

# SUCCESSFUL UNINFORMED BIDDING\*

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WP-AD 2001-06

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Editor: Instituto Valenciano de Investigaciones Económicas, S.A.  
First Edition March 2001  
Depósito Legal: V-1253-2001

Ivie working papers offer in advance the results of economic research under way in order to encourage a discussion process before sending them to scientific journals for their final publication.

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\* This paper is a revised version of Chapter 2 of my PhD dissertation, University College London, supervised by Prof. Tilman Börgers. I thank him, Murali Agastya, Jacques Cremer, Arup Daripa, Philippe Jehiel, Ian Jewitt, Benny Moldovanu, Marco Ottaviani, Thomas Tröger and the participants in seminars at University College London, University of Toulouse, Bristol University, the ESRC Game Theory Conference 1999 in Kenilworth, and Universitat Pompeu Fabra for their helpful comments. I also acknowledge the financial support from Fundación Ramón Areces, Banco de España, Ministerio de Educación y Ciencia (project CICYT PB-970131), and the hospitality of ELSE. The only responsibility of errors remains my own.

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## ABSTRACT

This paper provides some striking results that arise in the unique symmetric equilibrium of common value multiunit auction in which some bidders are better informed than others. We show that bidders with worse information can do surprisingly well: They can win with higher probability than better informed bidders, and sometimes, even with higher expected utility. We also find a positive relationship between the success of worse informed bidders and the number of units for sale. Finally we argue that the correct intuitive explanation of these results relies on the balance of the *winner's curse* and the *loser's curse* effects.

JEL classification number: D44, D82.

Keywords: asymmetric bidders, common value, winner's curse, loser's curse.

# 1 Introduction

In this paper we study auctions of one or more homogeneous units of a common value good in which some bidders have better information about this common value than the other bidders. Such auctions are of theoretical importance because they model reasonably well a number of real life auctions. Examples are auctions of oil and gas leases, treasury bill auctions, and auctions of parts of the radio spectrum.

We shall provide several instances in which the equilibrium bid behaviour is quite surprising. We show that under natural assumptions worse informed bidders can bid very aggressively and win to better informed bidders with high probability. We also provide an intuitive explanation based on the interaction of two effects: The winner's curse and the loser's curse.

In order to illustrate our results we start with the following example. Suppose that one auctioneer puts up for sale one unit of a good through a sealed bid second price auction to a pool of three bidders. One of the bidders, say bidder A, knows the common value of the good, whereas the other two bidders, say bidder B and bidder C, only know that this value is drawn from a given bounded set according to a probability distribution.

The same reasons as in a second price, private value auction show that bidder A has a unique weakly dominant strategy, to bid the true value of the good. Assuming that bidder A follows this strategy, bidder B and C's unique weakly dominant strategy is to bid the minimum value of the good, i.e. bidder A's minimum bid. Suppose, that bidder B submits a bid above bidder A's minimum bid. Then, bidder B can win under two events: (i) when bidder B bids above bidder A, and bidder A bids above bidder C, and (ii) when bidder B bids above bidder C, and bidder C bids above bidder A. In (i) bidder B pays a price that equals bidder A's bid, i.e. the true value of the good. Whereas in (ii) bidder B pays a price that equals bidder C's bid, i.e. a price above the true value of the good. In this sense, we can say that bidder B suffers a winner's curse.

Suppose next that the auctioneer puts up for sale two identical units of the good instead of one. In this case, we assume that the auction format is a generalisation of the sealed bid second price auction to a two unit sale. The bidders with the two highest bids win one unit each and the price that they pay is the third highest bid, this is the loser's bid. Note that this auction set-up is in fact the Vickrey auction for multiunit sales and bidders with single unit demand.

Again, bidder A's unique weakly dominant strategy is to bid the true value of the good. But, in this case, if bidder A follows this strategy, bidder B and bidder C's unique weakly dominant strategy is to bid the maximum value of the good, i.e. bidder A's maximum bid. Suppose that bidder B bids below bidder A's maximum bid. She can lose under two events: (i) when bidder C bids above bidder A, and bidder A bids above bidder B, and (ii) when bidder A bids above bidder C and bidder C bids above bidder B. In (i), had bidder B bid high enough, she would have won at a price equal to the true value of the good. Whereas in (ii), had bidder B bid high enough, she would have won at a price below the true value of the good. Thus, we can say that bidder B suffers a loser's curse.

Hence, if there is one unit for sale, the uninformed bidders (bidder B and bidder C) lose with probability one and the perfectly informed bidder (bidder A) wins with probability one at a minimum price in equilibrium. Whereas, if there are two units for sale, the uninformed bidders win with probability one at a price equal to the true value of the good, and the informed bidder loses with probability one. Note that the auctioneer's revenue is zero with one unit for sale, whereas the auctioneer gets full surplus extraction per unit when he sells two units.

In this paper we show how the results of this simple example extend to more general situations. We learn from our models that when there are enough number of units for sale, worse informed bidders tend to bid very aggressively and win more often than better informed bidders. Moreover, completely uninformed bidders get higher expected utility than noisily informed bidders in some cases. We also provide some results on auctioneer's expected revenue and on the informational content of the price. Finally, we show that our results are not a pathological equilibrium of the game but rather the unique symmetric equilibrium. Symmetric in the sense that bidders of the same class use the same strategy.

Our results have direct implications for an important application of auction theory, advising in bidding contests. Moreover, our models arise new questions on the optimal design and the efficiency of auctions.<sup>1</sup> For instance, our model suggests that if the auctioneer can choose into how many "lots" to divide what he has for sale, increasing the number of units will allocate a bigger share of the good in expected terms to worse informed bidders.

For the case that bidders have unit-demand, and that the number of units for sale is smaller than the number of well-informed bidders, Milgrom (1981) has displayed an equilibrium of a generalisation of the second price auction in which bidders without relevant private information lose out to better informed bidders with probability one. In this paper, we focus on the opposite case, that there are at least as many units for sale as there are well-informed bidders. It is in this case where we show that Milgrom's result is in some sense reversed. Actually, we find a kind of monotonicity, increasing the number of units for sale increases the probability that worse informed bidders bid higher than better informed bidders. In practice, for example in the auctions cited in the first paragraph of this Introduction, it often seems realistic that well-informed bidders form only a small fraction of the total market.

It is important to emphasise that, although we consider multiunit auctions we maintain the assumption that each bidder individually demands only one unit. Thus, our results are unrelated to the difficult problems arising in auctions in which bidders are allowed to submit multiunit-demands. Because we maintain the unit-demand assumption, it is also obvious how the second price auction needs to be defined in

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<sup>1</sup>Although efficiency is not an issue in a pure common value set-up, if there are small private value differences the final allocation matters. Moreover, we can think of situations in which the auctioneer can have preferences for bidders that are either better or worse informed. One example is when the auctioneer wants to encourage (or discourage) information acquisition. Another possibility is auctions in which there are "incumbents" and "entrants". Incumbents will be typically better informed than entrants. Hence, we could conjecture that the more successful worse informed bidders are the more attractive will be the auction to entrants.

the multiunit case, say with  $k$  units for sale: The bidders with the  $k$  highest bids win and pay the  $k + 1$ -th highest bid.

The observation that uninformed bidders may win auctions is not original to this paper. In fact, Engelbrecht-Wiggans, Milgrom, and Weber (1983) showed that this may happen in the single unit case if the format is a first price auction. Daripa (1998) extended Engelbrecht-Wiggans, Milgrom and Weber's result to a multiunit set-up, using a generalisation of the first price auction. Daripa also shows that a completely uninformed bidder can get higher expected utility than a perfectly informed bidder.

The auction format of Daripa is more difficult to analyse than ours. His analysis is also complicated by the fact that he allows for multiunit-demand. As a consequence, we obtain a more clear-cut analysis than Daripa. For example, we do not face as severe problems of multiplicity of equilibria as Daripa does.

Another reason for our interest in the second price format is that it allows us to develop particularly clearly the intuition for our findings. We explain the relatively good performance of poorly informed or uninformed bidders with respect to informed bidders in terms of the effect of the winner's curse and the loser's curse on the incentives to bid of bidders with different quality of information.

In the (generalised) second price auction a bidder will want to raise his bid by a small amount, say from  $b$  to  $b + \epsilon$ , if the expected value of a unit, conditional on its price being  $p \in (b; b + \epsilon)$ , is larger than  $p$ . The price is  $p$  if and only if the  $k$ -th highest bid of the other bidders is  $p$ . This event is the intersection of two events, one of which implies good news whereas the other implies bad news for the bidder. The good news is that at least  $k$  other bidders have been willing to bid  $p$  or more. If these bidders had any private information at all, it must have been favourable. This is good news. This effect has been called the loser's curse as a bidder who neglects this effect will regret losing. The bad news is that at least  $m - k$  other bidders (where  $m$  denotes the total number of bidders) have bid  $p$  or less, and hence, if they had any private information at all, this must have been unfavourable. This effect has been called the winner's curse as a bidder who neglects this effect will regret winning.<sup>2</sup>

The winner's curse reduces the incentives to bid higher, whereas the loser's curse raises the incentives to bid higher. Moreover, both effects are stronger for less informed bidders because of two reasons. Better informed bidders' estimation is more accurate and hence it is less sensitive to new information.<sup>3</sup> The average informational content of the other bidders' signals is of less quality from the view point of a better informed bidder than from the view point of a worse informed bidder. A better informed bidder faces one less better informed bidder and one more worse informed bidder than a worse informed bidder.

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<sup>2</sup>The winner's curse is well-known in the auction literature, see for instance the survey by Milgrom (1989). The concept of the loser's curse is less established. It was first used by Holt and Sherman (1994) in the context of a bargaining model. The concept was introduced in auction models by Pesendorfer and Swinkels (1997). They also presented a formal definition of the meaning of the winner's curse and the loser's curse in the spirit of that given in our paper.

<sup>3</sup>This is true only under the usual assumption in auction theory that the bidders' signals are informational substitutes, see the brief discussion in Milgrom and Weber (1982).

If the loser's curse is sufficiently strong in comparison to the winner's curse we can expect that in equilibrium bidders with less information win more often than bidders with more information. Moreover, we can also expect that the stronger the loser's curse is in comparison to the winner's curse, the more often less informed bidders win. This explains the increase aggressiveness of the uninformed or poorly informed bidders' behaviour with respect to the number of units. The more units there are for sale, the more winners and the fewer losers there are in the auction, thus the loser's curse will be stronger and the winner's curse will be weaker.

Note that when there is only one unit for sale the good news of the loser's curse are completely offset by the bad news of the winner's curse. In this case we can say that the loser's curse plays no role. At the opposite extreme is the case when the number of units for sale equals the number of bidders minus one. Then the winner's curse is completely offset by the loser's curse. The winner's curse thus plays no role, and it can only be the loser's curse that affects the incentives to bid higher.

The most closely related papers are those of Milgrom (1981), Engelbrecht-Wiggans and Daripa (1998) which were already discussed above. Another related study is that of Pesendorfer and Swinkels (1997). This paper, like ours, studies the generalisation of the second price auction to the multiunit case when bidders have unit-demand. Pesendorfer and Swinkels (1997) differs from our paper in two respects. Firstly, they assume that all bidders have signals of equal informativeness, whereas our focus is on the case that some bidders have more informative signals than others. Secondly, they focus on the case that the number of units for sale and the number of bidders are large. By contrast, our focus is on the case of a finite number.

This paper is structured as follows: In Section 2, we study a basic model in which there are one bidder with relevant, although potentially incomplete information, and several other, completely uninformed bidders. Section 3 extends the model and analyses a case in which there are several bidders who hold relevant information whereas other bidders are completely uninformed. In Section 4, we extend the model of Section 2 into a different direction, and allow the bidders who were uninformed in Section 2 to hold some pieces of information. We only assume that their information is less significant than that of the well-informed bidder. We show that the equilibria in this set-up converge in an appropriate sense to the equilibrium in Section 2 as the significance of the less informed bidders' signals tends to zero.

## 2 An Auction with One Informed and Many Uninformed Bidders

An auctioneer puts up for sale through auction  $k$  indivisible units of a good. There are  $n + 1$  bidders,<sup>4</sup>  $n \geq 2$ . Each bidder can bid for one or zero units of the good.<sup>5</sup>

<sup>4</sup>In the case  $n = k = 1$  the auction game which we are considering has very many equilibria. Since an analysis of these equilibria would distract from the main point of this paper, we restrict attention to the case  $n \geq 2$ .

<sup>5</sup>Equivalently we could assume that a perfectly divisible good is for sale, all bidders have constant marginal utility for the good, and the auctioneer splits the good into  $k$  identical lots and allows

We assume that the number of bidders is greater than the number of units for sale,  $n + 1 > k$ .

Each bidder obtains a von Neumann Morgenstern utility of  $v_j - p$  if she obtains one unit of the good, and she obtains a von Neumann Morgenstern utility of zero if she obtains no unit. The value  $v$  ( $v \in \mathbb{R}_+$ ) is common to all bidders. One bidder, the informed bidder, receives privately a signal  $s$  informative of  $v$ , whereas the other bidders, the uninformed bidders, do not receive any signal. For simplicity we assume that  $s$  is informative of  $v$  in the sense that the expected value of the good conditional on signal  $s$  is a continuous and strictly increasing<sup>6</sup> function,<sup>7</sup> say  $\phi(s)$ . We assume that  $s$  is drawn from the interval  $[\underline{s}; \bar{s}]$  with a continuous distribution function  $F(s)$ . This distribution is assumed to have support  $[\underline{s}; \bar{s}]$ .

We restrict to uniform price auction with neither a reserve bid nor an entry fee. In this auction format, all bidders submit simultaneously non-negative bids. The bidders who make the  $k$ -th highest bids win one unit each. The price which they have to pay is the  $k + 1$ -th highest bid. If the  $k$ -th highest bid and the  $k + 1$ -th highest bid have the same value  $b$ , then the price in the auction is  $b$ , all bidders who make a bid strictly higher than  $b$  get one unit with probability one, and the remaining winners are randomly selected among all bidders who have made bid  $b$ , whereby all such bidders have the same probability of being selected.

To analyse equilibrium bidding in this auction we begin with the following observation:

**Proposition 1.** The informed bidder has a unique weakly dominant strategy, to bid  $\phi(s)$  for all  $s \in [\underline{s}; \bar{s}]$ :

**Proof.** This follows from the standard argument that is used to show that in single object, private value, second price auctions bidding one's true value is a dominant strategy.  $\square$

The informed bidder's signal is the only information available to the bidders. Thus, the event winning does not convey any new information to the informed bidder about the value of the good, i.e. the informed bidder's incentives to bid are affected neither by the winner's curse nor the loser's curse. This explains the simplicity of the unique weakly dominant strategy of the informed bidder.

Given Proposition 1 we can focus on the behaviour of the uninformed bidders. We shall assume that all uninformed bidders play the same pure or mixed strategy. We shall describe this mixed strategy by its distribution function  $G : [\phi(\underline{s}); \phi(\bar{s})] \rightarrow [0; 1]$ . Notice that we rule out bids which are not in the interval  $[\phi(\underline{s}); \phi(\bar{s})]$ . Such bids are weakly dominated. We shall call a strategy of the uninformed bidders an equilibrium strategy if together with the weakly dominant strategy of the informed bidder it constitutes a Bayesian Nash equilibrium of the auction game.

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each bidder to bid for at most one of these lots.

<sup>6</sup>For instance, the conditional expected value  $\phi(s)$  is a continuous and strictly increasing function if  $v$  and  $s$  have a joint distribution such that (i) it has a density function that satisfies strictly the affiliation inequality, and (ii) it has full support.

<sup>7</sup>Note that the case that we study in the Introduction in which the informed bidder's signal is perfectly informative about the value of the good is a special case of this model.

We consider first two cases that allow for an analysis specially clear-cut. These two cases are generalisations of the examples in the Introduction. The first of these cases is when the number of units for sale is only one. Then, the leading effect on the incentives to bid is the winner's curse, whereas the loser's curse plays no role. Given that the winner's curse is bad news and the informed bidder does not suffer any winner's curse, the uninformed bidders have less incentives to bid than any of the types of the informed bidder. The next proposition states the corresponding result:<sup>8</sup>

**Proposition 2.** If there is only one unit for sale,  $k = 1$ , there is only one equilibrium strategy for the uninformed bidders, to bid  $\theta(\underline{s})$  with probability one.

**Proof that the proposed strategy is an equilibrium strategy:** In the proposed equilibrium the uninformed bidders get utility zero. The only possible deviation for uninformed bidders is to raise their bids. If all uninformed bidders except one bid  $\theta(\underline{s})$ , and one uninformed bidder raises her bid to some value  $b > \theta(\underline{s})$ , then this uninformed bidder wins if and only if the informed bidder's bid is between  $\theta(\underline{s})$  and  $b$ . Moreover, the price which the uninformed bidder has to pay is exactly the informed bidder's bid which equals the true value of one unit. Therefore, the expected utility from raising the bid is zero. Thus, there is no strict incentive for uninformed bidders to raise their bids.

**Proof that there are no other equilibrium strategies:** Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids above  $\theta(\underline{s})$ . Then each uninformed bidder can gain by changing her strategy, and bidding  $\theta(\underline{s})$  with probability one. To see this distinguish the following two events: (i) the highest of all other uninformed bidders' bids is greater than the informed bidders' bid; and (ii) the highest of all other uninformed bidders' bids is less than or equal to the informed bidders' bid. Observe that both events occur with positive probability. In event (ii) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (i), however, there is a strict incentive to be among the losers of the auction, this is, there is a winner's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is among the winners. Thus, she can strictly gain by deviating to  $\theta(\underline{s})$ . ¥

**Corollary 1.** If  $k = 1$  : (i) The price is completely uninformative, since it is always equal to  $\theta(\underline{s})$ . (ii) The informed bidder wins with probability one the unique unit for sale. (iii) The informed bidder has positive expected utility whereas the uninformed bidders have expected utility zero.

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<sup>8</sup>Note that in second price auctions there always exist equilibria in which a number of bidders equal to the number of units for sale bid very high, and all the other bidders bid very low, see Bikhchandani and Riley (1991). It implies that we could construct asymmetric equilibria in the model of this section such that one uninformed bidder wins with probability one at a very low price. However, these equilibria in general require a lot of co-ordination among the bidders, and in the case of the model of this section, these equilibria are in weakly dominated strategies.

The other specially simple case is when the number of units for sale equals the number of uninformed bidders. Then, the leading effect is the loser's curse, whereas the winner's curse plays no role. Since the loser's curse is bad news and the informed bidder does not suffer any loser's curse, the uninformed bidders have greater incentives to bid than any of the types of the informed bidder:

Proposition 3. If there are  $n$  units for sale,  $k = n$ , there is only one equilibrium strategy for the uninformed bidders, to bid  $\rho(\underline{s})$  with probability one.

Proof that the proposed strategy is an equilibrium strategy: In the proposed equilibrium the uninformed bidders have utility zero. This is because they all win with probability one, but the price equals the bid of the informed bidder, i.e. the value of the good. If an uninformed bidder lowers her bid, she loses the auction whenever the informed bidder's bid is above her lower bid. Otherwise she wins, but at a price which equals the informed bidder's bid. Hence her expected utility is again zero. Thus, no uninformed bidder can gain by deviating.

Proof that there are no other equilibrium strategies: Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids below  $\rho(\underline{s})$ . Then each uninformed bidder can gain by changing her strategy, and bidding  $\rho(\underline{s})$  with probability one. To see this distinguish the following two events: (i) the lowest of all other uninformed bidders' bids is greater than or equal to the informed bidders' bid; and (ii) the lowest of all other uninformed bidders' bids is less than the informed bidders' bid. Observe that both events occur with positive probability. In event (i) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (ii), however, there is a strict incentive to be among the winners of the auction, this is, there is a loser's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is not among the winners. Thus, she can strictly gain by deviating to  $\rho(\underline{s})$ .  $\text{¥}$

Corollary 2. If  $k = n$ : (i) The price reveals the conditional true value ( $\rho(s)$ ). (ii) With probability one all units are won by uninformed bidders. (iii) All bidders have expected utility zero.

In other cases, namely when  $1 < k < n$ , both the winner's curse and the loser's curse affect the uninformed bidder's incentives to bid. The study of the interaction of these two effects requires a slightly different analysis than that of the previous cases. This analysis is done in the next proposition:

Proposition 4. If  $1 < k < n$ , then there exists a unique equilibrium strategy for the uninformed bidders:

$$G^u(b) = \frac{(n - k) \int_{\underline{s}}^{\rho^{-1}(b)} F(s) ds}{(n - k) \int_{\underline{s}}^{\rho^{-1}(b)} F(s) ds + (k - 1) \int_{\rho^{-1}(b)}^{\bar{s}} [1 - F(s)] ds};$$

for all  $b \in [\rho(\underline{s}); \rho(\bar{s})]$ . This equilibrium distribution function is continuous and has support  $[\rho(\underline{s}); \rho(\bar{s})]$ .

Proof. This proof is broken down into two steps.

Step 1. In the first step we consider mixed strategies of the uninformed bidder that have a continuous distribution function  $G$ . A necessary condition for such strategies to be an equilibrium is that each uninformed bidder is indifferent between all the bids in the support, if she takes as given that all the other uninformed bidders adopt the proposed strategy, and that the informed bidder plays her weakly dominant strategy. This is just the standard indifference condition characterising Nash equilibria in mixed strategies, extended to the case of infinite strategy spaces. We shall show that this indifference condition implies that if  $G$  is an equilibrium distribution of bids, it must be equal to  $G^*$ .

The above indifference condition is satisfied only if each uninformed bidder gets zero expected utility. To see why note that the number of units for sale is less than the number of uninformed bidders, thus the lowest bid in the support of the uninformed bidders' strategy must lose with probability one, i.e. it gives zero expected utility.

To apply this condition we distinguish two events under which an uninformed bidder can win the auction: (i) the price in the auction equals the bid of the informed bidder, and (ii) the price in the auction equals the bid of another uninformed bidder. Under event (i), the expected utility of winning is trivially zero, the price equals the value of the good. Hence, the expected utility of winning must also be zero under event (ii). Since all the bids in the support of  $G$  must give zero expected utility, this means that the expected utility of winning at a given price  $b$  in the support of  $G$  conditional on event (ii) must also be zero.

To formalise the last necessary condition, we introduce for an arbitrary  $b$  in the support of  $G$ , the notation  $\frac{1}{2}(b)$ . This stands for the probability that the informed bidder's bid,  $v^o(s)$ , is greater than  $b$ , conditional on the following event: There are exactly  $k_i - 1$  bids above  $b$  among  $n_i - 2$  uninformed bidders' bids and the informed bidder's bid. This is the probability that an uninformed bidder suffers a loser's curse at price  $b$ . Similarly,  $1 - \frac{1}{2}(b)$  is the probability that an uninformed bidder suffers a winner's curse at price  $b$ . Using this notation, we can write our necessary condition as:<sup>9</sup>

$$\frac{1}{2}(b)E[v^o(s) \mid b] + (1 - \frac{1}{2}(b))E[v^o(s) \mid b] - b = 0; \quad (1)$$

where if  $G$  is the distribution of bids of each uninformed bidder, then  $\frac{1}{2}(b)$  equals by definition:

$$\frac{\int_{k_i-2}^{n_i-2} [1 - F]^{k_i-2} G(b)^{n_i-k} \frac{1}{k_i-2} [1 - F]^{k_i-2} G(b)^{n_i-k}}{\int_{k_i-2}^{n_i-2} [1 - F]^{k_i-2} G(b)^{n_i-k} + \int_{k_i-1}^{n_i-2} F [1 - G(b)]^{k_i-1} G(b)^{n_i-k-1}}; \quad (2)$$

with  $F = F(v^{-1}(b))$  to simplify the expression.

After some algebra and with the help of the following equality which is derived

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<sup>9</sup>Here and in the following  $E[\cdot \mid \cdot]$  denotes the expected value of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.

using the iterated expectation law:

$$F^{(i-1)}(b)E[v_j^\circ(s) \cdot b] = F^{(i-1)}(b)E[\int_{\underline{s}}^{\infty} \circ(s) dF(s)];$$

and a similar relationship for  $(1 - F^{(i-1)}(b))E[v_j^\circ(s) \cdot b]$ , we can show that the unique function  $G$  that solves equation (1) for a given  $b$  is  $G^\pi(b)$ . This function is continuous, strictly increasing and satisfies  $G^\pi(\underline{s}) = 0$  and  $G^\pi(\bar{s}) = 1$ . Hence, if the distribution of the mixed strategies is to be continuous, then it must be  $G^\pi(b)$  for all  $b \in [\underline{s}; \bar{s}]$ .

It only remains to be shown that  $G^\pi$  characterises in fact an equilibrium strategy. It is sufficient to show that each of the bids  $b \in [\underline{s}; \bar{s}]$  gives zero expected utility to an uninformed bidder, if all the other uninformed bidders adopt  $G^\pi$ , and the informed bidder plays her weakly dominant strategy. This holds since as we have already shown,  $G^\pi$  is the unique distribution function such that the expected utility of winning at a price  $b \in [\underline{s}; \bar{s}]$  for each uninformed bidder is zero, if all the other uninformed bidders adopt  $G^\pi$ , and the informed bidder plays her weakly dominant strategy.

Step 2. In this second step we study mixed strategies that have a discontinuous distribution function. Assume that  $G$  is one of such strategies with an atom at  $\hat{b}$ . We focus on the incentives to deviate of an uninformed bidder, say bidder  $l$ . Let  $b_{(k)}$  be the  $k$ -th highest bid of all the bidders but  $l$ . Define the event " $\hat{b}$  wins" to be the event in which bidder  $l$  when making a bid  $\hat{b}$  wins one unit, and the event " $\hat{b}$  loses" the complement of " $\hat{b}$  wins", this is the event in which bidder  $l$  when making a bid  $\hat{b}$  loses the auction.

We begin by arguing that we must have:  $E[v_j b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] \geq \hat{b}$ . Suppose instead  $E[v_j b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < \hat{b}$ . If this were the case, then bidder  $l$  could gain by shifting all probability mass that is placed on  $\hat{b}$  to some bid  $\hat{b} - \epsilon$  where  $\epsilon > 0$  is close to zero. This change would obviously make no difference to player  $l$ 's utility in the case that  $b_{(k)} > \hat{b}$ , nor would it affect  $l$ 's utility in the case that  $b_{(k)} = \hat{b}$  and  $\hat{b}$  loses. Finally, it would obviously also not make any difference in the case that  $b_{(k)} < \hat{b} - \epsilon$ . In the event that  $b_{(k)} = \hat{b}$  and  $\hat{b}$  wins, which has positive probability, the change in strategy would lead to a strict increase in player  $l$ 's utility. Finally, the probability of the event that  $\hat{b} - \epsilon < b_{(k)} < \hat{b}$  can be made arbitrarily small by choosing a sufficiently small  $\epsilon$ , so that it does not affect the advantageousness of the proposed deviation.

In a similar way it can be argued that we must have  $E[v_j b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}] \leq \hat{b}$ .

If  $\hat{b} = \bar{s}$ , the event  $b_{(k)} = \hat{b}$  means that the bid of the informed bidder is below  $\bar{s}$ . As a consequence the first of the conditions above cannot be satisfied. Similarly, it can be shown that  $\hat{b} = \underline{s}$  violates the second of the conditions above.

We can complete our indirect proof by arguing that if  $\underline{s} < \hat{b} < \bar{s}$ , then  $E[v_j b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < E[v_j b_{(k)} = \hat{b}] < E[v_j b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}]$ , this is, that there is a winner's and a loser's curse at price  $\hat{b}$ . This last inequality obviously contradicts the other two inequalities. Suppose you knew that  $b_{(k)} = \hat{b}$ , but you did



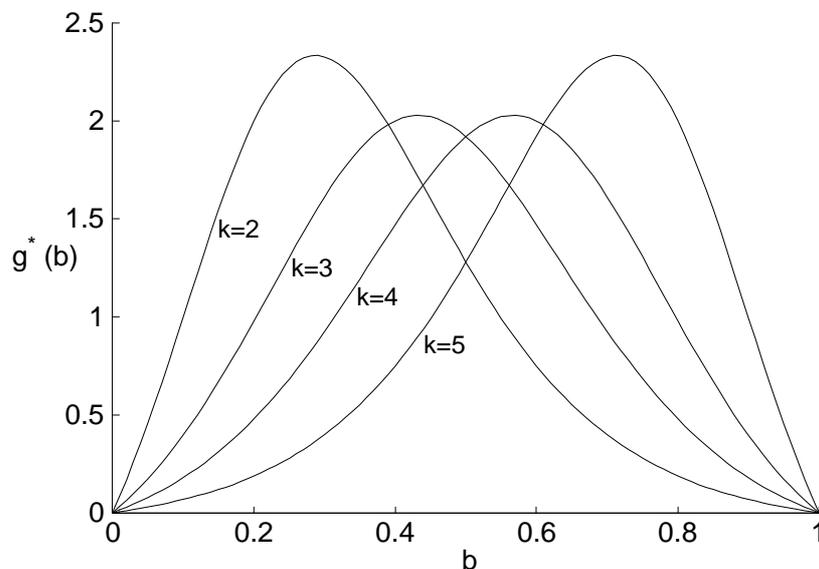


Figure 1: Plot of the density ( $g^a$ ) of  $G^a$  for  $n = 6$ .

the auction, there are more winners, but also an indirect effect. It changes the relative proportion with which bidders with different quality of information win. The consequence of this indirect effect is that increasing the number of units for sale will always increase the probability with which the uninformed bidders win. However, if the indirect effect is sufficiently strong, it can imply that an increase in the number of units for sale decreases the probability that the informed bidder wins. This happens in at least two cases. When we move from a situation in which there is one unit for sale, or when we move to a situation in which there are  $n$  units for sale.

### 3 An Auction in Which Uninformed Bidders have Positive Expected Utility

In the model of the previous section the uninformed bidders can win with positive probability but they always receive zero expected utility. The purpose of this section is to construct a model in which the uninformed bidders can win, and their expected payoff is strictly positive.

As before, we assume that there are  $k$  units of the same good for sale. The number of bidders is now assumed to be  $n_I + n_U$ , where  $n_I + n_U > k$ . Among the bidders,  $n_I$  bidders are called informed bidders. Each of these bidders receives privately a signal  $s_i$ . The other  $n_U$  bidders are called uninformed bidders. They receive no signal. We assume that the number of uninformed bidders is strictly greater than the number of informed bidders,  $n_I < n_U$ . If this assumption is not

satisfied our analysis is less clear due to multiplicity of equilibria.<sup>10</sup>

The value of the good,  $v$ , is assumed to be a simple arithmetic mean  $\frac{\sum_{i=1}^{n_I} s_i}{n_I}$ . Moreover, we assume that the signals  $s_i$  are independently drawn<sup>11</sup> from the set  $[\underline{s}; \bar{s}]$  ( $0 < \underline{s} < \bar{s}$ ) according to the same continuous distribution function  $F$  with support  $[\underline{s}; \bar{s}]$ . Bidders' preferences and the auction game are the same as in Section 2.

As in Section 2, we shall focus on symmetric equilibria. In this section, this will mean that all informed bidders play the same strategy, and all uninformed bidders play the same strategy. For simplicity, we shall focus on equilibria in pure strategies instead of allowing for mixed strategies as in Section 2. We shall denote by  $b_I : [\underline{s}; \bar{s}] \rightarrow \mathbb{R}_+$  the strategy of the informed bidders and by  $b_U \in \mathbb{R}_+$  the bid of the uninformed bidders. We shall further simplify our arguments by assuming that the informed bidders play a continuous and strictly increasing strategy.

Some of the results of the previous section generalise in natural ways to the model of the current section. For example, in the case  $k < n_I$ , it can be proved that in the unique equilibrium outcome the uninformed bidders lose with probability one. Such equilibria generalise the equilibrium in Proposition 2. For the case  $n_I < k < n_U$  one can show that there is no equilibrium in pure strategies. In this respect this case is similar to the case of Proposition 4.

We shall not deal explicitly in this paper with the two cases mentioned in the previous paragraph. We shall also omit the rather special case  $k = n_U$ . Instead, we shall focus on the case that  $k > n_U$ . This case yields for our purposes the most interesting result. The result is similar to Proposition 3.

**Proposition 5.** Suppose  $k > n_U$ . Then the bid functions  $(b_I; b_U)$  constitute an equilibrium if and only if:<sup>12</sup>

$$\begin{aligned} b_I(s) &= \mathbb{E} \left[ \sum_{j=1}^k v_j S_{(q)} \mid S_{(q)} = s \right] = s_{(q+1)} \\ b_U &= \mathbb{E} \left[ \sum_{j=1}^k v_j S_{(q)} \mid S_{(q)} = \bar{s} \right] \end{aligned}$$

Here, we define  $q = k - n_U$ .

These conditions are such that the uninformed bidders win with probability one in equilibrium.

We first provide an example of equilibrium bid functions. In this example we assume  $n_I = 16$ ,  $n_U = 18$ ,  $k = 30$  and  $F(x) = x^4$  with support  $[0; 1]$ . Then  $b_I(s) = \frac{22s}{5} + \frac{44(1+s+s^2+s^3+s^4)}{5(1+s+s^2+s^3)}$  and  $b_U = \frac{77}{5}$  satisfy the conditions given in Proposition 5. A plot of these equilibrium bid functions appears in Figure 2.

<sup>10</sup>If  $n_I < k < n_U$  there exist equilibria similar to those in Proposition 5 in which the uninformed bidders win with probability one. Moreover, there exists some other equilibria similar to those suggested by Milgrom (1981) in which uninformed bidders lose with probability one. Note that in fact the case that Milgrom studies is a special case of  $n_U < k < n_I$  in which  $n_U = 1$  and  $n_I > k$ .

<sup>11</sup>We assume that the value is a simple arithmetic mean of the signals and that the signals are stochastically independent for presentation convenience. Our analysis generalises directly to the usual information structure used in common value auctions in which signals and the true value are affiliated random variables.

<sup>12</sup>We denote by  $s_{(r)}$  the  $r$ -th highest signal of the signals  $s_i$ .

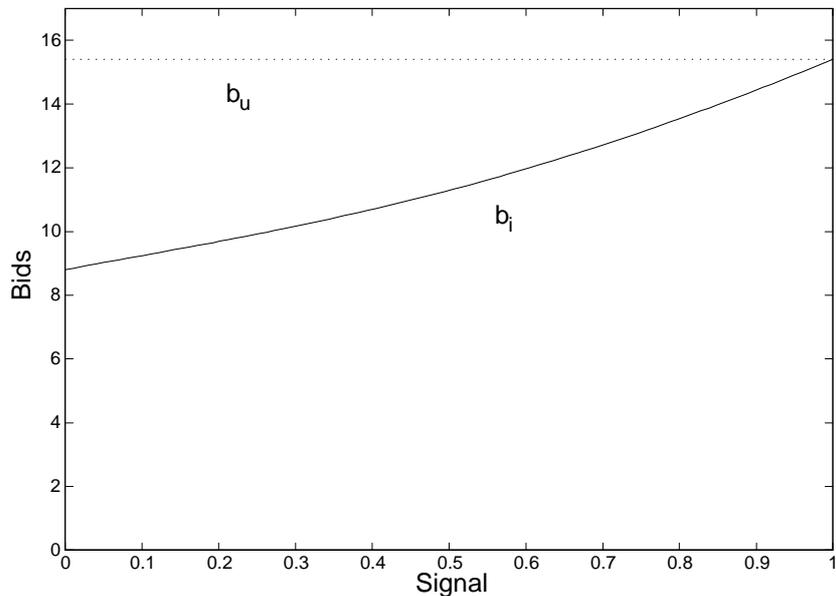


Figure 2: Equilibrium bid functions with  $n_I = 16$ ,  $n_U = 18$  and  $k = 30$ .

Proof that the conditions in the proposition are sufficient: The strategy of the informed bidders is the same as in the standard symmetric equilibrium in an auction of  $q$  units and no uninformed bidders. The arguments used for the standard symmetric equilibrium by Milgrom (1981) to show that the bidders do not have incentives to deviate also explain why the informed bidders do not have incentives to deviate if the conditions in the proposition are satisfied.

To see that the uninformed bidders do not have incentives to deviate suppose that all the bidders stick to some strategies that satisfy the proposed conditions. Under this assumption we can state the following arguments. First, each uninformed bidder does not have incentives to raise her bid since she is already winning with probability one. Second, if one uninformed bidder lowers her bid below  $b_U$  she does not improve her payoffs. This is because by deviating the bidder loses when the price in the auction is between her deviating bid and  $b_U$ , and the expected value of winning at a given price is positive for all the equilibrium prices. To see why, take an arbitrary price of the auction  $p$ . This price  $p$  must correspond to the equilibrium bid of a type  $s$  of the informed bidders. Then, the expected utility of winning at price  $p$  is the difference between the expected value of the good conditional on the  $q + 1$ -th highest signal equals  $s$ , and the price  $p$ . Since the price  $p$  equals the expected value of the good conditional on the  $q$ -th and  $q + 1$ -th highest signals equal  $s$  by definition of  $s$ , the difference is positive.

Proof that the conditions in the proposition are necessary: Assume that the bid of the uninformed bidders is above all the bids of the informed bidders, then the informed bidders' equilibrium strategy must be an equilibrium strategy of the same

auction game with  $q$  units for sale,  $n_I$  informed bidders and no uninformed bidders. In this last case, we can use a proof similar to that by Harstad and Levin (1986) to show that there is a unique equilibrium strategy. This equilibrium strategy is such that each informed bidder bids the expected value of the good given that the  $q$  and the  $q + 1$  highest signal of the informed bidders equal her own private signal. This is the equilibrium bid function that appears in the proposition.

In order to complete the proof it only remains to be shown that the uninformed bidder cannot lose the auction with positive probability in equilibrium. This proof follows a similar structure and notation to Step 2 of the proof of Proposition 4.

Assume that the uninformed bidders' bid,  $b_U$ , is below the maximum bid of the informed bidders. We focus on the incentives to deviate of an uninformed bidder, say bidder  $I$ . We reintroduce the notation  $b_{(k)}$  for the  $k$ -th highest bid of all the bidders but  $I$ . Define the event "I wins" to be the event in which bidder  $I$  wins when making the bid  $b_U$  and the event "I loses" as its complement, this is, the event in which bidder  $I$  loses when making the bid  $b_U$ .

We can use the same arguments as in Step 2 of the proof of Proposition 4 to show that two necessary conditions of equilibrium are:  $E[v|b_{(k)} = b_U \text{ and } I \text{ wins}] \geq b_U$  and  $E[v|b_{(k)} = b_U \text{ and } I \text{ loses}] \leq b_U$ .

We start showing that these inequalities cannot be met simultaneously if  $b_U$  is between the minimum and the maximum bid of the informed bidders. We prove this using a random variable<sup>13</sup>  $\tau$  that stands for the number of informed bidders that bid above  $b_U$ . Our indirect proof is to show that the conditional distribution of  $\tau$  shifts in the sense of strictly first order stochastic dominance, to the right when  $I$  loses and to the left when  $I$  wins. Since we restrict attention to equilibria in which the informed bidders' bid function is strictly increasing, this is sufficient for our claim.

By definition:<sup>14</sup>

$$\Pr_{\tau} \{ I \text{ wins} | \tau = l; k_j \leq \tau < k \} = \frac{k_j - l}{n_U}$$

$$\Pr_{\tau} \{ I \text{ loses} | \tau = l; k_j \leq \tau < k \} = \frac{n_U - k + l}{n_U}$$

where  $k_j \leq \tau < k$  is the same event than  $b_{(k)} = b_U$ . If  $\tau$  is less than  $k$  but more than  $k_j$ , the  $k$ -th highest bid of the other bidders is the bid of one of the uninformed bidders, all of which bid  $b_U$ .

Hence,

$$\frac{\Pr_{\tau} \{ I \text{ wins} | \tau = l; b_{(k)} = b_U \}}{\Pr_{\tau} \{ I \text{ loses} | \tau = l; b_{(k)} = b_U \}}$$

decreases strictly with  $l$ . Therefore,  $I$  wins and  $I$  loses, conditional on  $b_{(k)} = b_U$  can be interpreted as a pair of signals that satisfy the Monotone Likelihood Ratio Property. Consequently, the distribution of  $\tau$  conditional on  $I$  loses and  $b_{(k)} = b_U$

<sup>13</sup>We use the notation  $\tau$  for the random variable and  $l$  for its realisation.

<sup>14</sup>Here and in the following  $\Pr(\cdot | \cdot)$  denotes the expected probability of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.

strictly first order stochastically dominates the distribution of  $I$  conditional on  $I$  wins and  $b_{(k)} = b_U$ .

We complete our proof with the case in which  $b_U$  is equal or below the minimum bid of the informed bidders. In this case, the informed bidders and  $k - n_I$  of the poorly informed bidders get one unit each at a price equal to  $b_U$ . Since the events  $I$  wins,  $I$  loses and  $b_{(k)} = b_U$  are uninformative of  $v$ , the two inequalities that must satisfy  $b_U$  imply that  $b_U = E[v]$ . But this means that an informed bidder whose type is such that the expected value of the good conditional on her type is strictly less than  $E[v]$  has incentives to deviate. She gets strictly negative expected utility when she wins, and if she bids below  $b_U$  she loses with probability one and hence, she gets zero utility.  $\text{¥}$

**Proposition 6.** If  $k > n_U$ , then the expected utility of each uninformed bidder is strictly positive and strictly greater than the unconditional expected utility of each informed bidder in equilibrium.

**Proof.** The ex ante expected utility of winning is the same for both classes of bidders. They get one unit and pay a price equal to the bid of the  $q + 1$ -th highest signal of the informed bidders. Moreover, this ex ante expected utility is strictly positive because the price is below the expected value of the good conditional on the information contained in the price. The proposition then follows because each uninformed bidder wins with probability one and each informed bidder with probability  $q = n_I$ .  $\text{¥}$

This result shows that uninformed bidding can be quite attractive. Note, however, that this result does not necessarily imply that an uninformed bidder that wins with probability one does not have incentives to acquire information and become an informed bidder. The reason is that although when she becomes informed she loses with some positive probability, the other informed bidders can bid less aggressively, even if they are one more, because they compete for one more unit. This decreases the average price in the auction and can in fact, offset the decrease in the probability of winning. For instance, if there is one single informed bidder and  $n_U = k > 3$  uninformed bidders, we now from Section 2 that the each uninformed bidder wins with probability one but gets zero expected utility. However, if one of the uninformed bidder acquires information and becomes informed bidder, we now from this section that she will lose with positive probability but will get strictly positive expected utility.

## 4 An Auction with One Informed and Many Poorly Informed Bidders

In this section we extend the analysis of Section 2 by allowing the worse informed bidders to hold some relevant information although less informative than the informed bidder's information. We thus talk of one bidder with better information (the informed bidder) and some other bidders with worse information (the poorly

informed bidders), although not completely uninformed. The purpose of this extension is double. First, to show that similar results to those in Section 2 also hold in this more general set-up. Second, to check the robustness of the equilibria in Section 2, by showing that the equilibria in this extended model converge in an appropriate sense to the equilibria in the model of Section 2 when the informativeness of the poorly informed bidders' signals vanishes.

In this section we keep all the assumptions of Section 2 except the information structure. This is modified to allow for less informative signals. We assume that the value of the good  $v$  is a simple arithmetic mean of some  $n + 1$  signals  $s_i$  ( $i = 1; 2; \dots; n + 1$ ). The signals  $s_i$  are assumed to be statistically independent and to follow the same continuous distribution function  $F$  with a bounded support  $[\underline{s}; \bar{s}]$  ( $0 < \underline{s} < \bar{s}$ ).

We assume that the informed bidder observes one of these signals (say  $s_{n+1}$ ), that we call  $s^I$  in what follows, whereas each of the poorly informed bidders observes a garble of a different signal  $s_i$ , that we call  $s_i^P$  ( $i = 1; 2; \dots; n$ ). These garbles are generated by the following simple procedure: With a probability independent of the other random variables of the model, say  $\lambda$  ( $0 < \lambda < 1$ ),  $s_i^P$  equals  $s_i$ , and with the complementary probability  $1 - \lambda$ ,  $s_i^P$  equals another random variable statistically independent of the other random variables of the model and that follows the same distribution as  $s_i$ , i.e.  $F$ .

According to this procedure, the signal  $s^I$  is more informative of  $v$  than each of the signals  $s_i^P$  in Blackwell's sense. Moreover, when  $\lambda$  tends to one, each signal  $s_i^P$  becomes as informative of  $v$  as  $s^I$ , and when  $\lambda$  tends to zero, each signal  $s_i^P$  becomes completely uninformative of  $v$ . Note also that the signals  $s_i^P$  have a marginal distribution function  $F$ . We also introduce the following assumption:

Assumption 1.  $s_i \in E[s_i | s_i < s]$  and  $E[s_i | s_i > s] \in s$  are respectively strictly increasing and strictly decreasing in  $s$ .

Assumption 1 is satisfied by many distribution functions, e.g. the uniform distribution function. If  $F$  has a continuously differentiable density, see Lemma 3 in Bikhchandani and Riley (1991), a sufficient condition for the first part of the assumption is that  $F(s)$  is strictly log-concave. Similarly, if  $F$  has a continuously differentiable density, a sufficient condition for the second part is that  $1 - F(s)$  is strictly log-concave, this is that  $F(s)$  has an increasing hazard rate.

We explain in the paragraph before Lemma 1 the role that our simple information structure including this last assumption play in the model of this section.

We define an equilibrium of the game as a bid function  $b_I : [\underline{s}; \bar{s}] \rightarrow \mathbb{R}_+$  for the informed bidder and a bid function  $b_P : [\underline{s}; \bar{s}] \rightarrow \mathbb{R}_+$  for the poorly informed bidders that form a Bayesian Nash equilibrium of the game. Note that we study symmetric equilibrium in the sense that all poorly informed bidders use the same bid function.

For simplicity we shall only consider equilibria in continuous and strictly increasing strategies. In order to rule out some strange equilibria that exist in the case  $k = 1$  and  $k = n$  we also restrict attention to equilibria in which all the bidders have an unconditional positive probability of winning the auction.<sup>15</sup>

<sup>15</sup>For instance, if  $k = 1$ , there exists a set of equilibria where the informed bidder bids very high

We start our analysis proposing an indifference condition that we use to formulate some strategies. We shall show later that these strategies constitute the unique equilibrium of the game. Our condition says that each bidder's bid conditional on her type equals the expected value of the good conditional on: The bidder's private information, and the information that the bidder infers from the event that the  $k$ -th highest bid of the other bidders equals her own bid. Such condition assures that a bidder is indifferent between winning and losing when the price equals her bid and she wins. Note that this condition is satisfied by the bid function of the symmetric equilibrium that it is usually analysed when bidders are ex ante symmetric, see for instance Milgrom (1981).

However, the statement of the above indifference condition has an additional difficulty in asymmetric models like ours. We need to specify a function that gives the type of the informed bidder that submits the same bid that a given type of the poorly informed bidders. We solve this problem proposing one such function and we later show that this function is the one that corresponds to the unique equilibrium of the game.

Let  $\hat{A} : [\underline{s}; \bar{s}] \rightarrow [\underline{s}; \bar{s}]$  be a function implicitly defined by the condition that the expected value of the good conditional on the event that the  $k$ -th highest signal of the poorly informed bidders is  $s$ , and that the informed bidder's signal is  $\hat{A}(s)$  equals:

- (i) If  $k = 1$ , the expected value of the good conditional on the event that the two highest signals of the poorly informed bidders are equal to  $s$ , and the informed bidder's signal is below  $\hat{A}(s)$ . Formally:<sup>16</sup>

$$E[vjs^I = \hat{A}; s_{(1)}^P = s] = E[vjs^I \cdot \hat{A}; s_{(1)}^P = s_{(2)}^P = s]; \quad (3a)$$

- (ii) If  $k = n$ , the expected value of the good conditional on the event that the  $n-1$ -th and the  $n$ -th highest signals of the poorly informed bidders are equal to  $s$  and the informed bidder's signal is above  $\hat{A}(s)$ :

$$E[vjs^I = \hat{A}; s_{(n)}^P = s] = E[vjs^I \cdot \hat{A}; s_{(n-1)}^P = s_{(n)}^P = s]; \quad (3b)$$

- (iii) If  $1 < k < n$ , the expected value of the good conditional on the event that either the  $k-1$ -th and the  $k$ -th highest signals of the entrants' signals equal  $s^P$  and the informed bidder's signal is above  $\hat{A}(s)$ , or the  $k$ -th and the  $k+1$ -th highest signals of the poorly informed bidders equal  $s$  and the informed bidder's signal is below  $\hat{A}(s)$ :

$$E[vjs^I = \hat{A}; s_{(k)}^P = s] = \mu(\hat{A}; s)E[vjs^I \cdot \hat{A}; s_{(k-1)}^P = s_{(k)}^P = s] + (1 - \mu(\hat{A}; s))E[vjs^I \cdot \hat{A}; s_{(k)}^P = s_{(k+1)}^P = s]; \quad (3c)$$

and the poorly informed bidders bid very low. Thus, the informed bidder wins with probability one and each poorly informed bidder wins with probability zero. See Footnote 8.

<sup>16</sup>Again, we denote here and in what follows  $s_{(r)}^P$  the  $r$ -th highest signal of the poorly informed bidders. We also drop the dependence of  $\hat{A}$  on  $s$  in the equations (3a), (3b), and (3c) to simplify the notation.

where  $\mu(\hat{A}; s)$  is the probability that the  $k - 1$ -th and the  $k$ -th highest signals of the entrants' signals equal  $s$  and the informed bidder's signal is above  $\hat{A}$  conditional on the following event: Either the  $k - 1$ -th and the  $k$ -th highest signals of the entrants' signals equal  $s$  and the informed bidder's signal is above  $\hat{A}$ , or the  $k$ -th and the  $k + 1$ -th highest signals of the poorly informed bidders equal  $s$  and the informed bidder's signal is below  $\hat{A}(s)$ . Hence,

$$\mu(\hat{A}; s) = \frac{\int_{k-2}^{n_i} [1 - F(\hat{A})][1 - F(s)]^{k-2} F(s)^{n_i - k}}{\int_{k-2}^{n_i} [1 - F(\hat{A})][1 - F(s)]^{k-2} F(s)^{n_i - k} + \int_{k-1}^{n_i} F(\hat{A})[1 - F(s)]^{k-1} F(s)^{n_i - k - 1}}$$

In an equilibrium in continuous and strictly increasing bid functions, if  $\hat{A}$  is the function that relates types of the informed bidder and types of the poorly informed bidder that submit the same bid,  $\hat{A}$  must be continuous and strictly increasing. We use next our simple assumptions on the information structure to prove the ...rst part of the following result. Under more general information structures, e.g. a general affiliated model, we would need more additional and probably more complex assumptions to assure that the next result holds.

Lemma 1. Each equation (3a), (3b), and (3c) defines implicitly a unique function  $\hat{A}(s)$ . This function  $\hat{A}$  is continuous and strictly increasing in  $s$ . Moreover,

- (i) If  $k = 1$ , then  $\hat{A}(\underline{s}) = \underline{s}$  and  $\hat{A}(\bar{s}) < \bar{s}$ .
- (ii) If  $k = n$ , then  $\hat{A}(\underline{s}) > \underline{s}$  and  $\hat{A}(\bar{s}) = \bar{s}$ .
- (iii) If  $1 < k < n$ , then  $\hat{A}(\underline{s}) = \underline{s}$  and  $\hat{A}(\bar{s}) = \bar{s}$ .

Proof. See the Appendix. ¥

Another implication of this lemma is that the range of  $\hat{A}$  is  $[\underline{s}; \bar{s}]$  only if  $1 < k < n$ . This means that in the other cases, according to this function there are some types of the informed bidder that submit bids that are not in the range of the poorly informed bidder's strategy. We see below the implications of this property of  $\hat{A}$ .

If we assume that  $\hat{A}$  is actually the function that relates types of the informed bidder and of the poorly informed bidder that submit the same bid, we can use our indifference condition to define the informed bidder's bid function for types  $s \in [\hat{A}(\underline{s}); \hat{A}(\bar{s})]$ :

$$b_i(s) = E^h [v_j | s^j = s; s_{(k)}^P = \hat{A}^{-1}(s)] \quad (4)$$

Note that Lemma 1 means that this bid function is defined for all types of the informed bidders only if  $1 < k < n$ . If  $k = 1$  this function does not define the bid of types  $s \in (\hat{A}(\bar{s}); \bar{s}]$ . Similarly, if  $k = n$  this function does not define the bid of types  $s \in [\underline{s}; \hat{A}(\underline{s})$ ). We provide later conditions that restrict the bids of these types.

We next propose a bid function for the poorly informed bidders and we later discuss why in fact it satisfies our indifference condition:

$$b_P(s) = E[v_j | s^I = \hat{A}(s); s_{(k)}^P = s]; \quad (5)$$

for  $s \in [\underline{s}; \bar{s}]$ .

To see why  $b_P$  also satisfies our indifference condition note that the event that the  $k$ -th highest signal of the other bidders equals the bidder's bid can mean two things to a poorly informed bidder. Either the  $k$ -th highest bid of the other bidders is submitted by the informed bidder or by another poorly informed bidder. The function  $b_P$  verifies by definition our indifference condition under the first of the events. Moreover, the definition of the function  $\hat{A}$  guarantees that  $b_P$  also verifies our indifference condition under the latter event.

We complete the definition of the informed bidder's bid function. If  $k = 1$ , all bids of the informed bidder above  $b_I(\hat{A}(\bar{s}))$ , i.e. the maximum bid of the poorly informed bidders, are payoff equivalent to the informed bidder. However, these bids cannot be too low, otherwise, some poorly informed bidders could have incentives to deviate. We give in the next equation a condition that assures that this does not happen:

$$\int_{\hat{A}(\bar{s})}^{\bar{s}} (E[v_j | s^I = s; s_{(1)}^P = \bar{s}] - b_I(s)) dF(s) \geq 0; \quad (6)$$

for all  $s$  in  $(\hat{A}(\bar{s}); \bar{s}]$ .

Similarly, if  $k = n$ , all bids below  $b_I(\hat{A}(\underline{s}))$ , i.e. the minimum bid of the poorly informed bidders, are payoff equivalent to the informed bidder. All these bids lose with probability one. However, these bids cannot be too low, otherwise, some poorly informed bidders could have incentives to deviate. We give such condition below:

$$\int_{\underline{s}}^{\hat{A}(\underline{s})} (E[v_j | s^I = s; s_{(n)}^P = \underline{s}] - b_I(s)) dF(s) \leq 0; \quad (7)$$

for all  $s$  in  $[\underline{s}; \hat{A}(\underline{s})]$ .

Finally, note that both  $b_I$  and  $b_P$  satisfy that for all  $s \in [\underline{s}; \bar{s}]$ ,  $\hat{A}(s)$  is the type of the informed bidder that submits the same bid as a type  $s$  of the poorly informed bidders.

Figures 3, 4 and 5 show examples of bid functions that satisfy equations (4), (5), (6), and (7). All these examples are done assuming that  $F$  is a uniform distribution function with support  $[0; 1]$ .

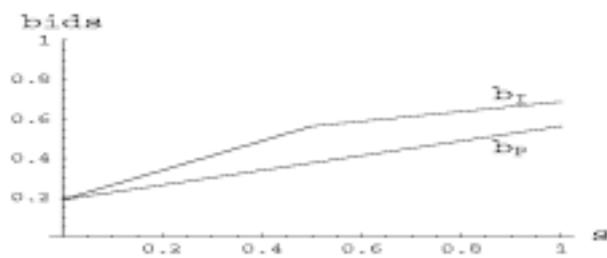


Figure 3: Equilibrium bid functions with  $k = 1$ ,  $n = 3$  and  $\alpha = 0.5$ .

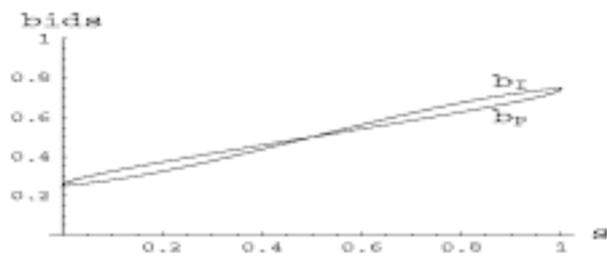


Figure 4: Equilibrium bid functions with  $k = 2$ ,  $n = 3$  and  $\alpha = 0.5$ .

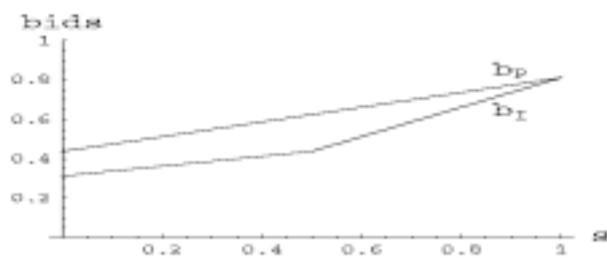


Figure 5: Equilibrium bid functions with  $k = 3$ ,  $n = 3$  and  $\alpha = 0.5$ .

Proposition 7. The pair of strategies  $(b_I; b_P)$  are an equilibrium of the game if and only if  $b_P$  is as defined in equation (5), and  $b_I$  as defined in (5), and satisfies equation (6) if  $k = 1$ , and equation (7) if  $k = n$ .

Proof. See the Appendix. ¥

In what follows we shall assume for the sake of simplicity that the inequalities (6), if  $k = 1$ , and (7), if  $k = n$ , hold with equality. It is easy to see that this assumption implies that there is a unique equilibrium bid function  $b_I$  in cases  $k = 1$  and  $k = n$ : If  $k = 1$ ,  $b_I(s) = E[v|s^I = s; s_{(1)}^P = \bar{s}]$ , for  $s \in ((\hat{A}(\bar{s}); \bar{s}]$ ; and if  $k = n$ ,  $b_I(s) = E[v|s^I = s; s_{(n)}^P = \underline{s}]$ , for  $s \in [\underline{s}; \hat{A}(\underline{s})$ .

Our next goal is to illustrate the robustness of the conclusions of Section 2. We start showing that the equilibrium in this section converges in an appropriate sense to the equilibrium in the model of Section 2, when the signals of the poorly informed bidders become fully uninformative of the value of the good.

In order to state this result we shall use the function  $\phi(s)$  of Section 2. This is the expected value of the good conditional on the informed bidder's signal. This is the only information available for the bidders in the model of Section 2, and in the model of this section when  $\delta$  tends to zero.

Proposition 8. When  $\delta$  goes to zero, the equilibrium bid function of the informed bidder converges (uniformly) to  $\phi(s)$ , and:

- (i) If  $k = 1$ , then the equilibrium bid function of the poorly informed bidders converges (uniformly) to  $\phi(\underline{s})$ .
- (ii) If  $k = n$ , then the equilibrium bid function of the poorly informed bidders converges (uniformly) to  $\phi(\bar{s})$ .
- (iii) The equilibrium distribution of bids of each of the poorly informed bidders converges (weakly) to  $G^*(b)$ , the equilibrium distribution of the bids of each uninformed bidder in Section 2.

Proof. See the Appendix. ¥

Note that this result is not only a robustness test of the equilibria in Section 2, but also a proof that the claim that worse informed bidders can do surprisingly well against a better informed bidder does not only hold for completely uninformed bidders but also for poorly informed bidders. To see why, we provide as a corollary of Proposition 8, the following result:

Corollary 5. (i) If  $k = 1$ , then the probability that a poorly informed bidder wins the auction converges to zero as  $\delta$  tends to zero. (ii) If  $k = n$ , then the probability that the poorly informed bidder win all the units for sale converges to one as  $\delta$  tends to zero.

We finally consider what is the effect of increasing the number of units for sale in this set-up.

Proposition 9. The probability that a poorly informed bidder with type  $s \in (\underline{s}; \bar{s})$  bids higher than the informed bidder strictly increases when the number of units for sale increases.

Proof. See the Appendix. ¥

This result is similar to Corollary 4 in the sense that it illustrates that increasing the number of units for sale makes the worse informed bidders bid relatively more aggressively than the better informed bidder. Note, however, that this result does not necessarily imply that increasing the number of units for sale shifts to the right in the sense of first order stochastic dominance, the distribution of bids of the poorly informed bidders.

As we mentioned after Corollary 4, one interesting implication of Proposition 9 is that increasing the number of units for sale has not only a direct effect on the bidders' probability of winning, but also an indirect effect. It changes the relative proportion with which bidders with different quality of information win. This means that increasing the number of units for sale always increases the probability with which poorly informed bidders win the auction. However, it is not clear that it increases the probability with which the informed bidder wins. In fact, if  $\delta$  is sufficiently close to zero, we can show that it decreases the probability with which the informed bidder wins the auction at least in two cases: When the initial number of units for sale is one, and when the final number of units for sale is  $n$ .

Our results have also a translation into expected utility comparisons. Note the following result.

Lemma 2. The expected utility of an informed bidder (poorly informed bidder) with type  $s$  is a continuous function with first derivative equal to the probability that she wins times  $\frac{1}{n+1} (\frac{1}{n+1})$ .

Proof. This result can be proved with a straightforward adaptation of the analysis of Myerson (1981). We can do this adaptation because of our assumption that the bidders' signal are independent and the value of the good is additive in the signals. ¥

This lemma says that each bidder's expected utility is a linear function of the probability of winning the auction for types below the bidder's type. Since the expected utility of each bidder's minimum type is invariant with respect to the number of units for sale, i.e. it is always zero,<sup>17</sup> this means that an increase in the number of units for sale has similar effects on the expected utility of the bidder to those on the probability of winning. There is one direct effect which increases the probability that each bidder wins the auction, and hence, the bidders' expected utility, and one indirect effect that changes the relative proportion with which bidders with different quality of information win. This indirect effect increases the expected utility of poorly informed bidders and decreases the expected utility of the informed bidder.

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<sup>17</sup>This is true under the assumption that constraint (7) is satisfied with equality when  $k = n$ . Otherwise, the minimum type of the poorly informed bidders gets strictly positive expected utility when  $k = n$ .

We can also argue that since both the direct and the indirect effect have the same direction for poorly informed bidders, their expected utility increases unambiguously with the number of units for sale. However, since the effects take opposite directions for the informed bidder, her expected utility could be decreasing in the number of units for sale. We can show that this is the case if  $\beta$  is close to zero and either the initial number of units for sale is one or the maximal number of units for sale is  $n$ .

## 5 Conclusions

In this paper we have provided some natural common value auction models with asymmetric bidders that have a striking equilibrium behaviour. Basically, we have shown that worse informed bidders can win with a surprisingly high probability, and some times even with high expected utility, when competing with better informed bidders. We have also shown that the number of units for sale is a key variable for the relative success of better informed-worse informed bidders: Increasing the number of units for sale makes the worse informed bidders more willing to outbid better informed bidders. We have also provided an intuitive explanation of these phenomena based on the interaction of the winners' curse and the loser's curse and their different qualitative effect on bidders with better or worse information.

We have seen how the increase in the loser's curse and the decrease in the winner's curse (in our model changing the number of units for sale) increases the relative success of worse informed bidders. We believe that this effect could have consequences in frameworks in which bidders' information acquisition is an issue or in situations in which entry of new bidders, probably less informed, is important for the auctioneer.

One of the limitations of our analysis is that we only consider one auction format, a generalised second price auction. The difficulty to analyse other auction formats comes from the difficulty to study auctions with asymmetric bidders. These difficulties are, however, less severe if we consider generalisations of the English auction. We provide this extension in a companion paper and compare the second price and the English auction when bidders are asymmetrically informed. We also believe that we could extend our analysis without many difficulties to generalisations of the first price auction if we limit to the case in which there is one informed bidder and several completely uninformed bidders. This extension could provide a new robustness test of our results.

## Appendix

In this appendix we provide the proofs of Section 4. We start introducing two functions that will be useful to simplify the notation:  $\hat{v} : [\underline{s}; \bar{s}] \rightarrow \mathbb{R}_+$ , where  $\hat{v}(s) = s_j \cdot E[s_i | s_i \geq s]$ , and  $\hat{p} : [\underline{s}; \bar{s}] \rightarrow \mathbb{R}_+$ , where  $\hat{p}(s) = E[s_i | s_i \leq s] \cdot s$ . These functions have the following properties:

Lemma 3. The function  $\hat{v}$  is continuous, strictly increasing and bounded, moreover,  $\hat{v}(\underline{s}) = 0$  and  $\hat{v}(\bar{s}) > 0$ . The function  $\hat{p}$  is continuous, strictly decreasing and bounded, moreover,  $\hat{p}(\bar{s}) = 0$  and  $\hat{p}(\underline{s}) > 0$ .



Similar arguments to those used in symmetric, common value, second price auctions show that a bidder does not have incentives to decrease her bid if the expected value of winning at each price below her bid is non negative. Suppose that a given bidder with type  $s$  wins at a price  $b$  that belongs to the range of her bid function. This means that there exists a type  $s^0 < s$ , such that if our bidder had received type  $s^0$ , she would have bid  $b$ . According to our indifference condition, this means that our bidder would be indifferent between winning and losing, if she had type  $s^0$  instead of type  $s$ . Since  $s > s^0$ , our bidder prefers winning, i.e. gets non negative expected utility. Finally, note that the price that a bidder pays can be fixed out of the range of her bid function only for the poorly informed bidders and when there are  $n$  units for sale. In all these cases, condition (7) assures that a bidder with type  $s \geq s^0$  weakly prefers winning. Hence, poorly informed bidders with a type  $s \geq s^0$  get non negative expected utility.

Necessary proof. We start the proof with the case  $1 < k < n$ . Suppose that the bid functions  $b_I; b_P$  are an equilibrium of the game. Let  $b$  be a bid that belongs to the intersection of the interior of the range of  $b_I$  and  $b_P$ , assuming by now that this intersection is not empty. The standard logic used in second price auctions shows that  $b_I$  and  $b_P$  must satisfy our indifference condition as otherwise bidders would have local incentives to deviate.

If we call the types of the informed bidder and the poorly informed bidders that submit bid  $b$ ,  $s$  and  $s^0$  respectively, then the indifference condition for the informed bidder implies that in an equilibrium in increasing strategies  $b = b_I(s) = E[v|s^1 = s; s_{(k)}^P = s^0]$ . Similarly, the indifference condition for the poorly informed bidders implies,

$$b = b_P(s^0) = \theta E[v|s^1 = s; s_{(k)}^P = s^0] + (1 - \theta) \mu(s; s^0) E[v|s^1 \geq s; s_{(k-1)}^P = s_{(k)}^P = s^0] + (1 - \mu(s; s^0)) E[v|s^1 < s; s_{(k)}^P = s_{(k+1)}^P = s^0]^\theta ;$$

where  $\theta$  is a number between 0 and 1 that corresponds to the probability that the  $k$ -th highest bid of the other bidders is the bid of the informed bidder given that the  $k$ -th highest bid of the other bidders equals  $b$ .

By definition of  $\hat{A}$ , the unique functions that satisfy these two necessary conditions for a given  $b$  are  $b_I$  in equation (4) and  $b_P$  in equation (5) for types  $s$  and  $s^0$ , and where  $s = \hat{A}(s^0)$ . Moreover, since we restrict to continuous and strictly increasing bid functions, equation (4), must be verified by all  $s$  in the range of  $\hat{A}$  and equation (5) by all  $s$  in the domain of  $\hat{A}$ , i.e. in the case of  $1 < k < n$  these two conditions completely specify the bid functions.

The proof is similar in the cases  $k = 1$  and  $k = n$ , the only novelty is that we need to show why condition (6) if  $k = 1$ , and condition (7) if  $k = n$  are necessary. Consider for instance, the case  $k = 1$ , and suppose that the poorly informed bidders' bid function and the informed bidder's bid function are as in equations (4) and (5). Then, if condition (6) is not satisfied for a type  $s \in (\hat{A}(s^0); \bar{s}]$  of the informed bidder,

a poorly informed bidder with a type  $\bar{s}$  (or arbitrary close to  $\bar{s}$ ) has incentives to deviate arising her bid to  $b_I(s)$ . This deviation only changes her payoffs when the informed bidder's bid is between  $[b_I(\bar{A}(\bar{s})); b_I(s))$ , and then she wins and gets strictly positive expected utility.

We finally show that the intersection of the range of  $b_I$  and  $b_P$  cannot be empty in equilibrium. Since we restrict to equilibrium in continuous and strictly increasing functions, this intersection can be empty if and only if either  $b_I(\underline{s}) > b_P(\bar{s})$  or  $b_P(\underline{s}) > b_I(\bar{s})$ . We only need to check that none of these two possibilities can happen in equilibrium.

We start with  $b_I(\underline{s}) > b_P(\bar{s})$ . If  $k = 1$  this possibility is rule out by our assumption that we restrict to equilibria in which all the bidders have an ex ante positive probability of winning. If  $k > 1$ , then the informed bidder gets one unit with probability one (independently of her signal) and the poorly informed bidders compete for the  $k - 1$  units left. We can use an analysis similar to Harstad and Levin (1986) to show that there is a unique symmetric equilibrium strategy for the poorly informed bidders and this is such that:  $b_P(s) = E[vj s_{(k-1)}^P = s_{(k)}^P = s]$ . If all the poorly informed bidders follow this strategy, the informed bidder with a type  $\underline{s}$  (or arbitrarily close to  $\underline{s}$ ) has incentives to deviate lowering her bid slightly below  $b_P(\bar{s})$ . This deviation only changes the informed bidder payoffs when the  $k$ -th highest type of the poorly informed bidders is arbitrarily close to  $\bar{s}$ . In this case, the informed bidder's expected utility of winning is strictly negative. Hence, the deviation is profitable since it allows losing.

The case  $b_P(\underline{s}) > b_I(\bar{s})$  is ruled out by assumption, the informed bidder loses with probability one. Note, however, that for the case  $1 < k < n$  we could also use a symmetric argument to the one in the former paragraph.  $\neq$

Proof of Proposition 8.

It is easy to show that since  $v = \frac{s^1 + \dots + (s_1^P + \dots + s_n^P) + (1 - \dots) n E[s_i]}{1 + n}$ ,  $j b_I(s) i^\circ(s) j \cdot \dots n \frac{s_i \underline{s}}{n+1}$  for all  $s \in [\underline{s}, \bar{s}]$ . Hence,  $b_I$  converges uniformly to  $^\circ$  as  $\delta$  tends to zero.

- (i) Case  $k = 1$ . Since  $b_P(s) = b_I(\bar{A}(s))$ , and  $b_I$  converges uniformly to  $^\circ$  when  $\delta$  goes to zero, we only need to prove that  $\bar{A}$  converges uniformly to  $\underline{s}$ .

$$\lim_{\delta \downarrow 0} \max_s j \bar{A}(s) i \delta j = \lim_{\delta \downarrow 0} j \bar{A}(\bar{s}) i \delta j = \lim_{\delta \downarrow 0} \bar{F}^{-1}(\bar{F}^{-1}(\bar{s})) i \delta j = \bar{F}^{-1}(0) i \delta j = 0;$$

where we have used that  $\bar{A}$  is increasing and  $\bar{A}(\underline{s}) = \underline{s}$  (Lemma 1) in the first step, and that  $\bar{F}$  is continuous and  $\bar{F}^{-1}(0) = \underline{s}$  (Lemma 3) in the last two steps.

- (ii) Case  $k = n$ . We proceed as in (i).

$$\lim_{\delta \downarrow 0} \max_s j \bar{A}(s) i \bar{s} j = \lim_{\delta \downarrow 0} j \bar{A}(\bar{s}) i \bar{s} j = \lim_{\delta \downarrow 0} \bar{F}^{-1}(\bar{F}^{-1}(\bar{s})) i \bar{s} j = \bar{F}^{-1}(0) i \bar{s} j = 0$$

where we have used that  $\bar{A}$  is increasing and  $\bar{A}(\bar{s}) = \bar{s}$  (Lemma 1) in the first step, and that  $\bar{F}$  is continuous and  $\bar{F}^{-1}(0) = \bar{s}$  (Lemma 3) in the last two steps.

(iii) Case  $1 < k < n$ . The probability that the bid of a poorly informed bidder is less than  $b$  equals  $F^{-1} \hat{A}^{-1}(b, i^{-1}(b))$ . We first prove the convergence of  $F(\hat{A}^{-1}(s^0))$ . Note that  $F(\hat{A}^{-1}(s^0))$  is a probability distribution function with support  $[\underline{s}; \bar{s}]$ . We can use equation (8) to show that  $F(\hat{A}^{-1}(s^0))$  satisfies:

$$F(\hat{A}^{-1}(s^0)) = \frac{(n_i - k)F(s^0) \int_{\hat{A}^{-1}(s^0)}^{\bar{s}} \hat{A}^{-1}(s^0)^\alpha}{(n_i - k)F(s^0) [\hat{A}^{-1}(s^0)] + (k_i - 1)(1 - F[s^0]) [\hat{A}^{-1}(s^0)]};$$

Thus,

$$\frac{(n_i - k)F(s^0) \int_{\hat{A}^{-1}(s^0)}^{\bar{s}} \hat{A}^{-1}(s^0)^\alpha}{(n_i - k)F(s^0) [\hat{A}^{-1}(s^0)] + (k_i - 1)(1 - F[s^0]) [\hat{A}^{-1}(s^0)]} \cdot F(\hat{A}^{-1}(s^0)) \cdot \frac{(n_i - k)F(s^0) \int_{\hat{A}^{-1}(s^0)}^{\bar{s}} \hat{A}^{-1}(s^0)^\alpha}{(n_i - k)F(s^0) [\hat{A}^{-1}(s^0)] + (k_i - 1)(1 - F[s^0]) [\hat{A}^{-1}(s^0)]};$$

where  $\underline{\hat{A}}$  and  $\bar{\hat{A}}$  are respectively a lower and an upper bound of  $\hat{A}$ , and  $\underline{s}$  and  $\bar{s}$  are respectively a lower and an upper bound of  $s$  (Lemma 3 says that  $\hat{A}$  and  $s$  are bounded). Hence,

$$\lim_{\delta \rightarrow 0} F(\hat{A}^{-1}(s^0)) = \frac{(n_i - k)F(s^0) \int_{\hat{A}^{-1}(s^0)}^{\bar{s}} \hat{A}^{-1}(s^0)^\alpha}{(n_i - k)F(s^0) [\hat{A}^{-1}(s^0)] + (k_i - 1)(1 - F[s^0]) [\hat{A}^{-1}(s^0)]} \\ = \frac{(n_i - k) \int_{\underline{s}}^{\bar{s}} F(s) ds}{(n_i - k) \int_{\underline{s}}^{\bar{s}} F(s) ds + (k_i - 1) \int_{\underline{s}}^{\bar{s}} [1 - F(s)] ds};$$

where the simplification of the last step is the same as that done in the proof of Proposition 4. Point-wise convergence implies that the distribution function  $F(\hat{A}^{-1}(s^0))$  converges weakly to the last expression. This result together with the uniform convergence of  $b_i$  to  $^\circ$  when  $\delta$  tends to zero, implies that  $F^{-1} \hat{A}^{-1}(b, i^{-1}(b))$  converges weakly to  $G^\alpha(b)$  when  $\delta$  tends to zero (see for instance Hildenbrand (1974), 38, page 51).

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Proof of Proposition 9. To prove the corollary it is enough to show that the type of the informed bidder that bids the same bid as a given type of the poorly informed bidders in equilibrium increases when the number of units increases. Since by Proposition 7,  $b_P(s) = b_I(\hat{A}(s))$ , the statement before follows if  $\hat{A}(s)$  shifts upwards when we increase  $k$ . We can use the same arguments as in the proof of Lemma 1 to show that the left hand side of equation (8) increases with  $k$  around the solutions of equation (8). The proof then follows since for a given  $s$ , the left hand side of equation (8) decreases with  $\hat{A}$ .

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