$, where D is the Szegő function defined in (9). This also implies that $1/K_n(\alpha, \alpha) = O(1/n)$. For $|\alpha| < 1$, notice that $\Phi_n(\alpha)$ and $\Phi_n^*(\alpha)$ are $O(\alpha^n)$ and $1/K_\infty(\alpha, \alpha) = \lambda_\infty(\alpha) > 0$, where λ_∞ is the Christoffel function associated with σ.
Then t_{n+1} is $O(1/n)$. Since $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ and t_{n+1} is $O(1/n)$, then $\sum_{n=0}^{\infty} |\widetilde{\Phi}_n(0)|^2 < \infty$.

(ii) Since $\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$, we only need to prove that

$$\sum_{n=0}^{\infty} |t_{n+1} - t_n| < \infty.$$

Notice that, from the recurrence relation

$$\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha) = \overline{\Phi_{n+1}(0)}\alpha\Phi_n(\alpha).$$

Then $|\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha)| = O(|\Phi_{n+1}(0)|)$ and therefore

$$\left| \frac{(\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha))\Phi_{n+1}^*(\alpha)}{\mathbf{k}_n K_n(\alpha, \alpha)} \right| = O\left(\frac{|\Phi_{n+1}(0)|}{n} \right). \tag{16}$$

On the other hand,

$$\left| \left(\frac{1}{K_{n+1}(\alpha, \alpha)} - \frac{1}{K_n(\alpha, \alpha)} \right) \frac{\Phi_n(\alpha)\overline{\Phi_{n-1}^*(\alpha)}}{\mathbf{k}_n} \right| = O\left(\frac{1}{n^2} \right). \tag{17}$$

Thus, from (16) and (17) we get

$$|t_{n+1} - t_n| = O\left(\frac{|\Phi_{n+1}(0)|}{n} \right) + O\left(\frac{1}{n^2} \right)$$

and, therefore,

$$\sum_{n=0}^{\infty} |t_{n+1} - t_n| < \infty.$$

■

3 The Uvarov transformation.

3.1 The bilinear functional $\langle p, q\rangle_{\mathcal{L}_U} := \langle p, q\rangle_{\mathcal{L}} + \boldsymbol{m} p(\alpha)\overline{q(\alpha)}$.

Now consider the bilinear functional

$$\langle p, q\rangle_{\mathcal{L}_U} := \langle p, q\rangle_{\mathcal{L}} + \boldsymbol{m} p(\alpha)\overline{q(\alpha)}, \quad p, q \in \mathbb{P}, \tag{18}$$

with $\boldsymbol{m} \in \mathbb{R}$ and $|\alpha| = 1$. Thus, $\mathcal{L}_U$ is also hermitian. This transformation of $\mathcal{L}$ is a particular case of the Uvarov transformation (see [12]).

We have

Proposition 8 *[12]*

(i) *$\mathcal{L}_U$ is quasi-definite if and only if $1 + \boldsymbol{m} K_{n-1}(\alpha, \alpha) \neq 0$ for every $n \geqslant 1$.*

(ii) *If $\{U_n\}_{n\geqslant 0}$ denotes the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_U$, then*

$$U_n(z) = \Phi_n(z) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1 + \boldsymbol{m} K_{n-1}(\alpha, \alpha)} K_{n-1}(z, \alpha). \tag{19}$$

Thus, it follows

Proposition 9 *Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the Verblunsky parameters associated with $\{\Phi_n(z)\}_{n\geqslant 0}$, the monic orthogonal polynomial sequence with respect to $\mathcal{L}$. Then, the Verblunsky parameters associated with $\{U_n(z)\}_{n\geqslant 0}$, are given by*

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1 + \boldsymbol{m} K_{n-1}(\alpha, \alpha))} \tag{20}$$

Proof. From (19), evaluating at $z = 0$, we obtain

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha)} K_{n-1}(0,\alpha), \tag{21}$$

$$= \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha)} \overline{\varphi^*_{n-1}(\alpha)}\varphi^*_{n-1}(0), \tag{22}$$

$$= \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha))}. \tag{23}$$

■

Using a formula for the Verblunsky parameters associated with $\tilde{\sigma}$ given by Simon in [17], this result was also proved in [20], as follows,

Theorem 10 *Suppose σ is a nontrivial probability measure on the unit circle and $0 < \gamma < 1$. Let $\tilde{\sigma}$ be the probability measure formed by adding a mass point $\zeta = e^{i\theta} \in \mathbb{T}$ to σ as follows*

$$d\tilde{\sigma} = (1-\gamma)d\sigma + \gamma\delta_\theta.$$

Then the Verblunsky parameters associated with $\tilde{\sigma}$ are

$$\widetilde{\Phi}_n(0) = \Phi_n(0) + \frac{(1 - |\Phi_{n+1}(0)|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)} \overline{\varphi_{n+1}(\zeta)}\varphi^*_n(\zeta). \tag{24}$$

Notice that in the above definition the probability character is preserved. Furthermore, $\mathbf{k}_n/\mathbf{k}_{n-1} = 1 - |\Phi_n(0)|^2$, so (20) is equivalent to the expression (24). There is also an analog of Proposition 7 for the Uvarov transformation on [20], which has been proved in a more general case with m masses.

In addition, observe than (23) also reads

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}[\alpha\Phi_{n-1}(\alpha) + \Phi_n(0)\Phi^*_{n-1}(\alpha)]\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha))},$$

or, in other words,

$$U_n(0) = A_U(\alpha;n)\Phi_n(0) + B_U(\alpha;n),$$

with

$$A_U(\alpha;n) = 1 - \frac{\boldsymbol{m}|\Phi^*_{n-1}(\alpha)|^2}{\mathbf{k}_{n-1}(1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha))},$$

$$B_U(\alpha;n) = -\frac{\boldsymbol{m}\alpha\Phi_{n-1}(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha))}.$$

3.2 *The bilinear functional* $\langle p,q\rangle_{\mathcal{L}_{\tilde{U}}} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}$.

Now consider the bilinear functional

$$\langle p,q\rangle_{\mathcal{L}_{\tilde{U}}} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}, \quad p,q \in \mathbb{P}, \tag{25}$$

with $\boldsymbol{m} \in \mathbb{C}$ and $|\alpha| \neq 1$. Thus, $\mathcal{L}_{\tilde{U}}$ is also hermitian.

Proposition 11 *[3] The bilinear functional associated with $\mathcal{L}_{\tilde{U}}$ is quasi-definite if and only if*

$$\Lambda_n := \begin{vmatrix} 1 + \boldsymbol{m}K_n(\alpha,\bar{\alpha}^{-1}) & \bar{\boldsymbol{m}}K_n(\alpha,\alpha) \\ \boldsymbol{m}K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) & 1 + \bar{\boldsymbol{m}}K_n(\bar{\alpha}^{-1},\alpha) \end{vmatrix} \neq 0,$$

for all $n \geqslant 0$.

Assuming the conditions of the above proposition we get

Proposition 12 *[3] The orthogonal polynomial sequence corresponding to $\mathcal{L}_{\tilde{U}}$, $\{V_n(z)\}_{n\geqslant 0}$, is given by*

$$V_n(z) = \Phi_n(z) - \boldsymbol{m}[A_n\Phi_n(\alpha) + B_n\Phi_n(\bar{\alpha}^{-1})]K_{n-1}(z,\bar{\alpha}^{-1}) - \bar{\boldsymbol{m}}[C_n\Phi_n(\alpha) + D_n\Phi_n(\bar{\alpha}^{-1})]K_{n-1}(z,\alpha). \tag{26}$$

where

$$A_n = \frac{-[1 + \bar{\boldsymbol{m}}K_{n-1}(\bar{\alpha}^{-1},\alpha)]}{\Lambda_{n-1}} \tag{27}$$

$$B_n = \frac{\bar{\boldsymbol{m}}K_{n-1}(\alpha,\alpha)}{\Lambda_{n-1}} \tag{28}$$

$$C_n = \frac{-\boldsymbol{m}K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})}{\Lambda_{n-1}} \tag{29}$$

$$D_n = \frac{1 + \boldsymbol{m}K_{n-1}(\alpha,\bar{\alpha}^{-1})}{\Lambda_{n-1}} \tag{30}$$

with $\Lambda_{n-1} = |\boldsymbol{m}|^2K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})K_{n-1}(\alpha,\alpha) - |1 + \boldsymbol{m}K_{n-1}(\alpha,\bar{\alpha}^{-1})|^2$.

Notice that $D_n = -\bar{A}_n$. Then, the Verblunsky parameters $\{V_n(0)\}_{n\geqslant 1}$ are

$$\begin{aligned} V_n(0) = \Phi_n(0) &- \boldsymbol{m}[A_n\Phi_n(\alpha) + B_n\Phi_n(\bar{\alpha}^{-1})]\varphi^*_{n-1}(0)\overline{\varphi^*_{n-1}(\bar{\alpha}^{-1})} \\ &- \bar{\boldsymbol{m}}[C_n\Phi_n(\alpha) + D_n\Phi_n(\bar{\alpha}^{-1})]\varphi^*_{n-1}(0)\overline{\varphi^*_{n-1}(\alpha)}. \end{aligned} \tag{31}$$

Assuming that $\sum_{n=0}^{\infty}|\Phi_n(0)|^2 < \infty$, we now study the behavior of $V_n(0)$ when $n \to \infty$. If $|\alpha| < 1$, then we know that $\lim_{n\to\infty} K_n(\alpha,\alpha) < \infty$ and $\lim_{n\to\infty} K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) =$

∞. Notice that, from [17]

$$\frac{K_n(\bar{\alpha}^{-1},\alpha)}{K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1})} = |\alpha|^{2n}\frac{\overline{K_n(\alpha,\bar{\alpha}^{-1})}}{K_n(\alpha,\alpha)}.$$

Thus

$$\lim_{n\to\infty}\frac{K_n(\bar{\alpha}^{-1},\alpha)}{K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1})} = \lim_{n\to\infty}|\alpha|^{2n}\frac{\overline{K_n(\alpha,\bar{\alpha}^{-1})}}{K_n(\alpha,\alpha)}$$

Since $K_n(\alpha,\bar{\alpha}^{-1})$ is $O(\bar{\alpha}^{-n})$ and $|\alpha| < 1$, we get

$$\lim_{n\to\infty}\frac{K_n(\bar{\alpha}^{-1},\alpha)}{K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1})} = 0.$$

We obtain the same result for $\frac{K_{n-1}(\alpha,\bar{\alpha}^{-1})}{K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})}$, since $K_n(\bar{\alpha}^{-1},\alpha) = \overline{K_n(\alpha,\bar{\alpha}^{-1})}$.

Therefore, if we divide by $K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})$ in the numerator and denominator of A_n, and take the limit when $n \to \infty$, then we observe that the numerator becomes 0, and only $|\boldsymbol{m}|^2 K_{n-1}(\alpha,\alpha)$ survives on the denominator. Hence, $A_n = 0$ when $n \to \infty$.

The same fact occurs with B_n and D_n. In a similar way, we obtain that $C_n \sim -\frac{1}{\bar{\boldsymbol{m}}K_\infty(\alpha,\alpha)}$ as $n \to \infty$.

As a conclusion, when $n \to \infty$

$$V_n(0) \sim \Phi_n(0) + \frac{\Phi_n(\alpha)}{K_{n-1}(\alpha,\alpha)}\varphi^*_{n-1}(0)\overline{\varphi^*_{n-1}(\alpha)} \tag{32}$$

$$= \Phi_n(0) + \frac{\Phi_n(\alpha)}{\mathbf{k}_{n-1}K_{n-1}(\alpha,\alpha)}\Phi^*_{n-1}(0)\overline{\Phi^*_{n-1}(\alpha)} \tag{33}$$

$$= \Phi_n(0) + \frac{\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}K_{n-1}(\alpha,\alpha)}. \tag{34}$$

Notice than (34) has the same form as (13). Therefore,

Proposition 13 *Suppose* $\sum_{n=0}^{\infty}|\Phi_n(0)|^2 < \infty$ *and* $\sum_{n=0}^{\infty}|\Phi_{n+1}(0) - \Phi_n(0)| < \infty$. *Then,*

(i) $\sum_{n=0}^{\infty}|V_n(0)|^2 < \infty$.

(ii) $\sum_{n=0}^{\infty}|V_{n+1}(0) - V_n(0)| < \infty$.

4 Examples.

4.1 The case $d\tilde{\sigma} = \frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$.

Proposition 14 *Let* $d\tilde{\sigma} = \frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$*. Then, the sequence of monic orthogonal polynomials with respect to* $\tilde{\sigma}$ *is given by*

$$V_n(z) = z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]\left(\frac{1-\alpha^{-n}z^n}{1-\alpha^{-1}z}\right) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]\left(\frac{1-\bar{\alpha}^n z^n}{1-\bar{\alpha}z}\right) \quad (35)$$

where

$$\begin{aligned} A_n &= -(1+n\bar{\boldsymbol{m}})/d_n(\alpha), \\ B_n &= \frac{\bar{\boldsymbol{m}}}{d_n(\alpha)}\sum_{k=0}^{n-1}|\alpha|^{2k}, \\ C_n &= -|\alpha|^{-2(n-1)}\bar{B}_n, \\ D_n &= -\bar{A}_n, \end{aligned}$$

and $d_n(\alpha) = |\boldsymbol{m}|^2|\alpha|^{-2(n-1)}\left[\sum_{k=0}^{n-1}|\alpha|^{2k}\right]^2 - |1+n\boldsymbol{m}|^2$.

Proof. It is well known that in this case $\Phi_n(z) = \varphi_n(z) = z^n$ as well as $\Phi_n(0) = 0$, $n \geqslant 1$. Then, from (26), we get

$$\begin{aligned} V_n(z) &= z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]K_{n-1}(z,\bar{\alpha}^{-1}) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]K_{n-1}(z,\alpha) \\ &= z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]\left(\frac{1-\alpha^{-n}z^n}{1-\alpha^{-1}z}\right) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]\left(\frac{1-\bar{\alpha}^n z^n}{1-\bar{\alpha}z}\right). \end{aligned}$$

The values of A_n, B_n, C_n, D_n, and $d_n(\alpha)$ follow from (28) - (30) since $K_{n-1}(\alpha,\alpha) = \sum_{k=0}^{n-1}|\alpha|^{2k}$, $K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) = \sum_{k=0}^{n-1}|\alpha|^{-2k}$, and $K_{n-1}(\bar{\alpha}^{-1},\alpha) = K_{n-1}(\alpha,\bar{\alpha}^{-1}) = n$. ■

Now we obtain a necessary and sufficient condition for the existence of $\{V_n(z)\}_{n\geqslant 0}$. The condition in Proposition 11 becomes

$$\begin{vmatrix} 1+\boldsymbol{m}(n+1) & \bar{\boldsymbol{m}}\sum_{k=0}^{n}|\alpha|^{2k} \\ \frac{\boldsymbol{m}}{|\alpha|^{2n}}\sum_{k=0}^{n}|\alpha|^{2k} & 1+\bar{\boldsymbol{m}}(n+1) \end{vmatrix}$$

$$= 1 + (\boldsymbol{m}+\bar{\boldsymbol{m}})(n+1) + |\boldsymbol{m}|^2(n+1)^2 - \frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}}\left[\sum_{k=0}^{n}|\alpha|^{2k}\right]^2,$$

with $|\alpha| < 1$. Notice that this expression is $\neq 0$ if and only if

$$\frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}}\left[\sum_{k=0}^{n}|\alpha|^{2k}\right]^2 \neq 1 + (\boldsymbol{m} + \bar{\boldsymbol{m}})(n+1) + |\boldsymbol{m}|^2(n+1)^2,$$

i.e.

$$\frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right)^2 \neq 1 + (\boldsymbol{m} + \bar{\boldsymbol{m}})(n+1) + |\boldsymbol{m}|^2(n+1)^2.$$

If $\boldsymbol{m} \in \mathbb{R}$, the above condition becomes

$$[\boldsymbol{m}(n+1)+1]^2 \neq \frac{\boldsymbol{m}^2}{|\alpha|^{2n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right)^2,$$

so we need

$$\boldsymbol{m}(n+1)+1 \neq \frac{\boldsymbol{m}}{|\alpha|^{n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right), \quad \text{for every } n \geqslant 0.$$

Corollary 15 *The Verblunsky parameters associated with* $\{V_n(z)\}_{n\geqslant 0}$ *are given by*

$$V_n(0) = -(\boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}] + \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]). \tag{36}$$

Proof. It follows immediately by evaluating (35) at $z = 0$. ■

Now we give an estimate for $V_n(0)$ when $n \to \infty$. Suppose $|\alpha| < 1$. We have

$$V_n(0) = -(\boldsymbol{m}A_n + \bar{\boldsymbol{m}}C_n)\alpha^n - (\boldsymbol{m}B_n + \bar{\boldsymbol{m}}D_n)\alpha^{-n}.$$

But

$$\begin{aligned}
-(\boldsymbol{m}A_n + \bar{\boldsymbol{m}}C_n)\alpha^n &= \frac{\alpha^n}{d_n(\alpha)}\left[\boldsymbol{m} + n|\boldsymbol{m}|^2 + |\boldsymbol{m}|^2|\alpha|^{-2(n-1)}\sum_{k=0}^{n-1}|\alpha|^{2k}\right],\\
&= \alpha^n\frac{|\alpha|^{2n-2}(\boldsymbol{m} + n|\boldsymbol{m}|^2) + |\boldsymbol{m}|^2\sum_{k=0}^{n-1}|\alpha|^{2k}}{|\boldsymbol{m}|^2\sum_{k=0}^{n-1}|\alpha|^{2k} - |\alpha|^{2n-2}|1+n\boldsymbol{m}|^2},\\
&\sim \frac{\alpha^n\frac{|\boldsymbol{m}|^2}{1-|\alpha|^2}}{\frac{|\boldsymbol{m}|^2}{1-|\alpha|^2}},\\
&= \alpha^n.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
-\bar{\alpha}^{-n}(\boldsymbol{m}B_n + \bar{\boldsymbol{m}}D_n) &= \bar{\alpha}^{-n}\frac{-|\boldsymbol{m}|^2\frac{|\alpha|^{2n}-1}{|\alpha|^2-1} - (\bar{\boldsymbol{m}} + n|\boldsymbol{m}|^2)}{|\boldsymbol{m}|^2\frac{1}{|\alpha|^{2n-2}}\frac{1-|\alpha|^{2n}}{1-|\alpha|^2} - |1 + n\boldsymbol{m}|^2},\\
&\sim \alpha^n\frac{-\frac{|\boldsymbol{m}|^2}{1-|\alpha|^2} - (\boldsymbol{m} + n|\boldsymbol{m}|^2)}{|\boldsymbol{m}|^2\frac{|\alpha|^2}{1-|\alpha|^2}},\\
&= -\frac{\alpha^n}{|\alpha|^2}\left(1 + \frac{(\bar{\boldsymbol{m}} + n|\boldsymbol{m}|^2)(1-|\alpha|^2)}{|\boldsymbol{m}|^2}\right),\\
&\sim -\frac{\alpha^n}{|\alpha|^2}\left(1 + n(1-|\alpha|^2)\right).
\end{aligned}
$$

As a conclusion,

$$V_n(0) \sim N_1(\alpha)n\alpha^n,$$

where $N_1(\alpha) = -\frac{1-|\alpha|^2}{|\alpha|^2}$.

4.2 The case $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1}), \boldsymbol{m} \neq 0$.

This is an example of a Geronimus canonical transformation in the sense that the Christoffel transform of this hermitian linear functional is the Lebesgue measure.

Proposition 16 *Let* $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$*, with* $|\alpha| < 1$*. Then, the sequence of monic orthogonal polynomials with respect to* $\tilde{\sigma}$ *is given by*

$$V_n(z) = z^n - \alpha z^{n-1} - \frac{\bar{\alpha}^{-n}(1-|\alpha|^2)^2[|\boldsymbol{m}|^2(1-|\alpha|^2)(\alpha^{-1}z)^{n-1} + \bar{\boldsymbol{m}} + |\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1 + \boldsymbol{m}(1-|\alpha|^2)|^2}, \tag{37}$$

for $n \geqslant 1$.

Proof. It is well known that the sequence of monic orthogonal polynomials with respect to $\frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi}$ is given by

$$\Phi_n(z) = z^n - \alpha z^{n-1}, \qquad |\alpha| < 1, n \geqslant 1.$$

Then, from (26) we have

$$V_n(z) = z^n - \alpha z^{n-1} - \boldsymbol{m}B_n\bar{\alpha}^{-n}(1-|\alpha|^2)K_{n-1}(z,\bar{\alpha}^{-1}) - \bar{\boldsymbol{m}}D_n\bar{\alpha}^{-n}(1-|\alpha|^2)K_{n-1}(z,\alpha). \tag{38}$$

since $\Phi_n(\alpha) = 0$, $n \geqslant 1$, and $\Phi_n(\bar{\alpha}^{-1}) = \bar{\alpha}^{-n}(1-|\alpha|^2)$. Notice that in this case $\mathbf{k}_0 = ||\Phi_0||^2 = \frac{1}{1-|\alpha|^2}$, as well as $\mathbf{k}_n = 1$, $n \geqslant 1$.

We also have $K_{n-1}(z,\alpha) = \frac{1}{\mathbf{k}_0}$ and, as a consequence,

$$K_{n-1}(\alpha,\alpha) = K_{n-1}(\alpha,\bar{\alpha}^{-1}) = K_{n-1}(\bar{\alpha}^{-1},\alpha) = \frac{1}{\mathbf{k}_0} = 1 - |\alpha|^2.$$

On the other hand, from the Christoffel-Darboux formula we get

$$\begin{aligned}
K_{n-1}(z,\bar{\alpha}^{-1}) &= \frac{\overline{\Phi_n^*(\bar{\alpha}^{-1})}\Phi_n^*(z) - \overline{\Phi_n(\bar{\alpha}^{-1})}\Phi_n(z)}{(1-\alpha^{-1}z)}, \\
&= \frac{-(\alpha^{-n} - \bar{\alpha}\alpha^{-n+1})(z^n - \alpha z^{n-1})}{(1-\alpha^{-1}z)}, \\
&= \frac{-\alpha^{-n}(z^n - \alpha z^{n-1})(1-|\alpha|^2)}{1-\alpha^{-1}z}, \\
&= (\alpha^{-1}z)^{n-1}(1-|\alpha|^2).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) &= \frac{\overline{\Phi_n^*(\bar{\alpha}^{-1})}\Phi_n^*(\bar{\alpha}^{-1})) - \overline{\Phi_n(\bar{\alpha}^{-1})}\Phi_n(\bar{\alpha}^{-1}))}{(1-\alpha^{-1}\bar{\alpha}^{-1}))}, \\
&= \frac{-(\alpha^{-n} - \bar{\alpha}\alpha^{-n+1})(\bar{\alpha}^{-n} - \alpha\bar{\alpha}^{-n+1})}{(1-|\alpha|^{-2})}, \\
&= -\frac{|\alpha|^{-2n}(1-|\alpha|^2)^2}{1-|\alpha|^{-2}}, \\
&= |\alpha|^{-2n+2}(1-|\alpha|^2),
\end{aligned}$$

since $\Phi_n^*(z) = 1 - \bar{\alpha}z$. Therefore

$$B_n = \frac{\bar{\boldsymbol{m}}(1-|\alpha|^2)}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2},$$

and

$$D_n = \frac{1+\boldsymbol{m}(1-|\alpha|^2)}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}.$$

Then, (38) becomes

$$V_n(z) = z^n - \alpha z^{n-1} - \frac{|\boldsymbol{m}|^2(1-|\alpha|^2)^3\bar{\alpha}^{-n}(\alpha^{-1}z)^{n-1} + \bar{\boldsymbol{m}}[1+\boldsymbol{m}(1-|\alpha|^2)]\bar{\alpha}^{-n}(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}$$

which is equivalent to (37). ∎

The existence of $\{V_n(z)\}_{n\geqslant 0}$ is determined by

$$\begin{vmatrix} 1+\boldsymbol{m}(1-|\alpha|^2) & \bar{\boldsymbol{m}}(1-|\alpha|^2) \\ \boldsymbol{m}|\alpha|^{-2n+2}(1-|\alpha|^2) & 1+\bar{\boldsymbol{m}}(1-|\alpha|^2) \end{vmatrix} =$$

$$1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2)+|\boldsymbol{m}|^2(1-|\alpha|^2)^2-|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2\neq 0$$

In other words,

$$\begin{aligned} |\alpha|^{-2n+2} &\neq \frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2)+|\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2}, \\ (-2n+2)\ln|\alpha| &\neq \ln\frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2)+|\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2}, \\ n &\neq 1-\frac{1}{2}\frac{\ln\frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2)+|\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2}}{\ln|\alpha|}. \end{aligned}$$

In particular, if $\boldsymbol{m}\in\mathbb{R}$ and $|\alpha|^2=\frac{1}{2}$, the above condition becomes

$$n\neq 1+\frac{\ln\left(1+\frac{2}{\boldsymbol{m}}\right)^2}{\ln 2},$$

i.e.

$$\frac{\ln\left(1+\frac{2}{\boldsymbol{m}}\right)^2}{\ln 2}\notin\mathbb{N}.$$

Corollary 17 *The Verblunsky parameters associated with* $\{V_n(z)\}_{n\geqslant 0}$ *are*

$$\begin{aligned} V_1(0) &= -\alpha-\frac{\bar{\alpha}^{-1}(1-|\alpha|^2)^2[\bar{\boldsymbol{m}}+2|\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2-|1+\boldsymbol{m}(1-|\alpha|^2)|^2} \\ V_n(0) &= -\frac{\bar{\alpha}^{-n}(1-|\alpha|^2)^2[\bar{\boldsymbol{m}}+|\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2-|1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \quad n\geqslant 2. \end{aligned}$$

Proof. It follows immediately evaluating (37) at $z=0$. ∎

Finally, we obtain an estimate for $V_n(0)$ when $n\to\infty$, From (37), the coefficient of z^{n-1} is

$$
\begin{aligned}
&= -\frac{\alpha[|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2]}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2} \\
&- \frac{\bar{\alpha}^{-1}(1-|\alpha|^2)^2[|\boldsymbol{m}|^2(1-|\alpha|^2)|\alpha|^{-2n+2}}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \\
&= \frac{|\boldsymbol{m}|^2\alpha|\alpha|^{-2n}(1-|\alpha|^2)^2 - \alpha|1+\boldsymbol{m}(1-|\alpha|^2)|^2}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \\
&= \frac{|\boldsymbol{m}|^2\alpha(1-|\alpha|^2)^2 - \alpha|\alpha|^{2n}|1+\boldsymbol{m}(1-|\alpha|^2)|^2}{|\boldsymbol{m}|^2|\alpha|^2(1-|\alpha|^2)^2 - |\alpha|^{2n}|1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \\
&\sim \frac{1}{\bar{\alpha}}.
\end{aligned}
$$

On the other hand, the independent term is

$$
\begin{aligned}
&= -\frac{\bar{\boldsymbol{m}}(1+\boldsymbol{m}(1-|\alpha|^2))(1-|\alpha|^2)^2\alpha^n}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}\frac{1}{|\alpha|^{2n}}, \\
&\sim -\frac{\bar{\boldsymbol{m}}(1+\boldsymbol{m}(1-|\alpha|^2))}{|\boldsymbol{m}|^2|\alpha|^2}\alpha^n, \\
&= -\frac{1}{|\alpha|^2\boldsymbol{m}}(1+\boldsymbol{m}(1-|\alpha|^2))\alpha^n
\end{aligned}
$$

In other words,

$$
V_n(0) \sim N_2(\alpha)\alpha^n,
$$

where $N_2(\alpha) = 1 - \frac{1+\boldsymbol{m}}{\boldsymbol{m}|\alpha|^2}$.

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