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Asymptotic properties of mixed type multiple orthogonal polynomials for Nikishin systems

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CONTENTS

1. INTRODUCTION

1.1 Our motivation

In the last four decades, the general theory of orthogonal polynomials has experienced a dramatic expansion, particularly so in connection with its analytic theory. An updated account in the sole direction of orthogonal polynomials on the unit circle may be obtained from [74], [75]. Potential theory [73], [77], Riemann-Hilbert analysis [17], operator theory [75], Riemann surfaces [61], [4], and the theory of boundary values of analytic functions [2], [3], have come into play, which together with the classical methods of real and complex analysis have produced deep and far reaching results within the standard theory of orthogonal polynomials.

Some examples of such outstanding results are: the theorem on the ratio asymptotics of orthogonal polynomials on the unit circle and a segment of the real line [62]-[65], [55], [16], the extension of Szegő's theory of orthogonal polynomials [54], [49]-[51], [65], the asymptotic behavior of orthogonal polynomials corresponding to general classes of measures supported on unbounded intervals of the real line [64], [52], [53], [35], and with it the solution of the Freud conjecture [44], and the 1/9 conjecture [28], the accurate description of the asymptotic behavior of orthogonal polynomials on the support of the measure and the complete asymptotic expansion of the orthogonal polynomials for special classes of measures [47], [48].

All this has been accompanied by a substantial advancement in the study of non standard models of orthogonality relations as in: orthogonal rational functions [11], polynomials orthogonal with respect to varying measures [38]-[40], [78], Sobolev orthogonal polynomials [46], [25], [42], matrix orthogonal polynomials [19], [79], [76], discrete orthogonal polynomials [66], [18], [7], and multiple orthogonal polynomials [60], [2], [3], [5].

This thesis is inscribed in the attempt of bringing new light to the analytic theory of orthogonal polynomials understood in a wide sense. More precisely, we will study several types of asymptotic properties of a certain class of multiple orthogonal polynomials. Such types of orthogonal polynomials are connected with vector rational approximation [60], [57], [12], simultaneous quadrature formulas [20], analytic number theory [59], [71], [80], and more recently in integrable systems, random matrix theory, and brownian motions of non-intersecting paths, [34], [15], [33]. Before describing our results, let us briefly review the sources which inspired our research.

Let f denote a formal power series at ∞. For each $n \in \mathbb{Z}_+$ (the set of all non negative integers) there exist polynomials Q_n, P_n satisfying:

i) $\deg P_n \leq n-1, \deg Q_n \leq n, Q_n \not\equiv 0$,

ii) $(Q_n f - P_n)(z) = \mathcal{O}(1/z^{n+1}), \quad z \to \infty$.

The quotient $\pi_n = \pi_n(f) = P_n/Q_n$ is uniquely determined and is called the n-th diagonal Padé approximant of f.

Let s denote a finite positive Borel measure with compact support $\operatorname{supp}(s)$ contained in the real line consisting of an infinite number of points. By

$$\widehat{s}(z) = \int \frac{ds(x)}{z-x}$$

we denote the Cauchy transform of the measure. Obviously, $\widehat{s}$ is holomorphic in the region $\overline{\mathbb{C}} \setminus \operatorname{supp}(s)$ and we write $\widehat{s} \in H(\overline{\mathbb{C}} \setminus \operatorname{supp}(s))$. The smallest interval which contains $\operatorname{supp}(s)$ will be denoted by Δ.

It is easy to verify that when $f = \widehat{s}$ then Q_n is an n-th orthogonal polynomial with respect to the measure s and P_n is the corresponding second type polynomial; that is,

$$0 = \int x^\nu Q_n(x) ds(x), \qquad \nu = 0, \ldots, n-1,$$

and

$$P_n(z) = \int \frac{Q_n(z) - Q_n(x)}{z-x} ds(x).$$

Consequently,

$$(Q_n \widehat{s} - P_n)(z) = \int \frac{Q_n(x) ds(x)}{z-x} = \frac{1}{Q_n(z)} \int \frac{Q_n^2(x) ds(x)}{z-x}. \tag{1.1}$$

In the sequel, we assume that Q_n is monic; that is, has leading coefficient equal to 1.

A classical result of A. A. Markov [45] may be restated as follows (originally it was expressed in terms of continued fractions).

Theorem (A. A. Markov). *For any measure s, the corresponding sequence of diagonal Padé approximants $\{\pi_n\}, n \in \mathbb{Z}_+$, converges to $\widehat{s}$ uniformly on each compact subset contained in $\overline{\mathbb{C}} \setminus \Delta$.*

We denote this by

$$\lim_n \pi_n = \widehat{s}, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \Delta. \tag{1.2}$$

(Throughout the thesis we will use this notation to express uniform convergence of different sequences of functions on compact subsets of the specified region.)

Using the maximum principle, it is not hard to deduce that convergence takes place with geometric rate on the indicated region measured in terms of the Green function $g_\Omega(z;\infty)$ of the region $\Omega = \overline{\mathbb{C}} \setminus \mathrm{supp}(s)$ with singularity at ∞. For a large class of measures this rate of convergence is exact.

We say that s is regular, and denote this by $s \in \mathbf{Reg}$, when

$$\lim_n \|Q_n\|_2^{1/n} = \mathrm{cap}(\mathrm{supp}(s)),$$

where $\mathrm{cap}(\cdot)$ denotes the logarithmic capacity of the set $(\cdot)$ and $\| * \|_2$ is the L_2 norm with respect to s of the function $*$. See [77, Section 3.1] for different forms of defining regular measures. Let $q_n = Q_n/\|Q_n\|_2$ denote the n-th orthonormal polynomial with respect to s. The regularity of s is equivalent to

$$\lim_n |q_n(z)|^{1/n} = e^{g_\Omega(z;\infty)}, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \Delta.$$

Formulas of this type receive the name of *logarithmic* or *n-th root asymptotics.* On account of (1.1), and trivial upper and lower bounds for the second integral expression in that formula, it follows that $s \in \mathbf{Reg}$ implies

$$\lim_n \|\widehat{s} - \pi_n(\widehat{s})\|_{\mathcal{K}}^{1/2n} = e^{-\kappa(\mathcal{K})} (< 1) \tag{1.3}$$

for every compact subset $\mathcal{K} \subset \overline{\mathbb{C}} \setminus \Delta$, where $\| \cdot \|_{\mathcal{K}}$ denotes the uniform norm and

$$\kappa(\mathcal{K}) = \min\{g_\Omega(z;\infty) : z \in \mathcal{K}\}.$$

In [26, Theorem 1] (see also pages 570-571 in that reference), A. A. Gonchar devised a way of proving an analogue of Markov's theorem for functions of the form

$$f(z) = \int \frac{\frac{p_1(x)}{q_1(x)}\, ds(x)}{z - x} + \frac{p_2(z)}{q_2(z)}, \qquad f(\infty) = 0, \tag{1.4}$$

where $(p_1, q_1) = 1$ and $(p_2, q_2) = 1$ (that is, are relatively prime), the zeros of p_1, q_1 and q_2 lie in $\mathbb{C} \setminus \Delta$, and the measure s is such that the corresponding sequence of orthogonal polynomials satisfies what is called ratio asymptotics.

In a series of two papers [62] and [63], E. A. Rakhmanov proved a theorem on *ratio asymptotics.*

Theorem (E. A. Rakhmanov). *Suppose that $s' > 0$ almost everywhere with respect to the Lebesgue measure on Δ, then*

$$\lim_n \frac{Q_{n+1}(z)}{Q_n(z)} = \frac{\varphi(z)}{\varphi'(\infty)}, \qquad \mathcal{K} \subset \mathbb{C} \setminus \Delta \tag{1.5}$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta$, where $\varphi(z)$ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta$ onto $\{w : |w| > 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$.

This result produced great impression because of its theoretical interest within the general theory of orthogonal polynomials and its applications to the theory of rational approximation of analytic functions. Simplified proofs of Rakhmanov's theorem may be found in [65] and [55].

A point $z_0 \in \mathbb{C}$ is said to be a d attraction point of zeros of a sequence of functions $\{g_n\}, n \in \Lambda \subset \mathbb{Z}_+$, if for each sufficiently small $\varepsilon > 0$ there exists N such that for all $n \in \Lambda, n > N$, the number of zeros (counting multiplicity) of g_n in $\{z : |z - z_0| < \varepsilon\}$ is d. A set E is an attractor of the zeros of $\{g_n\}, n \in \Lambda$, if for each $\varepsilon > 0$ there exists N_0 such that $n > N_0, n \in \Lambda$, implies that all the zeros of g_n lie in the ε neighborhood of E.

Taking into account E. A. Rakhmanov's theorem, we can state Gonchar's result as follows.

Theorem (A. A. Gonchar). *Let f be as in* (1.4). *Assume that $s' > 0$ almost everywhere with respect to the Lebesgue measure on Δ. Then*

$$\lim_n \|f - \pi_n(f)\|_{\mathcal{K}}^{1/2n} = e^{-\kappa(\mathcal{K})} (< 1) \tag{1.6}$$

on each compact subset $\mathcal{K} \subset \overline{\mathbb{C}} \setminus (\Delta \cup \{z : f(z) = \infty\})$. If Q_n denotes the n-th monic orthogonal polynomial with respect to s and $\widetilde{Q}_n$ is the denominator of $\pi_n(f)$ normalized to be monic, we have that $\deg \widetilde{Q}_n = n$ for all sufficiently large n, each zero a_ν of q_2 of multiplicity d_ν is a d_ν attraction point of the zeros of $\{\widetilde{Q}_n\}, n \in \mathbb{Z}_+$, and $\Delta \cup \{z : q_2(z) = 0\}$ is an attractor of the zeros of $\{\widetilde{Q}_n\}, n \in \mathbb{Z}_+$. If $p_1(z) = C\prod_{k=1}^{N_1}(z - \beta_k)$, $C \neq 0$, $q_1(z) = \prod_{k=1}^{N_2}(z - \alpha_k)$, and $q_2(z) = \prod_{k=1}^{N_3}(z - a_k)$, then

$$\lim_n \frac{\widetilde{Q}_n(z)}{Q_n(z)} = \Phi_1(z)\, \Phi_2(z), \tag{1.7}$$

where

$$\Phi_1(z) = \prod_{k=1}^{N_1} \frac{\varphi(z)\varphi(\beta_k)}{\varphi(z)\varphi(\beta_k) - 1} \prod_{k=1}^{N_2} \frac{\varphi(z)\varphi(\alpha_k) - 1}{\varphi(z)\,\varphi(\alpha_k)},$$

and

$$\Phi_2(z) = \prod_{k=1}^{N_3} \frac{\varphi(a_k)(\varphi(z) - \varphi(a_k))}{\varphi(z)\,\varphi(a_k) - 1}.$$

The expression given here for the functions on the right hand side of (1.7) are equivalent to those used in [26].

Formula (1.7) is of *relative asymptotics*. In its general form, relative asymptotics deals with the question of obtaining the limit $\lim_n \widetilde{Q}_n/Q_n$ in terms of $g, d\widetilde{s} = gds$, where $\widetilde{Q}_n$ is the n-th monic orthogonal polynomial with respect to the measure $\widetilde{s}$ and Q_n is the n-th monic orthogonal polynomial with respect to the measure s. The aim is to extend Szegő's theory to classes of measures not satisfying Szegő's condition. With this general purpose, the question was probably first raised by P. Nevai in [54]. The problem was later considerably developed by Maté, Nevai, and Totik in [49]-[51] and independently by Rakhmanov in [65], (see also [31], [32], and [75, Chapters 9 and 13]).

Rakhmanov's theorem has been extended in several directions. Orthogonal polynomials with respect to varying measures (depending on the degree of the polynomial) arise in the study of multipoint Padé approximation of Markov functions. In this context, in [38] and [39], an analogue of Rakhmanov's theorem for such sequences of orthogonal polynomials was proved. Recently, S. A. Denisov [16] (see also [56]) obtained a remarkable extension of Rakhmanov's result for the case when the support of s verifies $\operatorname{supp}(s) = \widetilde{\Delta} \cup e \subset \mathbb{R}$, where $\widetilde{\Delta}$ is a bounded interval, e is a set without accumulation points in $\mathbb{R} \setminus \widetilde{\Delta}$, and $s' > 0$ a.e. on $\widetilde{\Delta}$. A version for orthogonal polynomials with respect to varying Denisov type measures was given in [14].

In this thesis we obtain results on the logarithmic, ratio, and relative asymptotics of multiple orthogonal polynomials for Nikishin systems of measures. In the following section we define such systems of measures and their corresponding multiple orthogonal polynomials.

1.2 Nikishin systems, mixed multiple orthogonal polynomials, and normality

The notion of a Nikishin system of measures was introduced by E.M. Nikishin in [57]. He called them MT systems.

Let σ_α, σ_β be two measures with constant sign and support contained in $\mathbb{R}$. Let Δ_α, Δ_β denote the smallest intervals containing their supports, $\operatorname{supp}(\sigma_\alpha)$ and $\operatorname{supp}(\sigma_\beta)$, respectively. We write $\operatorname{Co}(\operatorname{supp}(\sigma_\alpha)) = \Delta_\alpha$. Assume that $\Delta_\alpha \cap \Delta_\beta = \emptyset$ and define

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) := \int \frac{d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x).$$

Therefore, $\langle \sigma_\alpha, \sigma_\beta \rangle$ is a measure with constant sign and support equal to that of σ_α.

For a system of bounded intervals $\Delta_0, \ldots, \Delta_m$ contained in $\mathbb{R}$ satisfying $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 0, \ldots, m-1$, and finite Borel measures $\sigma_0, \ldots, \sigma_m$ with constant sign in $\operatorname{Co}(\operatorname{supp}(\sigma_j)) = \Delta_j$, such that each one has infinitely many points in its support, we define recursively

$$\langle \sigma_0, \sigma_1, \ldots, \sigma_j \rangle = \langle \sigma_0, \langle \sigma_1, \ldots, \sigma_j \rangle \rangle, \quad j = 1, \ldots, m.$$

This special notation was introduced by Gonchar, Rakhmanov, and Sorokin in [29].

Definition 1.2.1. *We say that* $(s_0, \ldots, s_m) = \mathcal{N}(\sigma_0, \ldots, \sigma_m)$*, where*

$$s_0 = \langle \sigma_0 \rangle = \sigma_0, \quad s_1 = \langle \sigma_0, \sigma_1 \rangle, \ldots, \quad s_m = \langle \sigma_0, \ldots, \sigma_m \rangle,$$

is the Nikishin system of measures generated by $(\sigma_0, \ldots, \sigma_m)$.

Throughout this thesis, the generating measures of the Nikishin systems considered are understood to have constant sign, to be compactly supported on the real line, the supports contain infinitely many points, and $\Delta_j \cap \Delta_{j+1} = \emptyset, j =$

$0, \ldots, m-1$,. We will not repeat this each time. The results of Chapter 2 may be easily deduced when the support of the generating measures are unbounded and have finite moments, but for the rest of the work compactness is an essential assumption.

Notice that all the measures in a Nikishin system have the same support, namely $\mathrm{supp}(\sigma_0)$. We will denote ($s_{j,j} = \sigma_j$)

$$s_{j,k} = \langle \sigma_j, \ldots, \sigma_k \rangle, \qquad 0 \le j \le k \le m.$$

For simplicity, in what follows we use the notation $\widehat{s}_{j,k}$ (instead of $\widehat{s_{j,k}}$) to indicate the Cauchy transform of $s_{j,k}$.

Take two systems $S^1 = (s_0^1, \ldots, s_{m_1}^1) = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, $S^2 = (s_0^2, \ldots, s_{m_2}^2) = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ generated by $m_1 + 1$ and $m_2 + 1$ measures, respectively. The two systems need not coincide, but we will always assume that $\sigma_0^1 = \sigma_0^2$; that is, both systems stem from the same basis measure. The smallest interval containing $\mathrm{supp}(\sigma_j^i)$ will be denoted $\mathrm{Co}(\mathrm{supp}(\sigma_j^i)) = \Delta_j^i$.

Let $\mathbb{Z}_+$ denote the set of non-negative integers. Fix two multi-indices $\mathbf{n}_1 = (n_{1,0},\, n_{1,1}, \ldots,\, n_{1,m_1}) \in \mathbb{Z}_+^{m_1+1}$ and $\mathbf{n}_2 = (n_{2,0},\, n_{2,1}, \ldots,\, n_{2,m_2}) \in \mathbb{Z}_+^{m_2+1}$. Set $|\mathbf{n}_1| = n_{1,0} + \cdots + n_{1,m_1}$, $|\mathbf{n}_2| = n_{2,0} + \cdots + n_{2,m_2}$, and $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$. We always assume that $|\mathbf{n}_2| + 1 = |\mathbf{n}_1|$.

Let $|\mathbf{n}_1| \ge 1$. The system of polynomials $a_{\mathbf{n},0},\, a_{\mathbf{n},1}, \ldots,\, a_{\mathbf{n},m_1}$ satisfying

i) $\deg(a_{\mathbf{n},j}) \le n_{1,j} - 1$, $j = 0, \ldots,\, m_1$, not all identically equal to zero.

ii) For $k = 0, \ldots, m_2$ and $\nu = 0, \ldots, n_{2,k} - 1$,

$$\int x^\nu \Big(a_{\mathbf{n},0}(x) + \sum_{j=1}^{m_1} a_{\mathbf{n},j}(x) \widehat{s}_{1,j}^1(x) \Big) ds_{0,k}^2(x) = 0, \tag{1.8}$$

($\deg(a_{\mathbf{n},j}) \le -1$ means that $a_{\mathbf{n},j} \equiv 0$) is called a *system of mixed type multiple orthogonal polynomials associated to* $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$ *and* (S^1, S^2). In the context of pairs of Nikishin systems, this concept was first introduced by V.N. Sorokin in [70] (see also [69] for the general definition).

Finding $a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1}$ reduces to solving a homogeneous linear system of $|\mathbf{n}_2|$ equations on $|\mathbf{n}_1|$ unknowns. Since $|\mathbf{n}_2| = |\mathbf{n}_1| - 1$, a non-trivial solution is guaranteed. If we multiply all the polynomials giving a solution by the same constant, the new set of polynomials also solves the problem. Nevertheless, it is not a trivial fact to determine whether or not the class of all solutions is formed by collinear polynomial vectors. In more general settings this is known to be false. Recently, U. Fidalgo and G. López have claimed that the statement is true for mixed multiple orthogonal polynomials of two Nikishin systems.

In the particular case $m_2 = 0$, the polynomials $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1})$ are called *type I multiple orthogonal polynomials.* If $m_1 = 0$, $a_{\mathbf{n},0}$ is called a *type II multiple orthogonal polynomial.* The case $m_1 = m_2 = 0$ reduces to the usual definition of orthogonal polynomial.

We remark that in [57] Nikishin obtained a generalization of the classical Markov theorem for Nikishin systems consisting of two measures and multi-indices with two equal components for type II approximants. Later, J. Bustamante and G. López extended in [12] the theorem to arbitrary Nikishin systems and multi-indices whose components are “nearly” equal also in type II approximation. A substantial improvement, for quite general systems of multi-indices, was produced in [24].

In [57], E.M. Nikishin introduced the following definition.

Definition 1.2.2. *A set of real continuous functions $u_0(x), \dots, u_m(x)$ defined on an interval Δ, is called an* AT*-system for $\mathbf{n} = (n_0, \dots, n_m) \in \mathbb{Z}_+^{m+1}$, if for any polynomials $P_0, \dots, P_m$ such that $\deg(P_i) \leq n_i - 1$, $i = 0, \dots, m$, not simultaneously identically equal to zero, the function*

$$P_0(x)\, u_0(x) + \cdots + P_m(x)\, u_m(x),$$

has at most $|\mathbf{n}| - 1$ zeros on Δ ($\deg(P_j) \leq -1$ means that $P_j \equiv 0$).

We will consider several classes of multi-indices. Given an integer $m \geq 1$, we define

$$\mathbb{Z}_+^m(\bullet) = \{\mathbf{n} \in \mathbb{Z}_+^m : n_1 \geq \cdots \geq n_m\}, \tag{1.9}$$

$$\mathbb{Z}_+^m(*) = \{\mathbf{n} \in \mathbb{Z}_+^m : \nexists\, i < k < j \text{ such that } n_i < n_j < n_k\}. \tag{1.10}$$

More classes of multi-indices will be specified later. The following result was obtained by U. Fidalgo, J. Illán, and G. López in [20], and will be applied several times throughout the thesis.

Lemma 1.2.3. *Let $\mathbf{n} \in \mathbb{Z}_+^{m+1}(*)$ and $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$, then the system of functions $(1, \widehat{s}_1, \dots, \widehat{s}_m)$ defines an* AT*-system for $\mathbf{n} = (n_0, \dots, n_m)$ on any interval disjoint from* $\mathrm{Co}(\mathrm{supp}(\sigma_1))$.

We associate to the system of polynomials $(a_{\mathbf{n},0}, \dots, a_{\mathbf{n},m_1})$ the following functions ($\widehat{s}^1_{j+1,j}(z) \equiv 1, \mathcal{A}_{\mathbf{n},m_1} \equiv a_{\mathbf{n},m_1}$)

$$\mathcal{A}_{\mathbf{n},j}(z) := \sum_{k=j}^{m_1} a_{\mathbf{n},k}(z) \widehat{s}^1_{j+1,k}(z), \qquad j = 0, \dots, m_1, \tag{1.11}$$

$$\mathcal{A}_{\mathbf{n},-j-1}(z) := \int \frac{\mathcal{A}_{\mathbf{n},-j}(x)}{z - x} d\sigma_j^2(x), \qquad j = 0, \dots, m_2, \tag{1.12}$$

the latter being defined recursively. Notice that (1.8) indicates that $\mathcal{A}_{\mathbf{n},0}$ satisfies orthogonality conditions. We will show that in fact all the linear forms $\mathcal{A}_{\mathbf{n},j}$, $-m_2 \leq j \leq m_1$, satisfy certain orthogonality conditions and we will describe their logarithmic and ratio asymptotic properties.

For $j = 0, \dots, m_1$, let $Q_{\mathbf{n},j}$ be the monic polynomial whose zeros are those of the linear form $\mathcal{A}_{\mathbf{n},j}$ in the region $\mathbb{C} \setminus \Delta^1_{j+1}$, counting multiplicities ($\Delta^1_{m_1+1} = \emptyset$). In particular, $\mathcal{A}_{\mathbf{n},m_1} = a_{\mathbf{n},m_1} = Q_{\mathbf{n},m_1}$. In the hypothetical case that $\mathcal{A}_{\mathbf{n},j}$ had infinitely many zeros in the specified region, then $Q_{\mathbf{n},j}$ denotes a formal infinite product. In fact, on the basis of Lemma 1.2.3, in Proposition 2.1.5 we will show that if $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ then

$$\deg Q_{\mathbf{n},j} = N_{1,j} = N_{1,j}(\mathbf{n}) = n_{1,j} + \cdots + n_{1,m_1} - 1, \qquad j = 0, \ldots, m_1,$$

all the zeros of $Q_{\mathbf{n},j}$ are simple and lie in the interior of the interval Δ_j^1.

Analogously, for $j = 1, \ldots, m_2$, we let $Q_{\mathbf{n},-j}$ be the monic polynomial whose zeros are those of $\mathcal{A}_{\mathbf{n},-j}$ in the region $\mathbb{C} \setminus \Delta_{j-1}^2$, counting multiplicities. We will also prove in Proposition 2.1.7 that if $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ then

$$\deg Q_{\mathbf{n},-j} = N_{2,j} = N_{2,j}(\mathbf{n}) = n_{2,j} + \cdots + n_{2,m_2}, \qquad j = 0, \ldots, m_2,$$

all the zeros of $Q_{\mathbf{n},-j}$ are simple and lie in the interior of Δ_j^2. In this thesis we describe the logarithmic and ratio asymptotics of the polynomials $Q_{\mathbf{n},j}$, $j = -m_2, \ldots, m_1$.

A multi-index $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$ is said to be *normal* if every solution to i)-ii) satisfies $\deg a_{\mathbf{n},j} = n_{1,j} - 1, j = 0, \ldots, m$. If $\mathbf{n}$ is normal, it is easy to verify that the vector $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1})$ is uniquely determined except for a constant factor, and in that case we normalize it to be "monic" meaning by this that its last entry different from zero has leading coefficient equal to 1.

1.3 Main results

The results of this thesis appear in three papers: Chapters 2 through 4 are contained in [21], Chapter 5 corresponds to [37], and Chapter 6 develops the results of [36]. They have been exposed at various international meetings on orthogonal polynomials and their applications, as for instance: OPSFA, Luminy'07; Appopt, San Andres'08; and FoCM, Hong Kong'08. The methods employed are inscribed in the theory of functions of a real and complex variable as developed in [1] and [68], with elements of more advanced topics of logarithmic potential theory [67], and compact Riemann surfaces [9].

Let us describe briefly our main contributions. We will specify more details in the beginning of each chapter.

1.3.1 Logarithmic asymptotics

One of the main results in this thesis concerns the $|\mathbf{n}_1|$-root asymptotics of the linear forms $\mathcal{A}_{\mathbf{n},j}$, $j = -m_2 - 1, \ldots, m_1$, under mild conditions on the sequence of multi-indices and the measures generating both Nikishin systems. We remind the reader that a measure σ is said to be regular if

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1/\mathrm{cap}(\mathrm{supp}(\sigma)),$$

where $\mathrm{cap}(\cdot)$ denotes the logarithmic capacity of the Borel set $(\cdot)$ (see [77] or Section 3.1 for the definition of logarithmic capacity) and κ_n denotes the leading coefficient of the n-th orthonormal polynomial with respect to σ. For different equivalent forms of defining regular measures, see sections 3.1 to 3.3 in [77] (in particular Theorem 3.1.1). For short, we write $(S^1, S^2) \in \mathbf{Reg}$ to mean that all the measures which generate both Nikishin systems are regular and their supports are regular compact sets. Recall that a compact set $\mathcal{K}$ is said to be regular with respect to the Dirichlet problem when the Green function with singularity at infinity of the unbounded connected component of the complement of $\mathcal{K}$ can be extended continuously to all $\mathbb{C}$. Let us introduce some notation and results from potential theory which we need to formulate our findings of Chapter 3.

If E is a compact subset of the complex plane, we denote by $\mathcal{M}(E)$ the class of all finite, positive, Borel measures with support consisting of an infinite set of points contained in E, and $\mathcal{M}_1(E)$ is the subclass of probability measures of $\mathcal{M}(E)$.

Let E_k, $k = -m_2, \ldots, m_1$, be (not necessarily distinct) compact subsets of the real line and $\mathcal{C} = (c_{j,k})$, $-m_2 \le j, k \le m_1$, a real, positive definite, symmetric matrix of order $m_1 + m_2 + 1$. $\mathcal{C}$ will be called the interaction matrix. Set

$$\mathcal{M}_1 = \mathcal{M}_1(E_{-m_2}) \times \cdots \times \mathcal{M}_1(E_{m_1}) .$$

Given a vector measure $\mu = (\mu_{-m_2}, \ldots, \mu_{m_1}) \in \mathcal{M}_1$ and $j \in \{-m_2, \ldots, m_1\}$, we define the combined potential

$$W_j^{\mu}(x) = \sum_{k=-m_2}^{m_1} c_{j,k} V^{\mu_k}(x) , \tag{1.13}$$

where

$$V^{\mu_k}(x) = \int \log \frac{1}{|x-t|} \, d\mu_k(t) ,$$

denotes the standard logarithmic potential of μ_k. We denote

$$\omega_j^{\mu} = \inf\{W_j^{\mu}(x) : x \in E_j\} , \quad j = -m_2, \ldots, m_1 .$$

In [60, Chapter 5] the following important lemma is proved (we state the result in a form convenient for our purpose).

Lemma 1.3.1. *Assume that the compact sets $E_k, k = -m_2, \ldots, m_1$, are regular with respect to the Dirichlet problem. Let $\mathcal{C}$ be a real, positive definite, symmetric matrix of order $m_1 + m_2 + 1$. If there exists $\overline{\mu} = (\overline{\mu}_{-m_2}, \ldots, \overline{\mu}_{m_1}) \in \mathcal{M}_1$ such that for each $j = -m_2, \ldots, m_1$*

$$W_j^{\overline{\mu}}(x) = \omega_j^{\overline{\mu}} , \qquad x \in \operatorname{supp}(\overline{\mu}_j) ,$$

then $\overline{\mu}$ is unique. Moreover, if $c_{j,k} \ge 0$ when $E_j \cap E_k \neq \emptyset$, then $\overline{\mu}$ exists.

For details on how this lemma is derived from [60, Chapter 5] see [10, Section 4]. The vector measure $\overline{\mu} \in \mathcal{M}_1$ is called the equilibrium solution for the vector potential problem determined by the interaction matrix $\mathcal{C}$ on the system of compact sets E_j , $j = -m_2, \ldots, m_1$.

Let $\Lambda = \Lambda(p_{1,0},\ldots,p_{1,m_1};p_{2,0},\ldots,p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet)\times\mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that

$$\lim_{\mathbf{n}\in\Lambda}\frac{n_{1,j}}{|\mathbf{n}_1|} = p_{1,j}\in(0,1),\ j=0,\ldots,m_1, \quad \lim_{\mathbf{n}\in\Lambda}\frac{n_{2,j}}{|\mathbf{n}_1|} = p_{2,j}\in(0,1),\ j=0,\ldots,m_2.$$

Obviously, $p_{1,0}\geq\cdots\geq p_{1,m_1}, p_{2,0}\geq\cdots\geq p_{2,m_2}$, and $\sum_{j=0}^{m_1}p_{1,j}=\sum_{j=0}^{m_2}p_{2,j}=1$. Set

$$P_j=\sum_{k=j}^{m_1}p_{1,k},\ j=0,\ldots,m_1, \qquad P_{-j}=\sum_{k=j}^{m_2}p_{2,k},\ j=0,\ldots,m_2.$$

Let us define the interaction matrix $\mathcal{C}$ which is relevant in this thesis. Take the tri-diagonal matrix

$$\mathcal{C}=\begin{pmatrix} P^2_{-m_2} & -\frac{P_{-m_2}P_{-m_2+1}}{2} & 0 & \cdots & 0\\ -\frac{P_{-m_2}P_{-m_2+1}}{2} & P^2_{-m_2+1} & -\frac{P_{-m_2+1}P_{-m_2+2}}{2} & \cdots & 0\\ 0 & -\frac{P_{-m_2+1}P_{-m_2+2}}{2} & P^2_{-m_2+2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & P^2_{m_1}\end{pmatrix}. \tag{1.14}$$

This matrix satisfies all the assumptions of Lemma 1.3.1 on the compact sets $E_j=\operatorname{supp}(\sigma_j^1), j=0,1,\ldots,m_1, E_j=\operatorname{supp}(\sigma^2_{-j}), j=0,-1,\ldots,-m_2$, including $c_{j,k}\geq 0$ when $E_j\cap E_k\neq\emptyset$ (recall that $\sigma_0^1=\sigma_0^2$), and it is positive definite because the principal section $\mathcal{C}_r, r=1,\ldots,m_1+m_2+1$, of $\mathcal{C}$ satisfies

$$\det(\mathcal{C}_r)=P^2_{-m_2}\cdots P^2_{-m_2+r-1}\det\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0\\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0\\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2}\\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1\end{pmatrix}_{r\times r} > 0.$$

Let $\overline{\mu}(\mathcal{C})$ be the equilibrium solution for the corresponding vector potential problem.

Let $\{\mu_l\}\subset\mathcal{M}(E)$ be a sequence of positive measures and $\mu\in\mathcal{M}(E)$. We write

$$*\lim_l \mu_l=\mu\,,$$

if for every continuous function $f\in C(E)$

$$\lim_l\int f d\mu_l=\int f d\mu\,;$$

that is, when the sequence of measures converges to μ in the weak star topology.

Given a polynomial q_l of degree $l \geq 1$, we denote the associated normalized zero counting measure by

$$\mu_{q_l} = \frac{1}{l} \sum_{q_l(x)=0} \delta_x \,,$$

where δ_x is the Dirac measure with mass 1 at x (in the sum the zeros are repeated according to their multiplicity).

Frequently we will make use of the following renumbering of intervals and measures

$$\Delta_j = \Delta_j^1, \qquad \sigma_j = \sigma_j^1, \qquad j = 0, 1, \ldots, m_1,$$
$$\Delta_j = \Delta_{-j}^2, \qquad \sigma_j = \sigma_{-j}^2, \qquad j = 0, -1, \ldots, -m_2,$$

and

$$N_{\mathbf{n},j} = \begin{cases} N_{1,j}(\mathbf{n}) - 1, & j = 0, 1 \ldots, m_1, \\ \\ N_{2,-j}(\mathbf{n}), & j = 0, -1, \ldots, -m_2. \end{cases}$$

We have

Theorem 1.3.2. *Let* $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$, $(S^1, S^2) \in$ **Reg**, *and* $\Lambda = \Lambda(p_{1,0}, \ldots, p_{1,m_1}; p_{2,0}, \ldots, p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$, *be given. Then*

$$* \lim_{\mathbf{n}\in\Lambda} \mu_{Q_{\mathbf{n},j}} = \overline{\mu}_j, \quad j = -m_2, \ldots, m_1, \tag{1.15}$$

where $\overline{\mu} = \overline{\mu}(\mathcal{C}) \in \mathcal{M}_1$ *is the vector equilibrium measure determined by the matrix* $\mathcal{C}$ *in* (1.14) *on the system of compact sets* $E_j = \text{supp}(\sigma_j^1), j = 0, \ldots, m_1, E_j = \text{supp}(\sigma_{-j}^2), j = -m_2, \ldots, 0$. *Moreover,*

$$\lim_{\mathbf{n}\in\Lambda} \left(\int \frac{Q_{\mathbf{n},j}^2(x)}{|Q_{\mathbf{n},j-1}(x)|} \frac{|\mathcal{A}_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|} d|\sigma_j|(x) \right)^{1/2|\mathbf{n}_1|} = \exp\left(-\sum_{k=j}^{m_1} \omega_k^{\overline{\mu}}/P_k \right) , \tag{1.16}$$

where the $\omega_k^{\overline{\mu}}$ *denote the corresponding equilibrium constants.*

The previous result is demonstrated in Section 3.2. Concerning the logarithmic asymptotics of the functions $\mathcal{A}_{\mathbf{n},j}$, in Section 3.3 we obtain

Theorem 1.3.3. *Let* $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$, $(S^1, S^2) \in$ **Reg**, *and* $\Lambda = \Lambda(p_{1,0}, \ldots, p_{1,m_1}; p_{2,0}, \ldots, p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$, *be given. Let* $\{\mathcal{A}_{\mathbf{n},j}\}, \mathbf{n} \in \Lambda, j = -m_2 - 1, \ldots, m_1$, *be the sequences of "monic" linear forms associated with the corresponding mixed type orthogonal polynomials. Then, for each* $j = -m_2 - 1, \ldots, m_1$

$$\lim_{\mathbf{n}\in\Lambda} |\mathcal{A}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = G_j(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1}) \tag{1.17}$$

$(\Delta_{-m_2-1} = \Delta_{m_1+1} = \emptyset)$, *where*

$$G_j(z) = \exp\left(P_{j+1} V^{\overline{\mu}_{j+1}}(z) - P_j V^{\overline{\mu}_j}(z) - 2 \sum_{k=j+1}^{m_1} \frac{\omega_k^{\overline{\mu}}}{P_k} \right), \tag{1.18}$$

$(P_{-m_2-1} = P_{m_1+1} = 0)$ *when* $j = -m_2 - 1, \ldots, m_1 - 1$, *and*

$$G_{m_1}(z) = \exp\left(-P_{m_1} V^{\overline{\mu}_{m_1}}(z)\right). \tag{1.19}$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_{-m_2}, \ldots, \overline{\mu}_{m_1})$ *and* $(\omega^{\overline{\mu}}_{-m_2}, \ldots, \omega^{\overline{\mu}}_{m_1})$ *are the equilibrium vector measures and the system of equilibrium constants, respectively, for the vector potential problem determined by the interaction matrix* $\mathcal{C}$ *defined in* (1.14) *on the system of regular compact sets* $E_j = \operatorname{supp}(\sigma^1_j), j = 0, \ldots, m_1, E_j = \operatorname{supp}(\sigma^2_{-j}), j = -m_2, \ldots, 0$.

In [58], E. M. Nikishin proved a similar result for type I multiple orthogonal polynomials ($m_2 = 0$) and in Section 5.7 of [60] (see also [43]) the analogue for type II multiple orthogonal polynomials ($m_1 = 0$) is stated. V. N. Sorokin considered the mixed case in [70] where he stated the result in general but proved it only when $m_1 = m_2 = 1$. In these papers a stronger assumption is made on the generating measures. Namely, it is required that $|\sigma_k'| > 0, a.e.$ on $\Delta^i_k, k = 0, \ldots, m_i, i = 1, 2$. (As usual, by σ' we denote the Radon–Nikodym derivative of the measure σ.) We preserve this more restrictive hypothesis for the stronger Theorem 1.3.4. Weakening the hypothesis to class **Reg** is made possible using finer results from potential theory.

Regarding [70] we wish to point out the following. We arrived at the construction of mixed multiple orthogonal polynomials with respect to two Nikishin systems and the results in Chapters 2 to 4 without any knowledge of the existence of [70]. In a last attempt to update the references in [21] before submitting the paper, we discovered [69] and [70] and asked V. N. Sorokin to join the rest of us as co-author in due recognition for his contributions to the subject and he kindly accepted.

1.3.2 Ratio asymptotics

For the next result, we assume that $\operatorname{supp}(\sigma^i_j) = \widetilde{\Delta}^i_j \cup e^i_j, j = 0, \ldots, m_i, i = 1, 2$, where $\widetilde{\Delta}_j = [a^i_j, b^i_j]$ is a bounded interval of the real line, $|(\sigma^i_j)'| > 0$ a.e. on $\widetilde{\Delta}^i_j$, and e^i_j is a set without accumulation points in $\overline{\mathbb{R}} \setminus \widetilde{\Delta}^i_j$. We denote this writing $S^1 = \mathcal{N}'(\sigma^1_0, \ldots, \sigma^1_{m_1}), S^2 = \mathcal{N}'(\sigma^2_0, \ldots, \sigma^2_{m_2})$.

We need to introduce a Riemann surface that arises in the analysis of the ratio asymptotics of the family of polynomials $\{Q_{\mathbf{n},j}\}_{j=-m_2}^{m_1}$. Let us renumber the intervals $\widetilde{\Delta}^i_j$ as follows

$$\widetilde{\Delta}_j = \begin{cases} \widetilde{\Delta}^1_j, & j = 0, \ldots, m_1, \\ \widetilde{\Delta}^2_j, & j = -m_2, \ldots, 0. \end{cases}$$

Consider the $(m_1 + m_2 + 2)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=-m_2-1}^{m_1} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_{-m_2-1} := \overline{\mathbb{C}} \setminus \widetilde{\Delta}_{-m_2},\ \ \mathcal{R}_k := \overline{\mathbb{C}} \setminus (\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}),\ k = -m_2, \ldots, m_1 - 1,$$

$$\mathcal{R}_{m_1} := \overline{\mathbb{C}} \setminus \widetilde{\Delta}_{m_1},$$

where the upper and lower banks of the slits of two neighboring sheets are identified. It is easy to see that the surface $\mathcal{R}$ is orientable, compact, and has genus zero, the latter meaning that it is homeomorphic to the Riemann sphere.

Fix $l = (l_1, l_2)$, $0 \le l_1 \le m_1$, $0 \le l_2 \le m_2$. Let $\psi^{(l)}$ be a single valued function defined on $\mathcal{R}$ onto the extended complex plane satisfying

$$\psi^{(l)}(z) = \frac{C_1}{z} + \mathcal{O}\Big(\frac{1}{z^2}\Big), \quad z \to \infty^{(-l_2-1)},$$

$$\psi^{(l)}(z) = C_2\, z + \mathcal{O}(1), \quad z \to \infty^{(l_1)},$$

where C_1 and C_2 are nonzero constants. Since the genus of $\mathcal{R}$ is zero, $\psi^{(l)}$ exists and is uniquely determined up to a multiplicative constant. Consider the branches of $\psi^{(l)}$, corresponding to the different sheets $k = -m_2 - 1, \ldots, m_1$ of the surface $\mathcal{R}$

$$\psi^{(l)} := \{\psi_k^{(l)}\}_{k=-m_2-1}^{m_1}\,.$$

We normalize $\psi^{(l)}$ so that

$$\prod_{k=-m_2-1}^{m_1} |\psi_k^{(l)}(\infty)| = 1, \qquad C_1 \in \mathbb{R} \setminus \{0\}. \tag{1.20}$$

In fact there are only two $\psi^{(l)}$ verifying this normalization. To see this, assume that $\phi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is a single valued function satisfying

$$\phi(z) = \frac{D_1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \to \infty^{(-l_2-1)},$$

$$\phi(z) = D_2\, z + \mathcal{O}(1), \quad z \to \infty^{(l_1)},$$

where D_1 and D_2 are nonzero constants, and ϕ also satisfies (1.20). The functions ϕ and $\psi^{(l)}$ have the same divisor; consequently, there exists a constant C such that $\phi = C\,\psi^{(l)}$. This implies that $D_1 = C\,C_1$ and $C \in \mathbb{R}$. From

$$1 = \prod_{k=-m_2-1}^{m_1} |\phi_k(\infty)| = \prod_{k=-m_2-1}^{m_1} |C\,\psi_k^{(l)}(\infty)| = |C|^{m_1+m_2+1}$$

it follows that $C = \pm 1$.

Since the product of all the branches $\prod_{k=-m_2-1}^{m_1} \psi_k^{(l)}$ is a single valued analytic function in $\overline{\mathbb{C}}$ without singularities, by Liouville's theorem we know that $\prod_{k=-m_2-1}^{m_1} \psi_k^{(l)}$ is constant and because of the normalization (1.20) this constant is either 1 or -1.

The fact that $C_1 \in \mathbb{R} \setminus \{0\}$ implies in particular that

$$\psi^{(l)}(z) = \overline{\psi^{(l)}(\overline{z})}, \qquad z \in \mathcal{R}.$$

To justify this, let $\phi(z) := \overline{\psi^{(l)}(\overline{z})}$. ϕ and $\psi^{(l)}$ have the same divisor and therefore there exists a constant C such that $\phi = C\,\psi^{(l)}$. Comparing the leading coefficients of the Laurent expansion of these functions at $\infty^{(-l_2-1)}$, we conclude that $C = 1$ since $C_1 \in \mathbb{R} \setminus \{0\}$.

In terms of the branches of $\psi^{(l)}$, the symmetry formula above means that for each $k = -m_2 - 1, \ldots, m_1$,

$$\psi_k^{(l)} : \overline{\mathbb{R}} \setminus (\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}) \longrightarrow \overline{\mathbb{R}}$$

($\widetilde{\Delta}_{-m_2-1} = \widetilde{\Delta}_{m_1+1} = \emptyset$); therefore, the coefficients (in particular, the leading one) of the Laurent expansion at ∞ of these branches are real numbers, and

$$\psi_k^{(l)}(x_\pm) = \overline{\psi_k^{(l)}(x_\mp)} = \overline{\psi_{k+1}^{(l)}(x_\pm)}, \qquad x \in \widetilde{\Delta}_{k+1}. \tag{1.21}$$

Given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1}), z \to \infty, C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := F/C\,.$$

C is called the leading coefficient of F. When $C \in \mathbb{R}$, $\mathrm{sg}(F(\infty))$ will represent the sign of C.

Given $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$ and $l = (l_1, l_2)$, we associate a new multi-index $\mathbf{n}^l := (\mathbf{n}_1 + \mathbf{e}^{l_1}; \mathbf{n}_2 + \mathbf{e}^{l_2}) = (\mathbf{n}_1^{l_1}; \mathbf{n}_2^{l_2})$, where $\mathbf{e}^{l_i}$ denotes the unit vector of length $m_i + 1$ with all components equal to zero except the component $(l_i + 1)$ which equals 1.

We are ready to state the main result of Chapter 4.

Theorem 1.3.4. *Let $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ be given, and let $\mathbf{n} \in \Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\,n_{2,0} - |\mathbf{n}_2|) < \infty, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\,n_{1,0} - |\mathbf{n}_1|) < \infty.$$

Let us assume that there exists $l = (l_1; l_2), 0 \le l_1 \le m_1, 0 \le l_2 \le m_2$, such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$. Let $\{Q_{\mathbf{n},j}\}_{j=-m_2}^{m_1}$, $\mathbf{n} \in \Lambda$, be the corresponding sequences of polynomials defined in Section 1.2. *Then, for each fixed $j \in \{-m_2, \ldots, m_1\}$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} = \widetilde{F}_j^{(l)}(z), \qquad z \in \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_j) \tag{1.22}$$

where

$$F_j^{(l)} := sg\left(\prod_{k=j}^{m_1} \psi_k^{(l)}(\infty)\right) \prod_{k=j}^{m_1} \psi_k^{(l)}. \tag{1.23}$$

An interesting feature is that the functions $F_j^{(l)}$ are completely determined by the system of boundary conditions (4.14) (see Lemma 4.4.2 in Section 4.4 and the proof of Theorem 1.3.4 in Section 4.5). When $m_1 = m_2 = 0$, this result reduces to the Denisov–Rakhmanov Theorem.

For type II multiple orthogonal polynomials ($m_1 = 0$) and measures such that $|\sigma_k'| > 0$ on the whole interval $\Delta_k, k = -m_2, \ldots, 0$, this result was established in [5]. Later (also for type II multiple orthogonal polynomials), in [36] we improved the result to measures of Denisov type and more general classes of multi-indices than those considered here. The treatment of more general sequences of multi-indices introduces substantial technical difficulties. For this reason, we revisit ratio asymptotic for type II multiple orthogonal polynomials separately in Chapter 6. There we prove Theorem 6.5.2, which is an analogue of Theorem 1.3.4. The present result is already new when $m_2 = 0$ and $m_1 \geq 2$ since ratio asymptotics had not been proved before for liner forms (except the one that gives the remainder of Padé approximation).

Chapter 4 also contains results on the ratio asymptotic of the forms and the second type functions defined by them.

1.3.3 Relative asymptotics

In Chapters 5 and 6, we focus on type II multiple orthogonal polynomials. Now, the construction of the multiple orthogonal polynomials depends of only one Nikishin system so we drop the supra index on the generating and generated measures. Since we have no need to match systems at an initial measure as we did before ($\sigma_0^1 = \sigma_0^2$), the basis measure will be σ_1, following the notation employed in [36] and [37].

Given the collection of polynomials $(p_1, \ldots, p_m)$, we define

$$\mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k + \deg(p_{j+1} \cdots p_k) \leq n_j + 1\}.$$

In particular,

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k \leq n_j + 1\}.$$

Recall that a point $z_0 \in \mathbb{C}$ is said to be a d attraction point of zeros of a sequence of functions $\{\varphi_\mathbf{n}\}, \mathbf{n} \in \Lambda \subset \mathbb{Z}_+^m$, if for each sufficiently small $\varepsilon > 0$ there exists N such that for all $\mathbf{n} \in \Lambda, |\mathbf{n}| > N$, the number of zeros (counting multiplicity) of $\varphi_\mathbf{n}$ in $\{z : |z - z_0| < \varepsilon\}$ is d. A set E is an attractor of the zeros of $\{\varphi_\mathbf{n}\}, \mathbf{n} \in \Lambda$, if for each $\varepsilon > 0$ there exists N_0 such that $|\mathbf{n}| > N_0, \mathbf{n} \in \Lambda$, implies that all the zeros of $\varphi_\mathbf{n}$ lie in the ε neighborhood of E. In Section 5.4, we prove

Theorem 1.3.5. *Let $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1q_1, \ldots, p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda, n_1 - n_m \leq C$, where C is* *a constant. Let* $\widetilde{Q}_{\mathbf{n}}$ *be the monic multiple orthogonal polynomial of smallest degree relative to the Nikishin system* $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ *and* $\mathbf{n}$. *Then*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \frac{\mathcal{F}(z; p_1, \ldots, p_m)}{\mathcal{F}(z; q_1, \ldots, q_m)}, \quad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_1). \tag{1.24}$$

For all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, $\deg \widetilde{Q}_{\mathbf{n}} = |\mathbf{n}|$, $\operatorname{supp}(\sigma_1)$ *is an attractor of the zeros of* $\{\widetilde{Q}_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$, *and each point in* $\operatorname{supp}(\sigma_1) \setminus \widetilde{\Delta}_1$ *is a* 1 *attraction point of zeros of* $\{\widetilde{Q}_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$. *When the polynomials* $p_k, q_k, k = 1, \ldots, m$, *have real coefficients, the statements remain valid for* $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

An expression for $\mathcal{F}(z; p_1, \ldots, p_m)$ is given in (5.43) at the end of the proof of Theorem 1.3.5 in Section 5.4. Formula (1.24) reduces to (1.7) when $m_1 = m_2 = 0$ and $p_2 \equiv 0$. Relative asymptotics of type II multiple orthogonal polynomials of Nikishin systems had not been considered before. In Chapter 5 we also obtain the relative asymptotic behavior of the second type functions associated with the two Nikishin systems (the initial and perturbed ones). When the polynomials $p_k, q_k, k = 1, \ldots, m$, have real coefficients an analogue of Theorem 1.3.5 is obtained in Section 5.6 for sequences of the form $\{\widetilde{Q}_{\mathbf{n},j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$.

2. PRELIMINARY RESULTS

In this chapter, we prove a number of results which are later applied to derive the main asymptotic properties of the multiple orthogonal polynomials studied in this thesis. Section 2.1 describes the orthogonality relations satisfied by the linear forms $\mathcal{A}_{\mathbf{n},j}$ defined through (1.11)-(1.12), as well as the location of their zeros (Propositions 2.1.5, 2.1.6 and 2.1.7). Here, it is also proved that all the multi-indices belonging to the class $\mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ are normal (Proposition 2.1.2). Section 2.2 is devoted to the deduction of interlacing properties of the zeros of the functions $\mathcal{A}_{\mathbf{n},j}, -m_2 \leq j \leq m_1$, (see Theorem 2.2.5).

2.1 Orthogonality properties of the functions $\mathcal{A}_{\mathbf{n},j}$ and their zeros

Recall the notation $\mathrm{Co}(\mathrm{supp}(\sigma_j^i)) = \Delta_j^i$. We start this section by proving the following result.

Lemma 2.1.1. *Let $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$, $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, and $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$, be given. Then we have:*

a) For each $j = 0, \ldots, m_1$, the linear form $\mathcal{A}_{\mathbf{n},j}$ has at most $N_{1,j} - 1$ zeros on any interval disjoint from Δ_{j+1}^1 ($\Delta_{m_1+1}^1 = \emptyset$), where

$$N_{1,j} = N_{1,j}(\mathbf{n}) = n_{1,j} + \cdots + n_{1,m_1}. \tag{2.1}$$

b) $\mathcal{A}_{\mathbf{n},0}$ has at least $|\mathbf{n}_1| - 1 = |\mathbf{n}_2|$ sign changes on the interval $\Delta_0^1 = \Delta_0^2$.

Proof. For each $j = 0, \ldots, m_1 - 1$, we have that

$$(s_{j+1,j+1}^1, \ldots, s_{j+1,m_1}^1) = \mathcal{N}(\sigma_{j+1}^1, \ldots, \sigma_{m_1}^1).$$

Since $\mathbf{n}_1 = (n_{1,0}, \ldots, n_{1,m_1}) \in \mathbb{Z}_+^{m_1+1}(\bullet) \subset \mathbb{Z}_+^{m_1+1}(*)$, applying Lemma 1.2.3 it follows that $(1, \widehat{s}_{j+1,j+1}^1, \ldots, \widehat{s}_{j+1,m_1}^1)$ forms an AT-system with respect to $(n_{1,j}, \ldots, n_{1,m_1})$ on any interval disjoint from Δ_{j+1}^1. Therefore

$$\mathcal{A}_{\mathbf{n},j}(x) = a_{\mathbf{n},j}(x) + a_{\mathbf{n},j+1}(x)\widehat{s}_{j+1,j+1}^1(x) + \cdots + a_{\mathbf{n},m_1}(x)\widehat{s}_{j+1,m_1}^1(x)$$

has at most $N_{1,j} - 1$ zeros on any such interval. Obviously, the same is true for the polynomial $\mathcal{A}_{\mathbf{n},m_1} \equiv a_{\mathbf{n},m_1}$. This proves *a*).

Notice that $ds_{0,k}^2(x) = \widehat{s}_{1,k}^2(x)\, d\sigma_0^2(x)$. On the other hand, we can replace x^ν by any polynomial of degree $\leq n_{2,k} - 1$ inside the integral in (1.8). Set

$$\mathcal{B}_{\mathbf{n}_2}(z) = \sum_{k=0}^{m_2} b_{\mathbf{n}_2,k}(z)\widehat{s}_{1,k}^2(z), \quad \deg b_{\mathbf{n}_2,k} \leq n_{2,k} - 1, \quad k = 0, \ldots, m_2,$$

($\widehat{s}^2_{1,0}(z) \equiv 1$). Then (1.8) is equivalent to

$$\int \mathcal{B}_{\mathbf{n}_2}(x)\, \mathcal{A}_{\mathbf{n},0}(x)\, d\sigma_0^2(x) = 0, \tag{2.2}$$

for all $\mathcal{B}_{\mathbf{n}_2}$ as indicated.

Suppose that $\mathcal{A}_{\mathbf{n},0}$ has $N < |\mathbf{n}_1| - 1 = |\mathbf{n}_2|$ sign changes on the interval Δ_0^2. Choose polynomials $b_{\mathbf{n}_2,k}$ conveniently so that $\mathcal{B}_{\mathbf{n}_2}$ changes sign exactly at those points where $\mathcal{A}_{\mathbf{n},0}$ changes sign on Δ_0^2 and has a zero of order $|\mathbf{n}_2| - 1 - N$ at one of the extreme points of $\Delta_0^1 = \Delta_0^2$. By Lemma 1.2.3, the linear form $\mathcal{B}_{\mathbf{n}_2}$ has on Δ_0^2 at most $|\mathbf{n}_2| - 1$ zeros, thus it can only have those zeros which we have assigned to it. The continuous function $\mathcal{B}_{\mathbf{n}_2}\mathcal{A}_{\mathbf{n},0}$ has constant sign on Δ_0^2. This contradicts (2.2). □

We have proved that $\mathcal{A}_{\mathbf{n},0}$ has $|\mathbf{n}_1| - 1$ zeros with odd multiplicity in the interior of $\Delta_0^2 = \Delta_0^1$. In short, we shall see that $\mathcal{A}_{\mathbf{n},0}$ has no other zeros in $\mathbb{C} \setminus \Delta_1^1$ and that they are all simple. Before proving this, let us turn to the question of normality.

Proposition 2.1.2. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$, $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, *and* $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$, *be given. Then,* $\mathbf{n}$ *is normal and* $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1})$ *is uniquely determined except for a constant factor.*

Proof. Assume that there exists $j \in \{0, \ldots, m_1\}$ such that $\deg a_{\mathbf{n},j} \leq n_{1,j} - 2$. Then $\mathbf{n}_1 - \mathbf{e}^j \in \mathbb{Z}_+^{m_1+1}(*)$, where $\mathbf{e}^j$ denotes the $m_1 + 1$ dimensional unit vector with all components equal to zero except the component $j + 1$ which equals 1. According to Lemma 1.2.3 applied to $\mathbf{n}_1 - \mathbf{e}^j$, the linear form $\mathcal{A}_{\mathbf{n},0}$ has at most $|\mathbf{n}_1| - 2$ zeros on Δ_0^1, but from Lemma 2.1.1, we know that $\mathcal{A}_{\mathbf{n},0}$ has at least $|\mathbf{n}_1| - 1$ sign changes on this interval. This contradiction yields that for all $j \in \{0, \ldots, m_1\}$, $\deg a_{\mathbf{n},j} = n_{1,j} - 1$, which implies normality.

Now, let us assume that $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1})$ and $(a^*_{\mathbf{n},0}, \ldots, a^*_{\mathbf{n},m_1})$ solve i)-ii) and these vectors are not collinear. According to what we just proved, for all $j \in \{0, \ldots, m_1\}$, $\deg a_{\mathbf{n},j} = \deg a^*_{\mathbf{n},j} = n_{1,j} - 1$. Take $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\deg(a_{\mathbf{n},0} - \lambda\, a^*_{\mathbf{n},0}) \leq n_{1,0} - 2$. The vector $(a_{\mathbf{n},0} - \lambda\, a^*_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1} - \lambda\, a^*_{\mathbf{n},m_1})$ is not identically equal to zero and also solves i)-ii). This is not possible since all non-trivial solutions must have all components of maximal degree. □

Proposition 2.1.2 allows us to determine the "monic" $(a_{\mathbf{n},0}, a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m_1})$ uniquely and we impose this normalization. The next lemma will be used on several occasions.

Lemma 2.1.3. *Let* $s_k, k = 1, \ldots, m$, *be finite signed Borel measures such that* $\mathrm{Co}(\mathrm{supp}(s_k)) = \Delta \subset \mathbb{R}$. *Let* $F(z) = f_0(z) + \sum_{k=1}^m f_k(z)\widehat{s}_k(z) \in H(\overline{\mathbb{C}} \setminus \Delta)$, *where* $f_k \in H(V), k = 0, \ldots, m$, *and* V *is a neighborhood of* Δ. *If* $F(z) = \mathcal{O}(1/z^2), z \to \infty$, *then*

$$\sum_{k=1}^m \int f_k(x)\, ds_k(x) = 0 \tag{2.3}$$

and $F(z) = \mathcal{O}(1/z), z \to \infty$, *implies that*

$$F(z) = \sum_{k=1}^{m} \int \frac{f_k(x)\, ds_k(x)}{z-x}. \tag{2.4}$$

Proof. Let $\Gamma \subset V$ be a closed smooth Jordan curve that surrounds Δ. If $F(z) = \mathcal{O}(1/z^2), z \to \infty$, from Cauchy's theorem, Fubini's theorem and Cauchy's integral formula, it follows that

$$0 = \int_\Gamma F(z)\, dz = \sum_{k=1}^{m} \int_\Gamma f_k(z)\, \widehat{s}_k(z)\, dz$$

$$= \sum_{k=1}^{m} \int \int_\Gamma \frac{f_k(z)\, dz}{z-x}\, ds_k(x) = 2\pi i \sum_{k=1}^{m} \int f_k(x)\, ds_k(x),$$

and we obtain (2.3). On the other hand, if $F(z) = \mathcal{O}(1/z), z \to \infty$, and we assume that z is in the unbounded connected component of the complement of Γ, Cauchy's integral formula and Fubini's theorem render

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)\, d\zeta}{z-\zeta} = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_\Gamma \frac{f_k(\zeta)\, \widehat{s}_k(\zeta)\, d\zeta}{z-\zeta} =$$

$$\sum_{k=1}^{m} \int \frac{1}{2\pi i} \int_\Gamma \frac{f_k(\zeta)\, d\zeta}{(z-\zeta)(\zeta-x)} ds_k(x) = \sum_{k=1}^{m} \int \frac{f_k(x)\, ds_k(x)}{z-x}$$

which is (2.4). □

Remark 2.1.4. *If we assume that* $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *and* $n_{1,m_1} \geq 1$, *then by Proposition* 2.1.2 *we know that all the polynomials* $a_{\mathbf{n},j}$, $j = 0, \ldots, m_1$, *are nonzero. Therefore* ∞ *is not a zero of any of the linear forms* $\mathcal{A}_{\mathbf{n},j}$, $j = 0, \ldots, m_1$. *Though it is not the case, in principle, some of these linear forms may have an infinite number of zeros which accumulate on the boundary of the corresponding region of meromorphy.*

Recall that for $j \in \{0, \ldots, m_1\}$, $Q_{\mathbf{n},j}$ denotes the monic polynomial whose zeros are those of the linear form $\mathcal{A}_{\mathbf{n},j}$ in the region $\mathbb{C} \setminus \Delta_{j+1}^1$, counting multiplicities ($\Delta_{m_1+1}^1 = \emptyset$), and in the case that $\mathcal{A}_{\mathbf{n},j}$ had infinitely many zeros then $Q_{\mathbf{n},j}$ denotes a formal infinite product. The next proposition is adapted from [43].

Proposition 2.1.5. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *be a multi-index such that* $n_{1,m_1} \geq 1$, *and let* $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ *be given (recall that* $\sigma_0^1 = \sigma_0^2$*). Then,* $\deg Q_{\mathbf{n},j} = N_{1,j} - 1, j = 0, 1 \ldots, m_1$, *where* $N_{1,j}$ *is given by* (2.1)*, and all the zeros of* $Q_{\mathbf{n},j}$ *are simple and lie in the interior of* Δ_j^1. *If I* *denotes the closure of a connected component of* $\Delta_j^1 \setminus \operatorname{supp}(\sigma_j^1)$*, then* $Q_{\mathbf{n},j}$ *has at most one zero in* I*. Moreover,*

$$\int x^\nu \mathcal{A}_{\mathbf{n},j}(x) \frac{d\sigma_j^1(x)}{Q_{\mathbf{n},j-1}(x)} = 0, \qquad \nu = 0, \ldots, N_{1,j} - 2, \qquad j = 1, \ldots, m_1, \quad (2.5)$$

and for any polynomial $q, \deg q \le N_{1,j+1} - 1$*,*

$$\frac{q(z)\mathcal{A}_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} = \int \frac{q(x)\mathcal{A}_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{d\sigma_{j+1}^1(x)}{z - x}, \qquad j = 0, \ldots, m_1 - 1. \qquad (2.6)$$

Proof. Using induction on j, we will prove simultaneously the general statement concerning the zeros and (2.5). Then, we prove that on any interval I there is at most one zero of $Q_{\mathbf{n},j}$. Finally, we obtain (2.6). For $j = 0$, we already know by Lemma 2.1.1 that $\mathcal{A}_{\mathbf{n},0}$ has $N_{1,0} - 1 = |\mathbf{n}_1| - 1$ sign changes in the interior of $\Delta_0^1 = \Delta_0^2$. Therefore, $\deg Q_{\mathbf{n},0} \ge N_{1,0} - 1$. If $\deg Q_{\mathbf{n},0} = N_{1,0} - 1$ we conclude with the initial step.

Suppose that $\deg Q_{\mathbf{n},0} \ge N_{1,0}$ (including the possible case that $\deg Q_{\mathbf{n},0} = \infty$). It is easy to see that $\mathcal{A}_{\mathbf{n},0}(\overline{z}) = \overline{\mathcal{A}_{\mathbf{n},0}(z)}$, so the zeros of $Q_{\mathbf{n},0}$ come in conjugate pairs. Therefore, we can choose $N_{1,0}$ (or $N_{1,0} + 1$ if necessary) zeros of $Q_{\mathbf{n},0}$ in such a way that the monic polynomial $Q_{\mathbf{n},0}^*$ with this set of zeros has constant sign on Δ_1^1 ($\Delta_1^1 \cap \Delta_0^1 = \emptyset$). Notice that

$$\frac{\mathcal{A}_{\mathbf{n},0}}{Q_{\mathbf{n},0}^*} \in H(\overline{\mathbb{C}} \setminus \Delta_1^1)$$

is analytic in the indicated region and

$$\frac{z^\nu \mathcal{A}_{\mathbf{n},0}}{Q_{\mathbf{n},0}^*} = \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty, \qquad \nu = 0, \ldots, N_{1,1} - 1.$$

From (2.3), we get

$$0 = \int x^\nu \mathcal{A}_{\mathbf{n},1}(x) \frac{d\sigma_1^1(x)}{Q_{\mathbf{n},0}^*(x)}, \qquad \nu = 0, \ldots, N_{1,1} - 1.$$

This implies that $\mathcal{A}_{\mathbf{n},1}$ has at least $N_{1,1}$ zeros on Δ_1^1. According to Lemma 2.1.1 this linear form can only have $N_{1,1} - 1$ zeros on this interval. Consequently, our initial assumption is false and $\deg Q_{\mathbf{n},0} = N_{1,0} - 1$.

Suppose that we have proved that for some $j \in \{0, \ldots, m_1 - 1\}$, $\deg Q_{\mathbf{n},j} = N_{1,j} - 1$, all its zeros are simple and lie in the interior of Δ_j^1. Let us show that then, (2.5) and the statement concerning the zeros are valid for $j + 1$.

Indeed, the induction hypothesis implies that

$$\frac{\mathcal{A}_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in H(\overline{\mathbb{C}} \setminus \Delta_{j+1}^1), \qquad \frac{z^\nu \mathcal{A}_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^2}\right), \ z \to \infty, \quad \nu = 0, \ldots, N_{1,j+1} - 2.$$

From (2.3), it follows that

$$0 = \int x^\nu \mathcal{A}_{\mathbf{n},j+1}(x) \frac{d\sigma_{j+1}^1(x)}{Q_{\mathbf{n},j}(x)}, \qquad \nu = 0, \ldots, N_{1,j+1} - 2.$$

We have obtained (2.5) for $j+1$.

Formula (2.5) for $j+1$ implies that $Q_{\mathbf{n},j+1}$ has at least $N_{1,j+1}-1$ sign changes in the interior of Δ^1_{j+1}. If $\deg Q_{\mathbf{n},j+1} = N_{1,j+1}-1$, we have finished the proof (for example, this is the case when $j+1 = m_1$ because $\mathcal{A}_{\mathbf{n},m_1} \equiv a_{\mathbf{n},m_1}$). Let us suppose that $\deg Q_{\mathbf{n},j+1} \geq N_{1,j+1}$ (including the possible case that $\deg Q_{\mathbf{n},j+1} = \infty$, and of course $j \leq m_1 - 2$). Since $\mathcal{A}_{\mathbf{n},j+1}(\overline{z}) = \overline{\mathcal{A}_{\mathbf{n},j+1}(z)}$, we can choose $N_{1,j+1}$ (or $N_{1,j+1}+1$ if necessary) zeros of $Q_{\mathbf{n},j+1}$ so that the monic polynomial $Q^*_{\mathbf{n},j+1}$ with this set of zeros has constant sign on Δ^1_{j+2}. Then

$$\frac{\mathcal{A}_{\mathbf{n},j+1}}{Q^*_{\mathbf{n},j+1}} \in H(\overline{\mathbb{C}} \setminus \Delta^1_{j+2}), \quad \frac{z^\nu \mathcal{A}_{\mathbf{n},j+1}}{Q^*_{\mathbf{n},j+1}} = \mathcal{O}\left(\frac{1}{z^2}\right), z \to \infty, \quad \nu = 0, \ldots, N_{1,j+2}-1.$$

Using (2.3), it follows that

$$0 = \int x^\nu \mathcal{A}_{\mathbf{n},j+2}(x) \frac{d\sigma^1_{j+2}(x)}{Q^*_{\mathbf{n},j+1}(x)}, \qquad \nu = 0, \ldots, N_{1,j+2}-1.$$

This implies that $\mathcal{A}_{\mathbf{n},j+2}$ has at least $N_{1,j+2}$ zeros on Δ^1_{j+2}. According to Lemma 2.1.1 this linear form can only have $N_{1,j+2}-1$ zeros on this interval. This implies that our initial assumption is false; therefore, $\deg Q_{\mathbf{n},j+1} = N_{1,j+1}-1$ as stated.

Suppose that the interval I contains two zeros x_1, x_2 of $Q_{\mathbf{n},j}$; that is, of $\mathcal{A}_{\mathbf{n},j}$. According to (2.5)

$$\int x^\nu \frac{\mathcal{A}_{\mathbf{n},j}(x)}{(x-x_1)(x-x_2)} \frac{(x-x_1)(x-x_2)\, d\sigma^1_j(x)}{Q_{\mathbf{n},j-1}(x)} = 0, \qquad \nu = 0, \ldots, N_{1,j}-2.$$

The function $\mathcal{A}_{\mathbf{n},j}(x)/(x-x_1)(x-x_2)$ has $N_{1,j}-3$ sign changes on $\mathrm{supp}(\sigma^1_j)$, but notice that the measure $(x-x_1)(x-x_2)\, d\sigma^1_j(x)/Q_{\mathbf{n},j-1}(x)$ has constant sign on $\mathrm{supp}(\sigma^1_j)$. This is impossible because of the number of orthogonality relations.

Formula (2.6) follows from (2.4) since for any polynomial q such that $\deg q \leq N_{1,j+1}-1$, we have

$$\frac{q\mathcal{A}_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in H(\overline{\mathbb{C}} \setminus \Delta^1_{j+1}), \qquad \frac{q\mathcal{A}_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty.$$

With this we conclude the proof. □

Now we turn to the analysis of the orthogonality relations satisfied by the linear forms $\mathcal{A}_{\mathbf{n},-j}$, $j = 0, \ldots, m_2$. We start with the following result.

Proposition 2.1.6. *Let* $\mathbf{n} \in \mathbb{Z}^{m_1+1}_+(\bullet) \times \mathbb{Z}^{m_2+1}_+(\bullet)$ *be a multi-index such that* $n_{1,m_1} \geq 1$*, and let* $S^1 = \mathcal{N}(\sigma^1_0, \ldots, \sigma^1_{m_1})$ *and* $S^2 = \mathcal{N}(\sigma^2_0, \ldots, \sigma^2_{m_2})$ *be given. Then, for each* $j = 0, \ldots, m_2$

$$\int x^\nu \mathcal{A}_{\mathbf{n},-j}(x)\, ds^2_{j,k}(x) = 0, \qquad k = j, \ldots, m_2, \qquad \nu = 0, \ldots, n_{2,k}-1. \quad (2.7)$$

Proof. When $j = 0$ the statement reduces to the relations ii) which define $\mathcal{A}_{\mathbf{n},0}$. If $m_2 = 0$ we are done. Therefore, let us assume that $m_2 \geq 1$, that (2.7) holds for some $j \in \{0, \ldots, m_2 - 1\}$, and prove that it is also satisfied for $j + 1$.

Fix $j \in \{0, \ldots, m_2 - 1\}, k \in \{j+1, \ldots, m_2\}$, and $\nu \in \{0, \ldots, n_{2,k} - 1\}$. Using the definition (1.12) of $\mathcal{A}_{\mathbf{n},-j-1}$, Fubini's theorem, and the induction hypothesis, we obtain

$$\int x^\nu \mathcal{A}_{\mathbf{n},-j-1}(x)\, ds^2_{j+1,k}(x) = \int x^\nu \int \frac{\mathcal{A}_{\mathbf{n},-j}(t)}{x-t}\, d\sigma^2_j(t)\, ds^2_{j+1,k}(x) =$$

$$\int \mathcal{A}_{\mathbf{n},-j}(t) \int \frac{x^\nu \mp t^\nu}{x-t}\, ds^2_{j+1,k}(x)\, d\sigma^2_j(t) =$$

$$\int p_\nu(t)\, \mathcal{A}_{\mathbf{n},-j}(t)\, d\sigma^2_j(t) - \int t^\nu \mathcal{A}_{\mathbf{n},-j}(t)\, ds^2_{j,k}(t) = 0$$

since p_ν is a polynomial of degree $\leq n_{2,k} - 2$, and $n_{2,j+1} \geq n_{2,k}$. □

Observe that taking linear combinations of the relations (2.7), we obtain

$$\int \mathcal{B}_{\mathbf{n}_2,j}(x)\, \mathcal{A}_{\mathbf{n},-j}(x)\, d\sigma^2_j(x) = 0, \qquad j = 0, \ldots, m_2,$$

where $\mathcal{B}_{\mathbf{n}_2,j}$ is an arbitrary linear form of type

$$\mathcal{B}_{\mathbf{n}_2,j}(x) = \sum_{k=j}^{m_2} b_k(x)\, \widehat{s}^2_{j+1,k}(x), \qquad \deg b_k \leq n_{2,k} - 1.$$

Arguing exactly as in the proof of part *b*) from Lemma 2.1.1, it follows that $\mathcal{A}_{\mathbf{n},-j}$ has at least $N_{2,j}$ sign changes on $\Delta^2_j = \mathrm{Co}(\mathrm{supp}(\sigma^2_j))$, where

$$N_{2,j} = N_{2,j}(\mathbf{n}) = n_{2,j} + \cdots + n_{2,m_2}, \qquad j = 0, \ldots, m_2. \tag{2.8}$$

Recall that for $j = 1, \ldots, m_2$, $Q_{\mathbf{n},-j}$ was defined as the monic polynomial whose zeros are those of $\mathcal{A}_{\mathbf{n},-j}$ in the region $\mathbb{C} \setminus \Delta^2_{j-1}$. Consequently, $\deg Q_{\mathbf{n},-j} \geq N_{2,j}, j = 1, \ldots, m_2$. Also recall that for $j = 0$ we proved in Proposition 2.1.5 that $\deg Q_{\mathbf{n},0} = N_{2,0} = |\mathbf{n}_2| = |\mathbf{n}_1| - 1$, that the zeros of $Q_{\mathbf{n},0}$ are simple, and lie in the interior of $\Delta^2_0 = \Delta^1_0$.

Proposition 2.1.7. *Let $\mathbf{n} \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be a multi-index such that $n_{1,m_1} \geq 1$, and let $S^1 = \mathcal{N}(\sigma^1_0, \ldots, \sigma^1_{m_1})$, $S^2 = \mathcal{N}(\sigma^2_0, \ldots, \sigma^2_{m_2})$ be given. Then, $\deg Q_{\mathbf{n},-j} = N_{2,j}, j = 0, \ldots, m_2$, where $N_{2,j}$ is given by (2.8), all the zeros of $Q_{\mathbf{n},-j}$ are simple and lie in the interior of Δ^2_j, and $Q_{\mathbf{n},-m_2-1} \equiv 1$. If I denotes the closure of a connected component of $\Delta^2_j \setminus \mathrm{supp}(\sigma^2_j)$ then $Q_{\mathbf{n},-j}$ has at most one zero in I. Moreover, for each $j = 0, \ldots, m_2$, and $\nu = 0, \ldots, N_{2,j} - 1$,*

$$\int x^\nu \mathcal{A}_{\mathbf{n},-j}(x) \frac{d\sigma^2_j(x)}{Q_{\mathbf{n},-j-1}(x)} = 0, \tag{2.9}$$

and for any polynomial $q, \deg q \leq N_{2,j-1}$,

$$\frac{q(z)\mathcal{A}_{\mathbf{n},-j}(z)}{Q_{\mathbf{n},-j}(z)} = \int \frac{q(x)\mathcal{A}_{\mathbf{n},-j+1}(x)}{Q_{\mathbf{n},-j}(x)} \frac{d\sigma^2_{j-1}(x)}{z-x}, \qquad j = 1, \ldots, m_2 + 1. \tag{2.10}$$

Proof. Fix $j \in \{0, \dots, m_2\}$. From (2.7) we have that for each $q, \deg q \le n_{2,j}$,

$$\int \frac{q(z) - q(x)}{z - x} \mathcal{A}_{\mathbf{n},-j}(x)\, d\sigma_j^2(x) = 0.$$

It follows that

$$\mathcal{A}_{\mathbf{n},-j-1}(z) = \frac{1}{q(z)} \int \frac{q(x)}{z - x} \mathcal{A}_{\mathbf{n},-j}(x)\, d\sigma_j^2(x) = \mathcal{O}\left(1/z^{n_{2,j}+1}\right), \qquad z \to \infty.$$

We have shown that $\deg Q_{\mathbf{n},-j-1} \ge N_{2,j+1} (N_{2,m_2+1} = 0)$. The zeros of $Q_{\mathbf{n},-j-1}$ come in conjugate pairs since $\mathcal{A}_{\mathbf{n},-j-1}$ is also symmetric with respect to the real line. If $\deg Q_{\mathbf{n},-j-1} > N_{2,j+1}$ take $N_{2,j+1}+1$ (or $N_{2,j+1}+2$ if necessary) zeros from $Q_{\mathbf{n},-j-1}$ so that the monic polynomial $Q^*_{\mathbf{n},-j-1}$ with these zeros has constant sign on Δ_j^2. If $\deg Q_{\mathbf{n},-j-1} = N_{2,j+1}$ take $Q^*_{\mathbf{n},-j-1} = Q_{\mathbf{n},-j-1}$. Therefore,

$$\frac{\mathcal{A}_{\mathbf{n},-j-1}}{Q^*_{\mathbf{n},-j-1}} = \mathcal{O}\left(1/z^{n_{2,j}+\deg Q^*_{\mathbf{n},-j-1}+1}\right) \in H(\overline{\mathbb{C}} \setminus \Delta_j^2).$$

It follows that for all $\nu = 0, \dots, n_{2,j} + \deg Q^*_{\mathbf{n},-j-1} - 1$,

$$\frac{z^\nu \mathcal{A}_{\mathbf{n},-j-1}}{Q^*_{\mathbf{n},-j-1}} = \mathcal{O}\left(1/z^2\right) \in H(\overline{\mathbb{C}} \setminus \Delta_j^2), \qquad z \to \infty.$$

Using (2.3), we obtain

$$0 = \int x^\nu \mathcal{A}_{\mathbf{n},-j}(x) \frac{d\sigma_j^2(x)}{Q^*_{\mathbf{n},-j-1}(x)}, \qquad \nu = 0, \dots, n_{2,j} + \deg Q^*_{\mathbf{n},-j-1} - 1.$$

This formula implies that $\mathcal{A}_{\mathbf{n},-j}$ has at least $n_{2,j} + \deg Q^*_{\mathbf{n},-j-1} \ge N_{2,j}$ sign changes on Δ_j^2. In particular, we have proved that if for some $j \in \{0, \dots, m_2\}$, $\deg Q_{\mathbf{n},-j-1} > N_{2,j+1}$ then $\deg Q_{\mathbf{n},-j} > N_{2,j}$. Going downwards on the index j we would obtain that $\deg Q_{\mathbf{n},0} > N_{2,0} = |\mathbf{n}_2| = |\mathbf{n}_1| - 1$, which is false according to Proposition 2.1.5. Consequently, for all $j \in \{0, \dots, m_2\}, \deg Q_{\mathbf{n},-j-1} = N_{2,j+1}$ (in particular, $Q_{\mathbf{n},-m_2-1} \equiv 1$). Hence, $Q^*_{\mathbf{n},-j-1} = Q_{\mathbf{n},-j-1}$ and (2.9) follows. The proof that I contains at most one zero of $Q_{\mathbf{n},-j}$ is the same as in Proposition 2.1.5.

Now, fix $j \in \{1, \dots, m_2 + 1\}$. Notice that for any $q, \deg q \le N_{2,j-1}$,

$$\frac{q\mathcal{A}_{\mathbf{n},-j}}{Q_{\mathbf{n},-j}} \in H(\overline{\mathbb{C}} \setminus \Delta_{j-1}^2), \quad \frac{q\mathcal{A}_{\mathbf{n},-j}}{Q_{\mathbf{n},-j}} = \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

Using (2.4), (2.10) readily follows. □

2.2 Interlacing properties of the zeros of the functions $\mathcal{A}_{\mathbf{n},j}$

Fix a vector $l := (l_1; l_2)$ where $0 \le l_1 \le m_1$ and $0 \le l_2 \le m_2$. Given $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$, recall that $\mathbf{n}^l := (\mathbf{n}_1 + \mathbf{e}^{l_1}; \mathbf{n}_2 + \mathbf{e}^{l_2}) = (\mathbf{n}_1^{l_1}; \mathbf{n}_2^{l_2})$, where $\mathbf{e}^{l_i}$ denotes the unit vector of length m_i+1 with all components equal to zero except the component (l_i+1) which equals 1. In this section it is always assumed that both $\mathbf{n}$ and $\mathbf{n}^l$ belong to $\mathbb{Z}_+^{m_1+1}(\bullet)\times\mathbb{Z}_+^{m_2+1}(\bullet)$.

Fix real constants A,B such that $|A|+|B|>0$ and define

$$\mathcal{G}_{\mathbf{n},j} := A\mathcal{A}_{\mathbf{n},j} + B\mathcal{A}_{\mathbf{n}^l,j}, \qquad j=0,\ldots,m_1.$$

Before proving the interlacing property satisfied by the zeros of $\mathcal{A}_{\mathbf{n},j}$ and $\mathcal{A}_{\mathbf{n}^l,j}$, we need to analyze the zeros of these functions $\mathcal{G}_{\mathbf{n},j}$. Since $\deg a_{\mathbf{n}^l,l_1}=\deg a_{\mathbf{n},l_1}+1$ it is obvious that $\mathcal{G}_{\mathbf{n},j}\not\equiv 0, j\le l_1$. In particular, this is always true for $\mathcal{G}_{\mathbf{n},0}$.

Lemma 2.2.1. *Assume that $A,B\in\mathbb{R}, |A|+|B|>0$, and $n_{1,m_1}\ge 1$. Then for all $j\in\{0,\ldots,m_1\}$ such that $n_{1,j}\ge 2$, $\deg Aa_{\mathbf{n},j}+Ba_{\mathbf{n}^l,j}\ge n_{1,j}-2$ and $\mathcal{G}_{\mathbf{n},j}\not\equiv 0$.*

Proof. Assume that there exists $j\in\{0,\ldots,m_1\}$ such that $n_{1,j}\ge 2$ and $\deg Aa_{\mathbf{n},j}+Ba_{\mathbf{n}^l,j}\le n_{1,j}-3$ ($n_{1,j}-3=-1$ means that $Aa_{\mathbf{n},j}+Ba_{\mathbf{n}^l,j}\equiv 0$). Then $\mathbf{n}_1^{l_1}-2\,\mathbf{e}^j\in\mathbb{Z}_+^{m_1+1}(*)$, where $\mathbf{e}^j$ denotes the m_1+1 dimensional unit vector with all components equal to zero except the component $j+1$ which equals 1. According to Lemma 1.2.3 the linear form $\mathcal{G}_{\mathbf{n},0}$ has at most $|\mathbf{n}_1|-2$ zeros on Δ_0^1, but $\mathcal{G}_{\mathbf{n},0}$ satisfies the same orthogonality relations (1.8) as $\mathcal{A}_{\mathbf{n},0}$ and, therefore, it has at least $|\mathbf{n}_1|-1$ sign changes on this interval. This contradiction implies the statement. □

From this lemma it follows that if $n_{1,m_1}\ge 2$ then $\mathcal{G}_{\mathbf{n},j}\not\equiv 0, j\in\{0,\ldots,m_1\}$.

Lemma 2.2.2. *Assume that $A,B\in\mathbb{R}$ and $\mathcal{G}_{\mathbf{n},j}=A\mathcal{A}_{\mathbf{n},j}+B\mathcal{A}_{\mathbf{n}^l,j}\not\equiv 0$, for some $j\in\{0,\ldots,m_1\}$. If $j\le l_1$ then $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{1,j}$ zeros, counting multiplicities, on any interval disjoint from $\Delta_{j+1}^1(\Delta_{m_1+1}^1=\emptyset)$. If $j>l_1$ then $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{1,j}-1$ zeros, counting multiplicities, on any interval disjoint from Δ_{j+1}^1.*

Proof. We have

$$\mathcal{G}_{\mathbf{n},j}(z)=\sum_{k=j}^{m_1}(Aa_{\mathbf{n},k}(z)+Ba_{\mathbf{n}^l,k}(z))\widehat{s}_{j+1,k}^1(z),$$

where $\deg a_{\mathbf{n},k}=n_{1,k}-1$ and $\deg a_{\mathbf{n}^l,k}=n_{1,k}^{l_1}-1$. By Lemma 1.2.3, the functions $(1,\widehat{s}_{j+1,j+1}^1,\ldots,\widehat{s}_{j+1,m_1}^1)$ form an AT-system with respect to $(n_{1,j}^{l_1},\ldots,n_{1,m_1}^{l_1})$ on any interval disjoint from Δ_{j+1}^1, and the result follows immediately. □

Notice that for each $j\in\{0,\ldots,m_1\}$, $\mathcal{G}_{\mathbf{n},j}$ is a real function when it is restricted to the real line.

Proposition 2.2.3. *Let $n_{1,m_1}\ge 1$. Assume that $A,B\in\mathbb{R}, |A|+|B|>0$, and let $\kappa=\max\{k':\mathcal{G}_{\mathbf{n},k'}\not\equiv 0\}\le m_1$. Then, $\kappa\ge l_1$ and $\mathcal{G}_{\mathbf{n},j}\equiv 0, \kappa<j\le m_1$. If $j\le l_1$ then $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{1,j}$ zeros in $\mathbb{C}\setminus\Delta_{j+1}^1$, counting multiplicities, and at least $N_{1,j}-1$ sign changes in the interior of Δ_j^1. If $l_1<j\le\kappa$ then $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{1,j}-1$ zeros in $\mathbb{C}\setminus\Delta_{j+1}^1$ and at least $N_{1,j}-2$ sign changes in the interior of Δ_j^1. Therefore, all the zeros of $\mathcal{G}_{\mathbf{n},j}$ in $\mathbb{C}\setminus\Delta_{j+1}^1$ are real and simple.*

Proof. If $j \leq l_1$, since $\deg a_{\mathbf{n}^l,l_1} > \deg a_{\mathbf{n},l_1}$ it follows that $\mathcal{G}_{\mathbf{n},j} \not\equiv 0$. Consequently, $\kappa \geq l_1$. Obviously, from the definition of κ, $\mathcal{G}_{\mathbf{n},j} \equiv 0, \kappa < j \leq m_1$.

Assume that $\mathcal{G}_{\mathbf{n},j}, j \leq l_1$, has at least $N_{1,j}+1$ zeros in $\mathbb{C} \setminus \Delta^1_{j+1}$, counting multiplicities. Select $N_{1,j}+1$ or $N_{1,j}+2$ zeros of $\mathcal{G}_{\mathbf{n},j}$ which are symmetric with respect to the real axis, and let $Q^*_{\mathbf{n},j}$ be the monic polynomial whose zeros are those prescribed. If $j < l_1$ then

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},j}}{Q^*_{\mathbf{n},j}} = \mathcal{O}\Big(\frac{1}{z^2}\Big), \qquad z \to \infty, \qquad \nu = 0, \dots, N_{1,j+1}.$$

From (2.3), it follows that

$$0 = \int x^\nu \mathcal{G}_{\mathbf{n},j+1}(x) \frac{d\sigma^1_{j+1}(x)}{Q^*_{\mathbf{n},j}(x)}, \qquad \nu = 0, \dots, N_{1,j+1}.$$

These orthogonality relations imply that $\mathcal{G}_{\mathbf{n},j+1}$ has at least $N_{1,j+1}+1$ zeros on Δ^1_{j+1}. Since $\mathcal{G}_{\mathbf{n},j+1} \not\equiv 0$ we obtain a contradiction with Lemma 2.2.2.

If $j = l_1$ and $j < \kappa$, then

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},l_1}}{Q^*_{\mathbf{n},l_1}} = \mathcal{O}\Big(\frac{1}{z^2}\Big), \qquad z \to \infty, \qquad \nu = 0, \dots, N_{1,l_1+1} - 1.$$

Arguing as before, it follows that $\mathcal{G}_{\mathbf{n},l_1+1}$ has at least N_{1,l_1+1} zeros on $\Delta^1_{l_1+1}$, contradicting Lemma 2.2.2. If $j = l_1 = \kappa$ then $\mathcal{G}_{\mathbf{n},l_1+1} \equiv 0$ and $\mathcal{G}_{\mathbf{n},l_1} = Aa_{\mathbf{n},l_1} + Ba_{\mathbf{n}^l,l_1}$ is a polynomial of degree at most $n_{1,l_1} < N_{1,l_1}+1$ and thus it is identically equal to zero which is impossible. Consequently, when $j \leq l_1$, $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{1,j}$ zeros in $\mathbb{C} \setminus \Delta^1_{j+1}$ counting multiplicities.

Let $l_1 < j \leq \kappa$ and assume that $\mathcal{G}_{\mathbf{n},j}$ has at least $N_{1,j}$ zeros in $\mathbb{C} \setminus \Delta^1_{j+1}$, counting multiplicities. If $j = m_1$ we get immediately a contradiction because in this case $\mathcal{G}_{\mathbf{n},m_1}$ is a polynomial of degree at most $N_{1,m_1} - 1$. If $l_1 < j < m_1$, then there exists a polynomial $Q^*_{\mathbf{n},j}$ with real coefficients and degree at least $N_{1,j}$ such that

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},j}}{Q^*_{\mathbf{n},j}} = \mathcal{O}\Big(\frac{1}{z^2}\Big), \qquad z \to \infty, \qquad \nu = 0, \dots, N_{1,j+1} - 1.$$

This implies that $\mathcal{G}_{\mathbf{n},j+1}$ has at least $N_{1,j+1}$ zeros on Δ^1_{j+1} contradicting Lemma 2.2.2.

Now, let us analyze the sign changes. Notice that $\mathcal{G}_{\mathbf{n},0} \not\equiv 0$. Assume that $\mathcal{G}_{\mathbf{n},0}$ has $N < N_{1,0} - 1 = |\mathbf{n}_1| - 1 = |\mathbf{n}_2|$ sign changes on $\Delta^1_0 = \Delta^2_0$, choose a nonzero linear form

$$\mathcal{B}_{\mathbf{n}_2}(z) = \sum_{k=0}^{m_2} b_{\mathbf{n}_2,k}(z)\widehat{s}^2_{1,k}(z)\,, \quad \deg b_{\mathbf{n}_2,k} \leq n_{2,k} - 1\,, \quad k = 0, \dots, m_2\,,$$

such that $\mathcal{B}_{\mathbf{n}_2}$ has a simple zero at each point where $\mathcal{G}_{\mathbf{n},0}$ has a sign change, and a zero of order $|\mathbf{n}_2| - 1 - N$ at one of the extreme points of Δ^2_0. By Lemma

1.2.3, $\mathcal{B}_{\mathbf{n}_2}$ has at most $|\mathbf{n}_2|-1$ zeros on Δ_0^2. Thus, $\mathcal{B}_{\mathbf{n}_2}$ has exactly those zeros prescribed. By definition,

$$\int \mathcal{B}_{\mathbf{n}_2}(x)\,\mathcal{G}_{\mathbf{n},0}(x)\,d\sigma_0^2(x)=0,$$

which contradicts the fact that $\mathcal{B}_{\mathbf{n}_2}(x)\mathcal{G}_{\mathbf{n},0}(x)$ has constant sign on Δ_0^2.

Let us prove by induction that for all $j\leq l_1$, $\mathcal{G}_{\mathbf{n},j}$ has at least $N_{1,j}-1$ sign changes in the interior of Δ_j^1. For $j=0$ this was proved above and if $l_1=0$ we are done. Let us assume that for some $j<l_1$, $\mathcal{G}_{\mathbf{n},j}$ has at least $N_{1,j}-1$ sign changes on Δ_j^1, and let us show that $\mathcal{G}_{\mathbf{n},j+1}$ has at least $N_{1,j+1}-1$ sign changes on Δ_{j+1}^1.

Let $Q_{\mathbf{n},j}^*$ be a monic polynomial whose zeros are $N_{1,j}-1$ points where $\mathcal{G}_{\mathbf{n},j}$ has a sign change. Then

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},j}}{Q_{\mathbf{n},j}^*}=\mathcal{O}\Big(\frac{1}{z^2}\Big),\qquad z\to\infty,\qquad \nu=0,\ldots,N_{1,j+1}-2\,.$$

Using (2.3), this implies that

$$0=\int x^\nu \mathcal{G}_{\mathbf{n},j+1}(x)\frac{d\sigma_{j+1}^1(x)}{Q_{\mathbf{n},j}^*(x)}\,,\qquad \nu=0,\ldots,N_{1,j+1}-2\,.$$

Thus, $\mathcal{G}_{\mathbf{n},j+1}$ has at least $N_{1,j+1}-1$ sign changes in the interior of Δ_{j+1}^1 as claimed.

Finally, we prove that $\mathcal{G}_{\mathbf{n},j}, l_1<j\leq\kappa$, has at least $N_{1,j}-2$ sign changes in the interior of Δ_j^1. Let $Q_{\mathbf{n},l_1}^*$ be a monic polynomial of degree $N_{1,l_1}-1$ whose zeros are points where $\mathcal{G}_{\mathbf{n},l_1}$ changes sign in the interior of $\Delta_{l_1}^1$, then

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},l_1}}{Q_{\mathbf{n},l_1}^*}=\mathcal{O}\Big(\frac{1}{z^2}\Big),\qquad z\to\infty,\qquad \nu=0,\ldots,N_{1,l_1+1}-3.$$

From here we get orthogonality conditions that imply that $\mathcal{G}_{\mathbf{n},l_1+1}$ has at least $N_{1,l_1+1}-2$ sign changes in the interior of $\Delta_{l_1+1}^1$. One proceeds the same way until we arrive to $j=\kappa$.

From the upper bound on the number of zeros and the lower bound on the number of sign changes it follows that all the zeros are simple and lie on the real line. □

Let $j\in\{0,\ldots,m_2+1\}$. Given two real constants A,B, we define

$$\mathcal{G}_{\mathbf{n},-j}:=A\mathcal{A}_{\mathbf{n},-j}+B\mathcal{A}_{\mathbf{n}^l,-j}.$$

Thus, by (1.12),

$$\mathcal{G}_{\mathbf{n},-j-1}(z)=\int\frac{\mathcal{G}_{\mathbf{n},-j}(x)}{z-x}d\sigma_j^2(x),\qquad j=0,\ldots,m_2. \tag{2.11}$$

If $|A|+|B|>0$ then $\mathcal{G}_{\mathbf{n},0}\not\equiv 0$ and from (2.11) it follows that $\mathcal{G}_{\mathbf{n},-j}\not\equiv 0$ for all $j\in\{1,\ldots,m_2+1\}$.

Proposition 2.2.4. *Let $A, B \in \mathbb{R}, |A| + |B| > 0$. For every $j \in \{1, \ldots, m_2\}$, $\mathcal{G}_{\mathbf{n},-j}$ has at most $N_{2,j} + 1$ zeros on $\mathbb{C} \setminus \Delta^2_{j-1}$, counting multiplicities, and at least $N_{2,j}$ sign changes in the interior of Δ^2_j. Hence, all the zeros of $\mathcal{G}_{\mathbf{n},-j}$ on $\mathbb{C} \setminus \Delta^2_{j-1}$ are real and simple.*

Proof. Let $j \in \{0, \ldots, m_2\}$. By (2.7) we know that

$$\int x^\nu \mathcal{A}_{\mathbf{n}^l,-j}(x)\, ds^2_{j,k}(x) = 0, \qquad k = j, \ldots, m_2, \qquad \nu = 0, \ldots, n^{l_2}_{2,k} - 1.$$

Since $n_{2,k} \leq n^{l_2}_{2,k}$, it follows that

$$\int x^\nu \mathcal{G}_{\mathbf{n},-j}(x)\, ds^2_{j,k}(x) = 0, \qquad k = j, \ldots, m_2, \qquad \nu = 0, \ldots, n_{2,k} - 1. \quad (2.12)$$

Using the same arguments employed in the previous section to show that $\mathcal{A}_{\mathbf{n},-j}$ has at least $N_{2,j}$ sign changes in the interior of Δ^2_j (see the comments before Proposition 2.1.7), one obtains the same conclusion for $\mathcal{G}_{\mathbf{n},-j}$.

If q is a polynomial with $\deg q \leq n_{2,j}$, then from (2.12) we have

$$\int \frac{q(z) - q(x)}{z - x}\, \mathcal{G}_{\mathbf{n},-j}(x)\, d\sigma^2_j(x) = 0.$$

Hence, for every $j \in \{0, \ldots, m_2\}$,

$$\mathcal{G}_{\mathbf{n},-j-1}(z) = \frac{1}{q(z)} \int \frac{q(x)}{z - x}\, \mathcal{G}_{\mathbf{n},-j}(x)\, d\sigma^2_j(x) = \mathcal{O}\Big(\frac{1}{z^{n_{2,j}+1}}\Big), \quad z \to \infty.$$

Assume that for some $j \in \{0, \ldots, m_2 - 1\}$, $\mathcal{G}_{\mathbf{n},-j-1}$ has at least $N_{2,j+1} + 2$ zeros, counting multiplicities, on $\mathbb{C} \setminus \Delta^2_j$. Select at least $N_{2,j+1} + 2$ zeros of $\mathcal{G}_{\mathbf{n},-j-1}$, symmetric with respect to the real axis, and denote by $Q^*_{\mathbf{n},-j-1}$ the monic polynomial whose zeros are the points selected. Then,

$$\frac{z^\nu \mathcal{G}_{\mathbf{n},-j-1}}{Q^*_{\mathbf{n},-j-1}} = \mathcal{O}\Big(\frac{1}{z^2}\Big), \qquad z \to \infty, \qquad \nu = 0, \ldots, N_{2,j} + 1.$$

As before, this implies that $\mathcal{G}_{\mathbf{n},-j}$ has at least $N_{2,j} + 2$ zeros in the interior of Δ^2_j. Going downwards on the index j, we obtain that $\mathcal{G}_{\mathbf{n},0}$ has at least $N_{2,0} + 2 = N_{1,0} + 1$ zeros, which is impossible by Proposition 2.2.3. Therefore, for all $j \in \{1, \ldots, m_2 + 1\}$, $\mathcal{G}_{\mathbf{n},j}$ has at most $N_{2,j} + 1$ zeros in $\mathbb{C} \setminus \Delta^2_{j-1}$ and, therefore, they must be real and simple. □

We are now ready to prove the interlacing property satisfied by the zeros of $\mathcal{A}_{\mathbf{n},j}$ and $\mathcal{A}_{\mathbf{n}^l,j}$.

Theorem 2.2.5. *Let $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}^{m_1+1}_+(\bullet) \times \mathbb{Z}^{m_2+1}_+(\bullet), n_{1,m_1} \geq 2$. Then, for all $j \in \{-m_2, \ldots, m_1\}$ the zeros of $\mathcal{A}_{\mathbf{n},j}$ and $\mathcal{A}_{\mathbf{n}^l,j}$ interlace; that is, between two consecutive zeros of $\mathcal{A}_{\mathbf{n},j}$ there is one zero of $\mathcal{A}_{\mathbf{n}^l,j}$ and viceversa.*

Proof. Since $n_{1,m_1} \geq 2$, from Lemma 2.2.1 we know that for all $j \in \{0,\ldots,m_1\}$ and for all A, B real such that $|A|+|B| > 0$, the linear form $\mathcal{G}_{\mathbf{n},j}$ is not identically equal to zero. This is always true for $j \in \{-m_2,\ldots,-1\}$. Therefore, from Propositions 2.2.3 and 2.2.4 we know that for all real A, B, such that $|A|+|B| > 0$ the zeros of $\mathcal{G}_{\mathbf{n},j}, j \in \{-m_2,\ldots,m_1\}$, are real and simple. This is the basic fact we will use in the proof.

Fix $y \in \mathbb{R} \setminus \Delta^1_{j+1}$. It cannot occur that $\mathcal{A}_{\mathbf{n},j}(y) = \mathcal{A}_{\mathbf{n}^l,j}(y) = 0$. If so, y would be a simple zero of $\mathcal{A}_{\mathbf{n},j}$ and $\mathcal{A}_{\mathbf{n}^l,j}$. Thus, $\mathcal{A}'_{\mathbf{n},j}(y) \neq 0$ and $\mathcal{A}'_{\mathbf{n}^l,j}(y) \neq 0$. Take $A = 1$ and $B = -\mathcal{A}'_{\mathbf{n},j}(y)/\mathcal{A}'_{\mathbf{n}^l,j}(y)$ and consider $\mathcal{G}_{\mathbf{n},j} = A\mathcal{A}_{\mathbf{n},j} + B\mathcal{A}_{\mathbf{n}^l,j}$. With this choice of A and B, we have

$$\mathcal{G}_{\mathbf{n},j}(y) = \mathcal{G}'_{\mathbf{n},j}(y) = 0\,,$$

and we obtain a contradiction because the zeros of $\mathcal{G}_{\mathbf{n},j}$ are simple.

Now, taking $A = \mathcal{A}_{\mathbf{n}^l,j}(y)$ and $B = -\mathcal{A}_{\mathbf{n},j}(y)$, we have that $|A| + |B| > 0$. Since

$$\mathcal{A}_{\mathbf{n}^l,j}(y)\mathcal{A}_{\mathbf{n},j}(y) - \mathcal{A}_{\mathbf{n},j}(y)\mathcal{A}_{\mathbf{n}^l,j}(y) = 0\,,$$

and the zeros on $\mathbb{R} \setminus \Delta^1_{j+1}$ of $\mathcal{A}_{\mathbf{n}^l,j}(y)\mathcal{A}_{\mathbf{n},j}(x) - \mathcal{A}_{\mathbf{n},j}(y)\mathcal{A}_{\mathbf{n}^l,j}(x)$ with respect to x are simple, it follows that

$$\mathcal{A}_{\mathbf{n}^l,j}(y)\mathcal{A}'_{\mathbf{n},j}(y) - \mathcal{A}_{\mathbf{n},j}(y)\mathcal{A}'_{\mathbf{n}^l,j}(y) \neq 0\,.$$

But $\mathcal{A}_{\mathbf{n}^l,j}(y)\mathcal{A}'_{\mathbf{n},j}(y) - \mathcal{A}_{\mathbf{n},j}(y)\mathcal{A}'_{\mathbf{n}^l,j}(y)$ is a continuous real function on $\mathbb{R} \setminus \Delta^1_{j+1}$ in y so it must have constant sign on each one of the connected components of $\mathbb{R} \setminus \Delta^1_{j+1}$. In particular, its sign on Δ^1_j is constant.

Evaluating $\mathcal{A}_{\mathbf{n}^l,j}(y)\mathcal{A}'_{\mathbf{n},j}(y) - \mathcal{A}_{\mathbf{n},j}(y)\mathcal{A}'_{\mathbf{n}^l,j}(y)$ at two consecutive zeros of $\mathcal{A}_{\mathbf{n}^l,j}$, since the sign of $\mathcal{A}'_{\mathbf{n}^l,j}$ at these two points changes, the sign of $\mathcal{A}_{\mathbf{n},j}$ must also change. Using Bolzano's theorem we find that there must be an intermediate zero of $\mathcal{A}_{\mathbf{n},j}$. Analogously, one proves that between two consecutive zeros of $\mathcal{A}_{\mathbf{n},j}$ on Δ^1_j there is one of $\mathcal{A}_{\mathbf{n}^l,j}$. Thus, the interlacing property has been proved. □

3. LOGARITHMIC ASYMPTOTICS

Here, we treat the logarithmic asymptotic of mixed type multiple orthogonal polynomials generated by two Nikishin systems. The first section is dedicated to the introduction of the main concepts of potential theory and some results from the scalar case which will be needed for our work. In Section 3.2, we study the asymptotic distribution of the zeros of the sequences of polynomials $\{Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}, j \in \{-m_2,\ldots,m_1\}$, proving Theorem 1.3.2. This paves the way to obtain Theorem 1.3.3 in Section 3.3 on the logarithmic asymptotics of the mixed type multiple orthogonal polynomials and their associated linear forms. The final section of this chapter considers some application of the results obtained to the asymptotic behavior of mixed type Hermite-Padé approximants.

3.1 Preliminaries and notation

If E is a compact subset of the complex plane, recall that $\mathcal{M}(E)$ denotes the class of all finite, positive, Borel measures with support consisting of an infinite set of points contained in E, and $\mathcal{M}_1(E)$ is the subclass of probability measures of $\mathcal{M}(E)$.

Recall that given a polynomial q_l of degree $l \geq 1$, we denote the associated normalized zero counting measure by

$$\mu_{q_l} = \frac{1}{l} \sum_{q_l(x)=0} \delta_x \,,$$

where δ_x is the Dirac measure with mass 1 at x (in the sum the zeros are repeated according to their multiplicity).

If $\mu \in \mathcal{M}(E)$, the *logarithmic potential* associated to μ is given by

$$V^{\mu}(z) = \int \log \frac{1}{|z-x|}\, d\mu(x)\,,$$

whereas the *logarithmic energy* of μ is defined as

$$I(\mu) = \int V^{\mu}(z)\, d\mu(z) = \int\int \log \frac{1}{|z-x|}\, d\mu(x)\, d\mu(z)\,.$$

The quantity

$$I(E) := \inf\{I(\mu) : \mu \in \mathcal{M}_1(E)\}$$

is known as the *energy* of E, and

$$\mathrm{cap}(E) := e^{-I(E)}$$

is known as the *logarithmic capacity* of E. For an arbitrary set A, the *interior capacity* of A is by definition

$$\mathrm{cap}(A) := \sup\{\mathrm{cap}(E) : \ E \subset A, \ \ E \text{ compact}\}\,.$$

We will only use the interior logarithmic capacity of a set; therefore, in the sequel we will refer to it simply as the capacity.

Lemma 3.1.1. *Let $E \subset \mathbb{C}$ be a compact set which is regular with respect to the Dirichlet problem and ϕ a continuous function on E. Then there exists a unique $\overline{\mu} \in \mathcal{M}_1(E)$ and a constant w such that*

$$V^{\overline{\mu}}(z) + \phi(z) \begin{cases} \leq w, & z \in \mathrm{supp}(\overline{\mu})\,, \\ \geq w, & z \in E\,. \end{cases}$$

If the compact set E is not regular with respect to the Dirichlet problem then the second part of the statement is true except on a set e such that $\mathrm{cap}(e) = 0$. Theorem I.1.3 in [77] contains a proof of this lemma in this context. When E is regular, it is well known that this inequality except for a set of capacity zero implies the inequality for all points in the set. $\overline{\mu}$ is called the *equilibrium measure* in the presence of the external field ϕ on E and w is the *equilibrium constant.*

Recall that a measure $\sigma \in \mathcal{M}(E)$ is *regular* if

$$\lim_{n\to\infty} \kappa_n^{1/n} = \frac{1}{\mathrm{cap}(\mathrm{supp}(\sigma))}\,,$$

where κ_n is the leading coefficient of the n-th orthonormal polynomial with respect to σ.

In order to determine the asymptotic zero distribution of the polynomials $Q_{\mathbf{n},j}$ we use the following lemma. Different versions of it appear in [13], [27], and [77]. In [27], it was proved assuming that $\mathrm{supp}(\sigma)$ is an interval on which $\sigma' > 0$ a.e. Theorem 3.3.3 in [77] and Theorem 1 in [13], do not cover the type of external field we consider here. So, we will sketch a proof.

Lemma 3.1.2. *Let σ be a regular measure,* $\mathrm{supp}(\sigma) \subset \mathbb{R}$*, where* $\mathrm{supp}(\sigma)$ *is regular with respect to the Dirichlet problem. Let $\{\phi_l\}, l \in \Lambda \subset \mathbb{Z}_+$, be a sequence of positive continuous functions on* $\mathrm{supp}(\sigma)$ *such that*

$$\lim_{l\in\Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = \phi(x) > -\infty, \tag{3.1}$$

uniformly on $\mathrm{supp}(\sigma)$*. By $\{q_l\}, l \in \Lambda$, denote a sequence of monic polynomials such that* $\deg q_l = l$ *and*

$$\int x^k q_l(x)\phi_l(x)d\sigma(x) = 0, \qquad k = 0, \ldots, l-1. \tag{3.2}$$

Then

$$* \lim_{l \in \Lambda} \mu_{q_l} = \overline{\mu}, \tag{3.3}$$

and

$$\lim_{l \in \Lambda} \left(\int |q_l(x)|^2 \phi_l(x) d\sigma(x) \right)^{1/2l} = e^{-w}, \tag{3.4}$$

where $\overline{\mu}$ and w are the equilibrium measure and equilibrium constant in the presence of the external field ϕ on $E := \text{supp}(\sigma)$. We also have that

$$\lim_{l \in \Lambda} \left(\frac{|q_l(z)|}{\|q_l \phi_l^{1/2}\|_E} \right)^{1/l} = \exp(w - V^{\overline{\mu}}(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus \text{Co}(\text{supp}(\sigma)). \tag{3.5}$$

Proof. On account of (3.1) and Lemma 3.1.1, it follows that for any $\varepsilon > 0$ there exists l_0 such that for all $l \geq l_0, l \in \Lambda$, and $z \in \text{supp}(\overline{\mu}) \subset \text{supp}(\sigma) = E$

$$\frac{1}{l} \log \frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \leq \frac{1}{2l} \log \frac{1}{|\phi_l(z)|} \leq \phi(z) + \varepsilon \leq w - V^{\overline{\mu}}(z) + \varepsilon,$$

where $\{p_l\}, l \in \Lambda$, is any sequence of monic polynomials such that $\deg p_l = l$ and $\|p_l \phi_l^{1/2}\|_E = \max_{z \in E} |(p_l \phi_l^{1/2})(z)|$. Hence,

$$u_l(z) := V^{\overline{\mu}}(z) + \frac{1}{l} \log \frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \leq w + \varepsilon, \quad z \in \text{supp}(\overline{\mu}), \quad l \geq l_0.$$

Since u_l is subharmonic in $\overline{\mathbb{C}} \setminus \text{supp}(\overline{\mu})$, by the continuity and maximum principles, we have

$$u_l(z) \leq w + \varepsilon, \quad z \in \overline{\mathbb{C}}, \quad l \geq l_0.$$

In particular,

$$u_l(\infty) = \frac{1}{l} \log \frac{1}{\|p_l \phi_l^{1/2}\|_E} \leq w + \varepsilon.$$

The last two relations imply

$$\limsup_{l \in \Lambda} \left(\frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \right)^{1/l} \leq \exp(w - V^{\overline{\mu}}(z)), \qquad \mathcal{K} \subset \mathbb{C}, \tag{3.6}$$

and

$$\liminf_{l \in \Lambda} \|p_l \phi_l^{1/2}\|_E^{1/l} \geq \exp(-w). \tag{3.7}$$

In particular, these relations hold for the sequence of polynomials $\{q_l\}, l \in \Lambda$.

Let t_l be the weighted Fekete polynomial of degree l for the weight $e^{-\phi}$ on $\text{supp}(\sigma)$ and $|\sigma|$ be the total variation of σ. From the minimality property in the L_2 norm of q_l, we have

$$\|q_l \phi_l^{1/2}\|_2 := \left(\int |q_l(x)|^2 \phi_l(x) d\sigma(x) \right)^{1/2} \leq \|t_l \phi_l^{1/2}\|_2 \leq |\sigma|^{1/2} \|t_l \phi_l^{1/2}\|_E \leq$$

$$|\sigma|^{1/2}\|t_l e^{-l\phi}\|_E \|\phi_l^{1/2} e^{l\phi}\|_E.$$

Then, using (3.1) and Theorem III.1.9 in [77], we obtain that

$$\limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_2^{1/l} \le e^{-w}. \tag{3.8}$$

Since $\operatorname{supp}(\sigma)$ is regular with respect to the Dirichlet problem, Theorem 3.2.3 vi) in [77] yields

$$\limsup_{l\in\Lambda} \left(\frac{\|q_l\phi_l^{1/2}\|_E}{\|q_l\phi_l^{1/2}\|_2}\right)^{1/l} \le 1,$$

which combined with (3.7) (with $p_l = q_l$) and (3.8) implies

$$\lim_{l\in\Lambda} \left(\frac{\|q_l\phi_l^{1/2}\|_E}{\|q_l\phi_l^{1/2}\|_2}\right)^{1/l} = 1. \tag{3.9}$$

Thus, we obtain (3.4) since (3.7), (3.8), and (3.9) give

$$\limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_E^{1/l} = \limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_2^{1/l} = e^{-w}. \tag{3.10}$$

All the zeros of q_l lie in $\operatorname{Co}(\operatorname{supp}(\sigma)) \subset \mathbb{R}$. The unit ball in the weak star topology of measures is compact. Take any subsequence of indices $\Lambda' \subset \Lambda$ such that

$$* \lim_{l\in\Lambda'} \mu_{q_l} = \mu_{\Lambda'}.$$

Then,

$$\lim_{l\in\Lambda'} \frac{1}{l}\log|q_l(z)| = -\lim_{n\in\Lambda'} \int \log\frac{1}{|z-x|}\mu_{q_l}(x) = -V^{\mu_{\Lambda'}}(z),$$

uniformly on each compact $\mathcal{K} \subset \mathbb{C}\setminus\operatorname{Co}(\operatorname{supp}(\sigma))$. This, together with (3.4) and (3.6) (applied to $\{q_l\}, l\in\Lambda'$), implies

$$(V^{\overline{\mu}} - V^{\mu_{\Lambda'}})(z) \le 0, \qquad z \in \overline{\mathbb{C}}\setminus\operatorname{Co}(\operatorname{supp}(\sigma)).$$

Since $V^{\overline{\mu}} - V^{\mu_{\Lambda'}}$ is subharmonic in $\overline{\mathbb{C}}\setminus\operatorname{supp}(\overline{\mu})$ and $(V^{\overline{\mu}} - V^{\mu_{\Lambda'}})(\infty) = 0$, from the maximum principle, it follows that $V^{\overline{\mu}} \equiv V^{\mu_{\Lambda'}}$ in $\mathbb{C}\setminus\operatorname{Co}(\operatorname{supp}(\sigma))$ and thus $\mu_{\Lambda'} = \overline{\mu}$. Consequently, (3.3) holds. (3.3) and (3.4) imply (3.5). □

3.2 The asymptotic distribution of the zeros of $\{Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$

Using Lemma 3.1.2, we can obtain the asymptotic distribution of the zeros of the polynomials $Q_{\mathbf{n},j}$, $j = -m_2, \ldots, m_1$. We continue employing the following notation introduced in Subsection 1.3.1, namely

$$\Delta_j = \Delta_j^1, \qquad \sigma_j = \sigma_j^1, \qquad j = 0, 1, \ldots, m_1,$$

$$\Delta_j = \Delta^2_{-j}, \qquad \sigma_j = \sigma^2_{-j}, \qquad j = 0, -1, \ldots, -m_2,$$

and

$$N_{\mathbf{n},j} = \begin{cases} N_{1,j}(\mathbf{n}) - 1, & j = 0, 1 \ldots, m_1, \\ N_{2,-j}(\mathbf{n}), & j = 0, -1, \ldots, -m_2. \end{cases}$$

According to Propositions 2.1.5 and 2.1.7, for all $j = -m_2, \ldots, m_1$ the zeros of $Q_{\mathbf{n},j}$ are all simple, lie in the interior of Δ_j, and total $N_{\mathbf{n},j}$ points.

Proof of Theorem 1.3.2. The unit ball in the cone of positive Borel measures is weak star compact; therefore, it is sufficient to show that each one of the sequences of measures $\{\mu_{Q_{\mathbf{n},j}}\}$, $\mathbf{n} \in \Lambda$, $j = -m_2, \ldots, m_1$, has only one accumulation point which coincides with the corresponding component of the vector measure $\overline{\mu}(\mathcal{C})$. Let $\Lambda' \subset \Lambda$ be a subsequence of multi-indices such that for each $j = -m_2, \ldots, m_1$

$$* \lim_{\mathbf{n} \in \Lambda'} \mu_{Q_{\mathbf{n},j}} = \mu_j .$$

Notice that $\mu_j \in \mathcal{M}_1(E_j)$, $j = -m_2, \ldots, m_1$. Therefore,

$$\lim_{\mathbf{n} \in \Lambda'} |Q_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \exp(-P_j V^{\mu_j}(z)), \tag{3.11}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, where $P_j = \lim_{\mathbf{n} \in \Lambda'} N_{\mathbf{n},j}/|\mathbf{n}_1|$.

Because of the normalization adopted on $a_{\mathbf{n},m_1}$, $\mathcal{A}_{\mathbf{n},m_1} = Q_{\mathbf{n},m_1}$; consequently, when $j = m_1$, (2.5) takes the form

$$\int x^\nu Q_{\mathbf{n},m_1}(x) \frac{d|\sigma_{m_1}|(x)}{|Q_{\mathbf{n},m_1-1}(x)|} = 0 \,, \qquad \nu = 0, \ldots, N_{\mathbf{n},m_1} - 1 \,.$$

(By $|\sigma_k|$ we denote the total variation of the measure σ_k. Since our measures σ_k have constant sign, $|\sigma_k|$ is either equal to σ_k or to $-\sigma_k$.) According to (3.11)

$$\lim_{\mathbf{n} \in \Lambda'} \frac{1}{2N_{\mathbf{n},m_1}} \log |Q_{\mathbf{n},m_1-1}(x)| = -\frac{P_{m_1-1}}{2P_{m_1}} V^{\mu_{m_1-1}}(x) \,,$$

uniformly on Δ_{m_1}. Using Lemma 3.1.2, it follows that μ_{m_1} is the unique solution of the extremal problem

$$V^{\mu_{m_1}}(x) - \frac{P_{m_1-1}}{2P_{m_1}} V^{\mu_{m_1-1}}(x) \begin{cases} = \omega_{m_1}, & x \in \operatorname{supp}(\mu_{m_1}) \,, \\ \geq \omega_{m_1}, & x \in E_{m_1} \,, \end{cases} \tag{3.12}$$

and

$$\lim_{\mathbf{n} \in \Lambda'} \left(\int \frac{Q^2_{\mathbf{n},m_1}(x)}{|Q_{\mathbf{n},m_1-1}(x)|} d|\sigma_{m_1}|(x) \right)^{1/2N_{\mathbf{n},m_1}} = e^{-\omega_{m_1}} \,. \tag{3.13}$$

Let us show by induction on decreasing values of j, that if $j \in \{-m_2, \ldots, m_1\}$

$$V^{\mu_j}(x) - \frac{P_{j-1}}{2P_j} V^{\mu_{j-1}}(x) - \frac{P_{j+1}}{2P_j} V^{\mu_{j+1}}(x) + \frac{P_{j+1}}{P_j} \omega_{j+1} \begin{cases} = \omega_j, & x \in \operatorname{supp}(\mu_j) \,, \\ \geq \omega_j, & x \in E_j \,, \end{cases} \tag{3.14}$$

where $P_{-m_2-1} = P_{m_1+1} = 0$, and

$$\lim_{\mathbf{n}\in\Lambda'} \left(\int \frac{Q^2_{\mathbf{n},j}(x)}{|Q_{\mathbf{n},j-1}(x)|} \frac{|\mathcal{A}_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|} d|\sigma_j|(x) \right)^{1/2N_{\mathbf{n},j}} = e^{-\omega_j} , \qquad (3.15)$$

where $Q_{\mathbf{n},-m_2-1} \equiv 1$. For $j = m_1$ these relations are non other than (3.12)-(3.13) and the initial induction step is settled. Let us assume that the statement is true for $j+1 \in \{-m_2+1, \ldots, m_1\}$ and let us prove it for j.

It is easy to see that the orthogonality relations (2.5) and (2.9) can be expressed as

$$\int x^\nu Q_{\mathbf{n},j}(x) \frac{|Q_{\mathbf{n},j+1}(x)\mathcal{A}_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|} \frac{d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},j}-1 .$$

On account of (2.6) and (2.10) taking $q = Q_{\mathbf{n},j+1}$, this can be further transformed into

$$\int x^\nu Q_{\mathbf{n},j}(x) \left(\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \right) \frac{d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} = 0,$$

for $\nu = 0, \ldots, N_{\mathbf{n},j} - 1$.

Relation (3.11) implies that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{1}{2N_{\mathbf{n},j}} \log |Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)| = \qquad (3.16)$$

$$-\frac{P_{j-1}}{2P_j} V^{\mu_{j-1}}(x) - \frac{P_{j+1}}{2P_j} V^{\mu_{j+1}}(x) ,$$

uniformly on Δ_j. (Since $Q_{\mathbf{n},-m_2-1} \equiv 1$, when $j = -m_2$ we only get the second term on the right hand side of this limit; that is, $P_{-m_2-1} = 0$.)

Set

$$K_{\mathbf{n},j+1} = \left(\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} d|\sigma_{j+1}|(t) \right)^{-1/2} .$$

It follows that for all $x \in \Delta_j$

$$\frac{1}{\delta^*_{j+1} K^2_{\mathbf{n},j+1}} \leq \int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \leq \frac{1}{\delta_{j+1} K^2_{\mathbf{n},j+1}},$$

where $0 < \delta_{j+1} = \inf\{|x-t| : t \in \Delta_{j+1}, x \in \Delta_j\} \leq \max\{|x-t| : t \in \Delta_{j+1}, x \in \Delta_j\} = \delta^*_{j+1} < \infty$. Taking into consideration these inequalities, from the induction hypothesis, we obtain that

$$\lim_{\mathbf{n}\in\Lambda'} \left(\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \right)^{1/2N_{\mathbf{n},j}} = e^{-P_{j+1}\omega_{j+1}/P_j}. \qquad (3.17)$$

Taking (3.16) and (3.17) into account, Lemma 3.1.2 yields that μ_j is the unique solution of the extremal problem (3.14) and

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|}\frac{Q^2_{\mathbf{n},j}(x)d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|}\right)^{\frac{1}{2N_{\mathbf{n},j}}} = e^{-\omega_j}.$$

According to (2.6) and (2.10) with $q = Q_{\mathbf{n},j+1}$

$$\frac{1}{|Q_{\mathbf{n},j+1}(x)|}\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|\mathcal{A}_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|} = \frac{|\mathcal{A}_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}, \quad x\in\Delta_j\,,$$

which allows to reduce the previous formula to (3.15) thus concluding the induction.

Now, we can rewrite (3.14) multiplying through by P_j^2 and taking the constant term on the left to the right to obtain the system of boundary value equations

$$P_j^2 V^{\mu_j}(x) - \frac{P_{j-1}P_j}{2}V^{\mu_{j-1}}(x) - \frac{P_jP_{j+1}}{2}V^{\mu_{j+1}}(x)\begin{cases} = \omega'_j, & x\in \operatorname{supp}(\mu_j)\,,\\ \geq \omega'_j, & x\in E_j\,,\end{cases} \tag{3.18}$$

for $j = -m_2,\ldots,m_1$, where

$$\omega'_j = P_j^2\omega_j - P_jP_{j+1}\omega_{j+1}. \tag{3.19}$$

The terms with P_{-m_2-1} and P_{m_1+1} do not appear when $j = -m_2$ and $j = m_1$, respectively. By Lemma 1.3.1, $(\mu_{-m_2},\ldots,\mu_{m_1}) = (\overline{\mu}_{-m_2},\ldots,\overline{\mu}_{m_1})$ and $(\omega'_{-m_2},\ldots,\omega'_{m_1}) = (\omega^{\overline{\mu}}_{-m_2},\ldots,\omega^{\overline{\mu}}_{m_1})$ for any convergent subsequence showing the existence of the limits in (1.15) as stated.

Notice that (3.15) implies that

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int \frac{Q^2_{\mathbf{n},j}(x)}{|Q_{\mathbf{n},j-1}(x)|}\frac{|\mathcal{A}_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}d|\sigma_j|(x)\right)^{1/2|\mathbf{n}_1|} = e^{-P_j\omega_j}.$$

On the other hand, from (3.19) it follows that $P_{m_1}\omega_{m_1} = \omega^{\overline{\mu}}_{m_1}/P_{m_1}$ when $j = m_1$. Suppose that $P_{j+1}\omega_{j+1} = \sum_{k=j+1}^{m_1}\frac{\omega_k^{\overline{\mu}}}{P_k}, j+1\in\{-m_2+1,\ldots,m_1\}$. Then, according to (3.19)

$$P_j\omega_j = \frac{\omega_j^{\overline{\mu}}}{P_j} + P_{j+1}\omega_{j+1} = \sum_{k=j}^{m_1}\frac{\omega_k^{\overline{\mu}}}{P_k}$$

and (1.16) immediately follows. □

3.3 The n-th root asymptotics of $\{\mathcal{A}_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$ and $\{a_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$

Here, we maintain the change of notation introduced in the previous section.

Proof of Theorem 1.3.3. If $j = m_1, \mathcal{A}_{\mathbf{n},m_1} = Q_{\mathbf{n},m_1}$ and (1.15) directly implies that

$$\lim_{\mathbf{n}\in\Lambda} |\mathcal{A}_{\mathbf{n},m_1}(z)|^{1/|\mathbf{n}_1|} = \exp\left(-P_{m_1} V^{\overline{\mu}_{m_1}}(z)\right), \qquad \mathcal{K} \subset \mathbb{C} \setminus \Delta_{m_1}.$$

For $j \in \{-m_2 - 1, \ldots, m_1 - 1\}$, using (2.6) and (2.10) with $q = Q_{\mathbf{n},j+1}$, we obtain

$$\mathcal{A}_{\mathbf{n},j}(z) = \frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)} \int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{\mathcal{A}_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z - x}, \tag{3.20}$$

($Q_{\mathbf{n},-m_2-1} \equiv 1$.) From (1.15), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \left| \frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)} \right|^{1/|\mathbf{n}_1|} = \tag{3.21}$$

$$\exp\left(P_{j+1} V^{\overline{\mu}_{j+1}}(z) - P_j V^{\overline{\mu}_j}(z)\right), \qquad \mathcal{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1})$$

(we also use that the zeros of $Q_{\mathbf{n},j}$ and $Q_{\mathbf{n},j+1}$ lie in Δ_j and Δ_{j+1}, respectively). It remains to find the $|\mathbf{n}_1|$-th root asymptotic behavior of the integral.

Fix a compact set $\mathcal{K} \subset \mathbb{C} \setminus \Delta_{j+1}$. It is easy to verify that (for the definition of $K^2_{\mathbf{n},j+1}$ see proof of Theorem 1.3.2 above)

$$\frac{C_1}{K^2_{\mathbf{n},j+1}} \leq \left| \int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{\mathcal{A}_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z - x} \right| \leq \frac{C_2}{K^2_{\mathbf{n},j+1}},$$

where

$$C_1 = \frac{\min\{\max\{|u - x|, |v| : z = u + iv\} : z \in \mathcal{K}, x \in \Delta_{j+1}\}}{\max\{|z - x|^2 : z \in \mathcal{K}, x \in \Delta_{j+1}\}} > 0$$

and

$$C_2 = \frac{1}{\min\{|z - x| : z \in \mathcal{K}, x \in \Delta_{j+1}\}} < \infty.$$

Taking into account (1.16)

$$\lim_{\mathbf{n}\in\Lambda} \left| \int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{\mathcal{A}_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z - x} \right|^{1/|\mathbf{n}_1|} = \tag{3.22}$$

$$\exp\left(-2 \sum_{k=j+1}^{m_1} \omega_k^{\overline{\mu}} / P_k\right).$$

From (3.20)-(3.22), we obtain (1.17) and we are done. □

Remark 3.3.1. *Taking into consideration that the polynomials $Q_{\mathbf{n},j}$ (see Propositions* 2.1.5 *and* 2.1.7*) and the functions*

$$\int \frac{Q^2_{\mathbf{n},j}(x)}{Q_{\mathbf{n},j-1}(x)} \frac{\mathcal{A}_{\mathbf{n},j}(x)}{Q_{\mathbf{n},j}(x)} \frac{d\sigma_j(x)}{z - x},$$

may have at most one zero in each of the connected components of $\Delta_j \setminus E_j$*, where* $E_j = \operatorname{supp}(\sigma_j)$*, in place of* (1.17) *one can prove convergence in capacity on each compact subset* $\mathcal{K} \subset \mathbb{C} \setminus (E_j \cup E_{j+1})$*. More precisely, for any such compact set* $\mathcal{K}$ *and each* $\varepsilon > 0$

$$\lim_{\mathbf{n}\in\Lambda} \operatorname{cap}\left\{z \in \mathcal{K} : \left||\mathcal{A}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} - G_j(z)\right| > \varepsilon\right\} = 0.$$

Set

$$U_j^{\overline{\mu}}(z) = P_j V^{\overline{\mu}_j}(z) - P_{j+1} V^{\overline{\mu}_{j+1}}(z) + 2\sum_{k=j+1}^{m_1} \frac{\omega_k^{\overline{\mu}}}{P_k}, \quad j = -m_2 - 1, \ldots, m_1 - 1,$$

and

$$U_{m_1}^{\overline{\mu}}(z) = P_{m_1} V^{\overline{\mu}_{m_1}}(z).$$

Hence, $G_j(z) = \exp(-U_j^{\overline{\mu}}(z)), j = -m_2 - 1, \ldots, m_1$.

We have that for $j = -m_2, \ldots, m_1 (P_{-m_2-1} = P_{m_1+1} = 0)$

$$\frac{P_j}{2}(U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)) = -\frac{P_{j+1}P_j}{2} V^{\overline{\mu}_{j+1}}(z) + P_j^2 V^{\overline{\mu}_j}(z) - \frac{P_j P_{j-1}}{2} V^{\overline{\mu}_{j-1}}(z) - \omega_j^{\overline{\mu}}.$$

From the equilibrium property (see Lemma 1.3.1 and (3.18)), it follows that

$$U_j^{\overline{\mu}}(x) - U_{j-1}^{\overline{\mu}}(x) \begin{cases} = 0, & x \in \operatorname{supp}(\overline{\mu}_j), \\ \geq 0, & x \in E_j. \end{cases}$$

Define

$$p_j = \begin{cases} p_{1,j}, & j = 0, \ldots, m_1, \\ -p_{2,-j-1}, & j = -m_2 - 1, \ldots, -1. \end{cases}$$

It is easy to verify that for $j = -m_2, \ldots, m_1$

$$U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = \mathcal{O}((p_j - p_{j-1}) \log 1/|z|), \qquad z \to \infty. \tag{3.23}$$

In particular, $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = \mathcal{O}(1), z \to \infty$, whenever $p_j = p_{j-1}$. By assumption, $p_j - p_{j-1} \leq 0, j = -m_2, \ldots, m_1$ except for $p_0 - p_{-1} = p_{1,0} + p_{2,0} > 0$.

For all j, the function $U_j^{\overline{\mu}} - U_{j-1}^{\overline{\mu}}$ is subharmonic in $\mathbb{C} \setminus \operatorname{supp}(\overline{\mu}_j)$. If $p_j \geq p_{j-1}$, then it is subharmonic in all $\overline{\mathbb{C}} \setminus \operatorname{supp}(\overline{\mu}_j)$. According to what was said above, when $j = 0$ or $p_j = p_{j-1}$, from the equilibrium condition and the maximum principle, we have that $U_j^{\overline{\mu}} - U_{j-1}^{\overline{\mu}} \equiv 0$ on $\operatorname{supp}(\sigma_j) = E_j$ and $U_j^{\overline{\mu}} < U_{j-1}^{\overline{\mu}}$ on $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_j)$. In particular, in this case we have that $\operatorname{supp}(\overline{\mu}_j) = \operatorname{supp}(\sigma_j)$.

When $p_{j-1} > p_j$, (3.23) implies that in a neighborhood of $z = \infty, U_j^{\overline{\mu}} > U_{j-1}^{\overline{\mu}}$. Let $\gamma_j = \{z \in \mathbb{C} : U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = 0\}$. The equilibrium condition entails that $\gamma_j \supset \operatorname{supp}(\overline{\mu}_j)$ and the initial remark indicates that γ_j is bounded. Consider any bounded component of the complement of γ_j. On it, $U_j^{\overline{\mu}} - U_{j-1}^{\overline{\mu}}$ is subharmonic and on its boundary $U_j^{\overline{\mu}} - U_{j-1}^{\overline{\mu}} = 0$. Thus, on any bounded component of the complement of γ_j we have that $U_j^{\overline{\mu}} < U_{j-1}^{\overline{\mu}}$. From the initial remark it follows that on the unbounded component of the complement of $\gamma_j, U_j^{\overline{\mu}} > U_{j-1}^{\overline{\mu}}$.

Fix $j \in \{0, \ldots, m_1\}$. For each $k \in \{j, \ldots, m_1\}$ define

$$D_k^j := \{z \in \mathbb{C} \setminus \cup_{i=j}^{m_1} \Delta_i : U_k^{\overline{\mu}}(z) < U_i^{\overline{\mu}}(z), i = j, \ldots, m_1, i \neq k\}, \ \ D_{m_1}^{m_1} := \mathbb{C} \setminus \Delta_{m_1}.$$

Let

$$\zeta_j(z) = \min\{U_k^{\overline{\mu}}(z) : k = j, \ldots, m_1\}.$$

Corollary 3.3.2. *Let* $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1)$, $S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2)$, $(S^1, S^2) \in$ **Reg**, *and* $\Lambda = \Lambda(p_{1,0}, \ldots, p_{1,m_1}; p_{2,0}, \ldots, p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$, *be given. Let* $(a_{\mathbf{n},0}, a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m_1}), \mathbf{n} \in \Lambda$, *be the associated sequence of "monic" mixed type multiple orthogonal polynomials. Then, for* $j = 0, \ldots, m_1$

$$\lim_{\mathbf{n}\in\Lambda} |a_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \exp(-\zeta_j(z)), \qquad \mathcal{K} \subset \cup_{k=j}^{m_1} D_k^j, \tag{3.24}$$

and

$$\limsup_{\mathbf{n}\in\Lambda} |a_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} \leq \exp(-\zeta_j(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus \cup_{k=j}^{m_1} \Delta_k. \tag{3.25}$$

In particular, if $p_{1,0} = \cdots = p_{1,m_1} = 1/(m_1+1)$, *then*

$$\lim_{\mathbf{n}\in\Lambda} |a_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \exp(-U_{m_1}^{\overline{\mu}}(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus \cup_{k=j}^{m_1} \Delta_k. \tag{3.26}$$

Proof. For $j = m_1$, $\mathcal{A}_{\mathbf{n},m_1} = a_{\mathbf{n},m_1}$, $D_{m_1}^{m_1} = \mathbb{C} \setminus \Delta_{m_1}$ and $\zeta_{m_1} = U_{m_1}^{\overline{\mu}}$. Therefore, (3.24) reduces to (1.17) and implies (3.25). Let us prove these relations for $j = 0, \ldots, m_1 - 1$.

The $\mathcal{A}_{\mathbf{n},j}$ are expressed in terms of the $a_{\mathbf{n},k}, k = j, \ldots, m_1$, through a linear triangular scheme of equations with function coefficients which do not depend on $\mathbf{n}$. Using this system, we can solve for $a_{\mathbf{n},j}$, in terms of $\mathcal{A}_{\mathbf{n},k}, k = j, \ldots, m_1$.

Given $j \in \{1, \ldots, m_1\}$ and $0 \leq i < j$, we have

$$(-1)^{j-i} \langle \sigma_i^1, \ldots, \sigma_j^1 \widehat{\rangle}(z) = \int \cdots \int \frac{d\sigma_i^1(x_i) \cdots d\sigma_j^1(x_j)}{(z - x_i)(x_{i+1} - x_i) \cdots (x_j - x_{j-1})},$$

where $\langle \cdot \widehat{\rangle}(z)$ denotes the Cauchy transform of the indicated measure, and

$$\langle \sigma_j^1, \ldots, \sigma_i^1 \widehat{\rangle}(z) = \int \cdots \int \frac{d\sigma_i^1(x_i) \cdots d\sigma_j^1(x_j)}{(x_{i+1} - x_i) \cdots (x_j - x_{j-1})(z - x_j)}.$$

Consequently,

$$(-1)^{j-i} \langle \sigma_i^1, \ldots, \sigma_j^1 \widehat{\rangle}(z) - \langle \sigma_j^1, \ldots, \sigma_i^1 \widehat{\rangle}(z) =$$

$$\int \cdots \int \frac{-(x_j - x_i) d\sigma_i^1(x_i) \cdots d\sigma_j^1(x_j)}{(z - x_i)(x_{i+1} - x_i) \cdots (x_j - x_{j-1})(z - x_j)}.$$

Since $x_j - x_i = x_j - x_{j-1} + x_{j-1} - \cdots - x_{i+1} + x_{i+1} - x_i$, substituting this in the previous formula, we obtain

$$\langle \sigma_j^1, \ldots, \sigma_i^1 \widehat{\rangle}(z) = \tag{3.27}$$

$$\sum_{k=i}^{j-1} (-1)^{k-i} \langle \sigma_i^1, \ldots, \sigma_k^1 \widehat{\rangle}(z) \langle \sigma_j^1, \ldots, \sigma_{k+1}^1 \widehat{\rangle}(z) + (-1)^{j-i} \langle \sigma_i^1, \ldots, \sigma_j^1 \widehat{\rangle}(z).$$

(This formula is applicable to any Nikishin system. We will use it on S^2 in the last section.)

Using formula (3.27) it is easy to deduce that (the sum is empty when $j = m_1$)

$$a_{\mathbf{n},j}(z) = \mathcal{A}_{\mathbf{n},j}(z) + \sum_{k=j+1}^{m_1} (-1)^{k-j} \langle \sigma_k^1, \ldots, \sigma_{j+1}^1 \widehat{\rangle}(z) \mathcal{A}_{\mathbf{n},k}(z).$$

Taking (1.17) into consideration, on D_k^j the term containing $\mathcal{A}_{\mathbf{n},k}$ dominates the sum (notice that $\langle \sigma_k^1, \ldots, \sigma_{j+1}^1 \widehat{\rangle}(z) \neq 0, z \in \mathbb{C} \setminus \Delta_k$) and (3.24) immediately follows. On the complement of $\cup_{k=j}^{m_1} D_k^j$ there is no dominating term and all we can conclude from the previous equality is (3.25).

Let $p_{1,0} = \cdots = p_{1,m_1} = 1/(m_1+1)$. In this case, on $\mathbb{C} \setminus \cup_{k=j}^{m_1} \Delta_k$ we have that $U_{m_1}^{\overline{\mu}}(z) < U_{m_1-1}^{\overline{\mu}}(z) < \cdots < U_j^{\overline{\mu}}(z)$ and (3.26) follows from (3.24). □

3.4 Application to mixed type Hermite-Padé approximation

Let $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2), \sigma_0^1 = \sigma_0^2$ be given. Let us introduce the row vectors

$$\mathbb{U} = (1, \widehat{s}_{1,1}^2, \ldots, \widehat{s}_{1,m_2}^2), \qquad \mathbb{V} = (1, \widehat{s}_{1,1}^1, \ldots, \widehat{s}_{1,m_1}^1)$$

and the $(m_2+1) \times (m_1+1)$ dimensional matrix

$$\mathbb{W} = \mathbb{U}^t \mathbb{V},$$

where the super-index t means taking transpose. Define the matrix Markov type function

$$\widehat{\mathbb{S}}(z) = \int \frac{\mathbb{W}(x) d\sigma_0^2(x)}{z - x}$$

understanding that integration is carried out entry by entry on the matrix $\mathbb{W}$.

Fix $\mathbf{n}_1 = (n_{1,0}, n_{1,1}, \ldots, n_{1,m_1}) \in \mathbb{Z}_+^{m_1+1}$ and $\mathbf{n}_2 = (n_{2,0}, n_{2,1}, \ldots, n_{2,m_2}) \in \mathbb{Z}_+^{m_2+1}, |\mathbf{n}_2| = |\mathbf{n}_1| - 1$. It is easy to see that there exists a non zero vector polynomial

$$\mathbb{A}_{\mathbf{n}} = (a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1}), \qquad \deg(a_{\mathbf{n},k}) \leq n_{1,k} - 1, \qquad k = 0, \ldots, m_1,$$

such that

$$\widehat{\mathbb{S}}(z) \mathbb{A}_{\mathbf{n}}^t(z) - \mathbb{D}_{\mathbf{n}}^t(z) = \tag{3.28}$$

$$(\mathcal{O}(1/z^{n_{2,0}+1}),\ldots,\mathcal{O}(1/z^{n_{2,m_2}+1}))^t =: \mathcal{O}(1/z^{\mathbf{n}_2+1}), \qquad z\to\infty,$$

where $\mathbb{D}_{\mathbf{n}} = (d_{\mathbf{n},0},\ldots,d_{\mathbf{n},m_2})$ is some vector polynomial. When $m_2 = 0$, this construction is called type I Hermite-Padé approximation. If $m_1 = 0$ it is called of type II. When $m_1 = m_2 = 0$ it reduces to diagonal Padé approximation. This definition is of mixed type.

Lemma 3.4.1. *For $j = 0,\ldots,m_2$, $(\widehat{s}^1_{1,0}\equiv 1)$*

$$\int x^\nu \sum_{k=0}^{m_1} a_{\mathbf{n},k}(x)\widehat{s}^1_{1,k}(x)ds^2_j(x) = 0, \qquad \nu = 0,\ldots,n_{2,j}-1. \tag{3.29}$$

Proof. In fact, notice that according to (3.28), for each $\nu, 0\le\nu\le n_{2,j}-1, j = 0,\ldots,m_2$,

$$z^\nu\left(\sum_{k=0}^{m_1} a_{\mathbf{n},k}(z)\int\frac{\widehat{s}^2_{1,j}(x)\widehat{s}^1_{1,k}(x)d\sigma^2_0(x)}{z-x} - d_{\mathbf{n},j}(z)\right) = \mathcal{O}\left(1/z^2\right), \qquad z\to\infty,$$

$(\widehat{s}^2_{1,0}\equiv 1)$ and the function on the left hand side is analytic in $\overline{\mathbb{C}}\setminus \mathrm{Co}(\mathrm{supp}(\sigma^2_0))$. Using Lemma 2.1.3, we obtain (3.29). □

Because of this Lemma, we see that $\mathbb{A}_{\mathbf{n}}$ is an $\mathbf{n}$-th mixed type multiple orthogonal polynomial with respect to the pair (S^1, S^2) and in the sequel we assume that it is "monic". If

$$\mathbb{B}_{\mathbf{n}} = (b_{\mathbf{n},0},\ldots,b_{\mathbf{n},m_2}), \qquad \deg(b_{\mathbf{n},j})\le n_{2,j}-1, \qquad j = 0,\ldots,m_2,$$

denotes a generic vector polynomial with the indicated degrees, (3.29) may be rewritten in matrix form as

$$\int \mathbb{B}_{\mathbf{n}}(x)\mathbb{W}(x)\mathbb{A}^t_{\mathbf{n}}(x)d\sigma^2_0(x) = 0, \qquad \text{for all } \mathbb{B}_{\mathbf{n}}. \tag{3.30}$$

Fix $j\in\{0,\ldots,m_2\}$. For each $k\in\{-1,\ldots,-j-1\}$ define

$$\Omega^j_k = \{z\in\mathbb{C}\setminus\cup_{i=0}^{-j-1}\Delta_i : U^{\overline{\mu}}_k(z) < U^{\overline{\mu}}_i(z), i = -1,\ldots,-j-1, i\ne k\},$$

and

$$\Omega^0_{-1} = \mathbb{C}\setminus(\Delta_0\cup\Delta_{-1}).$$

Set

$$\chi_j(z) := \min\{U^{\overline{\mu}}_k(z) : k = -1,\ldots,-j-1\}$$

and

$$(\mathcal{R}_{\mathbf{n},0},\ldots,\mathcal{R}_{\mathbf{n},m_2})^t := \widehat{\mathbb{S}}(z)\mathbb{A}^t_{\mathbf{n}}(z) - \mathbb{D}^t_{\mathbf{n}}(z).$$

Theorem 3.4.2. *Let $S^1 = \mathcal{N}(\sigma^1_0,\ldots,\sigma^1_{m_1}), S^2 = \mathcal{N}(\sigma^2_0,\ldots,\sigma^2_{m_2})$, $(S^1,S^2)\in$ **Reg**, and $\Lambda = \Lambda(p_{1,0},\ldots,p_{1,m_1};p_{2,0},\ldots,p_{2,m_2})\subset\mathbb{Z}^{m_1+1}_+(\bullet)\times\mathbb{Z}^{m_2+1}_+(\bullet)$ be given. Then for each $j\in\{0,\ldots,m_2\}$*

$$\lim_{\mathbf{n}\in\Lambda}|\mathcal{R}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \exp(-\chi_j(z)), \qquad \mathcal{K}\subset\cup_{k=-1}^{-j-1}\Omega^j_k, \tag{3.31}$$

and

$$\lim_{\mathbf{n}\in\Lambda} |\mathcal{R}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} \leq \exp(-\chi_j(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus (\cup_{k=0}^{-j-1} \Delta_k). \tag{3.32}$$

In particular, if $p_{2,0} = \cdots = p_{2,m_2} = 1/(m_2+1)$, *then*

$$\lim_{\mathbf{n}\in\Lambda} |\mathcal{R}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \exp(-U_{-1}^{\overline{\mu}}(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus (\cup_{k=0}^{-j-1} \Delta_k). \tag{3.33}$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_{-m_2}, \ldots, \overline{\mu}_{m_1})$ *and* $(\omega_{-m_2}^{\overline{\mu}}, \ldots, \omega_{m_1}^{\overline{\mu}})$ *are the equilibrium vector measure and the system of equilibrium constants, respectively, for the vector potential problem determined by the interaction matrix* $\mathcal{C}$ *defined in* (1.14) *on the system of compact sets* $E_k = \operatorname{supp}(\sigma_k^1), k = 0, \ldots, m_1, E_k = \operatorname{supp}(\sigma_{-k}^2), k = -m_2, \ldots, 0.$

Proof. Notice that (3.30) implies that

$$\widehat{\mathbb{S}}(z)\mathbb{A}_{\mathbf{n}}^t(z) - \int \frac{\mathbb{W}(x)(\mathbb{A}_{\mathbf{n}}^t(z) - \mathbb{A}_{\mathbf{n}}^t(x))d\sigma_0^2(x)}{z-x} = \int \frac{\mathbb{W}(x)\mathbb{A}_{\mathbf{n}}^t(x)d\sigma_0^2(x)}{z-x},$$

where the right hand is $\mathcal{O}(1/z^{\mathbf{n}_2+1}), z \to \infty$. Taking

$$\mathbb{D}_{\mathbf{n}}^t(z) = \int \frac{\mathbb{W}(x)(\mathbb{A}_{\mathbf{n}}^t(z) - \mathbb{A}_{\mathbf{n}}^t(x))d\sigma_0^2(x)}{z-x}$$

we obtain an integral expression for the remainder in (3.28).

Then

$$(\mathcal{R}_{\mathbf{n},0}(z), \ldots, \mathcal{R}_{\mathbf{n},m_2}(z))^t = \int \frac{\mathbb{W}(x)\mathbb{A}_{\mathbf{n}}^t(x)d\sigma_0^2(x)}{z-x}.$$

In scalar form this says that

$$\mathcal{R}_{\mathbf{n},j}(z) = \int \frac{\mathcal{A}_{\mathbf{n},0}(x)}{z-x} ds_j^2(x), \qquad j = 0, \ldots, m_2.$$

Notice that (see (1.12))

$$\mathcal{R}_{\mathbf{n},0}(z) = \mathcal{A}_{\mathbf{n},-1}(z).$$

Let us establish a connection between the remainders $\mathcal{R}_{\mathbf{n},j}(z)$ and the forms $\mathcal{A}_{\mathbf{n},k}(z)$ with negative indices $k \in \{-1, \ldots, -j-1\}$.

Fix $j \in \{1, \ldots, m_2\}$. We have

$$(-1)^j \mathcal{R}_{\mathbf{n},j}(z) = \int \cdots \int \frac{\mathcal{A}_{\mathbf{n},0}(x_0)\, d\sigma_0^2(x_0) \cdots d\sigma_j^2(x_j)}{(z-x_0)(x_1-x_0)\cdots(x_j-x_{j-1})},$$

and

$$\mathcal{A}_{\mathbf{n},-j-1}(z) = \int \cdots \int \frac{\mathcal{A}_{\mathbf{n},0}(x_0)\, d\sigma_0^2(x_0) \cdots d\sigma_j^2(x_j)}{(x_1-x_0)\cdots(x_j-x_{j-1})(z-x_j)}.$$

Consequently,

$$(-1)^j \mathcal{R}_{\mathbf{n},j}(z) - \mathcal{A}_{\mathbf{n},-j-1}(z) = \int \cdots \int \frac{-(x_j-x_0)\mathcal{A}_{\mathbf{n},0}(x_0)\, d\sigma_0^2(x_0) \cdots d\sigma_j^2(x_j)}{(z-x_0)(x_1-x_0)\cdots(x_j-x_{j-1})(z-x_j)}.$$

Since $x_j - x_0 = x_j - x_{j-1} + x_{j-1} - \cdots - x_1 + x_1 - x_0$, substituting this in the previous formula, we obtain

$$\mathcal{A}_{\mathbf{n},-j-1}(z) = \sum_{k=0}^{j-1} (-1)^k \langle \sigma_j^2, \ldots, \sigma_{k+1}^2 \widehat{\rangle}(z) \mathcal{R}_{\mathbf{n},k}(z) + (-1)^j \mathcal{R}_{\mathbf{n},j}(z).$$

We have a triangular scheme of linear equations whose coefficients do not depend on $\mathbf{n}$. We can solve for $\mathcal{R}_{\mathbf{n},j}$ in terms of $\mathcal{A}_{\mathbf{n},-1}, \ldots, \mathcal{A}_{\mathbf{n},-j-1}$. Using (3.27) one obtains that for each $j \in \{0, \ldots, m_2\}$ (when $j = 0$ the sum below is empty)

$$\mathcal{R}_{\mathbf{n},j}(z) = \sum_{k=1}^{j} (-1)^{k-1} \langle \sigma_k^2, \ldots, \sigma_j^2 \widehat{\rangle}(z) \mathcal{A}_{\mathbf{n},-k}(z) + (-1)^j \mathcal{A}_{\mathbf{n},-j-1}(z).$$

Taking (1.17) into consideration, on Ω_{-k}^j the term containing $\mathcal{A}_{\mathbf{n},-k}$ dominates the sum (notice that $\langle \sigma_k^2, \ldots, \sigma_j^2 \widehat{\rangle}(z) \neq 0, z \in \mathbb{C} \setminus \Delta_{-k}$) and (3.31) immediately follows . On the complement of $\cup_{k=-1}^{-j-1} \Omega_k^j$ there is no dominating term and all we can conclude from the previous equality is (3.32).

Let $p_{2,0} = \cdots = p_{2,m_2} = 1/(m_2 + 1)$. In this case, on $\mathbb{C} \setminus \cup_{k=0}^{-j-1} \Delta_k$ we have that $U_{-1}^{\overline{\mu}}(z) < U_{-2}^{\overline{\mu}}(z) < \cdots < U_{-j-1}^{\overline{\mu}}(z)$ (see third sentence before Corollary 3.3.2) and (3.33) follows from (3.31). □

Remark 3.4.3. *Fix $j \in \{0, \ldots, m_2\}$. For each $k \in \{-1, \ldots, -j-1\}$ we could have defined*

$$\Omega_k^j = \{z \in \mathbb{C} \setminus \cup_{i=0}^{-j-1} E_i : U_k^{\overline{\mu}}(z) < U_i^{\overline{\mu}}(z), i = -1, \ldots, -j-1, i \neq k\},$$

$$\Omega_{-1}^0 = \mathbb{C} \setminus (E_0 \cup E_{-1}).$$

Taking into account that the polynomials $Q_{\mathbf{n},i}$ and the forms $\mathcal{A}_{\mathbf{n},i}$ may have at most one zero in each of the connected components of $\Delta_i \setminus E_i$, one can prove in place of (3.31) − (3.33) *convergence in capacity on each compact subset of the corresponding regions.*

We say that $\mathcal{I}_1 \subset \mathbb{Z}_+^{m_1+1}(\bullet)$ is a complete, ordered, sequence of multi-indices if:

a) For each $n \in \mathbb{Z}_+$, there exists a unique $\mathbf{n}_1 \in \mathcal{I}_1$ such that $|\mathbf{n}_1| = n$.

b) Any two multi-indices in $\mathcal{I}_1$ are ordered in the sense that all components of one of them are less than or equal to the corresponding components of the other one, or they are identical.

Remark 3.4.4. *Fix $\mathcal{I}_1 \subset \mathbb{Z}_+^{m_1+1}(\bullet), \mathcal{I}_2 \subset \mathbb{Z}_+^{m_2+1}(\bullet)$, two complete, ordered sequences of multi-indices. Each $n \in \mathbb{Z}_+$ determines a unique $\mathbf{n}_1 \in \mathcal{I}_1$ and $\mathbf{n}_2 \in \mathcal{I}_2$ such that $n = |\mathbf{n}_1| = |\mathbf{n}_2| + 1$. The corresponding "monic" mixed type multiple orthogonal polynomials we denote by $\mathbb{A}_n$. We can interchange the roles of the Nikishin systems S^1, S^2, and determine a sequence of "monic" mixed type* *multiple orthogonal polynomials which we denote* $\mathbb{B}_n$. *It is easy to verify that the sequences* $\{\mathbb{A}_n\}, \{\mathbb{B}_n\}, n \in \mathbb{Z}_+$ *are bi-orthogonal. That is,*

$$\int \mathbb{B}_{n'}(x)\mathbb{W}(x)\mathbb{A}_n^t(x)d\sigma_0^2(x) \begin{cases} = 0, & n \neq n', \\ \neq 0, & n = n'. \end{cases} \tag{3.34}$$

The inequality in (3.34) *is a consequence of Lemma* 1.2.3. *With the same hypothesis, all the results of this subsection hold true for the sequence* $\{\mathbb{B}_n\}, n \in \mathbb{Z}_+$.

4. RATIO ASYMPTOTICS

In Section 4.2 of this chapter, for convenience of the reader, we briefly review some results known about the asymptotic behavior of sequences of orthogonal polynomials with respect to varying measures which are essential for the proof of the results we obtain here. Then, in Section 4.3 we go on to prove the weak asymptotics of sequences of varying measures which appear in the study of the ratio asymptotics of multiple orthogonal polynomials. This is used, in particular, to obtain the normality of sequences of the form $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}, j \in \{-m_2, \ldots, m_1\}$. The next step, taken in Section 4.4, consists in showing that any convergent subsequence of $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$ satisfies a system of boundary value equations which turns out to have a unique solution. This already implies the existence of limit in (1.22). The expression of the limit functions, in terms of the Riemann surface introduced in Subsection 1.3.2, is deduced in Section 4.5, concluding the proof of Theorem 1.3.4. In this section, we also prove a complementary result on the ratio asymptotics of the corresponding orthonormal polynomials and their leading coefficients. Finally, in Section 4.6, we investigate the ratio asymptotics of the linear forms and the connection of the present result on ratio asymptotics with the one obtained in the previous chapter on logarithmic asymptotics.

4.1 *Preliminaries and notation*

In this chapter we study the convergence of the sequences $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$ and $\{\mathcal{A}_{\mathbf{n}^l,j}/\mathcal{A}_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$, where $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ is assumed to be an infinite sequence of multi-indices. Recall that given a multi-index $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$ and a vector $l = (l_1; l_2)$ such that $0 \le l_1 \le m_1$, $0 \le l_2 \le m_2$, by $\mathbf{n}^l = (\mathbf{n}_1^{l_1}; \mathbf{n}_2^{l_2})$ we denote the multi-index obtained by adding one to the component $l_i + 1$ of $\mathbf{n}_i$. As before, we will always assume that $l = (l_1; l_2)$ is a fixed vector and $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ for all $\mathbf{n} \in \Lambda$. Recall that

$$\Delta_j = \Delta_j^1, \qquad \sigma_j = \sigma_j^1, \qquad j = 0, 1, \ldots, m_1,$$

$$\Delta_j = \Delta_{-j}^2, \qquad \sigma_j = \sigma_{-j}^2, \qquad j = 0, -1, \ldots, -m_2,$$

and

$$N_{\mathbf{n},j} = \begin{cases} N_{1,j}(\mathbf{n}) - 1, & j = 0, 1 \ldots, m_1, \\ N_{2,-j}(\mathbf{n}), & j = 0, -1, \ldots, -m_2. \end{cases} \tag{4.1}$$

We will keep this notation throughout this chapter. Set

$$\mathcal{H}_{\mathbf{n},j} = \frac{Q_{\mathbf{n},j+1}\mathcal{A}_{\mathbf{n},j}}{Q_{\mathbf{n},j}}, \qquad j = -m_2 - 1, \ldots, m_1,$$

($Q_{\mathbf{n},-m_2-1} \equiv Q_{\mathbf{n},m_1+1} \equiv 1$ and $\mathcal{H}_{\mathbf{n},m_1} \equiv 1$). With these notations, relations (2.5), (2.9), (2.6), and (2.10) (replacing general q by $Q_{\mathbf{n},j+1}$ and shifting the index j by -1) can be rewritten as follows

$$\int x^\nu Q_{\mathbf{n},j}(x) \frac{|\mathcal{H}_{\mathbf{n},j}(x)|\, d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},j} - 1, \tag{4.2}$$

for each $j = -m_2, \ldots, m_1$, and

$$\mathcal{H}_{\mathbf{n},j-1}(z) = \int \frac{Q^2_{\mathbf{n},j}(x)}{z-x} \frac{\mathcal{H}_{\mathbf{n},j}(x) d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)}, \qquad j = -m_2, \ldots, m_1. \tag{4.3}$$

Since on the interval Δ_j the measure σ_j and the functions $\mathcal{H}_{\mathbf{n},j}, Q_{\mathbf{n},j-1}Q_{\mathbf{n},j+1}$, preserve a constant sign, we can take their absolute values in (4.2) without altering the orthogonality relations.

For each $j = -m_2, \ldots, m_1$, define

$$K_{\mathbf{n},j} = \left(\int_{\Delta_j} Q^2_{\mathbf{n},j}(x) \frac{|\mathcal{H}_{\mathbf{n},j}(x)|\, d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} \right)^{-1/2}. \tag{4.4}$$

Take

$$K_{\mathbf{n},m_1+1} = 1 \ , \quad \kappa_{\mathbf{n},j} = \frac{K_{\mathbf{n},j}}{K_{\mathbf{n},j+1}} \ , \quad j = -m_2, \ldots, m_1 \ .$$

Define

$$q_{\mathbf{n},j} = \kappa_{\mathbf{n},j} Q_{\mathbf{n},j} \ , \quad h_{\mathbf{n},j-1} = K^2_{\mathbf{n},j}\mathcal{H}_{\mathbf{n},j-1} \ , \tag{4.5}$$

and

$$d\rho_{\mathbf{n},j}(x) = \frac{h_{\mathbf{n},j}(x) d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)} \ . \tag{4.6}$$

From (4.2) and the notation introduced above, we obtain

$$\int_{\Delta_j} x^\nu Q_{\mathbf{n},j}(x) d|\rho_{\mathbf{n},j}|(x) = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},j} - 1, \quad j = -m_2, \ldots, m_1 \ , \tag{4.7}$$

and $q_{n,j}$ is orthonormal with respect to the varying measure $|\rho_{\mathbf{n},j}|$. On the other hand, using (4.3) it follows that

$$h_{\mathbf{n},j-1}(z) = \varepsilon_{\mathbf{n},j} \int_{\Delta_j} \frac{q^2_{\mathbf{n},j}(x)}{z-x} d|\rho_{\mathbf{n},j}|(x) \ , \quad j = -m_2, \ldots, m_1, \tag{4.8}$$

where $\varepsilon_{\mathbf{n},j}$ denotes the sign of the varying measure $\rho_{\mathbf{n},j}$.

In order to study the convergence of the sequence $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$, it is necessary to follow the following steps. First, we show that for each $j \in$

$\{-m_2,\dots,m_1\}$ the sequence $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}$ is uniformly bounded on each compact subset contained in $\mathbb{C}\setminus\operatorname{supp}(\sigma_j)$ (for all sufficiently large $|\mathbf{n}_1|$). Taking a subsequence of multi-indices such that all the sequences of ratios of polynomials have limit, we show that the limit functions must satisfy a system of boundary value problems. This system happens to have a unique solution from which we derive that all convergent subsequences have the same limit. Finally, we show that the limit functions can be expressed in terms of the branches of certain conformal representations of a related compact Riemann surface onto the extended complex plane.

In this chapter, we assume that $\operatorname{supp}(\sigma_j)=\widetilde{\Delta}_j\cup e_j$, $j=-m_2,\dots,m_1$, where $\widetilde{\Delta}_j=[a_j,b_j]$ is a bounded interval of the real line, $|\sigma_j'|>0$ a.e. on $\widetilde{\Delta}_j$, and e_j is a set without accumulation points in $\overline{\mathbb{R}}\setminus\widetilde{\Delta}_j$. We denote this writing $S^1=\mathcal{N}'(\sigma_0^1,\dots,\sigma_{m_1}^1)$, $S^2=\mathcal{N}'(\sigma_0^2,\dots,\sigma_{m_2}^2)$.

4.2 Ratio and relative asymptotics of orthogonal polynomials with respect to varying measures

For convenience of the reader, in this section we will present in the form of lemmas some results from [8] on the asymptotics of polynomials orthogonal with respect to varying measures which will be used in the proof of the theorems of this chapter and of Chapter 6.

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures whose supports $\operatorname{supp}(\mu_n)$ contain infinitely many points and are all contained in a fixed compact set $S\subset\mathbb{R}$. Assume also that $\{w_{2n}\}_{n\in\mathbb{N}}$ is a sequence of polynomials with real coefficients such that, for each $n\in\mathbb{N}$, $\deg w_{2n}=i_n$, $0\le i_n\le 2n$, w_{2n} is non-negative on S and

$$\int\frac{d\mu_n}{w_{2n}}<\infty.$$

We denote by $\{x_{n,i}\}_{i=1}^{2n}$ the set of zeros of w_{2n} whenever $\deg w_{2n}=2n$. If $\deg w_{2n}<2n$, we define $x_{n,i}=\infty$ for $i=1,\dots,2n-i_n$ and denote by $\{x_{n,i}\}_{i=2n-i_n+1}^{2n}$ the set of zeros of w_{2n}. We assume that the zeros are enumerated so that $|x_{n,i}|\ge|x_{n,i+1}|$.

Let $\{l_{n,j}\}$, $\deg l_{n,j}=j$, $j\in\mathbb{Z}_+$, denote the orthonormal polynomials associated to the varying measure $d\mu_n/w_{2n}$, i.e. these polynomials have positive leading coefficient and satisfy

$$\int l_{n,k}\,l_{n,j}\frac{d\mu_n}{w_{2n}}=\delta_{k,j},\qquad k,j\in\mathbb{Z}_+,$$

where $\delta_{k,j}$ denotes the Kronecker delta.

Given any compact interval Δ of the real line, we will denote by φ_Δ the conformal mapping of $\mathbb{C}\setminus\Delta$ onto $\{|z|>1\}$, such that $\varphi_\Delta(\infty)=\infty$ and $\varphi_\Delta'(\infty)>0$.

Let f be a Borel measurable function on $[0,2\pi]$, such that $\log f\in L^1[0,2\pi]$.

The Szegő function $D(f, \cdot)$ associated with f is given by

$$D(f,z) = \exp\Big(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log f(t)\,dt\Big), \qquad |z|<1$$

Definition 4.2.1. *Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S if*

a) There exists a finite Borel measure μ on $\mathbb{R}$, such that $\mu_n \stackrel{*}{\longrightarrow} \mu$, $n \to \infty$.

b) In case that k is negative, then

$$\int_S \prod_{i=1}^{-k} |1 - x/x_{n,i}|^{-1} d\mu_n \le M_k < \infty\,,$$

where $x/x_{n,i} = 0$ if $x_{n,i} = \infty$.

c) If Δ denotes the convex hull of S, then

$$\lim_{n\to\infty}\sum_{i=1}^{2n}(1 - |\varphi_\Delta(x_{n,i})|^{-1}) = \infty\,.$$

d) If μ_n', μ' denote the Radon-Nikodym derivatives of μ_n and μ, respectively, then

$$\lim_{n\to\infty}\int_S |\mu_n'(x) - \mu'(x)|dx\,.$$

The following definition was introduced in [8].

Definition 4.2.2. *Let $\{\mu_n\}_{n\in\mathbb{N}}$, be a sequence of finite positive Borel measures supported on the compact set $S \subset \mathbb{R}$. We say that $\{\mu_n\}$ is a Denisov-type sequence on S if*

1) There exists a finite positive Borel measure μ, such that $\mathrm{supp}(\mu) = S$ and $\mu_n \stackrel{*}{\longrightarrow} \mu$, $n \to \infty$.

2) There exists an interval $[a, b] \subset S$ such that for each $\epsilon > 0$, $S \setminus (a-\epsilon, b+\epsilon)$ is a finite set.

3) $\mu'(x) > 0$ a.e. on $[a, b]$ and for all sufficiently large n, $\mu_n'(x) > 0$ a.e. on $[a, b]$.

The following result on ratio asymptotics of orthogonal polynomials with respect to varying measures takes place. In [8, Theorem 1] it is proved

Lemma 4.2.3. *Suppose that, for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S and $\{\mu_n\}$ is a Denisov-type sequence on S. Then, for each fixed $k \in \mathbb{Z}$,*

$$\lim_{n\to\infty}\frac{l_{n,n+k}(z)}{l_{n,n+k-1}(z)} = \varphi_{\widetilde{\Delta}}(z)\,,$$

uniformly on compact subsets of $\mathbb{C} \setminus S$, where $\widetilde{\Delta} = [a, b]$ is the interval that appears in part 2) of Definition 4.2.2.

Regarding relative asymptotics of orthogonal polynomials with respect to varying measures, in [8, Theorem 2] the authors prove

Lemma 4.2.4. *Suppose that for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, 2k)$ is strongly admissible on S and $\{d\mu_n\}$ is a Denisov-type sequence on S. Let h be a non-negative Borel measurable function on S verifying:*

1) There exists an algebraic polynomial Q, such that $Q\,h^{\pm 1} \in L^\infty(S)$.

2) For each $k \in \mathbb{Z}$, $(\{h\,d\mu_n\}, \{w_{2n}\}, 2k)$ is strongly admissible on S.

Let $\{g_n\}_n \in \mathbb{N}$ be a sequence of continuous functions on S which converges to $g > 0$ uniformly on S. For each $n \in \mathbb{N}$, set $h_n = h\,g_n$ and let $\{q_{n,m}\}_{m\in\mathbb{N}}$, be the sequence of orthonormal polynomials with respect to $h_n\,d\mu_n/w_{2n}$. Then, for each fixed $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{q_{n,n+k}(z)}{l_{n,n+k}(z)} = \frac{1}{D(\widetilde{h}\,\widetilde{g}, 1/\varphi_{\widetilde{\Delta}}(z))}$$

uniformly on compact subsets of $\mathbb{C} \setminus S$, where $\widetilde{\Delta} = [a, b]$ is the interval that appears in part 2) of Definition 4.2.2, and $\widetilde{h}(\theta) = h(\tau(\cos\theta))$, $\widetilde{g}(\theta) = g(\tau(\cos\theta))$, τ being the affine-linear transformation that maps $[-1, 1]$ onto $\widetilde{\Delta}$.

In connection with the weak limit of sequences of varying measures Corollary 3 in [8] states the following. For the proof the authors refer to [41, Theorem 9] and [14, Theorem 8].

Lemma 4.2.5. *Suppose that, for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S and $\{\mu_n\}$ is a Denisov type sequence on S. Then, for each $k \in \mathbb{Z}$, and any function f continuous on S, we have*

$$\lim_{n\to\infty} \int f(x) \frac{l^2_{n,n+k}(x)d\mu_n(x)}{w_{2n}(x)} = \frac{1}{\pi}\int_a^b f(x)\frac{dx}{\sqrt{(b-x)(x-a)}}\,,$$

where $\widetilde{\Delta} = [a, b]$.

4.3 Weak convergence of the varying measures $q^2_{\mathbf{n},j}(x)\,d|\rho_{\mathbf{n},j}|(x)$ and uniform boundedness of the sequences $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$

In order to prove that for each $j = -m_2, \ldots, m_1$, the sequence $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}$ is uniformly bounded on each compact subset of $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_j)$, Theorem 2.2.5 would be sufficient if $\Delta_j = \widetilde{\Delta}_j, j = -m_2, \ldots, m_1$. To allow the compact sets to enter the connected components of $\Delta_j \setminus \operatorname{supp}(\sigma_j)$, we need to show that the zeros of $Q_{\mathbf{n},j}$ falling in the intervals I (see Propositions 2.1.5 and 2.1.7) are attracted to points in $\operatorname{supp}(\sigma_j) \setminus \widetilde{\Delta}_j$. In our aid comes the next result.

Lemma 4.3.1. *Let* $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ *be given, and let* $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\, n_{2,0} - |\mathbf{n}_2|) < \infty\,, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\, n_{1,0} - |\mathbf{n}_1|) < \infty\,. \tag{4.9}$$

For any continuous function f *on* $\operatorname{supp}(\sigma_j)$

$$\lim_{\mathbf{n}\in\Lambda} \int f(x)\, q_{\mathbf{n},j}^2(x)\, d|\rho_{\mathbf{n},j}|(x) = \frac{1}{\pi}\int_{a_j}^{b_j} f(x) \frac{dx}{\sqrt{(b_j-x)(x-a_j)}}\,, \tag{4.10}$$

where $\widetilde{\Delta}_j = [a_j, b_j]$, $-m_2 \le j \le m_1$. *In particular,*

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n},j}\, h_{\mathbf{n},j-1}(z) = \frac{1}{\sqrt{(z-b_j)(z-a_j)}}\,, \quad \mathcal{K} \subset \mathbb{C}\setminus \operatorname{supp}(\sigma_j)\,, \tag{4.11}$$

where $\sqrt{(z-b_j)(z-a_j)} > 0$ *if* $z > b_j$. *Consequently, for* $j = -m_2, \ldots, m_1$, *each point of* $\operatorname{supp}(\sigma_j) \setminus \widetilde{\Delta}_j$ *is a limit of zeros of* $\{Q_{\mathbf{n},j}\}, \mathbf{n}\in\Lambda$.

Proof. We will prove this by induction on j. For $j = m_1$, using Lemma 4.2.5 (see also Corollary 3 in [8]) and the second condition in (4.9), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \int_{\Delta_{m_1}} f(x) q_{\mathbf{n},m_1}^2(x) \frac{d|\sigma_{m_1}|(x)}{|Q_{\mathbf{n},m_1-1}(x)|} = \frac{1}{\pi}\int_{\widetilde{\Delta}_{m_1}} f(x) \frac{dx}{\sqrt{(b_{m_1}-x)(x-a_{m_1})}}\,,$$

where f is continuous on $\operatorname{supp}(\sigma_{m_1})$. Take $f(x) = (z-x)^{-1}$ where $z \in \mathbb{C}\setminus \operatorname{supp}(\sigma_{m_1})$. According to (4.8) and the previous limit one obtains that

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n},m_1} h_{\mathbf{n},m_1-1}(z) = \frac{1}{\sqrt{(z-b_{m_1})(z-a_{m_1})}} =: h_{m_1}(z)\,,$$

pointwise on $\mathbb{C}\setminus\operatorname{supp}(\sigma_{m_1})$. Since

$$\left| \int_{\Delta_{m_1}} \frac{q_{\mathbf{n},m_1}^2(x)}{z-x} \frac{d|\sigma_{m_1}|(x)}{|Q_{\mathbf{n},m_1-1}(x)|} \right| \le \frac{1}{d(\mathcal{K}, \operatorname{supp}(\sigma_{m_1}))}\,, \quad z \in \mathcal{K} \subset \mathbb{C}\setminus\operatorname{supp}(\sigma_{m_1})\,,$$

where $d(\mathcal{K}, \operatorname{supp}(\sigma_{m_1}))$ denotes the distance between the two compact sets, the sequence $\{h_{\mathbf{n},m_1-1}\}$, $\mathbf{n}\in\Lambda$, is uniformly bounded on compact subsets of $\mathbb{C}\setminus\operatorname{supp}(\sigma_{m_1})$ and (4.11) follows for $j = m_1$.

Let $\zeta \in \operatorname{supp}(\sigma_{m_1}) \setminus \widetilde{\Delta}_{m_1}$. Take $r > 0$ sufficiently small so that the circle $C_r = \{z : |z-\zeta| = r\}$ surrounds no other point of $\operatorname{supp}(\sigma_{m_1})\setminus\widetilde{\Delta}_{m_1}$ and contains no zero of $q_{\mathbf{n},m_1}, \mathbf{n}\in\Lambda$. From (4.11) for $j = m_1$

$$\lim_{\mathbf{n}\in\Lambda} \frac{1}{2\pi i}\int_{C_r} \frac{\varepsilon_{\mathbf{n},m_1} h'_{\mathbf{n},m_1-1}(z)}{\varepsilon_{\mathbf{n},m_1} h_{\mathbf{n},m_1-1}(z)}\, dz = \frac{1}{2\pi i}\int_{C_r} \frac{h'_{m_1}(z)}{h_{m_1}(z)}\, dz = 0\,. \tag{4.12}$$

Since ζ is a mass point of σ_{m_1}, formula (4.8) indicates that either $h_{\mathbf{n},m_1-1}$ has a simple pole at ζ or $Q_{\mathbf{n},m_1}(\zeta) = 0$. In any case, from (4.12) and the argument principle, it follows that for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, $Q_{\mathbf{n},m_1}$ must have a simple zero inside C_r. The parameter r can be taken arbitrarily small; therefore, the last statement of the lemma readily follows and the basis of induction is fulfilled.

Let us assume that the lemma is satisfied for $j \in \{k+1,\ldots,m_1\}, -m_2 \leq k \leq m_1 - 1$, and let us prove that it is also true for k. From (4.11) applied to $j = k+1$, we have that

$$\lim_{\mathbf{n}\in\Lambda} |h_{\mathbf{n},k}(x)| = \frac{1}{\sqrt{|(x-b_{k+1})(x-a_{k+1})|}}\,,$$

uniformly on $\Delta_k \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_{k+1})$. It follows that $\{|h_{\mathbf{n},k}|d|\sigma_k|\}, \mathbf{n} \in \Lambda$, is a sequence of Denisov type measures according to Definition 4.2.2. Additionally, $(\{|h_{\mathbf{n},k}|d|\sigma_k|\}, \{|Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}|\}, l), \mathbf{n} \in \Lambda$, is strongly admissible as in Definition 4.2.1 for each $l \in \mathbb{Z}$. Therefore, we can apply Lemma 4.2.5 of which (4.10) for $j = k$ is a particular case. In the proof of Lemma 4.2.5 (see [8, Corollary 3], [14, Theorem 8], and [41, Theorem 9]), it is required that the inequality $\deg(Q_{\mathbf{n},j-1}Q_{\mathbf{n},j+1}) - 2\deg(Q_{\mathbf{n},j}) \leq C$ holds for every $\mathbf{n} \in \Lambda$, where $C \geq 0$ is a constant. It is straightforward to check that this condition is satisfied under the assumption (4.9).

Now we return to the induction argument. From (4.10) for $j = k$, (4.11) and the rest of the statements of the lemma immediately follow just as in the case when $j = m_1$. With this we conclude the proof. □

Now we are ready to prove the uniform boundedness of the sequences of ratios $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}$.

Lemma 4.3.2. *Let $S^1 = \mathcal{N}'(\sigma_0^1,\ldots,\sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2,\ldots,\sigma_{m_2}^2)$ be given, and let $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\,n_{2,0} - |\mathbf{n}_2|) < \infty\,, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\,n_{1,0} - |\mathbf{n}_1|) < \infty\,.$$

Let us assume that there exists $l = (l_1; l_2), 0 \leq l_1 \leq m_1, 0 \leq l_2 \leq m_2$, such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$. Then, for each $j = -m_2,\ldots,m_1$, and each compact set $\mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)$ there exist positive constants $C_{j,1}(\mathcal{K}), C_{j,2}(\mathcal{K})$ such that

$$C_{j,1}(\mathcal{K}) \leq \inf_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq \sup_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq C_{j,2}(\mathcal{K}),$$

for all sufficiently large $|\mathbf{n}_1|, \mathbf{n} \in \Lambda$.

Proof. The uniform bound from above and below on each fixed compact subset $\mathcal{K} \subset \mathbb{C} \setminus \Delta_j$ (for all $\mathbf{n} \in \Lambda$) is a direct consequence of the interlacing property of the zeros of $Q_{\mathbf{n}^l,j}$ and $Q_{\mathbf{n},j}$. In fact, comparing distances to $z \in \mathcal{K}$ of consecutive interlacing zeros, it is easy to verify that

$$\min\left\{d_1, \frac{d_1}{d_2}\right\} \leq \inf_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq \sup_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq \frac{\max\{d_2, d_2^2\}}{d_1}\,,$$

where d_2 denotes the diameter of $\mathcal{K} \cup \Delta_j$ and d_1 denotes the distance between $\mathcal{K}$ and Δ_j. So, for such compact sets the assertion has been proved.

The additional restrictions made in the lemma guarantee that the zeros of the polynomials $Q_{\mathbf{n}^l,j}$ and $Q_{\mathbf{n},j}$ lying in $\Delta_j \setminus \operatorname{supp}(\sigma_j)$ converge to the mass points as results from Lemma 4.3.1. Let $\mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)$ and suppose that $\mathcal{K} \cap \Delta_j \neq \emptyset$. Notice that $\mathcal{K}$ can intersect at most a finite number of open intervals $I_1, \ldots, I_M$ forming the connected components of $\Delta_j \setminus \operatorname{supp}(\sigma_j)$. The polynomials $Q_{\mathbf{n}^l,j}$ and $Q_{\mathbf{n},j}$ can have at most one zero in each of those intervals. Consequently, for all $|\mathbf{n}_1|, \mathbf{n} \in \Lambda$, sufficiently large, the zeros of $Q_{\mathbf{n}^l,j}$ and $Q_{\mathbf{n},j}$ lie at a positive distance ε from $\mathcal{K}$. Now, it is easy to show that for all sufficiently large $|\mathbf{n}_1|$

$$\min\left\{\varepsilon, \frac{\varepsilon}{d_2}\right\} \leq \inf_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq \sup_{z\in\mathcal{K}} \left|\frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)}\right| \leq \frac{\max\{d_2, d_2^2\}}{\varepsilon}.$$

This concludes the proof. □

4.4 The system of boundary value problems

From Lemma 4.3.2 we know that the sequences

$$\left\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\right\}_{\mathbf{n}\in\Lambda}, \qquad j = -m_2, \ldots, m_1,$$

are uniformly bounded on each compact subset of $\mathbb{C} \setminus \operatorname{supp}(\sigma_j)$ for all sufficiently large $|\mathbf{n}_1|$. By Montel's theorem, there exists a subsequence of multi-indices $\Lambda' \subset \Lambda$ and a collection of functions $\widetilde{F}_j^{(l)}$, holomorphic in $\mathbb{C} \setminus \operatorname{supp}(\sigma_j)$, such that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} = \widetilde{F}_j^{(l)}(z), \quad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j), \quad j = -m_2, \ldots, m_1. \tag{4.13}$$

In principle, the functions $\widetilde{F}_j^{(l)}$ may depend on Λ'. We shall see that this is not the case and, therefore, the limit in (4.13) holds for $\mathbf{n} \in \Lambda$. First, let us obtain some general information on the functions $\widetilde{F}_j^{(l)}$.

Proposition 4.4.1. *The limiting functions $\widetilde{F}_j^{(l)}$ satisfy the following properties:*

a) For each $j = -m_2, \ldots, m_1$, the function $\widetilde{F}_j^{(l)}$ has no zeros in $\mathbb{C} \setminus \operatorname{supp}(\sigma_j)$.

b) For each $j = -m_2, \ldots, m_1$, the points in $\operatorname{supp}(\sigma_j) \setminus \widetilde{\Delta}_j$ are removable isolated singularities of $\widetilde{F}_j^{(l)}$. These points are not zeros of $\widetilde{F}_j^{(l)}$.

c) If $-l_2 \leq j \leq l_1$, the function $\widetilde{F}_j^{(l)}$ has a simple pole at infinity and $(\widetilde{F}_j^{(l)})'(\infty) = 1$, whereas, for $j \in \{-m_2, \ldots, -l_2 - 1\} \cup \{l_1 + 1, \ldots, m_1\}$, it is analytic at infinity and $\widetilde{F}_j^{(l)}(\infty) = 1$.

Proof. By Lemma 4.3.2 we know that the sequence $|Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}|, \mathbf{n} \in \Lambda$, is uniformly bounded on compact subsets of $\mathbb{C}\setminus \operatorname{supp}(\sigma_j)$ from below by a positive constant, for all sufficiently large $|\mathbf{n}_1|$. Therefore, the assertion a) follows.

It is clear that the points in $\operatorname{supp}(\sigma_j) \setminus \widetilde{\Delta}_j$ are isolated singularities of $\widetilde{F}_j^{(l)}$. Let $\zeta \in \operatorname{supp}(\sigma_j) \setminus \widetilde{\Delta}_j$. By Lemma 4.3.1, ζ is a limit of zeros of $Q_{\mathbf{n},j}$ and $Q_{\mathbf{n}^l,j}$ as $|\mathbf{n}_1| \to \infty, \mathbf{n} \in \Lambda$, and in a sufficiently small neighborhood of ζ, for large $|\mathbf{n}_1|, \mathbf{n} \in \Lambda$, there can be at most one zero of these polynomials (so there is exactly one, for all sufficiently large $|\mathbf{n}_1|$). Let $\lim_{\mathbf{n}\in\Lambda} \zeta_{\mathbf{n}} = \zeta$, where $Q_{\mathbf{n},j}(\zeta_{\mathbf{n}}) = 0$. From formula (4.13)

$$\lim_{\mathbf{n}\in\Lambda'} \frac{(z-\zeta_{\mathbf{n}})Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} = (z-\zeta)\widetilde{F}_j^{(l)}(z), \quad \mathcal{K} \subset (\mathbb{C} \setminus \operatorname{supp}(\sigma_j)) \cup \{\zeta\}\,,$$

and $(z-\zeta)\widetilde{F}_j^{(l)}(z)$ is analytic in a neighborhood of ζ. Hence ζ is not an essential singularity of $\widetilde{F}_j^{(l)}$. Taking into consideration that $Q_{\mathbf{n}^l,j}, \mathbf{n} \in \Lambda$, also has a sequence of zeros converging to ζ, from the argument principle it follows that ζ is a removable singularity of $\widetilde{F}_j^{(l)}$ which is not a zero.

According to the definitions of $Q_{\mathbf{n},j}, Q_{\mathbf{n}^l,j}$, and Propositions 2.1.5 and 2.1.7 (see also (4.1)), when $-l_2 \le j \le l_1$, we have that $\deg Q_{\mathbf{n}^l,j} = N_{\mathbf{n}^l,j} = N_{\mathbf{n},j}+1 = \deg Q_{\mathbf{n},j} + 1$ whereas, for $j \in \{-m_2, \ldots, -l_2-1\} \cup \{l_1+1, \ldots, m_1\}$, we obtain that $\deg Q_{\mathbf{n}^l,j} = N_{\mathbf{n}^l,j} = N_{\mathbf{n},j} = \deg Q_{\mathbf{n},j}$. This implies the assertion c). □

Now, let us prove that the functions $\widetilde{F}_j^{(l)}, j = -m_2, \ldots, m_1$, satisfy a system of boundary value problems.

Lemma 4.4.2. *Let $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ be given, and let $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\, n_{2,0} - |\mathbf{n}_2|) < \infty, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\, n_{1,0} - |\mathbf{n}_1|) < \infty.$$

Let us assume that there exists $l = (l_1; l_2), 0 \le l_1 \le m_1, 0 \le l_2 \le m_2$, such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$. Then, there exists a normalization $F_j^{(l)}$, $j = -m_2, \ldots, m_1$, by positive constants, of the functions $\widetilde{F}_j^{(l)}$ given in (4.13), which verifies the system of boundary value problems

$$\begin{aligned}
&1) \qquad F_j^{(l)},\ 1/F_j^{(l)} \in H(\mathbb{C} \setminus \widetilde{\Delta}_j)\,,\\
&2) \qquad (F_j^{(l)})'(\infty) > 0\,, \quad j \in \{-l_2, \ldots, l_1\}\,,\\
&2') \qquad F_j^{(l)}(\infty) > 0\,, \quad j \in \{-m_2, \ldots, -l_2-1\} \cup \{l_1+1, \ldots, m_1\}\,,\\
&3) \qquad |F_j^{(l)}(x)|^2 \frac{1}{|(F_{j-1}^{(l)}\, F_{j+1}^{(l)})(x)|} = 1,\ x \in \widetilde{\Delta}_j\,,
\end{aligned} \tag{4.14}$$

where $F_{-m_2-1}^{(l)} \equiv F_{m_1+1}^{(l)} \equiv 1$.

Proof. The assertions 1), 2), and 2') were proved above for the functions $\widetilde{F}_j^{(l)}$. Consequently, they are satisfied by any normalization of these functions by means of positive constants.

From (4.7) applied to $\mathbf{n}$ and $\mathbf{n}^l$, for each $j = -m_2, \ldots, m_1$, we have

$$\int x^\nu \, Q_{\mathbf{n},j}(x) \, d|\rho_{\mathbf{n},j}|(x) = 0, \qquad \nu = 0, \ldots, N_{\mathbf{n},j} - 1\,,$$

and

$$\int x^\nu \, Q_{\mathbf{n}^l,j}(x) \, g_{\mathbf{n},j}(x) \, d|\rho_{\mathbf{n},j}|(x) = 0\,, \qquad \nu = 0, \ldots, N_{\mathbf{n}^l,j} - 1\,,$$

where

$$g_{\mathbf{n},j}(x) = \frac{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|}{|Q_{\mathbf{n}^l,j-1}(x)Q_{\mathbf{n}^l,j+1}(x)|} \frac{|h_{\mathbf{n}^l,j}(x)|}{|h_{\mathbf{n},j}(x)|}\,, \quad d\rho_{\mathbf{n},j}(x) = \frac{h_{\mathbf{n},j}(x) d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)}\,.$$

From (4.11) and (4.13)

$$\lim_{\mathbf{n}\in\Lambda'} g_{\mathbf{n},j}(x) = |(\widetilde{F}_{j-1}^{(l)}\widetilde{F}_{j+1}^{(l)})(x)|^{-1} \tag{4.15}$$

uniformly on Δ_j.

Fix $j \in \{-m_2, \ldots, -l_2 - 1\} \cup \{l_1 + 1, \ldots, m_1\}$. As mentioned above, for this selection of j we have that $\deg Q_{\mathbf{n}^l,j} = \deg Q_{\mathbf{n},j} = N_{\mathbf{n},j}$. Due to (4.15) and (4.13), from Lemmas 4.2.3 and 4.2.4 (Theorems 1 and 2 of [8]), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} = \frac{S_j(z)}{S_j(\infty)} = \widetilde{S}_j(z) = \widetilde{F}_j^{(l)}(z)\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_j)\,, \tag{4.16}$$

where S_j is the Szegő function on $\overline{\mathbb{C}}\setminus\widetilde{\Delta}_j$ with respect to $|\widetilde{F}_{j-1}^{(l)}(x)\widetilde{F}_{j+1}^{(l)}(x)|^{-1}, x \in \widetilde{\Delta}_j$. The function S_j is uniquely determined by

$$\begin{aligned} &1) \quad S_j, 1/S_j \in H(\overline{\mathbb{C}} \setminus \widetilde{\Delta}_j)\,,\\ &2) \quad S_j(\infty) > 0\,,\\ &3) \quad |S_j(x)|^2 \frac{1}{\big|(\widetilde{F}_{j-1}^{(l)}\widetilde{F}_{j+1}^{(l)})(x)\big|} = 1, \qquad x \in \widetilde{\Delta}_j\,. \end{aligned} \tag{4.17}$$

Now, fix $j \in \{-l_2, \ldots, l_1\}$. In this situation $\deg Q_{\mathbf{n}^l,j} = \deg Q_{\mathbf{n},j} + 1 = N_{\mathbf{n},j} + 1$. Let $Q^*_{\mathbf{n},j}(x)$ be the monic polynomial of degree $N_{\mathbf{n},j}$ orthogonal with respect to the varying measure $g_{\mathbf{n},j}\, d|\rho_{\mathbf{n},j}|$. Using the same arguments as above, we have

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q^*_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} = \frac{S_j(z)}{S_j(\infty)} = \widetilde{S}_j(z)\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_j)\,. \tag{4.18}$$

On the other hand, since $\deg Q_{\mathbf{n}^l,j} = \deg Q^*_{\mathbf{n},j} + 1$ and both of these polynomials are orthogonal with respect to the same varying weight, by Lemma 4.2.3 [8, Theorem 1] and (4.11), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,j}(z)}{Q^*_{\mathbf{n},j}(z)} = \frac{\varphi_j(z)}{\varphi_j'(\infty)} = \widetilde{\varphi}_j(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)\,, \tag{4.19}$$

where φ_j denotes the conformal representation of $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_j$ onto $\{w : |w| > 1\}$ such that $\varphi_j(\infty) = \infty$ and $\varphi_j'(\infty) > 0$. The function φ_j is uniquely determined by

$$\begin{array}{ll} 1) & \varphi_j, 1/\varphi_j \in H(\mathbb{C} \setminus \widetilde{\Delta}_j)\,, \\ 2) & \varphi_j'(\infty) > 0\,, \\ 3) & |\varphi_j(x)| = 1, \quad x \in \widetilde{\Delta}_j\,. \end{array} \tag{4.20}$$

From (4.13), (4.18), and (4.19), we obtain

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} = (\widetilde{S}_j\widetilde{\varphi}_j)(z) = \widetilde{F}_j^{(l)}(z)\,, \quad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)\,. \tag{4.21}$$

Thus,

$$\widetilde{F}_j^{(l)} = \begin{cases} \widetilde{S}_j\widetilde{\varphi}_j\,, & j \in \{-l_2, \ldots, l_1\}\,, \\ \widetilde{S}_j\,, & j \in \{-m_2, \ldots, -l_2 - 1\} \cup \{l_1 + 1, \ldots, m_1\}\,, \end{cases} \tag{4.22}$$

and from (4.17) and (4.22) it follows that

$$|\widetilde{F}_j^{(l)}(x)|^2 \frac{1}{\left|(\widetilde{F}_{j-1}^{(l)}\widetilde{F}_{j+1}^{(l)})(x)\right|} = \frac{1}{\omega_j}\,, \qquad x \in \widetilde{\Delta}_j\,, \qquad j = -m_2, \ldots, m_1\,, \tag{4.23}$$

where

$$\omega_j = \begin{cases} (S_j\,\varphi_j')^2(\infty)\,, & j \in \{-l_2, \ldots, l_1\}\,, \\ S_j^2(\infty)\,, & j \in \{-m_2, \ldots, -l_2 - 1\} \cup \{l_1 + 1, \ldots, m_1\}\,. \end{cases} \tag{4.24}$$

Now, let us show that there exist positive constants $c_j, j = -m_2, \ldots, m_1$, such that the functions $F_j^{(l)} = c_j\widetilde{F}_j^{(l)}$ satisfy (4.14). In fact, according to (4.23) for any such constants c_j we have that

$$|F_j^{(l)}(x)|^2 \frac{1}{\left|(F_{j-1}^{(l)}F_{j+1}^{(l)})(x)\right|} = \frac{c_j^2}{c_{j-1}c_{j+1}\omega_j}\,, \qquad x \in \widetilde{\Delta}_j\,, \qquad j = -m_2, \ldots, m_1\,,$$

where $c_{-m_2-1} = c_{m_1+1} = 1$. The problem reduces to find appropriate constants c_j such that

$$\frac{c_j^2}{c_{j-1}c_{j+1}\omega_j} = 1\,, \qquad j = -m_2, \ldots, m_1\,. \tag{4.25}$$

Taking logarithm, we obtain the linear system of equations

$$2\log c_j - \log c_{j-1} - \log c_{j+1} = \log\omega_j\,, \qquad j = -m_2, \ldots, m_1 \tag{4.26}$$

($c_{-m_2-1} = c_{m_1+1} = 1$) on the unknowns $\log c_j$. This system has a unique solution with which we conclude the proof. □

4.5 The limiting functions of the sequences $\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$

Recall that given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1}), C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := F/C\,.$$

C is called the leading coefficient of F. When $C \in \mathbb{R}$, $\mathrm{sg}(F(\infty))$ represents the sign of C.

Proof of Theorem 1.3.4. Since the families of functions

$$\left\{Q_{\mathbf{n}^l,j}/Q_{\mathbf{n},j}\right\}_{\mathbf{n}\in\Lambda}\,, \qquad j = -m_2, \ldots, m_1,$$

are uniformly bounded on each compact subset $\mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_j)$ for all sufficiently large $|\mathbf{n}_1|, \mathbf{n} \in \Lambda$, uniform convergence on compact subsets of the indicated region follows from proving that any convergent subsequence has the same limit. According to Lemma 4.4.2 the limit functions, appropriately normalized, of a convergent subsequence satisfy the same system of boundary value problems (4.14). From Lemma 4.2 in [5] (see also [3]) this system has a unique solution.

It remains to show that the functions defined in (1.23) satisfy (4.14). When multiplying two consecutive branches, the singularities on the common slit cancel out by the Schwarz reflection principle; therefore, 1) takes place since only the singularities of $\psi_j^{(l)}$ on $\widetilde{\Delta}_j$ remain. From the definition of $\psi^{(l)}$ it also follows that for $j = -l_2, \ldots, l_1$, $F_j^{(l)}$ has at infinity a simple pole, whereas it is regular and different from zero at infinity when $j \in \{-m_2, \ldots, -l_2-1\} \cup \{l_1+1, \ldots, m_1\}$. The factor sign in front of (1.23) guarantees the positivity claimed in 2) and 2′).

In order to verify 3), notice that $F_j^{(l)}/F_{j-1}^{(l)} = \mathrm{sg}(\psi_{j-1}^{(l)}(\infty))/\psi_{j-1}^{(l)}$. Therefore, if $j = -m_2+1, \ldots, m_1$,

$$\frac{|F_j^{(l)}(x)|^2}{|F_{j-1}^{(l)}(x)F_{j+1}^{(l)}(x)|} = \frac{|\psi_j^{(l)}(x)|}{|\psi_{j-1}^{(l)}(x)|} = 1, \qquad x \in \widetilde{\Delta}_j,$$

on account of (1.21). For $j = -m_2$ and $x \in \widetilde{\Delta}_{-m_2}$, from the definition and (1.21)

$$\frac{|F_{-m_2}^{(l)}(x)|^2}{|F_{-m_2+1}^{(l)}(x)|} = |\psi_{-m_2}^{(l)}(x)|\,|\prod_{k=-m_2}^{m_1} \psi_k^{(l)}(x)| = |\prod_{k=-m_2-1}^{m_1} \psi_k^{(l)}(x)| = 1,$$

since $\prod_{k=-m_2-1}^{m_1} \psi_k^{(l)}$ is constantly equal to 1 or -1 on all $\overline{\mathbb{C}}$. The proof is complete. □

The following corollary complements Theorem 1.3.4. The proof is similar to that of Corollary 4.1 in [5].

Corollary 4.5.1. *Let* $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ *be given, and let* $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\,n_{2,0} - |\mathbf{n}_2|) < \infty, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\,n_{1,0} - |\mathbf{n}_1|) < \infty.$$

Let us assume that there exists $l = (l_1; l_2), 0 \le l_1 \le m_1, 0 \le l_2 \le m_2$, *such that for all* $\mathbf{n} \in \Lambda$ *we have that* $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$. *Let* $\{q_{\mathbf{n},j} = \kappa_{\mathbf{n},j}\, Q_{\mathbf{n},j}\}_{j=-m_2}^{m_1}, \mathbf{n} \in \Lambda$, *be the system of orthonormal polynomials as defined in* (4.5) *and* $\{K_{\mathbf{n},j}\}_{j=-m_2}^{m_1}, \mathbf{n} \in \Lambda$, *the values given by* (4.4). *Then, for each fixed* $j = -m_2, \ldots, m_1$, *we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}^l,j}}{\kappa_{\mathbf{n},j}} = \kappa_j^{(l)}\,, \tag{4.27}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{K_{\mathbf{n}^l,j}}{K_{\mathbf{n},j}} = \kappa_j^{(l)} \cdots \kappa_{m_1}^{(l)}\,, \tag{4.28}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,j}(z)}{q_{\mathbf{n},j}(z)} = \kappa_j^{(l)} \widetilde{F}_j^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)\,, \tag{4.29}$$

where

$$\kappa_j^{(l)} = \frac{c_j^{(l)}}{\sqrt{c_{j-1}^{(l)} c_{j+1}^{(l)}}}\,, \qquad c_j^{(l)} = \begin{cases} (F_j^{(l)})'(\infty)\,, & j \in \{-l_2, \ldots, l_1\}\,, \\[2ex] F_j^{(l)}(\infty)\,, & j \notin \{-l_2, \ldots, l_1\}\,, \end{cases} \tag{4.30}$$

($c_{-m_2-1}^{(l)} = c_{m_1+1}^{(l)} = 1$) *and the functions* $F_j^{(l)}$ *are defined by* (1.23).

Proof. By Theorem 1.3.4, we have limit in (4.15) along the whole sequence Λ. Reasoning as in the deduction of formulas (4.16) and (4.21), but now in connection with orthonormal polynomials (see Lemmas 4.2.3 and 4.2.4, [8, Theorems 1, 2]), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,j}(z)}{q_{\mathbf{n},j}(z)} = \begin{cases} (S_j\,\varphi_j)(z)\,, & j \in \{-l_2, \ldots, l_1\}\,, \\[2ex] S_j(z)\,, & j \in \{-m_2, \ldots, -l_2-1\} \cup \{l_1+1, \ldots, m_1\}\,, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\sigma_j)$, where S_j is defined in (4.17). This formula, divided by (4.16) or (4.21) according to the value of j gives

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}^l,j}}{\kappa_{\mathbf{n},j}} = \sqrt{\omega_j} = \frac{c_j}{\sqrt{c_{j-1}c_{j+1}}}\,,$$

where ω_j is defined in (4.24), and the c_j are the normalizing constants found in Lemma 4.4.2 solving the linear system of equations (4.26) which ensure that

$$F_j^{(l)} \equiv c_j \widetilde{F}_j^{(l)}\,, \qquad j = -m_2, \ldots, m_1\,,$$

with $F_j^{(l)}$ satisfying (4.14) and thus given by (1.23). Since $(\widetilde{F}_j^{(l)})'(\infty) = 1, j \in \{-l_2, \ldots, l_1\}$, and $(\widetilde{F}_j^{(l)})(\infty) = 1, j \in \{-m_2, \ldots, -l_2-1\} \cup \{l_1+1, \ldots, m_1\}$ formula (4.27) immediately follows with $\kappa_j^{(l)}$ as in (4.30).

From the definition of $\kappa_{\mathbf{n},j}$, we have that

$$K_{\mathbf{n},j} = \kappa_{\mathbf{n},j} \cdots \kappa_{\mathbf{n},m_1} .$$

Taking the ratio of these constants for the multi-indices $\mathbf{n}$ and $\mathbf{n}^l$ and using (4.27), we get (4.28). Formula (4.29) is an immediate consequence of (4.27) and (1.22). □

Let $\mathrm{lcm}(a, b)$ denote the least common multiple of two integers a and b, and define $m := \mathrm{lcm}(m_1+1, m_2+1)$, $d_1 := m/(m_1+1)$, $d_2 := m/(m_2+1)$. Within the class of pairs $l = (l_1; l_2)$ with $0 \le l_1 \le m_1$, $0 \le l_2 \le m_2$, we distinguish the subclass

$$L := \{(l_1; l_2) : l_1 \equiv r \bmod (m_1+1),\ l_2 \equiv r \bmod (m_2+1) \text{ for some } 0 \le r \le m-1\}.$$

It is easy to check that for different $r, 0 \le r \le m-1$, the pairs (l_1, l_2) in L are different. Let $\mathbf{p} := (\mathbf{p}_1; \mathbf{p}_2)$, where $\mathbf{p}_1 = (d_1, \ldots, d_1)$ and $\mathbf{p}_2 = (d_2, \ldots, d_2)$ have m_1+1 and m_2+1 components, respectively. By $\mathbf{n}+\mathbf{p}$ we denote the multi-index $(\mathbf{n}_1+\mathbf{p}_1; \mathbf{n}_2+\mathbf{p}_2)$.

Corollary 4.5.2. *Let $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ be given, and let $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\, n_{2,0} - |\mathbf{n}_2|) < \infty, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\, n_{1,0} - |\mathbf{n}_1|) < \infty.$$

Then, for each fixed $j \in \{-m_2, \ldots, m_1\}$, we have

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}+\mathbf{p},j}(z)}{Q_{\mathbf{n},j}(z)} = \prod_{l\in L} \widetilde{F}_j^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_j)\,, \tag{4.31}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}+\mathbf{p},j}}{\kappa_{\mathbf{n},j}} = \prod_{l\in L} \kappa_j^{(l)}\,, \tag{4.32}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}+\mathbf{p},j}(z)}{q_{\mathbf{n},j}(z)} = \prod_{l\in L} \kappa_j^{(l)} \widetilde{F}_j^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_j)\,. \tag{4.33}$$

Proof. Given $\mathbf{n} \in \Lambda$ and $0 \le r \le m$, let $\mathbf{n}(r) := \mathbf{n} + \mathbf{q}(r)$ where $\mathbf{q}(r) = (\mathbf{q}_1(r); \mathbf{q}_2(r))$ is the multi-index satisfying

$$\mathbf{q}_i(r) = (\underbrace{k+1, \ldots, k+1}_{s}, k, \ldots, k)\,, \quad r = k(m_i+1) + s\,,\ 0 \le s \le m_i\,.$$

Hence, $\mathbf{n}(0) = \mathbf{n}$, $\mathbf{n}(m) = \mathbf{n}+\mathbf{p}$ and $\mathbf{n}(r) \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ for every r.

We have

$$\frac{Q_{\mathbf{n}+\mathbf{p},j}(z)}{Q_{\mathbf{n},j}(z)} = \prod_{r=0}^{m-1} \frac{Q_{\mathbf{n}(r+1),j}(z)}{Q_{\mathbf{n}(r),j}(z)}\,.$$

In addition, by (1.22) we know that for each fixed $0 \leq r \leq m-1$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}(r+1),j}(z)}{Q_{\mathbf{n}(r),j}(z)} = \widetilde{F}_j^{(l)}(z), \qquad z \in \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_j)\,,$$

where $l = (l_1; l_2)$ is precisely the multi-index satisfying $l_1 \equiv r \bmod (m_1+1)$, $l_2 \equiv r \bmod (m_2+1)$. Therefore (4.31) follows. Relations (4.32) and (4.33) are proved analogously in view of (4.27) and (4.29). □

4.6 The limiting functions of the sequences $\{\mathcal{A}_{\mathbf{n}^l,j}/\mathcal{A}_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}$

At this point we need to introduce some notations. For $j \in \{-m_2, \ldots, m_1-1\}$, set

$$\delta_j := \begin{cases} 1, & \text{if} \quad \Delta_j \quad \text{is to the left of} \quad \Delta_{j+1}\,, \\ -1, & \text{if} \quad \Delta_j \quad \text{is to the right of} \quad \Delta_{j+1}\,. \end{cases}$$

For multi-indices $l = (l_1; l_2)$ such that $l_1 + l_2 \geq 2$, we define

$$\Delta_{j,\,l} := \begin{cases} 1, & \text{if} \quad j \geq l_1 + 2\,, \\ \delta_{j-1}, & \text{if} \quad j \in \{l_1, l_1+1\}\,, \\ -\delta_{j-1}\delta_j, & \text{if} \quad j \in \{-l_2+1, \ldots, l_1-1\}\,, \\ -\delta_j, & \text{if} \quad j \in \{-l_2-1, -l_2\}\,, \\ 1, & \text{if} \quad j \leq -l_2-2\,. \end{cases}$$

If $l_1 + l_2 = 1$ then

$$\Delta_{j,\,l} := \begin{cases} 1, & \text{if} \quad j \geq l_1 + 2\,, \\ \delta_{j-1}, & \text{if} \quad j \in \{l_1, l_1+1\}\,, \\ -\delta_j, & \text{if} \quad j \in \{-l_2-1, -l_2\}\,, \\ 1, & \text{if} \quad j \leq -l_2-2\,, \end{cases}$$

and for $l_1 = l_2 = 0$

$$\Delta_{j,(0;0)} := \begin{cases} 1, & \text{if} \quad j \geq 2\,, \\ \delta_0, & \text{if} \quad j = 1\,, \\ 1, & \text{if} \quad j = 0\,, \\ -\delta_{-1}, & \text{if} \quad j = -1\,, \\ 1, & \text{if} \quad j \leq -2\,. \end{cases}$$

Recall that $\varepsilon_{\mathbf{n},j}$ denotes the sign of the varying measure

$$d\rho_{\mathbf{n},j}(x) = \frac{h_{\mathbf{n},j}(x)d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)}\,.$$

Lemma 4.6.1. *For any* $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *and* $-m_2 \le j \le m_1$

$$\frac{\varepsilon_{\mathbf{n}^l,j}}{\varepsilon_{\mathbf{n},j}} = \prod_{k=j}^{m_1} \Delta_{k,l} \,. \tag{4.34}$$

Proof. We will denote by $\mathrm{sign}(f, \Delta)$ the sign of a function f on the interval Δ. Thus

$$\frac{\varepsilon_{\mathbf{n}^l,j}}{\varepsilon_{\mathbf{n},j}} = \mathrm{sign}(\frac{\mathcal{H}_{\mathbf{n}^l,j}\, Q_{\mathbf{n},j-1}\, Q_{\mathbf{n},j+1}}{\mathcal{H}_{\mathbf{n},j}\, Q_{\mathbf{n}^l,j-1}\, Q_{\mathbf{n}^l,j+1}}, \Delta_j)\,. \tag{4.35}$$

If $-l_2 \le j-1 \le l_1$, then $\deg(Q_{\mathbf{n}^l,j-1}) = 1 + \deg(Q_{\mathbf{n},j-1})$ and, therefore,

$$\mathrm{sign}(\frac{Q_{\mathbf{n},j-1}}{Q_{\mathbf{n}^l,j-1}}, \Delta_j) = \delta_{j-1}\,. \tag{4.36}$$

If $j-1 < -l_2$ or $j-1 > l_1$, then $\deg(Q_{\mathbf{n}^l,j-1}) = \deg(Q_{\mathbf{n},j-1})$, implying that

$$\mathrm{sign}(\frac{Q_{\mathbf{n},j-1}}{Q_{\mathbf{n}^l,j-1}}, \Delta_j) = 1\,. \tag{4.37}$$

Analogously, we have that for $-l_2 \le j+1 \le l_1$

$$\mathrm{sign}(\frac{Q_{\mathbf{n},j+1}}{Q_{\mathbf{n}^l,j+1}}, \Delta_j) = -\delta_j \tag{4.38}$$

and for $j+1 < -l_2$ or $j+1 > l_1$

$$\mathrm{sign}(\frac{Q_{\mathbf{n},j+1}}{Q_{\mathbf{n}^l,j+1}}, \Delta_j) = 1\,. \tag{4.39}$$

From (4.36)-(4.38) it follows that

$$\mathrm{sign}(\frac{Q_{\mathbf{n},j-1}\, Q_{\mathbf{n},j+1}}{Q_{\mathbf{n}^l,j-1}\, Q_{\mathbf{n}^l,j+1}}, \Delta_j) = \Delta_{j,\,l}\,. \tag{4.40}$$

Now, by (4.3)

$$\frac{\mathcal{H}_{\mathbf{n}^l,j}(x)}{\mathcal{H}_{\mathbf{n},j}(x)} = \frac{\int \frac{Q^2_{\mathbf{n}^l,j+1}(t)}{x-t} \frac{\mathcal{H}_{\mathbf{n}^l,j+1}(t)\, d\sigma_{j+1}(t)}{Q_{\mathbf{n}^l,j}(t) Q_{\mathbf{n}^l,j+2}(t)}}{\int \frac{Q^2_{\mathbf{n},j+1}(t)}{x-t} \frac{\mathcal{H}_{\mathbf{n},j+1}(t)\, d\sigma_{j+1}(t)}{Q_{\mathbf{n},j}(t) Q_{\mathbf{n},j+2}(t)}}\,.$$

Therefore,

$$\mathrm{sign}(\mathcal{H}_{\mathbf{n}^l,j}/\mathcal{H}_{\mathbf{n},j}, \Delta_j) = \frac{\varepsilon_{\mathbf{n}^l,j+1}}{\varepsilon_{\mathbf{n},j+1}}\,. \tag{4.41}$$

Since $\mathcal{H}_{\mathbf{n}^l,m_1} \equiv \mathcal{H}_{\mathbf{n},m_1} \equiv 1$, the right hand side of (4.41) equals 1 for $j = m_1$. Hence (4.34) follows from (4.35), (4.40) and (4.41). □

This lemma shows that $\varepsilon_{\mathbf{n}^l,j}/\varepsilon_{\mathbf{n},j}$ depends on j, l, and the relative positions of the intervals Δ_j but not on $\mathbf{n}$. Define the functions

$$\mathcal{A}_j^{(l)} := \widetilde{\psi}_j^{(l)} \prod_{k=j+1}^{m_1} \frac{\Delta_{k,\,l}}{(\kappa_k^{(l)})^2}$$

(the product should be understood to be equal to 1 when $j = m_1$).

Theorem 4.6.2. *Let* $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ *be given, and let* $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\,n_{2,0} - |\mathbf{n}_2|) < \infty\,, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\,n_{1,0} - |\mathbf{n}_1|) < \infty\,.$$

Let us assume that there exists $l = (l_1; l_2), 0 \le l_1 \le m_1, 0 \le l_2 \le m_2$, *such that for all* $\mathbf{n} \in \Lambda$ *we have that* $\mathbf{n}^l \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$. *Let* $\{\mathcal{A}_{\mathbf{n},j}\}_{j=-m_2-1}^{m_1}, \mathbf{n} \in \Lambda$, *be the associated sequences of "monic" linear forms defined by* (1.11)-(1.12). *Then, for each fixed* $j = -m_2-1, \ldots, m_1$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{\mathcal{A}_{\mathbf{n}^l,j}(z)}{\mathcal{A}_{\mathbf{n},j}(z)} = \mathcal{A}_j^{(l)}\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1})) \tag{4.42}$$

$(\operatorname{supp}(\sigma_{-m_2-1}) = \operatorname{supp}(\sigma_{m_1+1}) = \emptyset)$.

Proof. It follows from the definition of $\mathcal{H}_{\mathbf{n},j}$ and $\mathcal{H}_{\mathbf{n}^l,j}$ that

$$\frac{\mathcal{A}_{\mathbf{n}^l,j}(z)}{\mathcal{A}_{\mathbf{n},j}(z)} = \frac{\varepsilon_{\mathbf{n}^l,j+1} h_{\mathbf{n}^l,j}(z)}{\varepsilon_{\mathbf{n},j+1} h_{\mathbf{n},j}(z)} \frac{\varepsilon_{\mathbf{n},j+1}}{\varepsilon_{\mathbf{n}^l,j+1}} \frac{K^2_{\mathbf{n},j+1}}{K^2_{\mathbf{n}^l,j+1}} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} \frac{Q_{\mathbf{n},j+1}(z)}{Q_{\mathbf{n}^l,j+1}(z)}\,.$$

By Lemma 4.3.1,

$$\lim_{\mathbf{n}\in\Lambda} \frac{\varepsilon_{\mathbf{n}^l,j+1} h_{\mathbf{n}^l,j}(z)}{\varepsilon_{\mathbf{n},j+1} h_{\mathbf{n},j}(z)} = 1\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_{j+1})\,.$$

Using Lemma 4.6.1 and Corollary 4.5.1, we have

$$\lim_{\mathbf{n}\in\Lambda} \frac{\varepsilon_{\mathbf{n},j+1}}{\varepsilon_{\mathbf{n}^l,j+1}} \frac{K^2_{\mathbf{n},j+1}}{K^2_{\mathbf{n}^l,j+1}} = \prod_{k=j+1}^{m_1} \frac{\Delta_{k,\,l}}{(\kappa_k^{(l)})^2}\,.$$

Finally, applying (1.22) and (1.23) one obtains

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}^l,j}(z)}{Q_{\mathbf{n},j}(z)} \frac{Q_{\mathbf{n},j+1}(z)}{Q_{\mathbf{n}^l,j+1}(z)} = \widetilde{\psi}_j^{(l)}(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1}))\,.$$

Putting these relations together we get (4.42). □

Corollary 4.6.3. *Let* $S^1 = \mathcal{N}'(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}'(\sigma_0^2, \ldots, \sigma_{m_2}^2)$ *be given, and let* $\Lambda \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ *be an infinite sequence of distinct multi-indices such that*

$$\sup_{\mathbf{n}\in\Lambda}((m_2+1)\,n_{2,0} - |\mathbf{n}_2|) < \infty\,, \qquad \sup_{\mathbf{n}\in\Lambda}((m_1+1)\,n_{1,0} - |\mathbf{n}_1|) < \infty\,. \tag{4.43}$$

Then, for each fixed $j \in \{-m_2, \ldots, m_1\}$, *we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\mathcal{A}_{\mathbf{n}+\mathbf{p},j}(z)}{\mathcal{A}_{\mathbf{n},j}(z)} = \prod_{l\in L} \mathcal{A}_j^{(l)}(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1})) \tag{4.44}$$

$(\operatorname{supp}(\sigma_{-m_2-1}) = \operatorname{supp}(\sigma_{m_1+1}) = \emptyset)$. *Consequently, uniformly on each compact subset* $\mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1}))$,

$$\lim_{\mathbf{n}\in\Lambda} |\mathcal{A}_{\mathbf{n},j}(z)|^{1/|\mathbf{n}_1|} = \prod_{l\in L} |\mathcal{A}_j^{(l)}(z)|^{1/m}, \tag{4.45}$$

where $m = \operatorname{lcm}(m_1+1, m_2+1)$.

Proof. Using the same arguments employed to prove Corollary 4.5.2, we obtain (4.44). From (4.44) it is easy to deduce the $|\mathbf{n}_1|$-th root asymptotic of the linear forms.

In fact, it is easy to see that for each $\mathbf{n} \in \Lambda$ there exists $\mathbf{n}_0 \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ (which may depend on $\mathbf{n}$), whose entries are uniformly bounded by a constant C independent of $\mathbf{n}$ (condition (4.43) is used), such that $\mathbf{n} = r\mathbf{p} + \mathbf{n}_0$ for some $r \in \mathbb{Z}_+$. Write

$$\mathcal{A}_{\mathbf{n},j}(z) = \frac{\mathcal{A}_{\mathbf{n},j}(z)}{\mathcal{A}_{\mathbf{n}-\mathbf{p},j}(z)} \frac{\mathcal{A}_{\mathbf{n}-\mathbf{p},j}(z)}{\mathcal{A}_{\mathbf{n}-2\mathbf{p},j}(z)} \cdots \frac{\mathcal{A}_{\mathbf{n}_0+\mathbf{p},j}(z)}{\mathcal{A}_{\mathbf{n}_0,j}(z)} \mathcal{A}_{\mathbf{n}_0,j}(z).$$

Then

$$\frac{1}{|\mathbf{n}_1|} \log |\mathcal{A}_{\mathbf{n},j}(z)| = \frac{1}{|\mathbf{n}_1|} \log |\mathcal{A}_{\mathbf{n}_0,j}(z)| + \frac{1}{|\mathbf{n}_1|} \sum_{k=0}^{r-1} \log \left| \frac{\mathcal{A}_{\mathbf{n}_0+(k+1)\mathbf{p},j}(z)}{\mathcal{A}_{\mathbf{n}_0+k\mathbf{p},j}(z)} \right|.$$

Obviously,

$$\lim_{\mathbf{n}\in\Lambda} \frac{1}{|\mathbf{n}_1|} \log |\mathcal{A}_{\mathbf{n}_0,j}(z)| = 0, \qquad \mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1})),$$

and because of (4.44), uniformly on each compact subset $\mathcal{K} \subset \mathbb{C} \setminus (\operatorname{supp}(\sigma_j) \cup \operatorname{supp}(\sigma_{j+1}))$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{1}{|\mathbf{n}_1|} \sum_{k=0}^{r-1} \log \left| \frac{\mathcal{A}_{\mathbf{n}_0+(k+1)\mathbf{p},j}(z)}{\mathcal{A}_{\mathbf{n}_0+k\mathbf{p},j}(z)} \right| = \frac{1}{m} \log \left| \prod_{l\in L} \mathcal{A}_j^{(l)}(z) \right|,$$

since $|\mathbf{n}_1| = r|\mathbf{p}_1| + \mathcal{O}(1) = rm + \mathcal{O}(1), |\mathbf{n}_1| \to \infty$. □

The function appearing on the right hand side of (4.45) corresponds with the one on the right hand side of (1.17) associated to the vector equilibrium problem with interaction matrix $\mathcal{C}$ constructed taking $p_{1,k} = 1/(m_1+1)$, $0 \le k \le m_1$, and $p_{2,k} = 1/(m_2+1)$, $0 \le k \le m_2$. In that case, for each $j = -m_2-1, \ldots, m_1$, we have

$$G_j(z) = \prod_{l\in L} |\mathcal{A}_j^{(l)}(z)|^{1/m}, \qquad z \in \mathbb{C} \setminus (\widetilde{\Delta}_j \cup \widetilde{\Delta}_{j+1})$$

$(\Delta_{-m_2-1} = \Delta_{m_1+1} = \emptyset)$, where $m = \operatorname{lcm}(m_1+1, m_2+1)$.

5. RELATIVE ASYMPTOTICS

In this chapter and the next, we restrict our attention to type II multiple orthogonal polynomials. In Section 5.2, we establish some algebraic connections between the multiple orthogonal polynomials $Q_{\mathbf{n}}$ of the initial system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and the multiple orthogonal polynomials $\widetilde{Q}_{\mathbf{n}}$ of the perturbed system $\mathcal{N}(p_1\sigma_1, \ldots, p_m\sigma_m)$, where $p_1, \ldots, p_m$ denote polynomials with complex coefficients. Section 5.3 is used to explain some notational modifications we introduce in passing from mixed type multiple orthogonal polynomials to the more specific type II multiple orthogonal polynomials. Theorem 1.3.5 is proved in Section 5.4, first for polynomial perturbations from which the rational case is deduced. The relative asymptotic of the second type functions is obtained in Section 5.5 which is used in Section 5.6 to deduce the relative asymptotics for the sequences $\{Q_{\mathbf{n},j}\}_{\mathbf{n}\in\Lambda}, j \in \{1, \ldots, m\}$, as well.

5.1 Preliminaries and notation

Let $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$. Recall that the notation $(s_1, \ldots, s_m) = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$ is used to indicate that for each $k = 1, \ldots, m$, $\mathrm{supp}(\sigma_k) \subset \mathbb{R}$ consists of an interval $\widetilde{\Delta}_k$, on which $|\sigma_k'| > 0$ almost everywhere, and a discrete set without accumulation points in $\overline{\mathbb{R}} \setminus \widetilde{\Delta}_k$. Finally, let $(\widetilde{s}_1, \ldots, \widetilde{s}_m) = \mathcal{N}(p_1\sigma_1, \ldots, p_m\sigma_m)$, denote a "perturbation" of $(s_1, \ldots, s_m)$, where

$$\widetilde{s}_k = \langle p_1\sigma_1, \ldots, p_k\,\sigma_k\rangle, \qquad 1 \leq k \leq m,$$

and the $p_k, k = 1, \ldots, m$, are monic polynomials with complex coefficients whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Here, as before, $\Delta_k = \mathrm{Co}(\mathrm{supp}(\sigma_k))$. The system $(\widetilde{s}_1, \ldots, \widetilde{s}_m)$ is also regarded as a Nikishin system (even if the polynomials p_k have complex coefficients).

Let $Q_{\mathbf{n}}$ (resp. $\widetilde{Q}_{\mathbf{n}}$) be the monic polynomial of smallest degree (not identically equal to zero) such that

$$0 = \int x^\nu Q_{\mathbf{n}}(x)\, ds_k(x)\,, \quad \nu = 0, \ldots, n_k - 1\,, \quad k = 1, \ldots, m\,, \tag{5.1}$$

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)\, d\widetilde{s}_k(x)\,, \quad \nu = 0, \ldots, n_k - 1\,, \quad k = 1, \ldots, m\,, \tag{5.2}$$

where $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. One of the goals of this chapter is the study of the asymptotic behavior of the sequence $\{\widetilde{Q}_{\mathbf{n}}/Q_{\mathbf{n}}\}$, when $\mathbf{n}$ runs through an appropriate sequence of multi-indices contained in $\mathbb{Z}_+^m$.

Given the collection of polynomials $(p_1, \ldots, p_m)$, we define

$$\mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k + \deg(p_{j+1} \cdots p_k) \le n_j + 1\} .$$

In particular,

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k \le n_j + 1\} .$$

Obviously, $\mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m) \subset \mathbb{Z}_+^m(\circledast) \subset \mathbb{Z}_+^m(*)$ and $\mathbb{Z}_+^m(\bullet) \subset \mathbb{Z}_+^m(\circledast)$. Using Lemma 1.2.3, it follows that if $\mathbf{n} \in \mathbb{Z}_+^m(*)$, then the polynomial $Q_{\mathbf{n}}$ has degree $|\mathbf{n}|$, all its zeros are simple and lie in the interior of Δ_1.

5.2 Some algebraic relations

In this section we prove a number of auxiliary lemmas which are later applied in the analysis of the asymptotics of the sequence $\{\widetilde{Q}_{\mathbf{n}}/Q_{\mathbf{n}}\}$. Let us first express the orthogonality relations (5.2) satisfied by the polynomials $\widetilde{Q}_{\mathbf{n}}$ in terms of the measures in the initial system $(s_1, \ldots, s_m)$.

Lemma 5.2.1. *For each $k = 1, \ldots, m$, we have*

$$\widetilde{s}_k = p_1 l_{k,1} s_1 + p_1 p_2 l_{k,2} s_2 + \cdots + (p_1 \cdots p_k) l_{k,k} s_k , \tag{5.3}$$

where $l_{k,j}$ is a polynomial of degree $\deg l_{k,j} \le \deg(p_{j+1} \cdots p_k) - 1, j < k$, and $l_{k,k} \equiv 1$. In particular, if $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$, then for each $k = 1, \ldots, m$

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)(p_1 \cdots p_k)(x) ds_k(x) , \quad \nu = 0, \ldots, n_k - 1 . \tag{5.4}$$

Proof. To prove (5.3), we proceed by induction on m, the number of measures which generate the system. For $m = 1$, (5.3) is trivial, since $\widetilde{s}_1 = p_1\sigma_1 = p_1 s_1$. Assume that (5.3) is true for any Nikishin system with $m - 1 \ge 1$ generating measures and let us prove it when the number of generating measures is m.

Fix $k \in \{1, \ldots, m\}$. By definition,

$$\widetilde{s}_k = \langle p_1\sigma_1, \ldots, p_k\sigma_k\rangle = \langle p_1\sigma_1, \langle p_2\sigma_2, \ldots, p_k\sigma_k\rangle\rangle .$$

Consider the Nikishin system $\mathcal{N}(p_2\sigma_2, \ldots, p_k\sigma_k)$ which has at most $m-1$ generating measures. By the induction hypothesis, there exist polynomials $h_2, \ldots, h_k$, $\deg h_j \le \deg(p_{j+1} \cdots p_k) - 1, h_k \equiv 1$, such that

$$\langle p_2\sigma_2, \ldots, p_k\sigma_k\rangle = p_2 h_2 \sigma_2 + \cdots + (p_2 \cdots p_k) h_k \langle \sigma_2, \ldots, \sigma_k\rangle .$$

Inserting this relation above, we have

$$\widetilde{s}_k = \langle p_1\sigma_1, p_2 h_2 \sigma_2\rangle + \cdots + \langle p_1\sigma_1, (p_2 \cdots p_k) h_k \langle \sigma_2, \ldots, \sigma_k\rangle\rangle . \tag{5.5}$$

Given two measures $\sigma_\alpha, \sigma_\beta$, and a polynomial h, notice that

$$d\langle \sigma_\alpha, h\,\sigma_\beta\rangle(x) = \int \frac{(h(t) \mp h(x))d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x) =$$

$$h^*(x)d\sigma_\alpha(x) + h(x)d\langle \sigma_\alpha, \sigma_\beta\rangle(x)\,,$$

where $\deg h^* = \deg h - 1$. Making use of this property in each term of (5.5), it follows that

$$\widetilde{s}_k = p_1[(p_2h_2)^* + \cdots + (p_2\cdots p_k h_k)^*]\sigma_1 + (p_1p_2)h_2\langle \sigma_1, \sigma_2\rangle + \cdots$$

$$+ (p_1\cdots p_k)h_k\langle \sigma_1, \ldots, \sigma_k\rangle\,,$$

which establishes (5.3).

Using (5.3) and the orthogonality relations (5.2) satisfied by $\widetilde{Q}_{\mathbf{n}}$, it follows that for each $k \in \{1, \ldots, m\}$ and $\nu = 0, \ldots, n_k - 1$,

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)d\widetilde{s}_k(x) = \sum_{j=1}^{k} \int x^\nu l_{k,j}(x)\widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_j)(x)ds_j(x)\,. \tag{5.6}$$

In the rest of the proof we assume that $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. When $k = 1$ the last formula reduces to (5.4). Suppose that (5.4) holds up to $k-1, 1 \leq k-1 \leq m-1$, and let us show that it is also satisfied for k.

Let $j \in \{1, \ldots, k-1\}$ and $0 \leq \nu \leq n_k - 1$, then

$$\nu + \deg l_{k,j} \leq n_k - 1 + \deg(p_{j+1}\cdots p_k) - 1 \leq n_j - 1\,.$$

Therefore, according to the induction hypothesis

$$\int x^\nu l_{k,j}(x)\widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_j)(x)ds_j(x) = 0\,,$$

and (5.6) reduces to (5.4) for the index k. With this we conclude the proof. □

Lemma 5.2.2. *Let* $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. *Then, for each* $k = 1, \ldots, m$, *and* $\nu = 0, \ldots, n_k - \deg(p_{k+1}\cdots p_m) - 1$

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_m)(x)ds_k(x). \tag{5.7}$$

Proof. In place of x^ν we can put in (5.4) any polynomial of degree $\leq n_k - 1$. So, replacing x^ν by $x^\nu(p_{k+1}\cdots p_m)$ we obtain (5.7). □

Our next objective is to express the multiple orthogonal polynomials of the perturbed system in terms of multiple orthogonal polynomials of the initial system.

Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ and consider the multi-indices

$$\mathbf{n}_j = (n_1 - \deg(p_2\cdots p_m) + j, n_2 - \deg(p_3\cdots p_m), \ldots, n_m)\,, \quad j \geq 0\,.$$

It is easy to verify that

$$\mathbf{n}_j \in \mathbb{Z}_+^m(\circledast)\,, \qquad j \geq 0\,.$$

Therefore, $\deg Q_{\mathbf{n}_j} = |\mathbf{n}_j| = |\mathbf{n}| + \deg(p_2 p_3^2 \cdots p_m^{m-1}) + j$, all the $|\mathbf{n}_j|$ zeros of $Q_{\mathbf{n}_j}$ are simple and lie on Δ_1. Moreover, for each $j \geq 0$ and $k = 1, \ldots, m$,

$$0 = \int x^\nu Q_{\mathbf{n}_j}(x) ds_k(x)\,, \ \ \nu = 0, \ldots, n_k - \deg(p_{k+1} \cdots p_m) - 1\,. \tag{5.8}$$

Lemma 5.2.3. *Let* $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ *and set* $R_{\mathbf{n}} = \widetilde{Q}_{\mathbf{n}} p_1 \cdots p_m$. *There exist unique constants* $\lambda_{\mathbf{n},j}, j = 0, \ldots, N$, *such that*

$$R_{\mathbf{n}} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j}\,, \qquad N = \deg(p_1 p_2^2 \cdots p_m^m)\,. \tag{5.9}$$

If j' *is such that* $\deg R_{\mathbf{n}} = \deg Q_{\mathbf{n}_{j'}}$ *then* $\lambda_{\mathbf{n},j'} = 1$ *and* $\lambda_{\mathbf{n},j} = 0, j'+1 \leq j \leq N$. *In particular,* $\lambda_{\mathbf{n},N} = 1$ *if and only if* $\deg \widetilde{Q}_{\mathbf{n}} = |\mathbf{n}|$.

Proof. Since $\deg R_{\mathbf{n}} \leq |\mathbf{n}| + \deg(p_1 \cdots p_m)$, and $\{Q_{\mathbf{n}_j}\}, j = 0, \ldots, N$, has representatives of all degrees from $|\mathbf{n}| - \deg(p_2 p_3^2 \cdots p_m^{m-1})$ up to $|\mathbf{n}| + \deg(p_1 \cdots p_m)$, there exists a unique system of constants $\lambda_{\mathbf{n},j}, j = 0 \ldots, N$, such that

$$\deg(R_{\mathbf{n}} - \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j}) \leq |\mathbf{n}| - \deg(p_2 p_3^2 \cdots p_m^{m-1}) - 1\,.$$

From (5.7)-(5.8) it follows that for each $k = 1, \ldots, m$,

$$\int x^\nu (R_{\mathbf{n}} - \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j})\, ds_k(x)\,, \ \ \nu = 0, \ldots, n_k - \deg(p_{k+1} \cdots p_m) - 1\,.$$

By the normality of the multi-index

$$\mathbf{n}_0 = (n_1 - \deg(p_2 \cdots p_m), n_2 - \deg(p_3 \cdots p_m), \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)\,,$$

we obtain that

$$R_{\mathbf{n}} - \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j} \equiv 0\,,$$

which is (5.9). The rest of the statements follow because $R_{\mathbf{n}}$ is monic. □

Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. Define recursively the functions

$$R_{\mathbf{n},0}(z) = R_{\mathbf{n}}(z), \quad R_{\mathbf{n},k}(z) = \int \frac{R_{\mathbf{n},k-1}(x)}{z - x} d\sigma_k(x), \quad k = 1, \ldots, m. \tag{5.10}$$

In deriving (5.7), we lost some orthogonality relations. We will recover them in the form of analytic properties of the functions $R_{\mathbf{n},k}, k = 0, \ldots, m-1$.

Lemma 5.2.4. *Fix* $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. *The following relations take place: If* z_1 *is a zero of* $p_1 \cdots p_m$ *of multiplicity* τ_1, *then*

$$\Omega_{\mathbf{n}}^{(i)}(z_1) = \left(\frac{R_{\mathbf{n}}}{Q_{\mathbf{n}_0}}\right)^{(i)}(z_1) = 0\,, \qquad i = 0, \ldots, \tau_1 - 1\,. \tag{5.11}$$

If z_k *is a zero of* $p_k \cdots p_m, k = 2, \ldots, m,$ *of multiplicity* τ_k, *then*

$$R_{\mathbf{n},k-1}^{(i)}(z_k) = 0\,, \qquad i = 0, \ldots, \tau_k - 1\,. \tag{5.12}$$

Proof. The zeros of $p_1 \cdots p_m$ lie in $\mathbb{C} \setminus \Delta_1$, and those of $Q_{\mathbf{n}_0}$ in Δ_1. Therefore, $\Omega_{\mathbf{n}}$ has a zero at z_1 of multiplicity greater than or equal to τ_1 which implies (5.11).

For simplicity, first we will prove (5.12) for $k = 2$. By definition

$$R_{\mathbf{n},1}(z) = \int \frac{R_{\mathbf{n}}(x)}{z - x} d\sigma_1(x)\,.$$

Therefore, for each $i \geq 0$,

$$R_{\mathbf{n},1}^{(i)}(z) = (-1)^i i! \int \frac{R_{\mathbf{n}}(x)}{(z - x)^{i+1}} d\sigma_1(x)\,, \qquad z \in \mathbb{C} \setminus \Delta_1\,.$$

If z_2 is a zero of $p_2 \cdots p_m$ of multiplicity τ_2, using (5.4) with $k = 1$ we have that

$$0 = \int \frac{(p_2 \cdots p_m)(x)}{(z_2 - x)^{i+1}} \widetilde{Q}_{\mathbf{n}}(x) p_1(x) d\sigma_1(x) = \frac{(-1)^i R_{\mathbf{n},1}^{(i)}(z_2)}{i!}\,, \quad i = 0, \ldots, \tau_2 - 1\,,$$

which is (5.12) for $k = 2$. The proof of the general case uses basically the same arguments.

Consider the functions

$$\Phi_{\mathbf{n},k}(z) = \int \frac{R_{\mathbf{n}}(x)}{z - x} ds_k(x)\,, \qquad k = 1, \ldots, m\,.$$

Notice that $\Phi_{\mathbf{n},1} = R_{\mathbf{n},1}$. For each $i \geq 0$,

$$\Phi_{\mathbf{n},k}^{(i)}(z) = (-1)^i i! \int \frac{R_{\mathbf{n}}(x)}{(z - x)^{i+1}} ds_k(x)\,, \qquad k = 1, \ldots, m\,.$$

It is easy to verify that for each $k = 2, \ldots, m$,

$$\Phi_{\mathbf{n},k}(z) + (-1)^k R_{\mathbf{n},k}(z) = \int \cdots \int \frac{R_{\mathbf{n}}(x_1)(x_1 - x_k) d\sigma_1(x_1) \cdots d\sigma_k(x_k)}{(z - x_1)(x_1 - x_2) \cdots (x_{k-1} - x_k)(z - x_k)}\,.$$

Taking $x_1 - x_k = x_1 - x_2 + \cdots + x_{k-1} - x_k$, it follows that

$$R_{\mathbf{n},k}(z) = (-1)^{k-1} \Phi_{\mathbf{n},k}(z) + \sum_{l=1}^{k-1} (-1)^{l-1} \widehat{\vartheta}_{l,k}(z) \Phi_{\mathbf{n},l}(z), \tag{5.13}$$

for $z \in \mathbb{C} \setminus (\cup_{l=1}^m \Delta_l)$, where $\vartheta_{l,k} = \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_{l+1} \rangle$. If z_k is a zero of $p_k \cdots p_m$ of multiplicity $\tau_k (\leq \tau_{k-1} \leq \cdots \leq \tau_2)$, using (5.4) we obtain that for each $l = 2, \ldots, k$ and $i = 0, \ldots, \tau_k - 1$,

$$0 = \int \frac{(p_l \cdots p_m)(x)}{(z_k - x)^{i+1}} \widetilde{Q}_{\mathbf{n}}(x)(p_1 \cdots p_{l-1})(x) ds_{l-1}(x) = \frac{(-1)^i \Phi_{\mathbf{n},l-1}^{(i)}(z_k)}{i!} . \quad (5.14)$$

Now, (5.12) is a consequence of (5.13) (with k replaced by $k-1$), and (5.14). With this we conclude the proof. □

5.3 Some notational adjustments

The modifications we introduce here in the notation will be employed in this chapter and the next. For each $\mathbf{n} \in \mathbb{Z}_+^m(\circledast)$, define recursively the functions

$$\Psi_{\mathbf{n},0}(z) = Q_{\mathbf{n}}(z), \quad \Psi_{\mathbf{n},k}(z) = \int \frac{\Psi_{\mathbf{n},k-1}(x)}{z - x} d\sigma_k(x), \quad k = 1, \ldots, m. \quad (5.15)$$

These functions are the analogues of the functions $\mathcal{A}_{\mathbf{n},-j}, j = 0, \ldots, m_2 + 1$, defined in (1.12). In the previous chapters, we restricted our attention to multi-indices in $\mathbb{Z}_+^m(\bullet)$, which is strictly contained in $\mathbb{Z}_+^m(\circledast)$. For type II multiple orthogonal polynomials, the consideration of this slightly more general class causes no technical difficulty. In some instances, we will refer to previous results to deduce some formulas we need when $\mathbf{n} \in \mathbb{Z}_+^m(\circledast)$ instead of $\mathbb{Z}_+^m(\bullet)$. Their full proof would follow the arguments employed before. In any case, the reader can check the original source [37] for more details if needed.

By Proposition 2.1.6, for each $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$, $k = 1, \ldots, m$, and $k \leq k + r \leq m$,

$$\int \Psi_{\mathbf{n},k-1}(t)\, t^\nu d\langle \sigma_k, \ldots, \sigma_{k+r} \rangle(t) = 0, \quad \nu = 0, \ldots, n_{k+r} - 1. \quad (5.16)$$

Consequently, $\Psi_{\mathbf{n},k-1}, k = 1, \ldots, m$, has exactly $N_{\mathbf{n},k} := n_k + \cdots + n_m$ zeros in $\mathbb{C} \setminus \Delta_{k-1}$, they are all simple, and lie in the interior of Δ_k. Let $Q_{\mathbf{n},k}$ be the monic polynomial of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\Psi_{\mathbf{n},k-1}$ vanishes on Δ_k and let $Q_{\mathbf{n},m+1} \equiv 1$. From Proposition 2.1.7 it follows that

$$\int x^\nu \Psi_{\mathbf{n},k-1}(x) \frac{d\sigma_k(x)}{Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m . \quad (5.17)$$

As before, set

$$\mathcal{H}_{\mathbf{n},k}(z) := \frac{Q_{\mathbf{n},k-1}(z) \Psi_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n},k}(z)} , \qquad k = 1, \ldots, m+1 ,$$

($\mathcal{H}_{\mathbf{n},1}(z) \equiv 1$). The analogue of (4.3) (see also (50) in [14]) is

$$\mathcal{H}_{\mathbf{n},k+1}(z) = \int \frac{Q_{\mathbf{n},k}^2(x)}{z - x} \frac{\mathcal{H}_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} , \qquad k = 1, \ldots, m . \quad (5.18)$$

From (5.17), we have that for each multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$ there exists an associated system of polynomials

$$\{Q_{\mathbf{n},k}\}_{k=1}^m, \quad \deg Q_{\mathbf{n},k} = \sum_{\alpha=k}^m n_\alpha =: N_{\mathbf{n},k}, \qquad Q_{\mathbf{n},0} \equiv Q_{\mathbf{n},m+1} \equiv 1.$$

For each $k = 1, \ldots, m$, they satisfy the full system of orthogonality relation

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{\mathcal{H}_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \tag{5.19}$$

with respect to varying measures. Notice that $\mathcal{H}_{\mathbf{n},k}$ and $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ have constant sign on Δ_k.

Let $\varepsilon_{\mathbf{n},k}$ be the sign of the measure $\mathcal{H}_{\mathbf{n},k}(x)d\sigma_k(x)/Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)$ on $\operatorname{supp}(\sigma_k)$. For each $k = 1, \ldots, m$, set

$$K_{\mathbf{n},k} = \left(\int Q_{\mathbf{n},k}^2(x) \frac{\varepsilon_{\mathbf{n},k} \mathcal{H}_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} \right)^{-1/2} . \tag{5.20}$$

Take

$$K_{\mathbf{n},0} = 1 \, , \quad \kappa_{\mathbf{n},k} = \frac{K_{\mathbf{n},k}}{K_{\mathbf{n},k-1}} \, , \quad k = 1, \ldots, m \, .$$

Define

$$q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k} \, , \quad h_{\mathbf{n},k} = K_{\mathbf{n},k-1}^2 \mathcal{H}_{\mathbf{n},k} \, , \quad k = 1, \ldots, m \, . \tag{5.21}$$

From (5.19)

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{\varepsilon_{\mathbf{n},k} h_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m \, ,$$

and with the notation introduced above it follows that $q_{n,k}$ is orthonormal with respect to the varying measure

$$\frac{\varepsilon_{\mathbf{n},k} h_{\mathbf{n},k}(x) d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = d\rho_{\mathbf{n},k}(x) \, .$$

In the present context Lemma 4.3.1 implies (see also [36, Lemma 3.3] or [8, Corollary 3])

Lemma 5.3.1. *Let $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \leq C$, where C is a constant. Then, for each fixed $k = 1, \ldots, m$, we have*

$$\lim_{\mathbf{n} \in \Lambda} \varepsilon_{\mathbf{n},k} h_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z - b_k)(z - a_k)}} \, , \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k) \, , \tag{5.22}$$

where $[a_k, b_k] = \widetilde{\Delta}_k$. The square root is taken so that $\sqrt{(z - b_k)(z - a_k)} > 0$ for $z = x > b_k$. $\operatorname{supp}(\sigma_k)$ is an attractor of the zeros of $\{Q_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$, and each point of $\operatorname{supp}(\sigma_k) \setminus \widetilde{\Delta}_k$ is a 1 attraction point of zeros of $\{Q_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$.

In the proof of our main result, we use the asymptotic behavior of the polynomials $Q_{\mathbf{n},k}, k = 1, \ldots, m$, and the functions $\Psi_{\mathbf{n},k}, k = 1, \ldots, m$, when $\mathbf{n}$ runs through a sequence of multi-indices $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

The relevant Riemann surface has now $m+1$ sheets and is given by

$$\mathcal{R} = \overline{\bigcup_{k=0}^{m} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \widetilde{\Delta}_1, \quad \mathcal{R}_k := \overline{\mathbb{C}} \setminus \{\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}\}, \qquad k = 1, \ldots, m-1, \qquad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \widetilde{\Delta}_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \ldots, m\}$. Let $\psi^{(l)}, l = 1, \ldots, m$, be a single valued rational function on $\mathcal{R}$ whose divisor consists of a simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and a simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. Therefore,

$$\psi^{(l)}(z) = C_1/z + \mathcal{O}(1/z^2)\,, \; z \to \infty^{(0)}\,, \qquad \psi^{(l)}(z) = C_2 z + \mathcal{O}(1)\,, \; z \to \infty^{(l)}\,,$$

where C_1 and C_2 are constants different from zero. We denote the branches of the algebraic function $\psi^{(l)}$, corresponding to the different sheets $k = 0, \ldots, m$ of $\mathcal{R}$ by

$$\psi^{(l)} := \{\psi_k^{(l)}\}_{k=0}^m\,.$$

We normalize $\psi^{(l)}$ so that

$$\prod_{k=0}^{m} |\psi_k^{(l)}(\infty)| = 1, \qquad C_1 \in \mathbb{R} \setminus \{0\}. \tag{5.23}$$

The symmetry formula, $\psi^{(l)}(z) = \overline{\psi^{(l)}(\overline{z})}, z \in \mathcal{R}$, satisfied by the functions $\psi^{(l)}$, imply that for each $k = 0, 1, \ldots, m$

$$\psi_k^{(l)} : \overline{\mathbb{R}} \setminus (\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}) \longrightarrow \overline{\mathbb{R}} \tag{5.24}$$

($\widetilde{\Delta}_0 = \widetilde{\Delta}_{m+1} = \emptyset$). In particular, the coefficients of the Laurent expansion at ∞ of these branches are real numbers and $\mathrm{sg}(\psi_k^{(l)}(\infty))$ is defined. It also expresses that

$$\psi_k^{(l)}(x_\pm) = \overline{\psi_k^{(l)}(x_\mp)} = \overline{\psi_{k+1}^{(l)}(x_\pm)}, \qquad x \in \widetilde{\Delta}_{k+1}. \tag{5.25}$$

For any fixed multi-index $\mathbf{n} = (n_1, \ldots, n_m)$, set

$$\mathbf{n}^l := (n_1, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m)\,.$$

Corollary 4.5.1 (see also [36] or [5]) may be rewritten as

Corollary 5.3.2. *Let $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+^m(\circledast)$ and $n_1 - n_m \leq C$, where C is a constant. Let $\{q_{\mathbf{n},k} =$*

$\kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, be the system of orthonormal polynomials defined in (5.21) *and $\{K_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, the values given by* (5.20). *Then, for each fixed $k = 1, \ldots, m$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}^l,k}}{\kappa_{\mathbf{n},k}} = \kappa_k^{(l)} , \tag{5.26}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{K_{\mathbf{n}^l,k}}{K_{\mathbf{n},k}} = \kappa_1^{(l)} \cdots \kappa_k^{(l)} , \tag{5.27}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,k}(z)}{q_{\mathbf{n},k}(z)} = \kappa_k^{(l)} \widetilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k) , \tag{5.28}$$

where

$$\kappa_k^{(l)} = \frac{c_k^{(l)}}{\sqrt{c_{k-1}^{(l)} c_{k+1}^{(l)}}} , \qquad c_k^{(l)} = \begin{cases} (F_k^{(l)})'(\infty) , & k = 1, \ldots, l , \\ F_k^{(l)}(\infty) , & k = l+1, \ldots, m , \end{cases} \tag{5.29}$$

($c_0^{(l)} = c_{m+1}^{(l)} = 1$) and

$$F_k^{(l)} := \delta_{k,l} \prod_{\nu=k}^m \psi_\nu^{(l)}, \tag{5.30}$$

with $\delta_{k,l} = sg\left(\prod_{\nu=k}^m \psi_\nu^{(l)}(\infty)\right)$.

5.4 Relative asymptotics for the polynomials $Q_{\mathbf{n}}$

We will first prove the theorem when the measures are modified by means of polynomials; that is, we will initially suppose that $q_j \equiv 1, j = 1, \ldots, m$. Once this is done, the rational case easily follows.

Proof of Theorem 1.3.5 in the polynomial case. When $l = 1$, it is possible to find an algebraic function $\psi^{(1)}$ satisfying

$$\prod_{k=0}^m \psi_k^{(1)}(\infty) = 1 , \qquad C_1 \in \mathbb{R} \setminus \{0\}. \tag{5.31}$$

Let $(a,b)_k$ denote the interval (a,b) on the sheet $\mathcal{R}_k$. We distinguish two cases. Suppose that $\widetilde{\Delta}_1 = [a_1, b_1]$ is to the left of $\widetilde{\Delta}_2 = [a_2, b_2]$. Take $\psi^{(1)}$ verifying (5.23) with $C_1 = \lim_{z\to\infty} z\psi_0^{(1)}(z) > 0$. Because of (5.24), the restriction of $\psi^{(1)}$ to $(-\infty, a_1]_0 \cup (-\infty, a_1]_1$ establishes a bicontinuous bijection onto the interval $(-\infty, 0)$ of the real line. It follows that $\psi_1^{(1)}(x) \to -\infty, x \to -\infty, x \in \mathbb{R}$, which means that $C_2 > 0$, and $\psi_k^{(1)}(\infty) > 0, k = 2, \ldots, m$. Therefore, $\prod_{k=0}^m \psi_k^{(1)}(\infty) > 0$. If $\widetilde{\Delta}_1$ is to the right of $\widetilde{\Delta}_2$, take $\psi^{(1)}$ satisfying (5.23) with $C_1 < 0$. Now, the restriction of $\psi^{(1)}$ to $[b_1, +\infty)_0 \cup [b_1, +\infty)_1$ establishes a bicontinuous bijection onto $(-\infty, 0)$. It follows that $\psi_1^{(1)}(x) \to -\infty, x \to +\infty, x \in \mathbb{R}$, which means that $C_2 < 0$, and $\psi_k^{(1)}(\infty) > 0, k = 2, \ldots, m$. Again, $\prod_{k=0}^m \psi_k^{(1)}(\infty) > 0$.

Throughout the rest of this chapter, when $\widetilde{\Delta}_1$ is to the left of $\widetilde{\Delta}_2$, we will select $\psi^{(1)}$ so that $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1$, for all $k = 0, \ldots, m$. If $\widetilde{\Delta}_1$ is to the right of $\widetilde{\Delta}_2$, we will take $\psi^{(1)}$ so that $\mathrm{sg}(\psi_0^{(1)}(\infty)) = \mathrm{sg}(\psi_1^{(1)}(\infty)) = -1$ and $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1$, for all $k = 2, \ldots, m$.

In general, for any $l \in \{1, \ldots, m\}$ and $\psi^{(l)}$ verifying (5.23), we know that

$$\prod_{\nu=0}^{m} \psi_\nu^{(l)}(\infty) \in \{-1, 1\}\,.$$

Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ be an infinite sequence of distinct multi-indices such that $n_1 - n_m \leq C, \mathbf{n} \in \Lambda$. According to (5.26)-(5.30), for each fixed $j \geq 0$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_{j+1}}(z)}{Q_{\mathbf{n}_j}(z)} = \widetilde{F}_1^{(1)}(z) = \frac{\mathrm{sg}(\psi_0^{(1)}(\infty))}{c_1^{(1)}\psi_0^{(1)}(z)} =: \varphi_0(z), \quad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1)\,. \quad (5.32)$$

(Notice that (5.31) implies that $\prod_{\nu=0}^m \psi_\nu^{(1)}(z) \equiv 1$.)

Using (5.9),

$$\Omega_{\mathbf{n}} = \frac{R_{\mathbf{n}}}{Q_{\mathbf{n}_0}} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j} \frac{Q_{\mathbf{n}_j}}{Q_{\mathbf{n}_0}}\,, \qquad N = \deg(p_1 p_2^2 \cdots p_m^m)\,.$$

Set

$$\lambda_{\mathbf{n}}^* = (\sum_{j=0}^{N} |\lambda_{\mathbf{n},j}|)^{-1}\,.$$

At least one of the numbers in the sum is 1 so $\lambda_{\mathbf{n}}^*$ is finite. Define

$$\lambda_{\mathbf{n}}^*\Omega_{\mathbf{n}} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j}^* \frac{Q_{\mathbf{n}_j}}{Q_{\mathbf{n}_0}}\,, \qquad \sum_{j=0}^{N} |\lambda_{\mathbf{n},j}^*| = 1\,. \quad (5.33)$$

Because of (5.32) and (5.33), the family $\{\lambda_{\mathbf{n}}^*\Omega_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$, is normal in $\mathbb{C} \setminus \mathrm{supp}(\sigma_1)$, and any convergent subsequence $\{\lambda_{\mathbf{n}}^*\Omega_{\mathbf{n}}\}, \mathbf{n} \in \Lambda' \subset \Lambda$, converges to

$$\lim_{\mathbf{n}\in\Lambda'} \lambda_{\mathbf{n}}^*\Omega_{\mathbf{n}}(z) = p_{\Lambda'}(\varphi_0(z)) = \sum_{j=0}^{N} \lambda_j \varphi_0^j(z)\,, \quad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1)\,.$$

That is, $p_{\Lambda'}(w)$ is a polynomial of degree $\leq N$, not identically equal to zero since $\sum_{j=0}^N |\lambda_j| = 1$. We will show that $p_{\Lambda'}$ does not depend on the subsequence taken. This implies the existence of limit along all Λ. To this aim, we will uniquely determine N zeros of $p_{\Lambda'}$.

Let z_1 be one of the zeros of $p_1 \cdots p_m$ and τ_1 its multiplicity. Using (5.11) and the Weierstrass theorem, it follows that

$$(p_{\Lambda'} \circ \varphi_0)^{(i)}(z_1) = 0\,, \qquad i = 0, \ldots, \tau_1 - 1\,.$$

Since φ_0 is one to one in $\mathbb{C} \setminus \widetilde{\Delta}_1$, we conclude that $p_{\Lambda'}(w)$ is divisible by

$$(w - \varphi_0(z_1))^{\tau_1} .$$

We will detect the rest of the zeros of $p_{\Lambda'}(w)$ in virtue of (5.12). Consider the sequence $\{\lambda^*_{\mathbf{n}} R_{\mathbf{n},k-1}\}, \mathbf{n} \in \Lambda'$. From (5.9), (5.10) and (5.15)

$$\lambda^*_{\mathbf{n}} R_{\mathbf{n},k-1}(z) = \sum_{j=0}^{N} \lambda^*_{\mathbf{n},j} \Psi_{\mathbf{n}_j,k-1}(z) .$$

Multiplying this equation by $\varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} Q_{\mathbf{n}_0,k-1}/Q_{\mathbf{n}_0,k}$ and using the definition of $h_{\mathbf{n},k}$, we obtain

$$\frac{\lambda^*_{\mathbf{n}} \varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} (Q_{\mathbf{n}_0,k-1} R_{\mathbf{n},k-1})(z)}{Q_{\mathbf{n}_0,k}(z)}$$

$$= \sum_{j=0}^{N} \lambda^*_{\mathbf{n},j} \frac{K^2_{\mathbf{n}_0,k-1}}{K^2_{\mathbf{n}_j,k-1}} \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n}_j,k-1}(z)} \frac{Q_{\mathbf{n}_j,k}(z)}{Q_{\mathbf{n}_0,k}(z)} \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\varepsilon_{\mathbf{n}_j,k-1}} \varepsilon_{\mathbf{n}_j,k-1} h_{\mathbf{n}_j,k}(z) .$$

From (5.26)-(5.28), for each $j \geq 0$ and $k = 2, \ldots, m$,

$$\lim_{\mathbf{n} \in \Lambda'} \frac{K^2_{\mathbf{n}_j,k-1}}{K^2_{\mathbf{n}_{j+1},k-1}} \frac{Q_{\mathbf{n}_j,k-1}(z)}{Q_{\mathbf{n}_{j+1},k-1}(z)} \frac{Q_{\mathbf{n}_{j+1},k}(z)}{Q_{\mathbf{n}_j,k}(z)} = \frac{\widetilde{F}^{(1)}_k(z)}{(\kappa^{(1)}_1 \cdots \kappa^{(1)}_{k-1})^2 \widetilde{F}^{(1)}_{k-1}(z)} ,$$

uniformly on compact subsets of $\mathbb{C} \setminus (\text{supp}(\sigma_{k-1}) \cup \text{supp}(\sigma_k))$. On account of (5.29) and the expression of the functions $F^{(1)}_k$,

$$\frac{\widetilde{F}^{(1)}_k(z)}{(\kappa^{(1)}_1 \cdots \kappa^{(1)}_{k-1})^2 \widetilde{F}^{(1)}_{k-1}(z)} = \frac{\text{sg}(\psi^{(1)}_{k-1}(\infty))}{c^{(1)}_1 \psi^{(1)}_{k-1}(z)} =: \varphi_{k-1}(z). \tag{5.34}$$

Let us consider the ratios $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k}, k = 1, \ldots, m-1, j \geq 0$. Recall that $\varepsilon_{\mathbf{n},k}$ is by definition the sign of the measure $\mathcal{H}_{\mathbf{n},k}(x) d\sigma_k(x)/(Q_{\mathbf{n},k-1} Q_{\mathbf{n},k+1})(x)$ on Δ_k. Notice that for each fixed $k = 2, \ldots, m$ the polynomials $Q_{\mathbf{n}_j,k}$ have the same degree for all $j \geq 0$; therefore, they all have the same sign on any interval disjoint from Δ_k. On the other hand, the polynomials $Q_{\mathbf{n}_j,1}$ have degrees that increase one by one with j. Hence, if Δ_1 is to the left of Δ_2, all the polynomials $Q_{\mathbf{n}_j,1}$ have the same sign on Δ_2 whereas, if Δ_1 is to the right of Δ_2, the sign of these polynomials alternate on Δ_2 as j increases one by one. Taking these facts into consideration, it is easy to see that for all $j \geq 0$, the measures $\mathcal{H}_{\mathbf{n}_j,1}(x) d\sigma_1(x)/Q_{\mathbf{n}_j,2}(x) = d\sigma_1(x)/Q_{\mathbf{n}_j,2}(x)$, have the same sign; therefore, for all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},1}/\varepsilon_{\mathbf{n}_j,1} = 1$ and the functions $\mathcal{H}_{\mathbf{n}_j,2}$ have the same sign on Δ_2 (see (5.18)). Hence, the measures $\mathcal{H}_{\mathbf{n}_j,2}(x) d\sigma_2(x)/(Q_{\mathbf{n}_j,1} Q_{\mathbf{n}_j,3})(x)$ have the same sign if Δ_1 is to the left of Δ_2 and alternate signs as j increases when Δ_1 is to the right of Δ_2. Thus, for all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},2}/\varepsilon_{\mathbf{n}_j,2} = 1$ when Δ_1 is to the left of Δ_2 and $\varepsilon_{\mathbf{n}_{j+1},2}/\varepsilon_{\mathbf{n}_j,2} = -1$ when Δ_1 is to the right of Δ_2. By the same token (see (5.18)), for all $j \geq 0$ the functions $\mathcal{H}_{\mathbf{n}_j,3}$ have the same sign on Δ_3 when Δ_1 is to the left of Δ_2 and alternate sign when Δ_1 is to the right of Δ_2. From now on the situation repeats and for each fixed $k = 2, \ldots, m-1$, and all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k} = 1$ when Δ_1 is to the left of Δ_2 while $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k} = -1$ when Δ_1 is to the right of Δ_2.

Let $\delta = 1$ when Δ_1 is to the left of Δ_2 and $\delta = -1$ if Δ_1 is to the right of Δ_2. Using (5.22) and (5.26)-(5.29), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \lambda^*_{\mathbf{n}} \varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} \frac{Q_{\mathbf{n}_0,k-1}(z) R_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n}_0,k}(z)} =$$

$$\begin{cases} \frac{1}{\sqrt{(z-b_1)(z-a_1)}} \sum_{j=0}^N \lambda_j \varphi_1^j(z)\,, & k=2\,, \\ \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} \sum_{j=0}^N \lambda_j (\delta\varphi_{k-1})^j(z)\,, & k=3,\ldots,m\,, \end{cases} = \tag{5.35}$$

$$\begin{cases} \frac{1}{\sqrt{(z-b_1)(z-a_1)}} p_{\Lambda'}(\varphi_1(z))\,, & k=2\,, \\ \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_{\Lambda'}(\delta\varphi_{k-1}(z))\,, & k=3,\ldots,m\,, \end{cases}$$

uniformly on each compact subset $\mathcal{K}$ of $\mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$.

Let z_k be one of the zeros of $p_k \cdots p_m, k = 2, \ldots, m$, and τ_k its multiplicity. Using (5.35), (5.12), and Weierstrass theorem, it follows that

$$(p_{\Lambda'} \circ \varphi_1)^{(i)}(z_2) = 0\,, \qquad i = 0, \ldots, \tau_2 - 1\,,$$

and

$$(p_{\Lambda'} \circ (\delta\varphi_{k-1}))^{(i)}(z_k) = 0\,, \qquad i = 0, \ldots, \tau_k - 1\,, \qquad k = 3, \ldots, m\,.$$

Since φ_{k-1} is one to one in $\mathbb{C} \setminus (\widetilde{\Delta}_{k-1} \cup \widetilde{\Delta}_k)$, we conclude that $p_{\Lambda'}(w)$ is divisible by

$$(w - \varphi_1(z_2))^{\tau_2}\,,$$

and

$$(w - \delta\varphi_{k-1}(z_k))^{\tau_k}\,, \qquad k = 3, \ldots, m\,.$$

Therefore, the following sets are formed by zeros of $p_{\Lambda'}$:

$$\mathcal{Z}_0 := \{\varphi_0(z_1) : z_1 \,\text{is a zero of}\, p_1 \cdots p_m\}\,,$$

$$\mathcal{Z}_1 := \{\varphi_1(z_2) : z_2 \,\text{is a zero of}\, p_2 \cdots p_m\}\,,$$

$$\mathcal{Z}_k := \{\delta\varphi_k(z_{k+1}) : z_{k+1} \,\text{is a zero of}\, p_{k+1} \cdots p_m\}\,, \quad 2 \leq k \leq m-1\,.$$

Assume first that $\delta = 1$. Recall that in this case we selected $\psi^{(1)}$ so that $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1$ for all $0 \leq k \leq m$. Therefore the functions $\varphi_0, \varphi_1, \delta\varphi_k, 2 \leq k \leq m-1$, are the first m branches of $1/c_1^{(1)}\psi^{(1)}$. If $\delta = -1$, since $\psi^{(1)}$ was chosen so that $\mathrm{sg}(\psi_0^{(1)}(\infty)) = \mathrm{sg}(\psi_1^{(1)}(\infty)) = -1$ and $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1, 2 \leq k \leq m$, the functions $\varphi_0, \varphi_1, \delta\varphi_k, 2 \leq k \leq m-1$, are now the first m branches of $-1/c_1^{(1)}\psi^{(1)}$. In any case, since $\psi^{(1)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is bijective it follows that the zero sets $\mathcal{Z}_k, 0 \leq k \leq m-1$ are pairwise disjoint. Therefore, we have detected $N = \deg(p_1 p_2^2 \cdots p_m^m)$ zeros (counting multiplicities) of the polynomial $p_{\Lambda'}$ and their location does not depend on the subsequence $\Lambda' \subset \Lambda$.

Let

$$(p_k \cdots p_m)(z) = \prod_{\nu=1}^{l_k} (z - z_{k,\nu})^{\tau_{k,\nu}} ,$$

where $\{z_{k,1}, \ldots, z_{k,l_k}\}$ are the distinct zeros of $p_k \cdots p_m$. Then

$$p_{\Lambda'}(w) = c \prod_{k=1}^{2} \prod_{\nu=1}^{l_k} (w - \varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} \prod_{k=3}^{m} \prod_{\nu=1}^{l_k} (w - \delta\varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} ,$$

where c is uniquely defined by the conditions that it is a positive constant such that the sum of the moduli of the coefficients of $p_{\Lambda'}$ equals one; moreover,

$$0 < c = \lim_{\mathrm{n}\in\Lambda} \lambda^*_{\mathbf{n}} < \infty .$$

Consequently, uniformly on each compact subset $\mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1)$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{R_{\mathbf{n}}(z)}{Q_{\mathbf{n}_0}(z)} =$$

$$\prod_{k=1}^{2} \prod_{\nu=1}^{l_k} (\varphi_0(z) - \varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} \prod_{k=3}^{m} \prod_{\nu=1}^{l_k} (\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} . \tag{5.36}$$

From (5.26) and (5.28), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}}(z)}{Q_{\mathbf{n}_0}(z)} = (\widetilde{F}_1^{(1)}(z))^{\deg(p_2\cdots p_m)} \cdots (\widetilde{F}_1^{(m-1)}(z))^{\deg(p_m)} . \tag{5.37}$$

Combining (5.36) and (5.37), we get

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \mathcal{F}(z; p_1, \ldots, p_m) , \qquad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1) ,$$

where $(\varphi_0(z) = \widetilde{F}_1^{(1)}(z))$

$$\mathcal{F}(z; p_1, \ldots, p_m) = \prod_{\nu=1}^{l_1} \left(\frac{\varphi_0(z) - \varphi_0(z_{1,\nu})}{z - z_{1,\nu}} \right)^{\tau_{1,\nu}} \prod_{\nu=1}^{l_2} \left(1 - \frac{\varphi_1(z_{2,\nu})}{\varphi_0(z)} \right)^{\tau_{2,\nu}} \times$$

$$\prod_{k=3}^{m} \prod_{\nu=1}^{l_k} \left(\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} \right)^{\tau_{k,\nu}} .$$

Let us simplify the expression above. From the definition of the functions φ_k, and taking into account that $\delta = \mathrm{sg}(\psi_0^{(1)}(\infty))$, it follows that

$$1 - \frac{\varphi_1(z_{2,\nu})}{\varphi_0(z)} = 1 - \frac{\psi_0^{(1)}(z)}{\psi_1^{(1)}(z_{2,\nu})} .$$

It is easy to see that for $l \geq 2$ the following equation holds:

$$\frac{1}{\psi^{(1)}(z)} - \frac{1}{\psi^{(1)}(\infty^{(l-1)})} = \frac{C_0^{(l-1)}}{C_0^{(1)}\psi^{(l-1)}(z)}\,, \tag{5.38}$$

where

$$\psi^{(1)}(z) = C_0^{(1)}/z + \mathcal{O}(1/z^2)\,,\ z \to \infty^{(0)}\,,$$
$$\psi^{(l-1)}(z) = C_0^{(l-1)}/z + \mathcal{O}(1/z^2)\,,\ z \to \infty^{(0)}\,.$$

For $k \geq 3$ (recall that $\prod_{\nu=0}^m \psi_\nu^{(l)}(\infty) \in \{-1, 1\}$ when $l \geq 2$), we have that

$$\widetilde{F}_1^{(k-1)}(z) = \frac{\operatorname{sg}(\psi_0^{(k-1)})(\infty)}{c_1^{(k-1)}\psi_0^{(k-1)}(z)}\,.$$

Thus

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = \tag{5.39}$$
$$\frac{c_1^{(k-1)}\psi_0^{(k-1)}(z)}{c_1^{(1)}\operatorname{sg}(\psi_0^{(k-1)}(\infty))}\left(\frac{\operatorname{sg}(\psi_0^{(1)}(\infty))}{\psi_0^{(1)}(z)} - \frac{\delta}{\psi_{k-1}^{(1)}(z_{k,\nu})}\right).$$

From (5.38), it follows that

$$\psi_0^{(k-1)}(z)\left(\frac{1}{\psi_0^{(1)}(z)} - \frac{1}{\psi_{k-1}^{(1)}(\infty)}\right) = \frac{C_0^{(k-1)}}{C_0^{(1)}}\,.$$

Therefore,

$$\psi_0^{(k-1)}(z)\left(\frac{1}{\psi_0^{(1)}(z)} - \frac{\delta}{\psi_{k-1}^{(1)}(z_{k,\nu})}\right) = \tag{5.40}$$
$$\frac{C_0^{(k-1)}}{C_0^{(1)}} + \left(\frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(1)}(\infty)} - \frac{\delta\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(1)}(z_{k,\nu})}\right).$$

It is straightforward to check that

$$\frac{c_1^{(k-1)}}{c_1^{(1)}}\frac{C_0^{(k-1)}}{C_0^{(1)}} = \frac{\operatorname{sg}(\psi_0^{(k-1)}(\infty))}{\operatorname{sg}(\psi_0^{(1)}(\infty))}\,. \tag{5.41}$$

Evaluating (5.38) at $z_{k,\nu}$ we obtain

$$\frac{1}{\psi_{k-1}^{(1)}(z_{k,\nu})} - \frac{1}{\psi_{k-1}^{(1)}(\infty)} = \frac{C_0^{(k-1)}}{C_0^{(1)}\psi_{k-1}^{(k-1)}(z_{k,\nu})}\,. \tag{5.42}$$

Assume that Δ_1 is to the left of Δ_2, then $\delta = \operatorname{sg}(\psi_0^{(1)}(\infty)) = 1$. From (5.39), (5.40), (5.41), and (5.42), we find that

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = 1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})}\,.$$

If Δ_1 is to the right of Δ_2, then $\delta = \mathrm{sg}(\psi_0^{(1)}(\infty)) = -1$. Applying (5.39)-(5.42), we obtain again

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = 1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})} .$$

Therefore,

$$\mathcal{F}(z; p_1, \ldots, p_m) = \tag{5.43}$$

$$\prod_{\nu=1}^{l_1} \left(\frac{\varphi_0(z) - \varphi_0(z_{1,\nu})}{z - z_{1,\nu}}\right)^{\tau_{1,\nu}} \prod_{k=2}^{m} \prod_{\nu=1}^{l_k} \left(1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})}\right)^{\tau_{k,\nu}} .$$

(We did not substitute φ_0 in terms of $\psi_0^{(1)}$ (see (5.32)) in the first group of products for simplicity in the final expression.)

We have proved (1.24) on compact subsets of $\mathbb{C} \setminus \mathrm{supp}(\sigma_1)$. Using the maximum principle it follows that the same is true on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_1)$. Notice that $\mathcal{F}$ is analytic and has no zero in $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_1$. For all $\mathbf{n} \in \Lambda$, $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, $\mathrm{supp}(\sigma_1)$ is an attractor of the zeros of $\{Q_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$, and each point in $\mathrm{supp}(\sigma_1) \setminus \widetilde{\Delta}_1$ is a 1 attraction point of zeros of $\{Q_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$; therefore, the statements concerning $\deg \widetilde{Q}_{\mathbf{n}}$ and the asymptotic behavior of the zeros of these polynomials follow from (1.24), on account of the argument principle and the corresponding behavior of the zeros of the polynomials $Q_{\mathbf{n}}$ described in Lemma 5.3.1.

In order to prove the last statement, let us assume that the polynomials $p_k, k = 1, \ldots, m$, have real coefficients and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$. Notice that in this case the polynomials $\widetilde{Q}_{\mathbf{n}}$ are the multiple orthogonal polynomials with respect to the Nikishin system $\mathcal{N}(p_1\sigma_1, \ldots, p_m\sigma_m)$ generated by real measures with constant sign. Thus, Corollary 5.3.2 can be applied to them. Given Λ we construct the auxiliary sequence $\Lambda(\diamond)$ as follows. To each $\mathbf{n} = (n_1, \ldots, n_m) \in \Lambda$ we associate $\mathbf{n}_\diamond = (n_1, n_2 - \deg(p_2), \ldots, n_m - \deg(p_2 \cdots p_m))$ (we disregard those multi-indices in Λ for which a component of $\mathbf{n}_\diamond$ would turn out to be negative, according to the assumptions on Λ there can be at most a finite number of such $\mathbf{n}$). It is easy to see that $\Lambda(\diamond) \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$.

Choose consecutive multi-indices running from $\mathbf{n}_\diamond$ to $\mathbf{n}$ so that each one of them belongs to $\mathbb{Z}_+^m(\circledast)$. We can write $Q_{\mathbf{n}}/Q_{\mathbf{n}_\diamond}$ as the product of quotients of the corresponding monic multiple orthogonal polynomials. The same can be done with $\widetilde{Q}_{\mathbf{n}}/\widetilde{Q}_{\mathbf{n}_\diamond}$. According to (5.26) and (5.28), there exists an analytic function $G(z)$ in $\mathbb{C} \setminus \widetilde{\Delta}_1$, which is never zero, such that

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}}(z)}{Q_{\mathbf{n}_\diamond}(z)} = \lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{\widetilde{Q}_{\mathbf{n}_\diamond}(z)} = G(z) , \qquad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1) .$$

Since

$$\frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \frac{\widetilde{Q}_{\mathbf{n}}(z)}{\widetilde{Q}_{\mathbf{n}_\diamond}(z)} \frac{\widetilde{Q}_{\mathbf{n}_\diamond}(z)}{Q_{\mathbf{n}_\diamond}(z)} \frac{Q_{\mathbf{n}_\diamond}(z)}{Q_{\mathbf{n}}(z)} ,$$

using Theorem 1.3.5 on the ratio appearing in the middle of the right hand side, and the previous limits on the other two ratios, the last statement readily follows. $\square$

Proof of Theorem 1.3.5 in general. Notice that $\mathcal{N}(\frac{p_1}{q_1}\sigma_1,\ldots,\frac{p_m}{q_m}\sigma_m) = \mathcal{N}(\frac{p_1\overline{q}_1}{|q_1|^2}\sigma_1,\ldots,\frac{p_m\overline{q}_m}{|q_m|^2}\sigma_m)$, where $\overline{q}_k$ denotes the polynomial obtained conjugating the coefficients of q_k. Let $Q^*_{\mathbf{n}}$ be the $\mathbf{n}$th monic multiple orthogonal polynomial with respect to the Nikishin system $\mathcal{N}(\frac{\sigma_1}{|q_1|^2},\ldots,\frac{\sigma_m}{|q_m|^2})$ generated by measures with constant sign.

Using Theorem 1.3.5 for the polynomial case, we have

$$\lim_{\mathbf{n}\in\Lambda}\frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q^*_{\mathbf{n}}(z)} = \mathcal{F}(z;p_1\overline{q}_1,\ldots,p_m\overline{q}_m)\,,\quad \mathcal{K}\subset\overline{\mathbb{C}}\setminus\operatorname{supp}(\sigma_1)$$

and, considering the last remark of the same theorem for the polynomial case, we also have

$$\lim_{\mathbf{n}\in\Lambda}\frac{Q_{\mathbf{n}}(z)}{Q^*_{\mathbf{n}}(z)} = \mathcal{F}(z;q_1\overline{q}_1,\ldots,q_m\overline{q}_m)\,,\quad \mathcal{K}\subset\overline{\mathbb{C}}\setminus\operatorname{supp}(\sigma_1)\,.$$

On the other hand,

$$\frac{\mathcal{F}(z;p_1\overline{q}_1,\ldots,p_m\overline{q}_m)}{\mathcal{F}(z;q_1\overline{q}_1,\ldots,q_m\overline{q}_m)} = \frac{\mathcal{F}(z;p_1,\ldots,p_m)}{\mathcal{F}(z;q_1,\ldots,q_m)}$$

because in the products defining the functions on the left hand side all the factors connected with the zeros of the $\overline{q}_k$ cancel out. Consequently, (1.24) takes place. The rest of the statements are proved following arguments similar to those employed in the proof for the polynomial case. $\square$

The previous results allow to derive ratio asymptotics for the multiple orthogonal polynomials of our perturbed Nikishin systems.

Corollary 5.4.1. *Let $S = \mathcal{N}'(\sigma_1,\ldots,\sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1,\ldots,\frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C}\setminus\cup_{k=1}^m\Delta_k$. Let $\Lambda\subset\mathbb{Z}_+^m(\circledast;p_1q_1,\ldots,p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n}\in\Lambda$ and some fixed $l\in\{1,\ldots,m\}$, we have that $\mathbf{n}^l\in\mathbb{Z}_+^m(\circledast;p_1q_1,\ldots,p_mq_m)$ and $n_1-n_m\le C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n}}$ be the monic multiple orthogonal polynomial of smallest degree with respect to the Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1,\ldots,\frac{p_m}{q_m}\sigma_m)$ and $\mathbf{n}$. Then*

$$\lim_{\mathbf{n}\in\Lambda}\frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{\widetilde{Q}_{\mathbf{n}}(z)} = \lim_{\mathbf{n}\in\Lambda}\frac{Q_{\mathbf{n}^l}(z)}{Q_{\mathbf{n}}(z)} = \widetilde{F}_1^{(l)}(z),\qquad \mathcal{K}\subset\mathbb{C}\setminus\operatorname{supp}(\sigma_1)\,.$$

Proof. Since

$$\frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{\widetilde{Q}_{\mathbf{n}}(z)} = \frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{Q_{\mathbf{n}^l}(z)}\frac{Q_{\mathbf{n}^l}(z)}{Q_{\mathbf{n}}(z)}\frac{Q_{\mathbf{n}}(z)}{\widetilde{Q}_{\mathbf{n}}(z)}\,,$$

the result follows immediately applying Corollary 5.3.2 and Theorem 1.3.5. $\square$

5.5 Relative asymptotics of second type functions

Let $\widetilde{Q}_{\mathbf{n}}$ be the monic polynomial of smallest degree satisfying (5.2). Set

$$\widetilde{\Psi}_{n,0}(z) := \widetilde{Q}_{\mathbf{n}}(z)\,,$$

$$\widetilde{\Psi}_{n,k}(z) := \int \frac{\widetilde{\Psi}_{n,k-1}(x)}{z-x} p_k(x)\, d\sigma_k(x)\,, \quad 1 \leq k \leq m\,. \tag{5.44}$$

Lemma 5.5.1. *If $n_j \geq \deg(p_{j+1}\cdots p_m)$, $j = 1,\ldots,m-1$, then $R_{\mathbf{n},k}(z) = (p_{k+1}\cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k}(z)$, $z \in \mathbb{C}\setminus \operatorname{supp}(\sigma_k)$, $k = 0,1,\ldots,m$, $(R_{\mathbf{n},m} = \widetilde{\Psi}_{\mathbf{n},m})$.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial since by definition, $R_{\mathbf{n},0}(z) = (p_1\cdots p_m)(z)\widetilde{Q}_{\mathbf{n}}(z)$. Assume that the result holds for $k-1$, and let us prove it for k. We have

$$R_{\mathbf{n},k}(z) = \int \frac{R_{\mathbf{n},k-1}(x)}{z-x} d\sigma_k(x) = \int \frac{\widetilde{\Psi}_{\mathbf{n},k-1}(x)(p_k\cdots p_m)(x)}{z-x} d\sigma_k(x) =$$

$$(p_{k+1}\cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k}(z) + \int \widetilde{\Psi}_{\mathbf{n},k-1}(x)l(x)p_k(x)d\sigma_k(x)\,,$$

where $l(x)$ is a polynomial of degree $\deg(p_{k+1}\cdots p_m)-1$. Now, for $k \leq k+r \leq m$, the functions $\widetilde{\Psi}_{\mathbf{n},k}$ satisfy the orthogonality relations (see in [29] that the proof presented there is also valid for complex measures)

$$\int \widetilde{\Psi}_{\mathbf{n},k-1}(t)t^{\nu} d\langle p_k\sigma_k,\ldots,p_{k+r}\sigma_{k+r}\rangle(t) = 0\,, \qquad \nu = 0,\ldots,n_{k+r}-1.$$

In particular, $\int \widetilde{\Psi}_{\mathbf{n},k-1}(t)t^{\nu}p_k(t)d\sigma_k(t) = 0$ if $\nu \leq n_k - 1$. Thus, since we are assuming that $n_k \geq \deg(p_{k+1}\cdots p_m)$, we get that

$$\int \widetilde{\Psi}_{\mathbf{n},k-1}(x)l(x)p_k(x)d\sigma_k(x) = 0$$

and the result follows. □

Remark 5.5.2. *The condition $n_k \geq \deg(p_{k+1}\cdots p_m), k = 1,\ldots,m-1$, is automatically satisfied by the components of multi-indices $\mathbf{n}$ with norm sufficiently large that belong to a sequence $\Lambda \subset \mathbb{Z}_+^m(\circledast;p_1,\ldots,p_m)$ such that for all $\mathbf{n} \in \Lambda, n_1 - n_m \leq C$, where C is a constant. In fact, it is satisfied for all $\mathbf{n} \in \mathbb{Z}_+^m(\circledast;p_1,\ldots,p_m)$ such that $n_m \geq 1$.*

Now, we need to introduce some notation similar to that presented in Section 4.6

$$\delta_k := \begin{cases} 1, & \text{if } \Delta_k \text{ is to the left of } \Delta_{k+1}\,, \\ -1, & \text{if } \Delta_k \text{ is to the right of } \Delta_{k+1}\,. \end{cases}$$

For $k \geq 2$, set

$$\Delta_{k,l} := \begin{cases} -\delta_k\delta_{k-1}, & \text{if } l \geq k+1\,, \\ \delta_{k-1}, & \text{if } l \in \{k-1,k\}\,, \\ 1, & \text{if } l \leq k-2\,. \end{cases}$$

If $k = 1$,

$$\Delta_{1,l} := \begin{cases} 1, & \text{if } \ l = 1\,, \\ -\delta_1, & \text{if } \ l \geq 2\,. \end{cases}$$

Recall that $\varepsilon_{\mathbf{n},k}$ denotes the sign of the measure $\frac{\mathcal{H}_{\mathbf{n},k}(x)d\sigma_k(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)}$ on $\operatorname{supp}(\sigma_k)$. Lemma 4.6.1 can be rewritten as

Lemma 5.5.3. *For any* $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_+^m(\circledast)$

$$\frac{\varepsilon_{\mathbf{n}^l,j}}{\varepsilon_{\mathbf{n},j}} = \prod_{k=1}^{j} \Delta_{k,l}\,. \tag{5.45}$$

Definition 5.5.4. *We define the following functions*

$$\varphi_{k-1}^{(j)}(z) := \frac{\operatorname{sg}(\psi_{k-1}^{(j)}(\infty))}{c_1^{(j)}\psi_{k-1}^{(j)}(z)}\,, \qquad 1 \leq j \leq m-1\,. \tag{5.46}$$

Notice that $\varphi_{k-1}^{(1)} = \varphi_{k-1}$, where φ_{k-1} was previously defined in (5.34).

Theorem 5.5.5. *Let* $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$ *and* $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ *be a sequence of distinct multi-indices such that for all* $\mathbf{n} \in \Lambda, n_1 - n_m \leq C$, *where* C *is a constant. Then, for each* $k \in \{0, 1, \ldots, m\}$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n},k}(z)}{\Psi_{\mathbf{n},k}(z)} = G_k(z; p_1, \ldots, p_m)\,, \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))\,, \tag{5.47}$$

where G_k *is analytic and never vanishes in the indicated region. For each* $k = \{0, \ldots, m-1\}$ *and all sufficiently large* $|\mathbf{n}|, \mathbf{n} \in \Lambda$, $\widetilde{\Psi}_{\mathbf{n},k}$ *has exactly* $N_{\mathbf{n},k+1} = n_{k+1} + \cdots + n_m$ *zeros in* $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k)$, $\operatorname{supp}(\sigma_{k+1})$ *is an attractor of the zeros of* $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$, *in this region, and each point in* $\operatorname{supp}(\sigma_{k+1}) \setminus \widetilde{\Delta}_{k+1}$ *is a* 1 *attraction point of zeros of* $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$. *When the coefficients of the polynomials* $p_k, k = 1, \ldots, m$, *are real, all the statements above remain valid for* $\Lambda \subset \mathbb{Z}_+^m(\circledast)$. *An expression for* G_k *is given in* (5.50)-(5.51) *below.*

Proof. For $k = 0$, (5.47) is (1.24) since $\widetilde{\Psi}_{\mathbf{n},0} = \widetilde{Q}_{\mathbf{n}}$ and $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$; therefore,

$$G_0(z; p_1, \ldots, p_m) = \mathcal{F}(z; p_1, \ldots, p_m)\,.$$

By (5.35), we know that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\lambda_{\mathbf{n}}^* \varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 (Q_{\mathbf{n}_0,k-1} R_{\mathbf{n},k-1})(z)}{Q_{\mathbf{n}_0,k}(z)} =$$

$$\begin{cases} \frac{1}{\sqrt{(z-b_1)(z-a_1)}} p_\Lambda(\varphi_1(z))\,, & k = 2\,, \\ \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_\Lambda(\delta\varphi_{k-1}(z))\,, & k = 3, \ldots, m\,, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$. Also, see (5.22),

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n}_0,k-1} h_{\mathbf{n}_0,k}(z) = \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}}\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_{k-1}).$$

Thus, since $\lim_{\mathbf{n}\in\Lambda} \lambda_{\mathbf{n}}^* = c$, we conclude that

$$\lim_{\mathbf{n}\in\Lambda} \frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n}_0,k-1}(z)} = \lim_{\mathbf{n}\in\Lambda} K^2_{\mathbf{n}_0,k-1} \frac{(Q_{\mathbf{n}_0,k-1}R_{\mathbf{n},k-1})(z)}{(h_{\mathbf{n}_0,k}Q_{\mathbf{n}_0,k})(z)} = \tag{5.48}$$

$$\begin{cases} p_\Lambda(\varphi_1(z))/c\,, & k=2\,, \\ p_\Lambda(\delta\varphi_{k-1}(z))/c\,, & k=3,\ldots,m\,, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$.

Recall that $\mathbf{n}_j = (n_1 - \deg(p_2 \cdots p_m) + j, n_2 - \deg(p_3 \cdots p_m), \ldots, n_m)$. It is easy to see that

$$\frac{\Psi_{\mathbf{n}_0,k-1}}{\Psi_{\mathbf{n}_j,k-1}} = \frac{Q_{\mathbf{n}_0,k}}{Q_{\mathbf{n}_j,k}} \frac{Q_{\mathbf{n}_j,k-1}}{Q_{\mathbf{n}_0,k-1}} \frac{\varepsilon_{\mathbf{n}_0,k-1}h_{\mathbf{n}_0,k}}{\varepsilon_{\mathbf{n}_j,k-1}h_{\mathbf{n}_j,k}} \frac{\varepsilon_{\mathbf{n}_j,k-1}}{\varepsilon_{\mathbf{n}_0,k-1}} \frac{K^2_{\mathbf{n}_j,k-1}}{K^2_{\mathbf{n}_0,k-1}}\,.$$

From this expression, applying Proposition 5.3.2 and (5.45), we obtain that the following limit holds uniformly on compact subsets of $\mathbb{C}\setminus(\mathrm{supp}(\sigma_{k-1})\cup\mathrm{supp}(\sigma_k))$

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n}_j,k-1}(z)} = (\Delta_{k-1,1}\cdots\Delta_{1,1})^j \Big(\frac{\widetilde{F}^{(1)}_{k-1}(z)}{\widetilde{F}^{(1)}_{k}(z)}\Big)^j (\kappa_1^{(1)}\cdots\kappa_{k-1}^{(1)})^{2j}\,.$$

Now, from (5.29) and (5.30), we have

$$\frac{\widetilde{F}^{(1)}_{k-1}(z)}{\widetilde{F}^{(1)}_{k}(z)} = \frac{c_k^{(1)}}{c_{k-1}^{(1)}} \mathrm{sg}(\psi^{(1)}_{k-1}(\infty))\psi^{(1)}_{k-1}(z)\,,$$

and from (5.29)

$$(\kappa_1^{(1)}\cdots\kappa_{k-1}^{(1)})^2 = c_1^{(1)} \frac{c_{k-1}^{(1)}}{c_k^{(1)}}\,.$$

Thus,

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n}_j,k-1}(z)} = (\Delta_{k-1,1}\cdots\Delta_{1,1})^j (c_1^{(1)}\mathrm{sg}(\psi^{(1)}_{k-1}(\infty))\psi^{(1)}_{k-1}(z))^j\,.$$

Set

$$\Xi_k := (\Delta_{k-1,1}\cdots\Delta_{1,1})^{\deg(p_2\cdots p_m)}\cdots(\Delta_{k-1,m-1}\cdots\Delta_{1,m-1})^{\deg(p_m)}\,. \tag{5.49}$$

Using the same arguments above, on an appropriate consecutive collection of multi-indices, one proves that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \Xi_k \prod_{j=1}^{m-1} \frac{1}{(\varphi^{(j)}_{k-1}(z))^{\deg(p_{j+1}\cdots p_m)}}\,,$$

uniformly on compact subsets of $\mathbb{C}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))$. Therefore, writing

$$\frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n}_0,k-1}(z)}\frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)}\,,$$

using the expression of p_Λ, applying (5.48), and Lemma 5.5.1, for $k=2$ we get

$$\lim_{\mathbf{n}\in\Lambda}\frac{\widetilde{\Psi}_{\mathbf{n},1}(z)}{\Psi_{\mathbf{n},1}(z)} = \Xi_2\prod_{\nu=1}^{l_1}(\varphi_1(z)-\varphi_0(z_{1,\nu}))^{\tau_{1,\nu}}\prod_{\nu=1}^{l_2}\Big(\frac{1}{\varphi_1(z)}\frac{\varphi_1(z)-\varphi_1(z_{2,\nu})}{z-z_{2,\nu}}\Big)^{\tau_{2,\nu}}$$

$$\times\prod_{j=3}^{m}\prod_{\nu=1}^{l_j}\Big(\frac{\varphi_1(z)-\delta\varphi_{j-1}(z_{j,\nu})}{\varphi_1^{(j-1)}(z)}\Big)^{\tau_{j,\nu}} \tag{5.50}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus(\operatorname{supp}(\sigma_1)\cup\operatorname{supp}(\sigma_2))$, and for $k\geq 3$ we obtain

$$\lim_{\mathbf{n}\in\Lambda}\frac{\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \Xi_k\prod_{\nu=1}^{l_1}(\delta\varphi_{k-1}(z)-\varphi_0(z_{1,\nu}))^{\tau_{1,\nu}}\prod_{\nu=1}^{l_2}\Big(\frac{\delta\varphi_{k-1}(z)-\varphi_1(z_{2,\nu})}{\varphi_{k-1}(z)}\Big)^{\tau_{2,\nu}}$$

$$\times\prod_{\nu=1}^{l_k}\Big(\frac{\delta\varphi_{k-1}(z)-\delta\varphi_{k-1}(z_{k,\nu})}{\varphi_{k-1}^{(k-1)}(z)(z-z_{k,\nu})}\Big)^{\tau_{k,\nu}}\prod_{j=3,j\neq k}^{m}\prod_{\nu=1}^{l_j}\Big(\frac{\delta\varphi_{k-1}(z)-\delta\varphi_{j-1}(z_{j,\nu})}{\varphi_{k-1}^{(j-1)}(z)}\Big)^{\tau_{j,\nu}} \tag{5.51}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))$. Therefore, (5.47) is proved.

From the expression of the limit functions one sees that G_k does not vanish in $\overline{\mathbb{C}}\setminus(\operatorname{supp}(\sigma_k)\cup\operatorname{supp}(\sigma_{k+1}))$. The statements concerning the number of zeros of $\widetilde{\Psi}_{\mathbf{n},k}$ for $k\in\{0,\ldots,m-1\}$ and their limit behavior follows at once from (5.47), on account of the argument principle and the corresponding behavior of the zeros of the polynomials $Q_{\mathbf{n},k+1}$ described in Proposition 5.3.1. Recall that the zeros of $Q_{\mathbf{n},k+1}$ are those of $\Psi_{\mathbf{n},k}$ in $\overline{\mathbb{C}}\setminus\operatorname{supp}(\sigma_k)$.

Now, let us assume that the coefficients of the polynomials p_k are real and $\Lambda\subset\mathbb{Z}_+^m(\circledast)$. Since

$$\frac{\Psi_{\mathbf{n}^l,k-1}}{\Psi_{\mathbf{n},k-1}} = \frac{Q_{\mathbf{n}^l,k}}{Q_{\mathbf{n},k}}\frac{Q_{\mathbf{n},k-1}}{Q_{\mathbf{n}^l,k-1}}\frac{\varepsilon_{\mathbf{n}^l,k-1}h_{\mathbf{n}^l,k}}{\varepsilon_{\mathbf{n},k-1}h_{\mathbf{n},k}}\frac{\varepsilon_{\mathbf{n},k-1}}{\varepsilon_{\mathbf{n}^l,k-1}}\frac{K^2_{\mathbf{n},k-1}}{K^2_{\mathbf{n}^l,k-1}}\,,$$

applying (5.27), (5.28), (5.22), and (5.45), we conclude that the ratio asymptotics

$$\lim_{\mathbf{n}\in\Lambda}\frac{\Psi_{\mathbf{n}^l,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)}\,,\qquad \mathcal{K}\subset\mathbb{C}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))\,,$$

holds and the limit does not vanish in the indicated region.

Since each measure $p_k\,\sigma_k$ is real with constant sign, we can define the polynomials $\widetilde{Q}_{\mathbf{n},k}, 1\leq k\leq m$, as the monic polynomials of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\widetilde{\Psi}_{n,k-1}$ vanishes on Δ_k. Let $\widetilde{Q}_{\mathbf{n},0} \equiv \widetilde{Q}_{\mathbf{n},m+1} \equiv 1$. We also introduce the associated notions

$$\widetilde{\mathcal{H}}_{\mathbf{n},k} := \frac{\widetilde{Q}_{\mathbf{n},k-1}\widetilde{\Psi}_{\mathbf{n},k-1}}{\widetilde{Q}_{\mathbf{n},k}}, \qquad k = 1, \ldots, m+1, \tag{5.52}$$

$\widetilde{\varepsilon}_{\mathbf{n},k}$ as the sign of $\widetilde{\mathcal{H}}_{\mathbf{n},k}(x)p_k(x)d\sigma_k(x)/\widetilde{Q}_{\mathbf{n},k-1}(x)\widetilde{Q}_{\mathbf{n},k+1}(x)$ on $\operatorname{supp}(\sigma_k)$, and

$$\widetilde{K}_{\mathbf{n},k} := \left(\int \widetilde{Q}^2_{\mathbf{n},k}(x) \frac{\widetilde{\varepsilon}_{\mathbf{n},k}\widetilde{\mathcal{H}}_{\mathbf{n},k}(x)p_k(x)d\sigma_k(x)}{\widetilde{Q}_{\mathbf{n},k-1}(x)\widetilde{Q}_{\mathbf{n},k+1}(x)} \right)^{-1/2}. \tag{5.53}$$

The formulas (5.27), (5.28), (5.22), and (5.45) are independent of the orthogonality measures, hence

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n}^l,k-1}(z)}{\widetilde{\Psi}_{\mathbf{n},k-1}(z)} = \lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}^l,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)}.$$

Applying the same argument used in the last two paragraphs of the proof of Theorem 1.3.5 for the polynomial case, we conclude that (5.47) is valid for $\Lambda \subset \mathbb{Z}^m_+(\circledast)$.

The rest of the statements regarding the zeros of $\widetilde{\Psi}_{\mathbf{n},k}$ and their limit behavior follows as in the case of polynomials with complex coefficients. □

Corollary 5.5.6. *Let $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}^m_+(\circledast; p_1q_1, \ldots, p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n}}$ be the monic multiple orthogonal polynomial of smallest degree relative to the Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ and $\mathbf{n}$, whereas $\widetilde{\Psi}_{\mathbf{n},k}, 0 \leq k \leq m$, denote the second type functions defined in (5.44), with p_k replaced by p_k/q_k. Then, for each $k \in \{0, \ldots, m\}$, and $\mathcal{K} \subset \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))$*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{n,k}(z)}{\Psi_{n,k}(z)} = \frac{G_k(z; p_1, \ldots, p_m)}{G_k(z; q_1, \ldots, q_m)}. \tag{5.54}$$

For each $k = \{0, \ldots, m-1\}$ and all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, $\widetilde{\Psi}_{\mathbf{n},k}$ has exactly $N_{\mathbf{n},k+1}$ zeros in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k)$, $\operatorname{supp}(\sigma_{k+1})$ is an attractor of the zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$, in this region, and each point in $\operatorname{supp}(\sigma_{k+1}) \setminus \widetilde{\Delta}_{k+1}$ is a 1 attraction point of zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$. When the polynomials $p_k, q_k, k = 1, \ldots, m$, have real coefficients, all the statements remain valid when $\Lambda \subset \mathbb{Z}^m_+(\circledast)$.

Proof. We consider the auxiliary Nikishin system

$$S_1 := \mathcal{N}\Big(\frac{\sigma_1}{|q_1|^2}, \ldots, \frac{\sigma_m}{|q_m|^2}\Big),$$

and define the related second type functions

$$\Psi^*_{n,0}(z) := Q^*_{\mathbf{n}}(z)\,,$$

$$\Psi^*_{n,k}(z) := \int \frac{\Psi^*_{n,k-1}(x)}{z-x}\frac{d\sigma_k(x)}{|q_k(x)|^2}\,, \quad 1 \le k \le m\,,$$

where $Q^*_{\mathbf{n}}$ denotes the multiple orthogonal polynomial associated to S_1 and $\mathbf{n}$.

Notice that if we perturb the generator of system S_1 multiplying the k-th measure by the real polynomial $|q_k|^2$ we get the generator of the original Nikishin system S. Thus, applying Theorem 5.5.5, we obtain that for all $k \in \{0,\dots,m\}$

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{n,k}(z)}{\Psi^*_{n,k}(z)} = G_k(z;|q_1|^2,\dots,|q_m|^2)\,, \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))\,.$$

The perturbed system $S_2 := \mathcal{N}(\frac{p_1}{q_1}\sigma_1,\dots,\frac{p_m}{q_m}\sigma_m)$ can be written as

$$S_2 = \mathcal{N}\Big(p_1\overline{q}_1\frac{\sigma_1}{|q_1|^2},\dots,p_m\overline{q}_m\frac{\sigma_m}{|q_m|^2}\Big)\,.$$

Therefore, employing the same argument

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{n,k}(z)}{\Psi^*_{n,k}(z)} = G_k(z;p_1\overline{q}_1,\dots,p_m\overline{q}_m)\,, \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))\,.$$

We conclude that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{n,k}(z)}{\Psi_{n,k}(z)} = \frac{G_k(z;p_1\overline{q}_1,\dots,p_m\overline{q}_m)}{G_k(z;q_1\overline{q}_1,\dots,q_m\overline{q}_m)} = \frac{G_k(z;p_1,\dots,p_m)}{G_k(z;q_1,\dots,q_m)}\,,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))$. The statements concerning the zeros can be proved as in the case of polynomial perturbation.

When the polynomials $p_k, q_k, k = 1,\dots,m$, have real coefficients, it follows from Theorem 5.5.5 that (5.54) remains valid for $\Lambda \subset \mathbb{Z}^m_+(\circledast)$. The statements concerning the zeros are derived immediately. □

5.6 Relative asymptotics for the polynomials $Q_{\mathbf{n},k}$

In this section, we will restrict our attention to the case when the polynomials $p_k, q_k, k = 1,\dots,m$, have real coefficients (and of course their zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$). Accordingly, we use the objects $\widetilde{Q}_{\mathbf{n},k}, \widetilde{\mathcal{H}}_{\mathbf{n},k}, \widetilde{K}_{\mathbf{n},k}$, and $\widetilde{\varepsilon}_{\mathbf{n},k}$, introduced at the end of the proof of Theorem 5.5.5 (see (5.52) and (5.53)). Here, we study the asymptotics of the ratios $\widetilde{Q}_{\mathbf{n},k}/Q_{\mathbf{n},k}$.

Lemma 5.6.1. *For any* $\mathbf{n} \in \mathbb{Z}^m_+(\circledast)$

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}} = \prod_{i=1}^k \operatorname{sign}(p_i, \operatorname{supp}(\sigma_i))\,. \quad (5.55)$$

Proof. By definition $\varepsilon_{\mathbf{n},k}$ is the sign of $\mathcal{H}_{\mathbf{n},k}(x)d\sigma_k(x)/Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$ and $\widetilde{\varepsilon}_{\mathbf{n},k}$ is the sign of $\widetilde{\mathcal{H}}_{\mathbf{n},k}(x)p_k(x)d\sigma_k(x)/\widetilde{Q}_{\mathbf{n},k-1}(x)\widetilde{Q}_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$. If $k=1$ these measures reduce respectively to $d\sigma_1(x)/Q_{\mathbf{n},2}(x)$ and $p_1(x)d\sigma_1(x)/\widetilde{Q}_{\mathbf{n},2}(x)$. Since $Q_{\mathbf{n},2}$ and $\widetilde{Q}_{\mathbf{n},2}$ are monic polynomials of the same degree and their zeros are located in Δ_2, which is disjoint with $\mathrm{supp}(\sigma_1)$, it follows that $Q_{\mathbf{n},2}$ and $\widetilde{Q}_{\mathbf{n},2}$ have the same sign on $\mathrm{supp}(\sigma_1)$. Therefore,

$$\frac{\varepsilon_{\mathbf{n},1}}{\widetilde{\varepsilon}_{\mathbf{n},1}} = \mathrm{sign}(p_1, \mathrm{supp}(\sigma_1))\,.$$

To conclude the proof we show that

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}} = \mathrm{sign}(p_k, \mathrm{supp}(\sigma_k))\frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}}\,.$$

Notice that $Q_{\mathbf{n},k-1}$ and $\widetilde{Q}_{\mathbf{n},k-1}$ have the same sign on $\mathrm{supp}(\sigma_k)$ by an argument similar to the one explained above. The same holds for $Q_{\mathbf{n},k+1}$ and $\widetilde{Q}_{\mathbf{n},k+1}$. Therefore

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}} = \frac{\mathrm{sign}(\mathcal{H}_{\mathbf{n},k}, \mathrm{supp}(\sigma_k))}{\mathrm{sign}(p_k\widetilde{\mathcal{H}}_{\mathbf{n},k}, \mathrm{supp}(\sigma_k))}\,.$$

By (5.18), we know that

$$\mathcal{H}_{\mathbf{n},k}(x) = \int_{\Delta_{k-1}} \frac{Q^2_{\mathbf{n},k-1}(t)}{x-t}\frac{\mathcal{H}_{\mathbf{n},k-1}(t)d\sigma_{k-1}(t)}{Q_{\mathbf{n},k-2}(t)Q_{\mathbf{n},k}(t)}\,,$$

and

$$\widetilde{\mathcal{H}}_{\mathbf{n},k}(x) = \int_{\Delta_{k-1}} \frac{\widetilde{Q}^2_{\mathbf{n},k-1}(t)}{x-t}\frac{\widetilde{\mathcal{H}}_{\mathbf{n},k-1}(t)p_{k-1}(t)d\sigma_{k-1}(t)}{\widetilde{Q}_{\mathbf{n},k-2}(t)\widetilde{Q}_{\mathbf{n},k}(t)}\,.$$

Consequently,

$$\frac{\mathrm{sign}(\mathcal{H}_{\mathbf{n},k}, \mathrm{supp}(\sigma_k))}{\mathrm{sign}(\widetilde{\mathcal{H}}_{\mathbf{n},k}, \mathrm{supp}(\sigma_k))} = \frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}}\,,$$

and the claim follows. □

We are ready to state and prove

Theorem 5.6.2. *Let $S = \mathcal{N}'(\sigma_1,\dots,\sigma_m)$ and $\Lambda \subset \mathbb{Z}^m_+(\circledast)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda, n_1 - n_m \le C$, where C is a constant. Assume that the polynomials $p_k, k=1,\dots,m$, have real coefficients. For each $k \in \{1,\dots,m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \mathcal{F}_k(z;p_1,\dots,p_m)\,, \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_k)\,, \tag{5.56}$$

where $\mathcal{F}_k(z;p_1,\dots,p_m)$ is analytic and never vanishes in $\overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_k)$ and

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{K}^2_{\mathbf{n},k}}{K^2_{\mathbf{n},k}} = \frac{\prod_{i=1}^k \mathrm{sign}(p_i, \mathrm{supp}(\sigma_i))}{G_k(\infty;p_1,\dots,p_m)}. \tag{5.57}$$

For $k \in \{1, \ldots, m-1\}$ *and* $z \in \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))$

$$\mathcal{F}_{k+1}(z; p_1, \ldots, p_m) = \prod_{i=0}^{k} \frac{G_i(z; p_1, \ldots, p_m)}{G_i(\infty; p_1, \ldots, p_m)}\,, \tag{5.58}$$

where $G_i(z; p_1, \ldots, p_m)$ *is the function given in* (5.47).

Proof. If $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$, from (5.35) and Lemma 5.5.1, we have that

$$\lim_{\mathbf{n}\in\Lambda} \lambda_{\mathbf{n}}^* \varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 \frac{Q_{\mathbf{n}_0,k-1}(z)(p_k \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n}_0,k}(z)} =$$

$$\begin{cases} \frac{1}{\sqrt{(z-b_1)(z-a_1)}} p_\Lambda(\varphi_1(z))\,, & k=2\,, \\ \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_\Lambda(\delta\varphi_{k-1}(z))\,, & k = 3, \ldots, m\,. \end{cases} \tag{5.59}$$

By Proposition 5.3.1, we know that

$$\lim_{\mathbf{n}\in\Lambda} \widetilde{\varepsilon}_{\mathbf{n},k} \widetilde{K}_{\mathbf{n},k}^2 \widetilde{\mathcal{H}}_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z-b_k)(z-a_k)}}\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k)\,, \tag{5.60}$$

where $[a_k, b_k] = \widetilde{\Delta}_k$. Formula (5.52) implies

$$\frac{\lambda_{\mathbf{n}}^* \varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 Q_{\mathbf{n}_0,k-1}(z)(p_k \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{\widetilde{\varepsilon}_{\mathbf{n},k-1} \widetilde{K}_{\mathbf{n},k-1}^2 \widetilde{\mathcal{H}}_{\mathbf{n},k}(z) Q_{\mathbf{n}_0,k}(z)} =$$

$$\lambda_{\mathbf{n}}^* \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K_{\mathbf{n}_0,k-1}^2}{\widetilde{K}_{\mathbf{n},k-1}^2} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k \cdots p_m)(z)\,. \tag{5.61}$$

Using (5.59), (5.60), and (5.61), we obtain

$$\lim_{\mathbf{n}\in\Lambda} \lambda_{\mathbf{n}}^* \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K_{\mathbf{n}_0,k-1}^2}{\widetilde{K}_{\mathbf{n},k-1}^2} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k \cdots p_m)(z) =$$

$$\begin{cases} p_\Lambda(\varphi_1(z))\,, & k=2\,, \\ p_\Lambda(\delta\varphi_{k-1}(z))\,, & k = 3, \ldots, m\,. \end{cases} \tag{5.62}$$

Using the results on ratio asymptotics for the constants $K_{\mathbf{n},k}, \widetilde{K}_{\mathbf{n},k}$ and the polynomials $Q_{\mathbf{n},k}, \widetilde{Q}_{\mathbf{n},k}$, it follows that (5.62) is also valid for $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

Since

$$\frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} = \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\varepsilon_{\mathbf{n},k-1}} \frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}}\,,$$

applying Lemma 5.6.1, (5.45), and (5.49), we obtain

$$\frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} = \Xi_k \prod_{i=1}^{k-1} \operatorname{sign}(p_i, \operatorname{supp}(\sigma_i))\,. \tag{5.63}$$

We have

$$\frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} = \frac{K^2_{\mathbf{n}_0,k-1}}{K^2_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}}, \tag{5.64}$$

and by (5.27)

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n}_0,k-1}}{K^2_{\mathbf{n},k-1}} = \prod_{i=1}^{m-1} (\kappa_1^{(i)} \cdots \kappa_{k-1}^{(i)})^{-2\deg(p_{i+1}\cdots p_m)}. \tag{5.65}$$

Write

$$\frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} = \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n},k-1}(z)} \frac{Q_{\mathbf{n},k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)}, \tag{5.66}$$

and

$$\frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} = \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} \frac{Q_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)}. \tag{5.67}$$

Notice that

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n},k-1}(z)} = \prod_{i=1}^{m-1} (\widetilde{F}_{k-1}^{(i)}(z))^{-deg(p_{i+1}\cdots p_m)}. \tag{5.68}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} = \prod_{i=1}^{m-1} (\widetilde{F}_{k}^{(i)}(z))^{deg(p_{i+1}\cdots p_m)}. \tag{5.69}$$

From (5.30) and (5.29) it follows that

$$\frac{\widetilde{F}_k^{(i)}(z)}{\widetilde{F}_{k-1}^{(i)}(z)} = \frac{c_{k-1}^{(i)}}{c_k^{(i)}} \frac{\mathrm{sg}(\psi_{k-1}^{(i)}(\infty))}{\psi_{k-1}^{(i)}(z)},$$

$$(\kappa_1^{(i)} \cdots \kappa_{k-1}^{(i)})^2 = \frac{c_1^{(i)}\, c_{k-1}^{(i)}}{c_k^{(i)}}.$$

Therefore, using (5.46), we get

$$\frac{\widetilde{F}_k^{(i)}(z)}{\widetilde{F}_{k-1}^{(i)}(z)(\kappa_1^{(i)} \cdots \kappa_{k-1}^{(i)})^2} = \varphi_{k-1}^{(i)}(z). \tag{5.70}$$

Taking into consideration (5.63)-(5.70), we conclude that

$$\lim_{\mathbf{n}\in\Lambda} \lambda^*_{\mathbf{n}} \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k \cdots p_m)(z) =$$

$$c\,\Xi_k \prod_{i=1}^{k-1} \mathrm{sign}(p_i, \mathrm{supp}(\sigma_i)) \prod_{i=1}^{m-1} (\varphi_{k-1}^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)} \lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)}$$

$$\times (p_k \cdots p_m)(z) \lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)}, \tag{5.71}$$

provided that the limits on the right-hand side exist.

In Theorem 1.3.5 we proved (5.56) for $k = 1$. Assume that $k = 2$. Equations (5.62) and (5.71) yield

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} \frac{\widetilde{Q}_{\mathbf{n},2}(z)}{Q_{\mathbf{n},2}(z)}$$

$$= \frac{p_\Lambda(\varphi_1(z))\mathcal{F}(z;p_1,\ldots,p_m)}{c\,\Xi_2\,\mathrm{sign}(p_1,\mathrm{supp}(\sigma_1))(p_2\cdots p_m)(z)\prod_{i=1}^{m-1}(\varphi_1^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}}\,,$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus\mathrm{supp}(\sigma_2)$. Using (5.50), we have

$$\frac{p_\Lambda(\varphi_1(z))}{c\,\Xi_2\,(p_2\cdots p_m)(z)\,\prod_{i=1}^{m-1}(\varphi_1^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}} = G_1(z;p_1,\ldots,p_m)\,.$$

Consequently,

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} \frac{\widetilde{Q}_{\mathbf{n},2}(z)}{Q_{\mathbf{n},2}(z)} = \frac{\mathcal{F}(z;p_1,\ldots,p_m)\,G_1(z;p_1,\ldots,p_m)}{\mathrm{sign}(p_1,\mathrm{supp}(\sigma_1))}\,.$$

Evaluating at infinity, we obtain ($\mathcal{F}(\infty;p_1,\ldots,p_m)=1$)

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} = \frac{G_1(\infty;p_1,\ldots,p_m)}{\mathrm{sign}(p_1,\mathrm{supp}(\sigma_1))}\,.$$

Therefore, (5.57) and (5.58) are satisfied for $k = 1$, since $G_0 = \mathcal{F}$.

Define the functions

$$\mathcal{F}_k(z;p_1,\ldots,p_m) := \lim_{n\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)}$$

provided the limit exists. From (5.51) it follows that for any $k \geq 3$,

$$\frac{p_\Lambda(\delta\varphi_{k-1}(z))}{c\,\Xi_k\,(p_k\cdots p_m)(z)\,\prod_{i=1}^{m-1}(\varphi_{k-1}^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}} = G_{k-1}(z;p_1,\ldots,p_m)\,.$$

As a consequence, using (5.71), we obtain that for any $k \geq 3$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{\mathcal{F}_{k-1}(z;p_1,\ldots,p_m)\,G_{k-1}(z;p_1,\ldots,p_m)}{\prod_{i=1}^{k-1}\mathrm{sign}(p_i,\mathrm{supp}(\sigma_i))}\,.$$

Therefore, using an induction process one proves (5.56)-(5.58). □

Corollary 5.6.3. *Let $S = \mathcal{N}'(\sigma_1,\ldots,\sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1,\ldots,\frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials with real coefficients whose zeros lie in $\mathbb{C}\setminus\cup_{k=1}^m\Delta_k$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of distinct multi-indices such that for all $\mathbf{n}\in\Lambda$, $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n},k}, 1 \leq k \leq m$, be the monic polynomials of degree*

$N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\widetilde{\Psi}_{n,k-1}$ vanishes on Δ_k, where $\widetilde{\Psi}_{\mathbf{n},k}, 0 \leq k \leq m$, denote the second type functions defined in (5.44)*, with p_k replaced by p_k/q_k. Let $\widetilde{K}_{\mathbf{n},k}, 1 \leq k \leq m$ be the constants defined in* (5.53)*, with p_k replaced by p_k/q_k. Then, for each $k \in \{1, \ldots, m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{\mathcal{F}_k(z; p_1, \ldots, p_m)}{\mathcal{F}_k(z; q_1, \ldots, q_m)}, \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k), \tag{5.72}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{K}^2_{\mathbf{n},k}}{K^2_{\mathbf{n},k}} = \prod_{i=1}^{k} \operatorname{sign}(p_i/q_i, \operatorname{supp}(\sigma_i)) \frac{G_k(\infty; q_1, \ldots, q_m)}{G_k(\infty; p_1, \ldots, p_m)}. \tag{5.73}$$

Proof. By $Q^*_{\mathbf{n},k}$ denote polynomials associated with the auxiliary Nikishin system $\mathcal{N}(\sigma_1/q_1, \ldots, \sigma_m/q_m)$, corresponding to the indices $\mathbf{n}, k$. On account of Theorem 5.6.2, we have that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q^*_{\mathbf{n},k}(z)} = \mathcal{F}_k(z; p_1, \ldots, p_m), \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k).$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k}(z)}{Q^*_{\mathbf{n},k}(z)} = \mathcal{F}_k(z; q_1, \ldots, q_m), \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k).$$

Therefore, (5.72) is obtained. Using the same idea, (5.73) follows from (5.57). □

Remark 5.6.4. *Theorem* 5.5.5 *and Corollary* 5.5.6 *allow to define polynomials $\widetilde{Q}_{\mathbf{n},k}, k = 1, \ldots, m$, in the case when p_k, q_k have complex coefficients as those monic polynomials which carry the zeros of $\widetilde{\Psi}_{\mathbf{n},k-1}$ lying in $\mathbb{C} \setminus \Delta_{k-1}$. For such polynomials $\widetilde{Q}_{\mathbf{n},k}$, results analogous to those expressed in Theorem* 5.6.2 *and Corollary* 5.6.3 *can be proved.*

6. RATIO ASYMPTOTICS REVISITED

This chapter is organized as follows. In Section 6.2 we introduce and study an auxiliary system of second type functions. These second type functions are specially reviewed in Section 6.3 when $m = 2, 3$ to exemplify their construction. An interlacing property for the zeros of the polynomials $Q_{\mathbf{n}}$ and of the second type functions is proved in Section 6.4. Using the interlacing property of zeros and results on ratio and relative asymptotics of polynomials orthogonal with respect to varying measures, in Section 6.5 a system of boundary value problems is derived which allows to conclude the proof of the main result of this chapter, Theorem 6.5.2.

6.1 Preliminaries and notation

Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$. Fix a multi-index $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$. Let $Q_{\mathbf{n}}$ be an $\mathbf{n}$-th monic multiple orthogonal polynomial with respect to S. That is, $Q_{\mathbf{n}} \not\equiv 0$ is monic, $\deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \cdots + n_m$, and

$$\int Q_{\mathbf{n}}(x) x^{\nu} ds_k(x) = 0, \quad \nu = 0, \dots, n_k - 1, \quad k = 1, \dots, m. \tag{6.1}$$

If (6.1) implies that $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, the multi–index $\mathbf{n}$ is *normal* and the corresponding monic multiple orthogonal polynomial is uniquely determined. In addition, if the zeros of $Q_{\mathbf{n}}$ are simple and lie in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$, the multi–index is said to be *strongly normal.* For Nikishin systems with $m = 1, 2, 3$, all multi-indices are strongly normal (see [22]). An open question is whether or not this is true for all $m \in \mathbb{N}$. (Recently, U. Fidalgo Prieto and G. López Lagomasino claimed to have proved that all multi-indices are strongly normal.) The best result known when $m \geq 4$ is that all multi-indices in the class

$$\mathbb{Z}_+^m(*) = \{\mathbf{n} \in \mathbb{Z}_+^m : \nexists\, 1 \leq i < j < k \leq m\,, \text{ with } n_i < n_j < n_k\}$$

are strongly normal (see [20]).

In [5], a Rakhmanov type theorem was proved for Nikishin systems such that $|\sigma_k'| > 0$ a.e. on $\mathrm{Co}(\mathrm{supp}(\sigma_k)), k = 1, \dots, m$, and sequences of multi-indices contained in

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : 1 \leq i < j \leq m \Rightarrow n_j \leq n_i + 1\}\,.$$

It is easy to see that $\mathbb{Z}_+^m(\circledast) \subset \mathbb{Z}_+^m(*)$. Here, we assume that $\mathrm{supp}(\sigma_k) = \widetilde{\Delta}_k \cup e_k, k = 1, \dots, m$, where $\widetilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$

a.e. on $\widetilde{\Delta}_k$, e_k is a set without accumulation points in $\mathbb{R} \setminus \widetilde{\Delta}_k$, and the sequence of multi-indices on which the limit is taken is in $\mathbb{Z}_+^m(*)$.

As in Theorem 1.3.4, the proof of the corresponding result under these weaker assumptions, Theorem 6.5.2 below, uses the construction of second type functions. Now, this construction depends on the relative value of the components of the multi-indices in $\mathbb{Z}_+^m(*)$ under consideration. As we saw before, a crucial step consists in proving an interlacing property for the zeros of the second type functions corresponding to "consecutive" multi-indices. For this purpose, we need to be sure that the second type functions are built using the same procedure. To distinguish different classes of multi-indices which respond for the same construction of second type functions, we introduce the following definition.

Definition 6.1.1. *Suppose that* $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$*. Let* $\tau_{\mathbf{n}}$ *denote the permutation of* $\{1, 2, \ldots, m\}$ *given by*

$$\tau_{\mathbf{n}}(i) = j \quad \textit{if} \quad \left\{ \begin{array}{lll} n_j > n_k & \textit{for} & k < j, \quad k \notin \{\tau_{\mathbf{n}}(1), \ldots, \tau_{\mathbf{n}}(i-1)\} \\ n_j \geq n_k & \textit{for} & k > j, \quad k \notin \{\tau_{\mathbf{n}}(1), \ldots, \tau_{\mathbf{n}}(i-1)\} \end{array} \right. .$$

In words, $\tau_{\mathbf{n}}(1)$ is the subindex of the first component of $\mathbf{n}$ (from left to right) which is greater or equal than the rest, $\tau_{\mathbf{n}}(2)$ is the subindex of the first component which is second largest, and so forth. For example, if $n_1 \geq \cdots \geq n_m$ then $\tau_{\mathbf{n}}$ is the identity.

Let τ denote a permutation of $\{1, 2, \ldots, m\}$. Set

$$\mathbb{Z}_+^m(*, \tau) = \{\mathbf{n} \in \mathbb{Z}_+^m(*) : \tau_{\mathbf{n}} = \tau\} .$$

Let $\mathbf{n} \in \mathbb{Z}_+^m$ and $l \in \{1, \ldots, m\}$. Define

$$\mathbf{n}_l := (n_1, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m) .$$

(In Chapters 1 to 4 this was denoted by $\mathbf{n}^l$ but here we will need to give that notation a different meaning.)

The relevant Riemann surface in this chapter coincides with the one presented in Chapter 5. The main result of this chapter is Theorem 6.5.2.

6.2 Functions of second type and orthogonality properties

Fix $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(*)$ and consider $Q_{\mathbf{n}}$ the $\mathbf{n}$-th multi-orthogonal polynomial with respect to a Nikishin system $S = \mathcal{N}(\Sigma)$, $\Sigma = (\sigma_1, \ldots, \sigma_m)$. As before, $\Delta_k = \mathrm{Co}(\mathrm{supp}(\sigma_k)), k = 1, \ldots, m$. Inductively, we define functions of second type $\Psi_{\mathbf{n},k}$, $k = 0, 1, \ldots, m$, systems of measures $\Sigma^k = (\sigma_{k+1}^k, \ldots, \sigma_m^k), k = 0, 1, \ldots, m-1, \mathrm{Co}(\mathrm{supp}(\sigma_j^k)) \subset \Delta_j$, which generate Nikishin systems, and multi-indices $\mathbf{n}^k \in \mathbb{Z}_+^{m-k}(*), k = 0, \ldots, m-1$. Take $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}, \mathbf{n}^0 = \mathbf{n}$, and $\Sigma^0 = \Sigma$.

Suppose that $\mathbf{n}^k = (n_{k+1}^k, \ldots, n_m^k)$, $\Sigma^k = (\sigma_{k+1}^k, \ldots, \sigma_m^k)$ and $\Psi_{\mathbf{n},k}$ have already been defined, where $0 \leq k \leq m-2$. Let

$$\mathbf{n}^{k+1} = (n_{k+2}^{k+1}, \ldots, n_m^{k+1}) \in \mathbb{Z}_+^{m-k-1}(*)$$

be the multi-index obtained deleting from $\mathbf{n}^k$ the first component $n^k_{r_k}$ which verifies

$$n^k_{r_k} = \max\{n^k_j : k+1 \le j \le m\}.$$

The components of $\mathbf{n}^{k+1}$ and $\mathbf{n}^k$ are related as follows

$$n^k_{k+1} = n^{k+1}_{k+2}, \ldots, n^k_{r_k-1} = n^{k+1}_{r_k}, n^k_{r_k+1} = n^{k+1}_{r_k+1}, \ldots, n^k_m = n^{k+1}_m.$$

Denote

$$\Psi_{\mathbf{n},k+1}(z) = \int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)}{z-x}\, ds^k_{r_k}(x)\,, \tag{6.2}$$

where $s^k_{r_k} = \langle \sigma^k_{k+1}, \ldots, \sigma^k_{r_k} \rangle$ is the corresponding component of the Nikishin system $S^k = \mathcal{N}(\Sigma^k) = (s^k_{k+1}, \ldots, s^k_m)$.

In order to define Σ^{k+1} we introduce the following notation. Set

$$s^k_{i,j} = \langle \sigma^k_i, \ldots, \sigma^k_j \rangle, \qquad k+1 \le i \le j \le m,$$

where $\sigma^k_i, \ldots, \sigma^k_j$ are measures in Σ^k. In page 390 of [30] it is proved that there exists a finite measure $\tau^k_{i,j}$ with constant sign such that

$$\mathrm{Co}(\mathrm{supp}(\tau^k_{i,j})) \subset \mathrm{Co}(\mathrm{supp}(s^k_{i,j}))$$

$$\frac{1}{\widehat{s}^k_{i,j}(z)} = l^k_{i,j}(z) + \widehat{\tau}^k_{i,j}(z)$$

where $l^k_{i,j}$ is a certain polynomial of degree 1. That $\mathrm{Co}(\mathrm{supp}(s^k_{i,j})) \subset \Delta_i$ easily follows by induction. We wish to remark that the continuous part of $\mathrm{supp}(s^k_{i,j})$ and $\mathrm{supp}(\tau^k_{i,j})$ coincide, but not their isolated parts. In fact, zeros of $\widehat{s}^k_{i,j}$ on Δ_i (there is one such zero between two consecutive mass points of $s^k_{i,j}$) become poles of $\widehat{\tau}^k_{i,j}$ (mass points of $\tau^k_{i,j}$).

Suppose that $r_k = k+1$. In this case, we take

$$\Sigma^{k+1} = (\sigma^k_{k+2}, \ldots, \sigma^k_m) = (\sigma^{k+1}_{k+2}, \ldots, \sigma^{k+1}_m)$$

deleting the first measure of Σ^k. If $r_k \ge k+2$, then Σ^{k+1} is defined by

$$(\tau^k_{k+2,r_k}, \widehat{s}^k_{k+2,r_k} d\tau^k_{k+3,r_k}, \ldots, \widehat{s}^k_{r_k-1,r_k} d\tau^k_{r_k,r_k}, \widehat{s}^k_{r_k,r_k} d\sigma^k_{r_k+1}, \sigma^k_{r_k+2}, \ldots, \sigma^k_m)\,,$$

where $\mathrm{Co}(\mathrm{supp}(\sigma^{k+1}_j)) \subset \Delta_j, j = k+2, \ldots, m$. Any two consecutive measures in the system Σ^{k+1} are supported on disjoint intervals; therefore, Σ^{k+1} generates a Nikishin system. To conclude we define

$$\Psi_{\mathbf{n},m}(z) = \int_{\Delta_m} \frac{\Psi_{\mathbf{n},m-1}(x)}{z-x}\, ds^{m-1}_m(x)\,.$$

If $n_1 \ge \cdots \ge n_m$, we have that $\mathbf{n}^k = (n_{k+1}, \ldots, n_m), \Sigma^k = (\sigma_{k+1}, \ldots, \sigma_m)$ and $\Psi_{\mathbf{n},k}(z) = \int_{\Delta_k} \frac{\Psi_{\mathbf{n},k-1}(x)}{z-x} d\sigma_k(x), k = 1, \ldots, m$. Basically, this is the situation considered in [5] and in Chapter 5.

6.3 Some examples when $m = 2, 3$

To fix ideas let us turn our attention to the cases $m = 2$ and $m = 3$. We denote by $\mathcal{C}(f;\mu)$ the Cauchy transform of $f d\mu$; that is,

$$\mathcal{C}(f;\mu)(z) = \int \frac{f(x)}{z-x}\, d\mu(x)\,.$$

In the tables below, we omit the line corresponding to $k = 0$ because by definition $\Sigma^0 = \Sigma$, $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$ and $\mathbf{n}^0 = \mathbf{n}$.

Tab. 6.1: m=2

$m = 2$	k	r_{k-1}	$\Psi_{\mathbf{n},k}$	Σ^k	$\mathbf{n}^k$
$n_1 \geq n_2$	1	1	$\mathcal{C}(Q_{\mathbf{n}};\sigma_1)$	(σ_2)	(n_2)
$n_1 < n_2$	1	2	$\mathcal{C}(Q_{\mathbf{n}};\langle\sigma_1,\sigma_2\rangle)$	(τ_2)	(n_1)

Tab. 6.2: $m = 3$

$m = 3$	k	r_{k-1}	$\Psi_{\mathbf{n},k}$	Σ^k	$\mathbf{n}^k$
$n_1 \geq n_2 \geq n_3$	1	1	$\mathcal{C}(Q_{\mathbf{n}};\sigma_1)$	(σ_2,σ_3)	(n_2,n_3)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1};\sigma_2)$	(σ_3)	(n_3)
$n_1 \geq n_3 > n_2$	1	1	$\mathcal{C}(Q_{\mathbf{n}};\sigma_1)$	(σ_2,σ_3)	(n_2,n_3)
	2	3	$\mathcal{C}(\Psi_{\mathbf{n},1};\langle\sigma_2,\sigma_3\rangle)$	(τ_3)	(n_2)
$n_2 > n_1 \geq n_3$	1	2	$\mathcal{C}(Q_{\mathbf{n}};\langle\sigma_1,\sigma_2\rangle)$	$(\tau_2,\langle\sigma_3,\sigma_2\rangle)$	(n_1,n_3)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1};\tau_2)$,	$(\langle\sigma_3,\sigma_2\rangle)$	(n_3)
$n_2 \geq n_3 > n_1$	1	2	$\mathcal{C}(Q_{\mathbf{n}};\langle\sigma_1,\sigma_2\rangle)$	$(\tau_2,\langle\sigma_3,\sigma_2\rangle)$	(n_1,n_3)
	2	3	$\mathcal{C}(\Psi_{\mathbf{n},1};\langle\tau_2,\sigma_3,\sigma_2\rangle)$	$(\tau_{3,2})$	(n_1)
$n_3 > n_1 \geq n_2$	1	3	$\mathcal{C}(Q_{\mathbf{n}};\langle\sigma_1,\sigma_2,\sigma_3\rangle)$	$(\tau_{2,3},\langle\tau_3,\sigma_2,\sigma_3\rangle)$	(n_1,n_2)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1};\tau_{2,3})$	$(\langle\tau_3,\sigma_2,\sigma_3\rangle)$	(n_2)

In Theorem 2 of [23] it was proved that the functions $\Psi_{\mathbf{n},k}$ verify the following orthogonality relations. For each $k = 0, 1, \ldots, m-1$,

$$\int_{\Delta_{k+1}} x^\nu \Psi_{\mathbf{n},k}(x)\, ds_i^k(x) = 0, \quad \nu = 0,1,\ldots,n_i^k - 1, \quad i = k+1,\ldots,m\,, \tag{6.3}$$

where $s_i^k = \langle\sigma_{k+1}^k, \ldots, \sigma_i^k\rangle$.

We wish to underline that since $\mathbb{Z}_+^2(*) = \mathbb{Z}_+^2$, all multi-indices with two components have associated functions of second type. However, for $m = 3$ the case $n_1 < n_2 < n_3$ has not been considered (see Table 6.2). The rest of this section will be devoted to the construction of certain functions $\Psi_{\mathbf{n},k}$ for this case and to the proof of the orthogonality relations they satisfy. We use the following auxiliary result.

Lemma 6.3.1. *Let $s_{3,2} = \langle\sigma_3,\sigma_2\rangle$. Then*

$$\int_{\Delta_2} \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} \frac{d\tau_{2,3}(x)}{z-x} + C_1 = \frac{\widehat{\sigma}_2(z)}{\widehat{s}_{2,3}(z)}\,, \quad z \in \mathbb{C} \setminus \operatorname{supp}(\sigma_2)\,, \tag{6.4}$$

where $C_1 = \sigma_2(\Delta_2)/s_{2,3}(\Delta_2)$.

Proof. We employ two useful relations. The first one is

$$\widehat{\sigma}_2(\zeta)\,\widehat{\sigma}_3(\zeta) = \widehat{s}_{2,3}(\zeta) + \widehat{s}_{3,2}(\zeta), \quad \zeta \in \mathbb{C} \setminus (\mathrm{supp}(\sigma_2) \cup \mathrm{supp}(\sigma_3)) . \tag{6.5}$$

The proof is straightforward and may be found in Lemma 4 of [22]. The second one was mentioned above and states that there exists a polynomial $l_{2,3}$ of degree 1 and a measure $\tau_{2,3}$ such that

$$\frac{1}{\widehat{s}_{2,3}(z)} = \widehat{\tau}_{2,3}(z) + l_{2,3}(z), \quad z \in \mathbb{C} \setminus \mathrm{supp}(\sigma_2) . \tag{6.6}$$

Notice that

$$\frac{\widehat{\sigma}_2(z)}{\widehat{s}_{2,3}(z)} - C_1 = \mathcal{O}\left(\frac{1}{z}\right) \in H(\overline{\mathbb{C}} \setminus \Delta_2) .$$

On the other hand, from (6.5) and (6.6) it follows that

$$\frac{\widehat{\sigma}_2}{\widehat{s}_{2,3}} = \frac{\widehat{\sigma}_2\,\widehat{\sigma}_3}{\widehat{\sigma}_3\,\widehat{s}_{2,3}} = \frac{\widehat{s}_{2,3} + \widehat{s}_{3,2}}{\widehat{\sigma}_3\,\widehat{s}_{2,3}} = \frac{1}{\widehat{\sigma}_3} + \frac{\widehat{s}_{3,2}}{\widehat{\sigma}_3}\,l_{2,3} + \frac{\widehat{s}_{3,2}}{\widehat{\sigma}_3}\,\widehat{\tau}_{2,3} .$$

Since $\frac{1}{\widehat{\sigma}_3} + \frac{\widehat{s}_{3,2}}{\widehat{\sigma}_3}\,l_{2,3}$ and $\frac{\widehat{s}_{3,2}}{\widehat{\sigma}_3}$ are analytic on a neighborhood of Δ_2, from (2.4) the thesis readily follows. □

We are ready to define the functions of second type and to prove the orthogonality properties they verify for multi-indices with 3 components not in $\mathbb{Z}_+^3(*)$ (with $n_1 < n_2 < n_3$).

Lemma 6.3.2. *Fix* $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3$ *where* $n_1 < n_2 < n_3$ *and consider* $Q_{\mathbf{n}}$ *the* $\mathbf{n}$*-th orthogonal polynomial associated to a Nikishin system* $S = (s_1, s_2, s_3) = \mathcal{N}(\sigma_1, \sigma_2, \sigma_3)$. *Set* $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$,

$$\Psi_{\mathbf{n},1}(z) = \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{z - x}\, d\,s_{1,3}(x) , \tag{6.7}$$

$$\Psi_{\mathbf{n},2}(z) = \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)}{z - x}\, \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)}\, d\,\tau_{2,3}(x) . \tag{6.8}$$

Then

$$\int_{\Delta_1} t^{\nu}\, \Psi_{\mathbf{n},0}(t)\, d\,s_{1,j}(t) = 0, \qquad 0 \le \nu \le n_j - 1, \quad 1 \le j \le 3 \tag{6.9}$$

$$\int_{\Delta_2} t^{\nu}\, \Psi_{\mathbf{n},1}(t)\, d\,\tau_{2,3}(t) = 0, \qquad 0 \le \nu \le n_1 - 1 \tag{6.10}$$

$$\int_{\Delta_2} t^{\nu}\, \Psi_{\mathbf{n},1}(t)\, \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(t)}\, d\,\tau_{2,3}(t) = 0, \qquad 0 \le \nu \le n_2 - 1 \tag{6.11}$$

$$\int_{\Delta_3} t^{\nu}\, \Psi_{\mathbf{n},2}(t)\, \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)}\, d\,\tau_{3,2}(t) = 0, \qquad 0 \le \nu \le n_1 - 1. \tag{6.12}$$

Remark 6.3.3. *The measure $\widehat{s}_{3,2}\, d\tau_{2,3}/\widehat{\sigma}_3$ supported on Δ_2 cannot be written in the form $\langle \tau_{2,3}, \mu\rangle$ for some measure μ supported on Δ_3, so there is no Σ^1 and S^1 in this case.*

Proof of Lemma 6.3.2. The relations (6.9) follow directly from the definition of $Q_{\mathbf{n}}$. Let us justify (6.10) and (6.11).

For $0 \le \nu \le n_1 - 1 (\le n_3 - 3)$, applying Fubini's theorem,

$$\int_{\Delta_2} t^\nu\, \Psi_{\mathbf{n},1}(t)\, d\tau_{2,3}(t) = \int_{\Delta_2} t^\nu \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{t-x}\, ds_{1,3}(x)\, d\tau_{2,3}(t)$$

$$= \int_{\Delta_1} Q_{\mathbf{n}}(x) \int_{\Delta_2} \frac{t^\nu - x^\nu + x^\nu}{t-x}\, d\tau_{2,3}(t)\, ds_{1,3}(x)$$

$$= \int_{\Delta_1} Q_{\mathbf{n}}(x)\, p_\nu(x)\, ds_{1,3}(x) - \int_{\Delta_1} x^\nu Q_{\mathbf{n}}(x)\, \widehat{\tau}_{2,3}(x)\, ds_{1,3}(x)\,,$$

where $p_\nu(x) = \int_{\Delta_2} \frac{t^\nu - x^\nu}{t-x}\, d\tau_{2,3}(t)$ is a polynomial of degree at most $n_1 - 2$. Since $ds_{1,3}(x) = \widehat{s}_{2,3}(x)\, d\sigma_1(x)$ and $\widehat{\tau}_{2,3}(x)\,\widehat{s}_{2,3}(x) = 1 - l_{2,3}(x)\,\widehat{s}_{2,3}(x)$, the measure $\widehat{\tau}_{2,3}(x)\, ds_{1,3}(x)$ is equal to $d\sigma_1(x) - l_{2,3}(x)\, ds_{1,3}(x)$. Therefore, applying (6.9) both integrals vanish and we obtain (6.10). Actually, we only needed that $n_1 \le n_3 - 1$.

If $0 \le \nu \le n_2 - 1 (\le n_3 - 2)$,

$$\int_{\Delta_2} t^\nu\, \Psi_{\mathbf{n},1}(t)\, \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(t)}\, d\tau_{2,3}(t) = \int_{\Delta_2} t^\nu \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(t)} \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{t-x}\, ds_{1,3}(x)\, d\tau_{2,3}(t)$$

$$= \int_{\Delta_1} Q_{\mathbf{n}}(x) \int_{\Delta_2} \frac{t^\nu - x^\nu + x^\nu}{t-x}\, \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(t)}\, d\tau_{2,3}(t)\, ds_{1,3}(x)$$

$$= \int_{\Delta_1} Q_{\mathbf{n}}(x)\, x^\nu \int_{\Delta_2} \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(t)}\, \frac{d\tau_{2,3}(t)}{t-x} ds_{1,3}(x)\,.$$

By Lemma 6.3.1, the last expression is equal to

$$C_1 \int_{\Delta_1} Q_{\mathbf{n}}(x)\, x^\nu\, ds_{1,3}(x) - \int_{\Delta_1} Q_{\mathbf{n}}(x)\, x^\nu\, \frac{\widehat{\sigma}_2(x)}{\widehat{s}_{2,3}(x)}\, ds_{1,3}(x)$$

$$= - \int_{\Delta_1} Q_{\mathbf{n}}(x)\, x^\nu\, ds_{1,2}(x) = 0\,,$$

taking into account that $ds_{1,3}(x) = \widehat{s}_{2,3}(x)\, d\sigma_1(x)$ and (6.9). This proves (6.11). It would have been sufficient to require $n_2 \le n_3$.

Let us prove (6.12). Take $0 \le \nu \le n_1 - 1$, we have

$$\int_{\Delta_3} t^\nu\, \Psi_{\mathbf{n},2}(t)\, \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)} d\tau_{3,2}(t) = \int_{\Delta_3} t^\nu \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)}{t-x}\, \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} d\tau_{2,3}(x) \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)} d\tau_{3,2}(t)$$

$$= \int_{\Delta_2} \Psi_{\mathbf{n},1}(x)\, \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} \int_{\Delta_3} \frac{t^\nu - x^\nu + x^\nu}{t-x}\, \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)} d\tau_{3,2}(t)\, d\tau_{2,3}(x)$$

$$= \int_{\Delta_2} p_\nu(x)\, \Psi_{\mathbf{n},1}(x)\, \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)}\, d\tau_{2,3}(x)$$
$$+ \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)\, x^\nu\, \widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} \int_{\Delta_3} \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)} \frac{d\tau_{3,2}(t)}{t-x} d\tau_{2,3}(x)$$

where $p_\nu(x)$ is the polynomial defined by

$$\int_{\Delta_3} \frac{t^\nu - x^\nu}{t-x}\, \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)}\, d\tau_{3,2}(t),$$

of degree $\leq n_1 - 2$. Applying (6.11), the first integral after the last equality equals zero since $n_1 < n_2$ (though $n_1 \leq n_2 + 1$ would have been sufficient). If we interchange the sub-indices 2 and 3 in Lemma 6.3.1, we obtain

$$\int_{\Delta_3} \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)}\, \frac{d\tau_{3,2}(t)}{t-x} = -\frac{\widehat{\sigma}_3(x)}{\widehat{s}_{3,2}(t)} + C_2\,, \qquad (6.13)$$

where $C_2 = \sigma_3(\Delta_3)/s_{3,2}(\Delta_3)$. Therefore, using (6.13), (6.11) and (6.10), it follows that

$$\int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)\, x^\nu\, \widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(x)} \int_{\Delta_3} \frac{\widehat{s}_{2,3}(t)}{\widehat{\sigma}_2(t)} \frac{d\tau_{3,2}(t)}{t-x}\, d\tau_{2,3}(x)$$
$$= \int_{\Delta_2} \Psi_{\mathbf{n},1}(x)\, x^\nu\, \frac{\widehat{s}_{3,2}(t)}{\widehat{\sigma}_3(x)} \left(C_2 - \frac{\widehat{\sigma}_3(x)}{\widehat{s}_{3,2}(t)} \right) d\tau_{2,3}(x) = 0\,,$$

since $n_1 \leq n_2$. This completes the proof. □

6.4 Interlacing property of zeros of polynomials and second type functions

As we have pointed out, from the definition $\mathbb{Z}_+^m(*) = \mathbb{Z}_+^m, m = 1, 2$. We have introduced adequate functions of second type also when $m = 3$ and $n_1 < n_2 < n_3$ which were the only multi-indices initially not in $\mathbb{Z}_+^3(*)$. To unify notation, in the rest of this chapter we will consider that $\mathbb{Z}_+^3(*) = \mathbb{Z}_+^3$.

In this section, we show that for $\mathbf{n} \in \mathbb{Z}_+^m(*), m \in \mathbb{N}$, the functions $\Psi_{\mathbf{n},k}, k = 0, \ldots, m-1$, have exactly $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1} and no other zeros on $\mathbb{C} \setminus \Delta_k$. The zeros of "consecutive" $\Psi_{\mathbf{n},k}$ satisfy an interlacing property. These properties are proved in Lemma 6.4.1 below which complements Theorem 2.1 (see also Lemma 2.1) in [5] and substantially enlarges the class of multi-indices for which it is applicable.

Theorem 1 of [22] proves that Lemma 1.2.3 remains valid for any $\mathbf{n} \in \mathbb{Z}_+^3$ and $m = 2$. Recall that in this chapter $\mathbf{n}_l$ denotes the multi-index obtained adding 1 to the l-th component of $\mathbf{n}$.

The following lemma resumes some properties proved in Chapter 2 for multi-indices in $\mathbb{Z}_+^m(\bullet)$ which we need to extend for the more general class of multi-indices $\mathbb{Z}_+^m(*)$. The proof follows the same guidelines employed before. For the sake of completeness we reproduce them here since there are some slight modifications.

Lemma 6.4.1. *Let* $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$. *Let* $\mathbf{n} \in \mathbb{Z}_+^m(*), m \in \mathbb{N}$, *then for each* $k = 0, \ldots, m-1$, *the function* $\Psi_{\mathbf{n},k}$ *has exactly* $|\mathbf{n}^k|$ *simple zeros in the interior of* Δ_{k+1} *and no other zeros on* $\mathbb{C} \setminus \Delta_k$. *Let* I *denote the closure of any one of the connected components of* $\Delta_{k+1} \setminus \operatorname{supp}(\sigma_{k+1}^k)$, *then* $\Psi_{\mathbf{n},k}$ *has at most one simple zero on* I. *Assume that* $l \in \{1, 2, \ldots, m\}$ *is such that* $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$ *for a fixed permutation* τ. *Then, for each* $k \in \{0, \ldots, m-1\}$ *between two consecutive zeros of* $\Psi_{\mathbf{n}_l,k}$ *lies exactly one zero of* $\Psi_{\mathbf{n},k}$ *and viceversa (that is, the zeros of* $\Psi_{\mathbf{n}_l,k}$ *and* $\Psi_{\mathbf{n},k}$ *on* Δ_{k+1} *interlace).*

Proof. Assume that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. We claim that for any real constants $A, B, |A| + |B| > 0$, and $k \in \{0, 1, \ldots, m-1\}$, the function

$$\mathcal{G}_{\mathbf{n},k}(x) = A\Psi_{\mathbf{n},k}(x) + B\Psi_{\mathbf{n}_l,k}(x)$$

has at most $|\mathbf{n}^k| + 1$ zeros in $\mathbb{C} \setminus \Delta_k$ (counting multiplicities) and at least $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1} ($\Delta_0 = \emptyset$). We prove this by induction on k.

Let $k = 0$. The polynomial $\mathcal{G}_{\mathbf{n},0} = A\Psi_{\mathbf{n},0} + B\Psi_{\mathbf{n}_l,0}$ is not identically equal to zero, and $|\mathbf{n}| \leq \deg(\mathcal{G}_{\mathbf{n},0}) \leq |\mathbf{n}| + 1$. Therefore, $\mathcal{G}_{\mathbf{n},0}$ has at most $|\mathbf{n}| + 1$ zeros in $\mathbb{C}$. Let $h_j, j = 1, \ldots, m$, denote polynomials, where $\deg(h_j) \leq n_j - 1$. According to (6.3),

$$\int_{\Delta_1} \mathcal{G}_{\mathbf{n},0}(x) \sum_{j=1}^{m} h_j(x)\widehat{s}_{2,j}(x)d\sigma_1(x) = 0 \tag{6.14}$$

($\widehat{s}_{2,1} \equiv 1$).

In the sequel, we call change knot a point on the real line where a function changes its sign. Notice that for each $k \in \{0, \ldots, m-1\}$, $\mathcal{G}_{\mathbf{n},k}$ is a real function when restricted to the real line. Assume that $\mathcal{G}_{\mathbf{n},0}$ has $N \leq |\mathbf{n}| - 1$ change knots in the interior of Δ_1. We can find polynomials $h_j, j = 1, \ldots, m, \deg(h_j) \leq n_j - 1$, such that $\sum_{j=1}^m h_j\widehat{s}_{2,j}$ has a simple zero at each change knot of $\mathcal{G}_{\mathbf{n},0}$ on Δ_1 and a zero of order $|\mathbf{n}| - 1 - N$ at one of the extreme points of Δ_1. By Lemma 1.2.3, $(1, \widehat{s}_{2,2}, \ldots, \widehat{s}_{2,m})$ forms an AT system with respect to $\mathbf{n} \in \mathbb{Z}_+^m(*)$; therefore, $\sum_{j=1}^m h_j\widehat{s}_{2,j}$ can have no other zero on Δ_1, but this contradicts (6.14) since $\mathcal{G}_{\mathbf{n},0}\sum_{j=1}^m h_j\widehat{s}_{2,j}$ would have a constant sign on Δ_1 (and $\operatorname{supp}(\sigma_1)$ contains infinitely many points). Therefore, $\mathcal{G}_{\mathbf{n},0}$ has at least $|\mathbf{n}|$ change knots in the interior of Δ_1. Consequently, all the zeros of $\mathcal{G}_{\mathbf{n},0}$ are simple and lie on $\mathbb{R}$ as claimed.

Assume that for each $k \in \{0, \ldots, \kappa - 1\}, 1 \leq \kappa \leq m-1$, the claim is satisfied whereas it is violated when $k = \kappa$. Let h_j denote polynomials such that $\deg(h_j) \leq n_j^\kappa - 1, \kappa + 1 \leq j \leq m$. Using (6.3) or (6.9)-(6.12) according to the situation (to simplify the writing we use the notation of (6.3) but the arguments are the same when $m = 3$ and $n_1 < n_2 < n_3$; in particular, in this case, $ds_{r_0}^0 = ds_{1,3}, ds_{r_1}^1 = \widehat{s}_{3,2}d\tau_{2,3}/\widehat{\sigma}_3$ and $ds_{r_2}^2 = \widehat{s}_{2,3}d\tau_{3,2}/\widehat{\sigma}_2$)

$$\int_{\Delta_{\kappa+1}} \mathcal{G}_{\mathbf{n},\kappa}(x) \sum_{j=\kappa+1}^{m} h_j(x)\widehat{s}_{\kappa+2,j}^{\kappa}(x)d\sigma_{\kappa+1}^{\kappa}(x) = 0 \tag{6.15}$$

($\widehat{s}^{\kappa}_{\kappa+2,\kappa+1} \equiv 1$). Arguing as above, since $(1, \widehat{s}^{\kappa}_{\kappa+2,\kappa+2}, \ldots, \widehat{s}^{\kappa}_{\kappa+2,m})$ forms an AT system with respect to $\mathbf{n}^{\kappa} \in \mathbb{Z}_+^{m-\kappa}(*)$, we conclude that $\mathcal{G}_{\mathbf{n},\kappa}$ has at least $|\mathbf{n}^{\kappa}|$ change knots in the interior of $\Delta_{\kappa+1}$.

Let us suppose that $\mathcal{G}_{\mathbf{n},\kappa}$ has at least $|\mathbf{n}^{\kappa}|+2$ zeros in $\mathbb{C}\setminus\Delta_{\kappa}$ and let $W_{\mathbf{n},\kappa}$ be the monic polynomial whose zeros are those points (counting multiplicities). The complex zeros of $\mathcal{G}_{\mathbf{n},\kappa}$ (if any) must appear in conjugate pairs since $\mathcal{G}_{\mathbf{n},\kappa}(\overline{z}) = \overline{\mathcal{G}_{\mathbf{n},\kappa}(z)}$; therefore, the coefficients of $W_{\mathbf{n},\kappa}$ are real numbers. On the other hand, from (6.3) ((6.9) or (6.11) when necessary)

$$0 = \int_{\Delta_{\kappa}} \mathcal{G}_{\mathbf{n},\kappa-1}(x) \frac{z^{n^{\kappa-1}_{r_{\kappa-1}}} - x^{n^{\kappa-1}_{r_{\kappa-1}}}}{z-x} ds^{\kappa-1}_{r_{\kappa-1}}(x) .$$

Therefore,

$$\mathcal{G}_{\mathbf{n},\kappa}(z) = \frac{1}{z^{n^{\kappa-1}_{r_{\kappa-1}}}} \int_{\Delta_{\kappa}} \frac{x^{n^{\kappa-1}_{r_{\kappa-1}}} \mathcal{G}_{\mathbf{n},\kappa-1}(x)}{z-x} ds^{\kappa-1}_{r_{\kappa-1}}(x) = \mathcal{O}\left(\frac{1}{z^{n^{\kappa-1}_{r_{\kappa-1}}+1}}\right), \quad z \to \infty ,$$

and taking into consideration the degree of $W_{\mathbf{n},\kappa}$, we obtain

$$\frac{z^j \mathcal{G}_{\mathbf{n},\kappa}}{W_{\mathbf{n},\kappa}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\mathbb{C}\setminus\Delta_{\kappa}) , \qquad j = 0, \ldots, |\mathbf{n}^{\kappa-1}| + 1 .$$

Let Γ be a closed Jordan curve which surrounds Δ_{κ} and such that all the zeros of $W_{\mathbf{n},\kappa}$ lie in the exterior of Γ. Using Cauchy's theorem, the integral expression for $\mathcal{G}_{\mathbf{n},\kappa}$, Fubini's theorem, and Cauchy's integral formula, for each $j = 0, \ldots, |\mathbf{n}^{\kappa-1}| + 1$, we have

$$0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j \mathcal{G}_{\mathbf{n},\kappa}(z)}{W_{\mathbf{n},\kappa}(z)} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j}{W_{\mathbf{n},\kappa}(z)} \int_{\Delta_{\kappa}} \frac{\mathcal{G}_{\mathbf{n},\kappa-1}(x)}{z-x} ds^{\kappa-1}_{r_{\kappa-1}}(x) dz =$$
$$\int_{\Delta_{\kappa}} \frac{x^j \mathcal{G}_{\mathbf{n},\kappa-1}(x)}{W_{\mathbf{n},\kappa}(x)} ds^{\kappa-1}_{r_{\kappa-1}}(x) ,$$

which implies that $\mathcal{G}_{\mathbf{n},\kappa-1}$ has at least $|\mathbf{n}^{\kappa-1}|+2$ change knots in the interior of Δ_{κ}. This contradicts our induction hypothesis since this function can have at most $|\mathbf{n}^{\kappa-1}|+1$ zeros in $\mathbb{C}\setminus\Delta_{\kappa-1} \supset \Delta_{\kappa}$. Hence $\mathcal{G}_{\mathbf{n},\kappa}$ has at most $|\mathbf{n}^{\kappa}|+1$ zeros in $\mathbb{C}\setminus\Delta_{\kappa}$ as claimed.

Taking $B = 0$ the assumption $\mathbf{n}_l \in \mathbb{Z}_+^m(*,\tau)$ is not required, and the arguments above lead to the proof that $\Psi_{\mathbf{n},k}$ has at most $|\mathbf{n}^k|$ zeros on $\mathbb{C}\setminus\Delta_k$ since $Q_{\mathbf{n}} = \Psi_{\mathbf{n},0}$ has at most $|\mathbf{n}|$ zeros on $\mathbb{C}$. Consequently, the zeros of $\Psi_{\mathbf{n},k}$ in $\mathbb{C}\setminus\Delta_k$ are exactly the $|\mathbf{n}^k|$ simple ones it has in the interior of Δ_{k+1}.

Let I be the closure of a connected component of $\Delta_{k+1}\setminus\mathrm{supp}(\sigma^k_{k+1})$ and let us assume that I contains two consecutive simple zeros x_1, x_2 of $\Psi_{\mathbf{n},k}$. Taking $B = 0$ and $A = 1$, we can rewrite (6.15) as follows

$$\int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)}{(x-x_1)(x-x_2)} \sum_{j=k+1}^{m} h_j(x) \widehat{s}^k_{k+2,j}(x)(x-x_1)(x-x_2) d\sigma^k_{k+1}(x) = 0 , \tag{6.16}$$

where $\deg(h_j) \leq n_j^k - 1, j = k+1, \ldots, m$. The measure $(x-x_1)(x-x_2)d\sigma_{k+1}^k(x)$ has a constant sign on Δ_{k+1} and $\Psi_{\mathbf{n},k}(x)/(x-x_1)(x-x_2)$ has $|\mathbf{n}^k| - 2$ change knots on Δ_{k+1}. Using again Lemma 1.2.3, we can construct appropriate polynomials h_j to contradict (6.16). Therefore, I contains at most one zero of $\Psi_{\mathbf{n},k}$.

Fix $y \in \mathbb{R} \setminus \Delta_k$ and $k \in \{0, 1, \ldots, m-1\}$. It cannot occur that $\Psi_{\mathbf{n}_l,k}(y) = \Psi_{\mathbf{n},k}(y) = 0$. If this was so, y would have to be a simple zero of $\Psi_{\mathbf{n}_l,k}$ and $\Psi_{\mathbf{n},k}$. Therefore, $(\Psi_{\mathbf{n}_l,k})'(y) \neq 0 \neq (\Psi_{\mathbf{n},k})'(y)$. Taking $A = 1, B = -\Psi'_{\mathbf{n},k}(y)/\Psi'_{\mathbf{n}_l,k}(y)$, we find that

$$\mathcal{G}_{\mathbf{n},k}(y) = (A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}_l,k})(y) = (\mathcal{G}_{\mathbf{n},k})'(y) = 0\,,$$

which means that $\mathcal{G}_{\mathbf{n},k}$ has at least a double zero at y against what we proved before.

Now, taking $A = \Psi_{\mathbf{n}_l,k}(y), B = -\Psi_{\mathbf{n},k}(y)$, we have that $|A| + |B| > 0$. Since

$$\Psi_{\mathbf{n}_l,k}(y)\Psi_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi_{\mathbf{n}_l,k}(y) = 0\,,$$

and the zeros on $\mathbb{R} \setminus \Delta_k$ of $\Psi_{\mathbf{n}_l,k}(y)\Psi_{\mathbf{n},k}(x) - \Psi_{\mathbf{n},k}(y)\Psi_{\mathbf{n}_l,k}(x)$ with respect to x are simple, using again what we proved above, it follows that

$$\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y) \neq 0\,.$$

But $\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y)$ is a continuous real function on $\mathbb{R} \setminus \Delta_k$ so it must have constant sign on each one of the intervals forming $\mathbb{R} \setminus \Delta_k$; in particular, its sign on Δ_{k+1} is constant.

We know that $\Psi_{\mathbf{n}_l,k}$ has at least $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1}. Evaluating $\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y)$ at two consecutive zeros of $\Psi_{\mathbf{n}_l,k}$, since the sign of $\Psi'_{\mathbf{n}_l,k}$ at these two points changes the sign of $\Psi_{\mathbf{n},k}$ must also change. Using Bolzano's theorem we find that there must be an intermediate zero of $\Psi_{\mathbf{n},k}$. Analogously, one proves that between two consecutive zeros of $\Psi_{\mathbf{n},k}$ on Δ_{k+1} there is one of $\Psi_{\mathbf{n}_l,k}$. Thus, the interlacing property has been proved. □

Let $Q_{\mathbf{n},k+1}, k = 0, \ldots, m-1$, denote the monic polynomial whose zeros are equal to those of $\Psi_{\mathbf{n},k}$ on Δ_{k+1}. From (6.3) ((6.9), (6.11), or (6.12) when necessary)

$$0 = \int_{\Delta_{k+1}} \Psi_{\mathbf{n},k}(x) \frac{z^{n^k_{r_k}} - x^{n^k_{r_k}}}{z - x} ds^k_{r_k}(x)$$

(Recall that when $m = 3$ and $n_1 < n_2 < n_3$, we take $ds^0_{r_0} = ds_{1,3}, ds^1_{r_1} = \widehat{s}_{3,2}d\tau_{2,3}/\widehat{\sigma}_3$ and $ds^2_{r_2} = \widehat{s}_{2,3}d\tau_{3,2}/\widehat{\sigma}_2$.) Therefore,

$$\Psi_{\mathbf{n},k+1}(z) = \frac{1}{z^{n^k_{r_k}}} \int_{\Delta_{k+1}} \frac{x^{n^k_{r_k}}\Psi_{\mathbf{n},k}(x)}{z - x} ds^k_{r_k}(x) = \mathcal{O}\left(\frac{1}{z^{n^k_{r_k}+1}}\right), \quad z \to \infty\,,$$

and taking into consideration the degree of $Q_{\mathbf{n},k+2}$ (by definition $Q_{\mathbf{n},m+1} \equiv 1$), we obtain

$$\frac{z^j \Psi_{\mathbf{n},k+1}}{Q_{\mathbf{n},k+2}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \Delta_{k+1})\,, \qquad j = 0, \ldots, |\mathbf{n}^k| - 1\,.$$

On account of (2.3), it follows that (take $Q_{\mathbf{n},0} \equiv 1$)

$$0 = \int_{\Delta_{k+1}} x^j Q_{\mathbf{n},k+1}(x) \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}\,, \quad k = 0, \ldots, m-1\,, \tag{6.17}$$

where

$$\mathcal{H}_{\mathbf{n},k+1} = \frac{Q_{\mathbf{n},k} \Psi_{\mathbf{n},k}}{Q_{\mathbf{n},k+1}}\,, \qquad k = 0, \ldots, m\,,$$

has constant sign on Δ_{k+1}.

This last relation implies that

$$\int_{\Delta_{k+1}} \frac{Q(z) - Q(x)}{z - x}\; Q_{\mathbf{n},k+1}(x)\, \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)} = 0\,,$$

where Q is any polynomial of degree $\leq |\mathbf{n}^k|$. If we use this formula with $Q = Q_{\mathbf{n},k+1}$ and $Q = Q_{\mathbf{n},k+2}$, respectively, we obtain

$$\int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}(x)}{z - x} \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)} =$$

$$\frac{1}{Q_{\mathbf{n},k+1}(z)} \int_{\Delta_{k+1}} \frac{Q^2_{\mathbf{n},k+1}(x)}{z - x} \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}$$

and

$$\int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}(x)}{z - x} \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)} = \frac{1}{Q_{\mathbf{n},k+2}(z)} \int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x) ds^k_{r_k}(x)}{z - x}\,.$$

Equating these two relations and using the definition of $\Psi_{\mathbf{n},k+1}$ and $\mathcal{H}_{\mathbf{n},k+2}$, we obtain

$$\mathcal{H}_{\mathbf{n},k+2}(z) = \int_{\Delta_{k+1}} \frac{Q^2_{\mathbf{n},k+1}(x)}{z - x} \frac{\mathcal{H}_{\mathbf{n},k+1}(x) ds^k_{r_k}(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}\,, \quad k = 0, \ldots, m-1\,. \tag{6.18}$$

Notice that from the definition $\mathcal{H}_{\mathbf{n},1} \equiv 1$.

For each $k = 1, \ldots, m$, set

$$K^{-2}_{\mathbf{n},k} = \int_{\Delta_k} Q^2_{\mathbf{n},k}(x) \left| \frac{Q_{\mathbf{n},k-1}(x) \Psi_{\mathbf{n},k-1}(x)}{Q_{\mathbf{n},k}(x)} \right| \frac{d|s^{k-1}_{r_{k-1}}|(x)}{|Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)|}\,, \tag{6.19}$$

where $|s|$ denotes the total variation of the measures s. Take

$$K_{\mathbf{n},0} = 1\,, \quad \kappa_{\mathbf{n},k} = \frac{K_{\mathbf{n},k}}{K_{\mathbf{n},k-1}}\,, \quad k = 1, \ldots, m\,.$$

Define

$$q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\,, \quad h_{\mathbf{n},k} = K^2_{\mathbf{n},k-1} \mathcal{H}_{\mathbf{n},k}\,, \tag{6.20}$$

and

$$d\rho_{\mathbf{n},k}(x) = \frac{h_{\mathbf{n},k}(x)ds^{k-1}_{r_{k-1}}(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)}\,. \tag{6.21}$$

Notice that the measure $\rho_{\mathbf{n},k}$ has constant sign on Δ_k. Let $\varepsilon_{\mathbf{n},k}$ be the sign of $\rho_{\mathbf{n},k}$. From (6.17) and the notation introduced above, we obtain

$$\int_{\Delta_k} x^\nu q_{\mathbf{n},k}(x)d|\rho_{\mathbf{n},k}|(x) = 0, \quad \nu = 0,\dots,|\mathbf{n}^{k-1}|-1, \quad k=1,\dots,m\,, \tag{6.22}$$

and $q_{n,k}$ is orthonormal with respect to the varying measure $|\rho_{\mathbf{n},k}|$. On the other hand, using (6.18) it follows that

$$h_{\mathbf{n},k+1}(z) = \varepsilon_{\mathbf{n},k}\int_{\Delta_k} \frac{q^2_{\mathbf{n},k}(x)}{z-x}d|\rho_{\mathbf{n},k}|(x)\,, \quad k=1,\dots,m\,. \tag{6.23}$$

Let us state the analogue of Lemma 4.3.1 in this context. The proof is similar, so we refrain from repeating it and refer to [36] for the details should it be necessary.

Lemma 6.4.2. *Let $S=\mathcal{N}'(\sigma_1,\dots,\sigma_m)$. Let $\Lambda\subset\mathbb{Z}^m_+(*)$ be an infinite sequence of distinct multi-indices with the property that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty$. For any continuous function f on $\mathrm{supp}(\sigma^{k-1}_k)$*

$$\lim_{\mathbf{n}\in\Lambda}\int_{\Delta_k} f(x)q^2_{\mathbf{n},k}(x)d|\rho_{\mathbf{n},k}|(x) = \frac{1}{\pi}\int_{\widetilde{\Delta}_k} f(x)\frac{dx}{\sqrt{(b_k-x)(x-a_k)}}\,, \tag{6.24}$$

where $\widetilde{\Delta}_k=[a_k,b_k]$. In particular,

$$\lim_{\mathbf{n}\in\Lambda}\varepsilon_{\mathbf{n},k}h_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z-b_k)(z-a_k)}}\,, \quad \mathcal{K}\subset\mathbb{C}\setminus\mathrm{supp}(\sigma^{k-1}_k)\,, \tag{6.25}$$

where $\sqrt{(z-b_k)(z-a_k)}>0$ if $z>b_k$. Consequently, for $k=1,\dots,m$, each point of $\mathrm{supp}(\sigma^{k-1}_k)\setminus\widetilde{\Delta}_k$, is a limit of zeros of $\{Q_{\mathbf{n},k}\},\mathbf{n}\in\Lambda$.

As in Section 4.3, from Lemma 6.4.2 we obtain the following analogue of Lemma 4.3.1 (see [36] for details).

Lemma 6.4.3. *Let $S=\mathcal{N}'(\sigma_1,\dots,\sigma_m)$. Let $\Lambda\subset\mathbb{Z}^m_+(*)$ be an infinite sequence of distinct multi-indices such that*

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty.$$

Assume that there exists $l\in\{1,\dots,m\}$ and a fixed permutation τ of $\{1,\dots,m\}$ such that for all $\mathbf{n}\in\Lambda$ we have that $\mathbf{n},\mathbf{n}_l\in\mathbb{Z}^m_+(*,\tau)$. Then, for each $k=1,\dots,m$, and each compact set $\mathcal{K}\subset\mathbb{C}\setminus\mathrm{supp}(\sigma^{k-1}_k)$ there exist positive constants $C_{k,1}(\mathcal{K}),C_{k,2}(\mathcal{K})$ such that

$$C_{k,1}(\mathcal{K}) \le \inf_{z\in\mathcal{K}}\left|\frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)}\right| \le \sup_{z\in\mathcal{K}}\left|\frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)}\right| \le C_{k,2}(\mathcal{K}),$$

for all sufficiently large $|\mathbf{n}|,\mathbf{n}\in\Lambda$.

6.5 Proof of Theorem 6.5.2

In this final section, $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$; that is, $\mathrm{supp}(\sigma_k) = \widetilde{\Delta}_k \cup e_k, k = 1, \ldots, m$, where $\widetilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\widetilde{\Delta}_k$, and e_k is a set without accumulation points in $\overline{\mathbb{R}} \setminus \widetilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices. Let us assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. From Lemma 6.4.3 we know that the sequences

$$\{Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}\}_{\mathbf{n}\in\Lambda}\,, \qquad k = 1, \ldots, m,$$

are uniformly bounded on each compact subset of $\mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1})$ for all sufficiently large $|\mathbf{n}|$. By Montel's theorem, there exists a subsequence of multi-indices $\Lambda' \subset \Lambda$ and a collection of functions $\widetilde{F}_k^l$, holomorphic in $\mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1})$, respectively, such that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F}_k^{(l)}(z), \quad \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1}), \quad k = 1, \ldots, m. \tag{6.26}$$

In principle, the functions $\widetilde{F}_k^{(l)}$ may depend on Λ'. We shall see that this is not the case and, therefore, the limit in (6.26) holds for $\mathbf{n} \in \Lambda$. First, let us obtain some general information on the functions $\widetilde{F}_k^{(l)}$.

The points in $\mathrm{supp}(\sigma_k^{k-1}) \setminus \widetilde{\Delta}_k$ are isolated singularities of $\widetilde{F}_k^{(l)}$. Let $\zeta \in \mathrm{supp}(\sigma_k^{k-1}) \setminus \widetilde{\Delta}_k$. By Lemma 6.4.2 each such point is a limit of zeros of $Q_{\mathbf{n},k}$ and $Q_{\mathbf{n}_l,k}$ as $|\mathbf{n}| \to \infty, \mathbf{n} \in \Lambda$, and in a sufficiently small neighborhood of them, for each $\mathbf{n} \in \Lambda$, there can be at most one such zero of these polynomials (so there is exactly one, for all sufficiently large $|\mathbf{n}|$). Let $\lim_{\mathbf{n}\in\Lambda} \zeta_{\mathbf{n}} = \zeta$ where $Q_{\mathbf{n},k}(\zeta_{\mathbf{n}}) = 0$. From (6.26)

$$\lim_{\mathbf{n}\in\Lambda'} \frac{(z-\zeta_{\mathbf{n}})Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = (z-\zeta)\widetilde{F}_k^{(l)}(z), \quad \mathcal{K} \subset (\mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1})) \cup \{\zeta\}\,,$$

and $(z-\zeta)\widetilde{F}_k^{(l)}(z)$ is analytic in a neighborhood of ζ. Hence ζ is not an essential singularity of $\widetilde{F}_k^{(l)}$. Taking into consideration that $Q_{\mathbf{n}_l,k}, \mathbf{n} \in \Lambda$ also has a sequence of zeros converging to ζ, from the argument principle it follows that ζ is a removable singularity of $\widetilde{F}_k^{(l)}$ which is not a zero. By Lemma 6.4.3 we also know that the sequence of functions $|Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}|, \mathbf{n} \in \Lambda$, is uniformly bounded from below by a positive constant for all sufficiently large $|\mathbf{n}|$. Therefore, in $\mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1})$ the function $\widetilde{F}_k^{(l)}$ is also different from zero. According to the definition of $Q_{\mathbf{n},k}$ and $Q_{\mathbf{n}_l,k}$ and Lemma 6.4.1, for $k = 1, \ldots, \tau^{-1}(l)$, we have that $\deg Q_{\mathbf{n}_l,k} = |\mathbf{n}_l^{k-1}| = |\mathbf{n}^{k-1}| + 1 = \deg Q_{\mathbf{n},k} + 1$ whereas, for $k = \tau^{-1}(l) + 1, \ldots, m$, we obtain that $\deg Q_{\mathbf{n}_l,k} = |\mathbf{n}_l^{k-1}| = |\mathbf{n}^{k-1}| = \deg Q_{\mathbf{n},k}$. Consequently, for $k = 1, \ldots, \tau^{-1}(l)$, the function $\widetilde{F}_k^{(l)}$ has a simple pole at infinity and $(\widetilde{F}_k^{(l)})'(\infty) = 1$, whereas, for $k = \tau^{-1}(l) + 1, \ldots, m$, it is analytic at infinity and $\widetilde{F}_k^{(l)}(\infty) = 1$.

Now let us prove that the functions $\widetilde{F}_k^{(l)}$ satisfy a system of boundary value problems.

Lemma 6.5.1. *Let $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi-indices such that*

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty.$$

Assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. Take $\Lambda' \subset \Lambda$ such that (6.26) holds. Then, there exists a normalization $F_k^{(l)}$, $k = 1, \ldots, m$, by positive constants, of the functions $\widetilde{F}_k^{(l)}$, $k = 1, \ldots, m$, given in (6.26), which verifies the system of boundary value problems

$$\begin{array}{ll} 1) & F_k^{(l)},\, 1/F_k^{(l)} \in H(\mathbb{C} \setminus \widetilde{\Delta}_k)\,, \\ 2) & (F_k^{(l)})'(\infty) > 0\,, \quad k = 1, \ldots, \tau^{-1}(l)\,, \\ 2') & F_k^{(l)}(\infty) > 0\,, \quad k = \tau^{-1}(l)+1, \ldots, m\,, \\ 3) & |F_k^{(l)}(x)|^2 \dfrac{1}{|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)|} = 1,\ x \in \widetilde{\Delta}_k\,, \end{array} \tag{6.27}$$

where $F_0^{(l)} \equiv F_{m+1}^{(l)} \equiv 1$.

Proof. The assertions 1), 2), and 2') were proved above for the functions $\widetilde{F}_k^{(l)}$. Consequently, they are satisfied for any normalization of these functions by means of positive constants.

From (6.22) applied to $\mathbf{n}$ and $\mathbf{n}_l$, for each $k = 1, \ldots, m$, we have

$$\int_{\Delta_k} x^\nu Q_{\mathbf{n},k}(x) d|\rho_{\mathbf{n},k}|(x) = 0, \qquad \nu = 0, \ldots, |\mathbf{n}^{k-1}| - 1\,,$$

and

$$\int_{\Delta_k} x^\nu Q_{\mathbf{n}_l,k}(x) g_{\mathbf{n},k}(x) d|\rho_{\mathbf{n},k}|(x) = 0\,, \qquad \nu = 0, \ldots, |\mathbf{n}_l^{k-1}| - 1\,,$$

where

$$g_{\mathbf{n},k}(x) = \frac{|Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)|}{|Q_{\mathbf{n}_l,k-1}(x) Q_{\mathbf{n}_l,k+1}(x)|} \frac{|h_{\mathbf{n}_l,k}(x)|}{|h_{\mathbf{n},k}(x)|}\,, \quad d\rho_{\mathbf{n},k}(x) = \frac{h_{\mathbf{n},k}(x) ds_{r_{k-1}}^{k-1}(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)}\,.$$

From (6.25) and (6.26)

$$\lim_{\mathbf{n}\in\Lambda'} g_{\mathbf{n},k}(x) = |(\widetilde{F}_{k-1}^{(l)} \widetilde{F}_{k+1}^{(l)})(x)|^{-1} \tag{6.28}$$

uniformly on Δ_k.

Fix $k \in \{\tau^{-1}(l)+1,\ldots,m\}$. As mentioned above, for this selection of k we have that $\deg Q_{\mathbf{n}_l,k} = \deg Q_{\mathbf{n},k} = |\mathbf{n}^{k-1}|$. Using Lemmas 4.2.3, 4.2.4, and (6.26), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{S_k(z)}{S_k(\infty)} = \widetilde{S}_k(z) = \widetilde{F}_k^{(l)}(z)\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k^{k-1})\,, \tag{6.29}$$

where S_k denotes the Szegő function on $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_k$ with respect to the weight $|\widetilde{F}_{k-1}^{(l)}(x)\widetilde{F}_{k+1}^{(l)}(x)|^{-1}, x \in \widetilde{\Delta}_k$. The function S_k is uniquely determined by

$$\begin{array}{ll} 1) & S_k, 1/S_k \in H(\overline{\mathbb{C}} \setminus \widetilde{\Delta}_k)\,, \\ 2) & S_k(\infty) > 0\,, \\ 3) & |S_k(x)|^2 \dfrac{1}{\left|(\widetilde{F}_{k-1}^{(l)}\widetilde{F}_{k+1}^{(l)})(x)\right|} = 1, \qquad x \in \widetilde{\Delta}_k\,. \end{array} \tag{6.30}$$

Now, fix $k \in \{1,\ldots,\tau^{-1}(l)\}$. In this situation $\deg Q_{\mathbf{n}_l,k} = \deg Q_{\mathbf{n},k} + 1 = |\mathbf{n}^{k-1}| + 1$. Let $Q^*_{\mathbf{n},k}(x)$ be the monic polynomial of degree $|\mathbf{n}^{k-1}|$ orthogonal with respect to the varying measure $g_{\mathbf{n},k} d|\rho_{\mathbf{n},k}|$. Using the same arguments as above, we have

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q^*_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{S_k(z)}{S_k(\infty)} = \widetilde{S}_k(z)\,, \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k^{k-1})\,. \tag{6.31}$$

On the other hand, since $\deg Q_{\mathbf{n}_l,k} = \deg Q^*_{\mathbf{n},k}+1$ and both of these polynomials are orthogonal with respect to the same varying weight, by Lemma 4.2.3 and (6.26), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q^*_{\mathbf{n},k}(z)} = \frac{\varphi_k(z)}{\varphi_k'(\infty)} = \widetilde{\varphi}_k(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,, \tag{6.32}$$

where φ_k denotes the conformal representation of $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_k$ onto $\{w : |w| > 1\}$ such that $\varphi_k(\infty) = \infty$ and $\varphi_k'(\infty) > 0$. The function φ_k is uniquely determined by

$$\begin{array}{ll} 1) & \varphi_k, 1/\varphi_k \in H(\mathbb{C} \setminus \widetilde{\Delta}_k)\,, \\ 2) & \varphi_k'(\infty) > 0\,, \\ 3) & |\varphi_k(x)| = 1, \quad x \in \widetilde{\Delta}_k\,. \end{array} \tag{6.33}$$

From (6.31) and (6.32), we obtain

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = (\widetilde{S}_k\widetilde{\varphi}_k)(z) = \widetilde{F}_k^{(l)}(z)\,, \quad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,. \tag{6.34}$$

Thus,

$$\widetilde{F}_k^{(l)} = \begin{cases} \widetilde{S}_k\widetilde{\varphi}_k\,, & k = 1,\ldots,\tau^{-1}(l)\,, \\ \widetilde{S}_k\,, & k = \tau^{-1}(l)+1,\ldots,m\,, \end{cases} \tag{6.35}$$

and from (6.30) and (6.35) it follows that

$$|\widetilde{F}_k^{(l)}(x)|^2 \frac{1}{\left|(\widetilde{F}_{k-1}^{(l)}\widetilde{F}_{k+1}^{(l)})(x)\right|} = \frac{1}{\omega_k}\,, \qquad x \in \widetilde{\Delta}_k\,, \qquad k = 1,\dots,m\,, \tag{6.36}$$

where

$$\omega_k = \begin{cases} (S_k\varphi_k')^2(\infty)\,, & k = 1,\dots,\tau^{-1}(l)\,, \\ S_k^2(\infty)\,, & k = \tau^{-1}(l)+1,\dots,m\,. \end{cases} \tag{6.37}$$

Now, let us show that there exist positive constants $c_k, k = 1,\dots,m$, such that the functions $F_k^{(l)} = c_k\widetilde{F}_k^{(l)}$ satisfy (6.27). In fact, according to (6.36) for any such constants c_k we have that

$$|F_k^{(l)}(x)|^2 \frac{1}{\left|(F_{k-1}^{(l)}F_{k+1}^{(l)})(x)\right|} = \frac{c_k^2}{c_{k-1}c_{k+1}\omega_k}\,, \qquad x \in \widetilde{\Delta}_k\,, \qquad k = 1,\dots,m\,,$$

where $c_0 = c_{m+1} = 1$. The problem reduces to finding appropriate constants c_k such that

$$\frac{c_k^2}{c_{k-1}c_{k+1}\omega_k} = 1\,, \qquad k = 1,\dots,m\,. \tag{6.38}$$

Taking logarithm, we obtain the linear system of equations

$$2\log c_k - \log c_{k-1} - \log c_{k+1} = \log \omega_k\,, \qquad k = 1,\dots,m \tag{6.39}$$

($c_0 = c_{m+1} = 1$) on the unknowns $\log c_k$. This system has a unique solution with which we conclude the proof. □

Consider the $(m+1)$-sheeted compact Riemann surface $\mathcal{R}$ introduced in Chapter 5 and a conformal representation of $\mathcal{R}$ onto the extended complex plane $\psi^{(l)}, l \in \{1,\dots,m\}$, such that

$$\psi^{(l)}(z) = \frac{C_1}{z} + \mathcal{O}\Big(\frac{1}{z^2}\Big), \quad z \to \infty^{(0)}$$

$$\psi^{(l)}(z) = C_2\, z + \mathcal{O}(1), \quad z \to \infty^{(l)}$$

where C_1 and C_2 are nonzero constants. As before, the branches of $\psi^{(l)}$, corresponding to the different sheets $k = 0,1,\dots,m$ of $\mathcal{R}$ are denoted

$$\psi^{(l)} := \{\psi_k^{(l)}\}_{k=0}^m\,,$$

and $\psi^{(l)}$ is normalized as in (5.23).

We are ready to state and prove

Theorem 6.5.2. *Let $S = \mathcal{N}'(\sigma_1,\dots,\sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices such that*

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty.$$

Let us assume that there exists $l \in \{1, \dots, m\}$ *and a fixed permutation* τ *of* $\{1, \dots, m\}$ *such that for all* $\mathbf{n} \in \Lambda$ *we have that* $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$*. Let* $\{Q_{\mathbf{n},k}\}_{k=1}^m$*,* $\mathbf{n} \in \Lambda$*, be the corresponding sequences of polynomials defined in Section* 6.4*. Then, for each fixed* $k \in \{1, \dots, m\}$*, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F}_k^{(l)}(z), \qquad z \in \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1}) \tag{6.40}$$

where

$$F_k^{(l)} := sg\left(\prod_{\nu=k}^{m} \psi_\nu^{(\tau^{-1}(l))}(\infty)\right) \prod_{\nu=k}^{m} \psi_\nu^{(\tau^{-1}(l))}. \tag{6.41}$$

Proof. Since the families of functions

$$\{Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}\}_{\mathbf{n}\in\Lambda}, \qquad k = 1, \dots, m,$$

are uniformly bounded on each compact subset $\mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$ for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, uniform convergence on compact subsets of the indicated region follows from proving that any convergent subsequence has the same limit. According to Lemma 6.5.1 the limit functions, appropriately normalized, of a convergent subsequence satisfy the same system of boundary value problems (6.27). According to Lemma 4.2 in [5] this system has a unique solution.

It remains to show that the functions defined in (6.41) satisfy (6.27). When multiplying two consecutive branches, the singularities on the common slit cancel out; therefore, 1) takes place since only the singularities of $\psi_k^{(\tau^{-1}(l))}$ on $\widetilde{\Delta}_k$ remain. From the definition of $\psi^{(\tau^{-1}(l))}$ it also follows that for $k = 1, \cdots, \tau^{-1}(l)$, $F_k^{(l)}$ has at infinity a simple pole, whereas it is regular and different from zero when $k = \tau^{-1}(l) + 1, \cdots, m$. The factor sign in front of (6.41) guarantees the positivity claimed in 2) and 2′).

In order to verify 3), notice that $F_k^{(l)}/F_{k-1}^{(l)} = \operatorname{sg}(\psi_{k-1}^{(\tau^{-1}(l))}(\infty))/\psi_{k-1}^{(\tau^{-1}(l))}$. Therefore, if $k = 2, \dots, m$,

$$\frac{|F_k^{(l)}(x)|^2}{|F_{k-1}^{(l)}(x)F_{k+1}^{(l)}(x)|} = \frac{|\psi_k^{(\tau^{-1}(l))}(x)|}{|\psi_{k-1}^{(\tau^{-1}(l))}(x)|} = 1, \qquad x \in \widetilde{\Delta}_k,$$

on account of (5.25). For $k = 1$, from the definition and (5.25)

$$\frac{|F_1^{(l)}(x)|^2}{|F_2^{(l)}(x)|} = |\psi_1^{(\tau^{-1}(l))}(x)|^2 |\prod_{\nu=2}^{m} \psi_\nu^{(\tau^{-1}(l))}(x)| = |\prod_{\nu=0}^{m} \psi_\nu^{(\tau^{-1}(l))}(x)| = 1, \;\; x \in \widetilde{\Delta}_1,$$

since $\prod_{\nu=0}^m \psi_\nu^{(\tau^{-1}(l))}$ is constantly equal to 1 or -1 on all $\overline{\mathbb{C}}$. □

Let $\mathbf{1} = (1, \dots, 1)$. An immediate consequence of Theorem 6.5.2 is

Corollary 6.5.3. *Let* $S = \mathcal{N}'(\sigma_1, \dots, \sigma_m)$*. Let* $\Lambda \subset \mathbb{Z}_+^m(*)$ *be an infinite sequence of distinct multi-indices such that*

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty.$$

Then, for each $k = 1, \ldots, m$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n+1},k}(z)}{Q_{\mathbf{n},k}(z)} = \prod_{l=1}^{m} \widetilde{F}_k^{(l)}(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,. \tag{6.42}$$

Proof. Let

$$\Lambda_\tau = \Lambda \cap \mathbb{Z}_+^m(*, \tau)\,,$$

where τ is a given permutation of $\{1, \ldots, m\}$. We are only interested in those Λ_τ with infinitely many terms. There are at most $m!$ such subsequences. For $\mathbf{n} \in \Lambda_\tau$ fixed, denote $\mathbf{n}_{\tau(j)}, j \in \{1, \ldots, m\}$, the multi-index obtained adding one to all j components $\tau(1), \ldots, \tau(j)$ of $\mathbf{n}$. (Notice that this notation differs from that introduced previously for $\mathbf{n}_l$.) In particular, $\mathbf{n}_{\tau(m)} = \mathbf{n} + \mathbf{1}$. It is easy to verify that for all $j \in \{1, \ldots, m\}$, $\mathbf{n}_{\tau(j)} \in \Lambda_\tau$. For all $\mathbf{n} \in \Lambda_\tau$ and each $k \in \{1, \ldots, m\}$, we have

$$\frac{Q_{\mathbf{n+1},k}}{Q_{\mathbf{n},k}} = \prod_{j=0}^{m-1} \frac{Q_{\mathbf{n}_{\tau(j+1)},k}}{Q_{\mathbf{n}_{\tau(j)},k}}\,,$$

where $Q_{\mathbf{n}_{\tau(0)},k} = Q_{\mathbf{n},k}$. From (6.40) it follows that

$$\lim_{\mathbf{n}\in\Lambda_\tau} \frac{Q_{\mathbf{n+1},k}(z)}{Q_{\mathbf{n},k}(z)} = \prod_{l=1}^{m} \widetilde{F}_k^{(l)}(z)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,.$$

The right side does not depend on l, since all possible values intervene. Therefore, the limit is the same for all τ and thus (6.42) is obtained. □

The following corollary complements Theorem 6.5.2. The proof is similar to that of Corollary 4.5.1.

Corollary 6.5.4. *Let* $S = \mathcal{N}'(\sigma_1, \ldots, \sigma_m)$ *and* $\Lambda \subset \mathbb{Z}_+^m(*)$ *be a sequence of distinct multi-indices such that*

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty.$$

Let us assume that there exists $l \in \{1, \ldots, m\}$ *and a fixed permutation* τ *of* $\{1, \ldots, m\}$ *such that for all* $\mathbf{n} \in \Lambda$ *we have that* $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. *Let* $\{q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, *be the system of orthonormal polynomials as defined in* (6.20) *and* $\{K_{\mathbf{n},k}\}_{k=1}^m, \mathbf{n} \in \Lambda$, *the values given by* (6.19). *Then, for each fixed* $k = 1, \ldots, m$, *we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}_l,k}}{\kappa_{\mathbf{n},k}} = \kappa_k^{(l)}\,, \tag{6.43}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{K_{\mathbf{n}_l,k}}{K_{\mathbf{n},k}} = \kappa_1^{(l)} \cdots \kappa_k^{(l)}\,, \tag{6.44}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}_l,k}(z)}{q_{\mathbf{n},k}(z)} = \kappa_k^{(l)} \widetilde{F}_k^{(l)}(z), \qquad z \in \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,, \tag{6.45}$$

where

$$\kappa_k^{(l)} = \frac{c_k^{(l)}}{\sqrt{c_{k-1}^{(l)} c_{k+1}^{(l)}}}\,, \qquad c_k^{(l)} = \begin{cases} (F_k^{(l)})'(\infty)\,, & k = 1, \ldots, \tau^{-1}(l)\,, \\ F_k^{(l)}(\infty)\,, & k = \tau^{-1}(l) + 1, \ldots, m\,, \end{cases} \tag{6.46}$$

and the $F_k^{(l)}$ *are defined by* (6.41).

Proof. By Theorem 6.5.2, we have limit in (6.28) along the whole sequence Λ. Reasoning as in the deduction of formulas (6.29) and (6.34), but now in connection with orthonormal polynomials, it follows that

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}_l,k}(z)}{q_{\mathbf{n},k}(z)} = \begin{cases} (S_k\varphi_k)(z)\,, & k = 1, \ldots, \tau^{-1}(l)\,, \\ S_k(z)\,, & k = \tau^{-1}(l) + 1, \ldots, m\,, \end{cases} \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})\,,$$

where S_k is defined in (6.30). This formula, divided by (6.29) or (6.34) according to the value of k gives

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}_l,k}}{\kappa_{\mathbf{n},k}} = \sqrt{\omega_k} = \frac{c_k}{\sqrt{c_{k-1}c_{k+1}}}\,,$$

where ω_k is defined in (6.37), and the c_k are the normalizing constants found solving the linear system of equations (6.39) which ensure that

$$F_k^{(l)} \equiv c_k \widetilde{F}_k^{(l)}\,, \qquad k = 1, \ldots, m\,,$$

with $F_k^{(l)}$ satisfying (6.27) and thus given by (6.41). Since $(\widetilde{F}_k^{(l)})'(\infty) = 1, k = 1, \ldots, \tau^{-1}(l)$, and $(\widetilde{F}_k^{(l)})(\infty) = 1, k = \tau^{-1}(l) + 1, \ldots, m$, formula (6.43) immediately follows with $\kappa_k^{(l)}$ as in (6.46).

From the definition of $\kappa_{\mathbf{n},k}$, we have that

$$K_{\mathbf{n},k} = \kappa_{\mathbf{n},1} \cdots \kappa_{\mathbf{n},k}\,.$$

Taking the ratio of these constants for the multi-indices $\mathbf{n}$ and $\mathbf{n}_l$ and using (6.43), we get (6.44). Formula (6.45) is an immediate consequence of (6.43) and (6.40). □

Remark 6.5.5. *We have imposed two types of restrictions on the class of multi-indices under consideration. The first one refers to being in* $\mathbb{Z}_+^m(*)$*. This is connected with the long standing question in the theory of multiple orthogonal polynomials of whether or not for any* m *all multi-indices of a Nikishin system are strongly normal. We have proved our results in the largest class of multi-indices known to be strongly normal. Should this conjecture be solved in the positive sense (as it appears to be the case), the methods exposed in this chapter should allow to eliminate this condition as we have done for the cases* $m = 1, 2, 3$.

The second restriction

$$\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$$

is connected with the use of Lemma 6.4.2. *This condition means that all components of the multi-indices are of the same order and that orthogonality is, basically, equally distributed between all the measures. The proof of* (6.24) *requires the density of certain classes of rational functions with fixed poles (in our case at the zeros of the polynomials* $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ *and numerator of degree twice the order of orthogonality) in the space of continuous functions on a given interval. In general, this is not true if the rational functions are such that the degree of the denominator is much larger in order than that of the numerator (as* $|\mathbf{n}| \to \infty$*). This is what may occur if we eliminate the restriction above. It can be relaxed to* $n_k = |\mathbf{n}|/m + \mathcal{O}(\log|\mathbf{n}|), k = 1, \ldots, m,$ *without changing the structure of the Riemann surface which describes the solution of the problem, but not much more. In applications, the diagonal case* ($n_k = |\mathbf{n}|/m, k = 1, \ldots, m$) *and nearby indices seem to be the most important.*

7. CONCLUDING REMARKS

We have obtained the logarithmic and ratio asymptotics of mixed type multiple orthogonal polynomials associated with two Nikishin systems of measures. The results have been proved under assumptions that match those required in the case of standard orthogonality as far as the measures is concerned. So in this sense they are sharp. Possible extensions would require relaxing the assumptions on the systems of multi-indices. In this connection see Remark 6.5.5 whose statements are valid for the mixed type case. The strong asymptotics for type II Nikishin multiple orthogonal polynomials was obtained in [3]. It would be nice to extend that result to mixed type orthogonality when the generating measures are in the Szegő class. The Riemann-Hilbert approach could also give new light to the strong asymptotics of mixed type Nikishin multiple orthogonal polynomials giving finer strong asymptotics for special classes of generating measures. In this direction little has been achieved. Another area of further research would be the consideration of measures with unbounded support and the study of the contracted asymptotics of the corresponding multiple orthogonal polynomials.

We also proved relative asymptotics for type II multiple orthogonal polynomials in which the perturbation is given by rational functions. This result allows to give a Markov type theorem for simultaneous Padé approximants of the corresponding system of Cauchy transforms. The result can be extended to mixed type multiple orthogonal polynomials combining the methods exhibited in Chapter 5 and the results from Chapter 4. Nevertheless, here we are far from achieving our initial goals. We would have liked to give the relative asymptotics under the assumption that the perturbation is due to functions $g_k, k = 1, \ldots, m$, such that $p_k g_k^{\pm 1} \in L^\infty(\sigma_k)$ for some appropriate polynomials $p_k, k = 1, \ldots, m$. This would correspond with what is known for standard orthogonality (see [50], [51], [65]). The technical difficulty here was of a different character. We were unable to provide the normality of the sequences of $\{\widetilde{Q}_{\mathbf{n},k}/Q_{\mathbf{n},k}\}_{\mathbf{n}\in\Lambda}$, under this weaker assumption on the perturbation functions. An immediate application would be the following. In [3], strong asymptotics was proved for measures given by weights in the Szegő class. This result could be extended to general measures in the Szegő class.

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