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Effective Scrappage Subsidies*

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Abstract

It is a common practice for governments to offer scrappage subsidies in order to stimulate the early removal of used cars and modify the distribution of vehicle holdings. In this paper, we analyze the market implications of such subsidies when producers have market power and face competition from a secondary used car market. One key result is that, with market power, a subsidy can induce scrappage even if it pays less than the price of a used car in the absence of the subsidy. We provide a full characterization of the effects of scrappage subsidies on primary and secondary markets for the case of a monopoly, and show that the subsidy that maximizes aggregate welfare lowers prices in the used car market. Our results contrast with the predictions derived from a model with perfect competition.

KEYWORDS: scrappage subsidy, secondary market, market power, automobile industry

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1 Introduction

It is a common practice for governments to offer scrappage subsidies in order to stimulate the early voluntary removal of used cars and modify the distribution of vehicle holdings.\(^1\) Typically, such subsidies are temporary and are offered in exchange for used cars of delineated vintages, which are characteristically more polluting.\(^2\) Nonetheless, since scrappage subsidies interfere with the workings of car markets, their implications are not confined to the environmental impact. In this paper, we analyze the market implications of scrappage subsidies when producers have market power and face competition from a secondary used car market.

There is little data and analysis identifying how scrappage subsidies affect car markets. In Canada and the United States, a few geographically-localized subsidies have been implemented, and most of the analysis has focused on measuring the effects on emission reduction.\(^3\) Nonetheless, the possibility that such programs will be expanded has evoked a debate surrounding their effects on car markets: some argue that they would jeopardize the auto-parts industry, benefit automobile manufacturers, and harm low income consumers by removing inexpensive cars from the marketplace.\(^4\)

In the European Union, scrappage subsidies were very popular during the 1990s. France, Greece, Hungary, Ireland, Italy, and Spain, most of them countries with established automobile manufacturing industries, offered subsidies that required purchasing a new vehicle as a replacement. Instead, Denmark and Norway (and also the United States and Canada) offered subsidies without any constraints.

A common pattern that arises from the data is an increase in the volume of sales of new cars. For instance, European Conference of Ministers of Transport (1999) documents the cases of France and Italy, and Data Resources Inc./McGraw-Hill (1991) identifies the increase in sales as a likely consequence of implementing scrappage subsidies in the United States. Another observed pattern that is particularly important for the arguments advanced in this paper is the one documented for Denmark. European Conference of Ministers of Transport (1999) reports that, during the first six months of the subsidy, 6% of the Danish car fleet was scrapped. Among participants, 11% bought a new car, 45% bought a used one, and 44% did

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\(^1\)Scrappage subsidies are also known as buy-back programs or accelerated vehicle retirement programs.

\(^2\)See Kahn (1996) for a study of vehicle emission trends across model years, makes, and sizes.


\(^4\)There has been opposition to scrappage subsidies because of their distributional effects. Other government policies, like taxes on gas, miles, or engine size, or subsidies to new cars, have distributional implications as well, as discussed in West (2004).
not replace the scrapped vehicle. Thus, although the volume of new car sales increased, most transactions took place in the secondary market. These observations suggest that accounting for an active secondary market might be critical for studying scrappage subsidies. Supporting this conjecture, the report also documents that, in a typical year, only 10% of the Danish consumers who own a car which is ten years or older replace it with a new one.

A useful benchmark for the analysis of scrappage subsidies is a replacement demand model with a competitive primary market. Adda and Cooper (2000) use such a model to characterize the effects of scrappage subsidies and tax credits in France, where the subsidy was tied to the replacement vehicle being new.5 In a replacement model, consumers buy a new car, keep it until they scrap it, and then return to the primary market to buy its replacement. Since the secondary market is never active, the subsidy works by advancing the time of car replacement, which creates a rapid and sharp increase in the sales of new cars. This replacement pattern would be consistent with subsidies that are tied to the purchase of a new vehicle, such as the cases of France and Spain, since they provide clear incentives to return to the primary market. Nonetheless, the trading pattern would diverge from the replacement decisions in Denmark, where the subsidy was offered without constraints (as it was also the case in Canada, Norway, and the United States).

In this paper, we examine the effects on car markets of an unconstrained scrappage subsidy in an environment with market power and active secondary markets. We focus on a monopolist that sells new cars and competes with a competitive secondary market in which used cars are transacted. Consumers are heterogeneous in their valuations of cars, and thus the secondary market also plays an allocative role.

The government seeks to induce scrappage by offering a subsidy to those who scrap a used car. Since scrappage is an alternative to selling in the secondary market, the subsidy constitutes a price floor in the used car market. However, whether the price floor binds—whether the subsidy is effective—depends on the payment offered, the number of used cars in the marketplace, and the choice of production by the firm. For instance, if the firm’s response was to increase production, the price in the primary market would decrease and this would induce additional scrappage.

In understanding the implications of scrappage subsidies, a natural question to address is how low a subsidy should be to induce scrappage. An obvious candidate is to offer the price of the used car in the absence of the subsidy (i.e., pay the without-subsidy price). However, as we will see, we can offer less than this subsidy and still induce scrappage.

Take as a benchmark the problem of the firm choosing its level of output when

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5Licandro and Sampayo (2004) also use a replacement demand model to analyze the effects of scrappage subsidies in Spain.
the subsidy is not offered. The substitutability in consumption between new and used cars implies that, if the firm increases production, prices in the secondary market decrease. This decrease also lowers the willingness to pay for new cars and their prices, since the price of the alternative—the price of a used car—has decreased. Thus, through the secondary market, the firm creates a negative externality on itself when it increases output as it further reduces the new car price.

Suppose, instead, there is a subsidy that pays the without-subsidy price for a used car. Obviously, since the subsidy matches the used car price, a feasible choice for the firm is to keep the price and profit unchanged by simply producing the without-subsidy output. However, the firm can do better. The subsidy is a price floor in the secondary market that truncates the dependence of new car prices on new car production through the secondary market. Consequently, the subsidy suppresses the negative externality that the monopolist creates on itself and makes an increase in production profitable for the firm. Then, since the without-subsidy profit level remains feasible by producing the without-subsidy output, simple revealed preference arguments tell us that the subsidy making the firm indifferent must pay less for a used car than its without-subsidy price.

What drives this result is the market power of the firm and the heterogeneity of consumers. Without market power, a subsidy induces scrappage only if it offers (strictly) more for a used car than its without-subsidy price. To see this, consider the consumer who, without a subsidy, would be indifferent between purchasing a used car and not purchasing one. By definition, the indifferent consumer pays his full willingness to pay for the car, while the remaining buyers derive strictly positive utility from their purchases. Now suppose the subsidy equals the without-subsidy price of a used car. Since the primary market is competitive, the subsidy does not modify prices. Since all consumers—but the marginal one—value the used car more than the subsidy, they continue to purchase from the secondary market and their cars are not scrapped. Only the marginal consumer would be indifferent to scrapping his car, but this consumer has measure zero. Thus, to induce scrappage, the subsidy must pay (strictly) more for a used car than its without-subsidy price.

Our model generates a number of implications for transactions and prices in the primary and secondary markets. If the subsidy pays less for a used car that its without-subsidy price, the subsidy lowers prices in the secondary market and increases the fleet of cars. Otherwise, if it pays more than this price, used car prices increase and the car fleet decreases. We also provide a characterization of the welfare effects of scrappage subsidies and show that aggregate welfare is maximized if the subsidy equals the lowest effective subsidy. That is to say, if the subsidy was welfare maximizing, prices in the secondary used car market would decrease.

Despite being able to capture many aspects of car markets, our model is quite parsimonious: it is static, there is a single firm in the primary market, and the car
vintages are only new and used. These assumptions provide us with tractability and engender no loss of generality since our main result is robust to the additions of dynamics, imperfect competition, and multiple vintages. In Section 4, we study the three generalizations and show that the lowest effective subsidy is always less than the without-subsidy price of a used car.\(^6\)

Our focus on car markets leads us to abstract away from modeling and evaluating the environmental consequences of scrappage subsidies. Recent literature has evaluated the effectiveness of different policies that aim to control vehicle pollution such as taxes on gasoline, miles, or engine size (see, among others, Innes (1996), Fullerton and West (2000), and Fullerton and West (2002)), corporate average fuel economy (CAFE) standards (see, among others, Goldberg (1998)), and scrappage subsidies (Hahn (1995), Baltas and Xepapadeas (1999), Alberini, Harrington, and McConnell (1995), Alberini, Harrington, and McConnell (1996)).

The paper proceeds as follows. In Section 2 we derive the optimal production rule for the monopolist and characterize the effects of a subsidy on the primary and secondary markets. In Section 3 we derive the welfare-maximizing subsidy. In Section 4 we show how our key result is robust to the addition of imperfect competition, multiple vintages, and forward-looking dynamics. We conclude in Section 5.

2 Model

A quantity-setting monopolist produces homogenous cars at a constant marginal cost of \(c \geq 0\) and competes with a competitive secondary used car market. To induce scrappage, a government offers a monetary transfer \(S \geq 0\) to each consumer who scraps a used car instead of selling it in the secondary market or keeping it for his own use. After observing \(S\), the monopolist chooses its level of production to maximize its profit. We refer to \(S\) as a scrappage subsidy, which is in effect a price floor in the secondary market.

Consumers are heterogeneous in their willingness to pay for a car and want to consume at most one unit of one car, which can be either new or used. The heterogeneity amongst consumers is parameterized by \(\theta\), and \(\theta \sim U[0, 1]\) with a unit mass. Without loss of generality, we assume that cars are not durable. With \(^6\)The generalized dynamic model predicts that the distribution of vehicle holdings is not modified in the long run, since the distribution of car ownership slowly converges to the steady state values. This prediction contrasts with the implications of a replacement demand model with competitive pricing, constant marginal costs, and no shocks. In such a model, a subsidy modifies the age distribution of vehicle holdings permanently. See Section 4 for more explanation.
this assumption, the maximization problems of consumers and firm are static.\footnote{Section 4 shows that the elimination of durability does not engender any loss of generality. The intuition is that the subsidy works by modifying the interaction between contemporary primary and secondary markets, and does not affect the interaction between these markets and future secondary markets. On the other hand, the advantage of modeling the problem as static is that it makes the framework tractable and allows us to isolate the effects of the subsidy on prices and output, while obtaining the same qualitative results as in a fully dynamic model.}

We assume that consumers face no transaction costs in the primary and secondary markets, and their utility functions are quasi-linear in income. The utility of a type-\(\theta\) consumer is given by \(U = \theta q + y\theta\), where \(q\) is the quality of the car and \(y\theta\) is his residual income. We assume \(q = 1\) for a new car, \(q = \alpha \in (0, 1)\) for a used car, and \(q = 0\) for the outside option of not buying a car. An implication of vertical differentiation in the quality of the product and heterogeneity in \(\theta\) is that those consumers who are willing to pay less for a car allocate to the secondary market.

Each consumer \(\theta\) determines her optimal consumption choice among the two cars available, new and used, and the option of not consuming a car, to maximize her utility. Whether a type-\(\theta\) consumer owns a used car or not is immaterial in our problem, since the assumptions of quasi-linearity in income and no transaction costs imply that each consumer’s decision is independent of his income and thus of the used car’s resale value. Then, a consumer of type \(\theta\) chooses the option that gives her utility of

\[
\max \{\theta - p_N, \alpha \theta - p_U, 0\},
\]

where \(p_N\) is the price of a new car, \(p_U\) is the price of a used car, and \(0\) is the normalized price of the outside good. Obviously, \(p_U \geq S\), since \(S\) is a price floor in the secondary market.

We let \(x_N\) denote the production of new cars and \(x_U\) denote the stock of used cars. To differentiate between scrappage and consumption, we denote by \(s_U\) the used cars that are scrapped and by \(c_U \equiv x_U - s_U\) those that are consumed and thus are transacted in the secondary market.

A way to classify subsidies is by the amount of scrappage they induce. Given a stock of used cars \(x_U\), we say that a subsidy is effective if it induces the scrappage of used cars. We say that it is minimum, which we denote by \(S_{\min}(x_U)\), if it is the smallest of the effective subsidies. We say that it is maximum, which we denote by \(S_{\max}(x_U)\), if it is the smallest of the subsidies inducing the scrappage of all used cars. Thus, a maximum subsidy is an effective subsidy that closes down the secondary market. Obviously, for \(S \geq S_{\max}(x_U)\), \(s_U = x_U\) (i.e., \(c_U = 0\)). For \(S < S_{\min}(x_U)\), \(s_U = 0\). The case of partial scrappage, which is \(S_U \in (0, x_U)\), corresponds to \(S \in [S_{\min}(x_U), S_{\max}(x_U)]\).
To derive the demand functions, we consider the cases of active and inactive secondary markets separately. We begin with an active secondary market, which requires $S < S_{\text{max}}(x_U)$. Since the secondary market is active, the demand functions for new and used cars are characterized by the two consumers, $\theta_N$ and $\theta_U$, who are indifferent between purchasing a new and a used car, and purchasing a used car and not purchasing a car, respectively. These are identified by the indifference conditions

$$\theta_N - p_N = \alpha \theta_N - p_U$$

and

$$\alpha \theta_U - p_U = 0.$$  

These indifferent consumers are cutoffs in determining the demand for each car: types $\theta \geq \theta_N$ consume a new car, types $\theta \in [\theta_U, \theta_N]$ consume a used car, and types $\theta \leq \theta_U$ do not make any purchase.\(^8\) Therefore, new and used car sales are given by $x_N = 1 - \theta_N$ and $c_U = \theta_N - \theta_U$, respectively, which implies that the corresponding cutoffs are $\theta_N = 1 - x_N$ and $\theta_U = 1 - x_N - c_U$. Since the secondary market is active, $c_U > 0$.

An active secondary market is consistent with two types of subsidies: subsidies that are effective but smaller than the maximum subsidy and subsidies that are not effective. For effective subsidies, the price floor in the secondary market binds, i.e., $p_U = S$. The amount of scrappage is obtained from the condition $\alpha \theta_U = S$, where $\theta_U = 1 - x_N - c_U$ and $c_U < x_U$. Thus, for any $\theta \in [1 - x_N - x_U, \theta_U]$, we have $\alpha \theta - S < 0$, which implies that type-$\theta$ prefers the outside option to purchasing a used car. In other words, those consumers in the interval $[1 - x_N - x_U, \theta_U]$ are willing to pay less for a used car than its scrappage value, and thus the used cars $x_U - c_U$ are scrapped. Lastly, for ineffective subsidies, the price floor does not bind and all used cars are consumed, that is, $c_U = x_U$. In this case, the price of a used car is simply $p_U = \alpha (1 - x_N - x_U)$.

More formally, if the secondary market is active, we can compactly express the inverse demand functions for new and used cars as

$$p_N = (1 - \alpha)(1 - x_N) + p_U$$

and

$$p_U = \max \{\alpha (1 - x_N - x_U), S\}.$$  

Notice that the condition $\alpha (1 - x_N) - p_U > 0$ must hold for the secondary market to be active. What this condition says is that for (some) types with $\theta < \theta_N$ to be

\(^8\)Sales of new cars are positive if $\theta_N > \theta_U$, which can be expressed as $(p_N - p_U)/(1 - \alpha) > p_U/\alpha$ by solving for $\theta_N$ and $\theta_U$ in (2) and (3). This inequality is implied by profit maximization, since profits are zero otherwise.

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purchasing used cars, the cut-off consumer $\theta_N$ in (2) must prefer buying a used car to the outside option. Otherwise, the neighboring lower types to $\theta_N$, who derive even less utility from consuming a car, do not purchase a used car either.

The case of an inactive secondary market requires a sufficiently large subsidy so that all used cars are scrapped. That is, if $S \geq S_{\text{max}}(x_U)$, the demand function for new cars is characterized by the single cut-off consumer, $\theta_N$, who is indifferent between purchasing a new car and the outside option. This cut-off type is identified by the condition

$$\tilde{\theta}_N - p_N = 0. \quad (6)$$

Since $\tilde{\theta}_N = 1 - x_N$, (6) implies

$$p_N = 1 - x_N. \quad (7)$$

Then, the condition $\alpha(1 - x_N) - S \leq 0$ must hold for the subsidy to close down the secondary market.

We place the following constraint on the parameter values:

**Assumption 1.** $1 - \alpha - c > 0$.

This assumption is intuitive and says that type $\theta = 1$ (the highest type) derives a utility gain from consuming a new car rather than a used one that is more than the marginal cost of production.\(^9\)

We can now turn to the problem of the firm.

### 2.1 Monopolist’s Problem

The monopolist chooses production to maximize its profit given a subsidy $S > 0$. With its choice of $x_N$, the firm endogenously determines the effectiveness of the subsidy since scrappage depends on how $\alpha(1 - x_N - x_U)$, $\alpha(1 - x_N)$, and $S$ compare.

Formally, given $S$ and $x_U$, the problem of the monopolist is to choose $x_N$ that maximizes

$$(p_N(x_N, x_U, S) - c)x_N, \quad (8)$$

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\(^9\)Another reading of this assumption is that the monopolist’s profit is positive when the market is covered with used cars. That is, when $x_U = 1$, the price of a used car equals 0 and the consumer with highest valuation, $\theta = 1$, derives a utility gain from consuming a new car rather than a used one that equals $1 - p_N - (\alpha - p_U) = 1 - p_N - \alpha$. Therefore, $p_N \leq 1 - \alpha$ must hold for some consumers to purchase new cars. We then require that choosing $p_N = 1 - \alpha$ results in positive profits for the firm, which is the condition $1 - \alpha - c > 0$. 
where, by (4), (5), and (7),

\[
  p_N(x_N, x_U, S) = \begin{cases} 
  (1 - \alpha)(1 - x_N) + \alpha(1 - x_N - x_U) & \text{if } \alpha(1 - x_N - x_U) \geq S, \\
  (1 - \alpha)(1 - x_N) + S & \text{if } \alpha(1 - x_N - x_U) \leq S \text{ and } \alpha(1 - x_N) \geq S, \\
  1 - x_N & \text{if } \alpha(1 - x_N) \leq S.
  \end{cases}
\]

(9)

The three cases in (9) correspond to three different levels of effectiveness: ineffective subsidies (no scrappage), effective but less than maximum subsidies (partial scrappage), and at least maximum subsidies (full scrappage).

Before solving this maximization problem, it is useful to gauge some intuition for how a subsidy affects the workings of car markets. A question we want to ask is how large a subsidy should be to be effective. An obvious candidate is for the subsidy to equal the price of the used car in the absence of the subsidy (i.e., to equal its without-subsidy price). As we next show, we can offer less than this price and still induce scrappage.

We address this question by considering the empirically relevant case, which corresponds to \(x_U\) such that \(S_{\text{max}}(x_U) > S_{\text{min}}(x_U)\). This inequality says that any subsidy between \(S_{\text{min}}(x_U)\) and \(S_{\text{max}}(x_U)\) will be effective but will not induce the scrappage of all cars. In contrast, if \(S_{\text{max}}(x_U) = S_{\text{min}}(x_U)\), a subsidy will be effective only if it induces full scrappage. We also consider \(S_{\text{min}}(x_U) > 0\), which says that a subsidy is necessary for some used cars to be scrapped (else, the minimum subsidy trivially equals zero). As we later show in Proposition 2, restricting to \(S_{\text{max}}(x_U) > S_{\text{min}}(x_U) > 0\) is equivalent to ruling out stocks of used cars that are either too large or too small. Finally, in what follows, we use a hat to identify without-subsidy prices and production.

Suppose the lowest effective subsidy equals the without-subsidy used car price, which we denote by \(\hat{p}_U\). Our claim is that the monopolist is strictly better off with this subsidy. First, since the subsidy matches the used car price, a feasible choice for the firm is to keep the price and profit unchanged by simply producing the without-subsidy output. That is, \(p_N = (1 - \alpha)(1 - x_N) + S = \hat{p}_N = (1 - \alpha)(1 - \hat{x}_N) + \alpha(1 - \hat{x}_N - x_U)\) since \(x_N = \hat{x}_N\) and \(S = \hat{p}_U\). Figure 1 depicts the inverse demand functions with and without a subsidy, where the darker line is the inverse demand with a subsidy. The two inverse demand functions intersect at \(\hat{x}_N\) since prices are the same. We can now show that, with the subsidy, the monopolist can do strictly better.

Consider the two price equations for \(p_N\) and \(\hat{p}_N\). Without the subsidy, the price of a new car depends on the supply of new cars through the secondary market, which
Figure 1: A subsidy equal to $\hat{p}_U$ is given by the term $-\alpha x_N$. This implies that increasing the level of production lowers the used car price, which feedbacks into the primary market by lowering the consumers' willingness to pay for new cars. That is, through the secondary market, the firm creates a negative externality on itself.

Now suppose we offer a subsidy equal to the without-subsidy price. If the subsidy is effective (i.e., the price floor binds), it truncates the dependence of new car prices on new car production through the secondary market, which suppresses the negative externality that the monopolist creates on itself and makes an increase in output profitable for the firm. Graphically, this corresponds to the new car price being higher with a subsidy than without one to the right of $x^N$, which is the same as the with-subsidy inverse demand function being flatter. Given this incentive to raise output, the monopolist chooses $x^P_N$ in the figure. Then, since the without-subsidy profit is feasible, standard revealed-preference arguments imply that offering $S = \hat{p}_U$ increases the profit of the firm. Thus, the subsidy that makes the firm indifferent to not having a subsidy—the minimum subsidy—is less than $\hat{p}_U$.

**Proposition 1.** Let $x_U$ be such that $S_{\text{max}}(x_U) > S_{\text{min}}(x_U) > 0$. Then, the minimum subsidy is less than the without-subsidy price of a used car: $S_{\text{min}}(x_U) < \hat{p}_U$.

We next solve the monopolist's maximization problem in (8). Notice that the monopolist does not face any competition from the secondary market if $x_U = 0$. In this case, the monopolist's maximization problem is the standard monopoly problem. Since the secondary market jeopardizes the profit of the firm, the profit earned when $x_U = 0$ constitutes an upper bound on the attainable profit. We call this case the standard monopoly problem and identify it with the superscript $M$. It is immediate to verify that $x^M_N = \frac{1-c}{2}$ since $p_N = 1 - x_N$. 
In the next proposition, we use the superscripts $N$, $P$, and $F$ to denote the cases of no scrappage, partial scrappage, and full scrappage, respectively.

**Proposition 2.** Given $x_U$ and $S$, the production rule that solves the monopolist’s problem in (8) is given by:

\[
x_N = \begin{cases} 
\frac{1-c}{2} - \frac{a}{2} x_U & \text{if } S \leq S_{\min}(x_U), \\
\frac{1-a-c+S}{2(1-a)} & \text{if } S \in [S_{\min}(x_U), S_{\max}(x_U)] \text{ and } S_{\max}(x_U) > S_{\min}(x_U), \\
1 - \frac{S}{a} & \text{if } S \in [S_{\max}(x_U), S_M], \\
\frac{1-c}{2} & \text{if } S \geq S_M, 
\end{cases}
\]  

where

\[
S_{\min}(x_U) = \begin{cases} 
\frac{1+c}{a} - \alpha \sqrt{\left(\frac{1-c}{2}\right)^2 - \left(\frac{1-c}{2} - \frac{a}{2} x_U\right)^2} & \text{if } x_U < \tilde{x}_U, \\
\sqrt{1 - \alpha (1 - a - \alpha x_U) - (1 - \alpha - c)} & \text{if } x_U \in [\tilde{x}_U, \bar{x}_U], \\
0 & \text{if } x_U \geq \bar{x}_U, 
\end{cases}
\]  

\[
S_{\max}(x_U) = \begin{cases} 
\frac{1-c}{2} & \text{if } x_U \geq \bar{x}_U, \\
S_{\min}(x_U) & \text{if } x_U < \bar{x}_U, 
\end{cases}
\]  

\[
S_M = \frac{1+c}{\alpha}, \quad \frac{1-a-c+\alpha \frac{1+c}{2} - \alpha \sqrt{1 - \alpha}}{\alpha \sqrt{1 - \alpha}}, \quad (14)
\]

and

\[
\tilde{x}_U = \frac{(1-c) \sqrt{1 - \alpha} - (1 - \alpha - c)}{\alpha \sqrt{1 - \alpha}}. \quad (15)
\]

**Proof.** To solve the monopolist’s maximization problem, we proceed in two steps. In the first step, we divide the monopolist’s maximization in (8) into three constrained problems and derive the respective optimal production rules. In the second step, we solve the overall problem in (8) by comparing profits in the three subproblems.

**Step 1. Division: Three Constrained Maximization Problems**

We build the three subproblems, denoted (P1)–(P3), by requiring that in each subproblem the subsidy be effective at a different (qualitative) level. The three
In (P1), the subsidy is not effective (no scrappage): \( x_N \) is such that \( \alpha (1 - x_N - x_U) \geq S \). In (P2), the subsidy is effective but not maximum (partial scrappage): \( x_N \) is such that \( S \geq \alpha (1 - x_N - x_U) \) and \( S \leq \alpha (1 - x_N) \). In (P3), the subsidy is at least maximum (full scrappage): \( x_N \) is such that \( \alpha (1 - x_N) \leq S \). Together (P1)–(P3) cover all the feasible choices of production for the firm in (8). Thus, we can solve the overall problem by simply comparing profits in (P1)–(P3).

In order to facilitate the exposition, Figure 2.2 depicts the optimal production rules for (P1)–(P3) when standard monopoly profits are not attainable for the firm. Formally, this figure corresponds to the case of \( S \leq \alpha \frac{1+\epsilon}{2} \equiv S^P3 \), which can be expressed as \( \frac{1-\epsilon}{2} < 1 - S \frac{a}{\alpha} \) by rearranging terms. Notice that this inequality determines the ranking of intersections in the vertical axis of Figure 2.2. The meaning and derivation of this inequality will be clear when we solve (P3).

The problem of the firm in (P1) is to choose \( x_N \) that solves

\[
\max_{x_N} \quad ((1 - \alpha)(1 - x_N) + \alpha (1 - x_N - x_U) - c)x_N, \quad \text{(P1)}
\]

s.t. \( \alpha (1 - x_N - x_U) \geq S \).
The solution is given by
\[
x_N^{P1} = \begin{cases} 
\frac{1-c - \frac{\alpha}{2} x_U}{2} & \text{if } x_U < \tilde{x}_U^{P1}, \\
1 - x_U - \frac{S}{\alpha} & \text{otherwise},
\end{cases}
\] (16)
where the threshold \( \tilde{x}_U^{P1} \) is obtained from the constraint \( \alpha(1 - (\frac{1-c}{2} - \frac{\alpha}{2} \tilde{x}_U^{P1}) - \tilde{x}_U^{P1}) = S \) and is given by
\[
\tilde{x}_U^{P1} = \left( \frac{1 + c - \frac{2S}{\alpha}}{2 - \alpha} \right) \frac{1}{\alpha}.
\] (17)

This production rule says that if the constraint determining the effectiveness of the subsidy does not bind, the optimal level of output depends negatively on the stock of used cars, which follows from new and used cars being substitutes for consumers. Otherwise, production is given by \( x_N = 1 - x_U - \frac{S}{\alpha} \).

**Remark 1.** The following observations will be useful: (i) when \( x_U \leq \tilde{x}_U^{P1} \), the choice of production in (16) equals \( x_N^N(x_U) \) in the proposition, which corresponds to the case of no scrappage; (ii) the substitutability between new and used cars implies that profits in \( P1 \) are decreasing in \( x_U \); and (iii) if \( x_U = 0 \), the monopolist attains standard profits in \( P1 \). Since standard profits are an upper bound on the profit of the firm, we can establish that the solutions to \( P1 \) and the overall problem coincide if \( x_U = 0 \).

We turn next to the case of partial scrappage. The problem of the firm in \( P2 \) is to choose \( x_N \) that solves
\[
\max_{x_N} \quad ((1 - \alpha)(1 - x_N) + S - c)x_N,
\] s.t. \[ \alpha(1 - x_N - x_U) \leq S \text{ and } \alpha(1 - x_N) \geq S. \] (P2)
The solution is given by
\[
x_N^{P2} = \begin{cases} 
1 - x_U - \frac{S}{\alpha} & \text{if } x_U \leq \tilde{x}_U^{P2}, \\
\tilde{x}_U^{P2} & \text{if } x_U \geq \tilde{x}_U^{P2} \text{ and } S \leq \alpha(1-\frac{\alpha+c}{2-\alpha}) = \tilde{S}^{P2}, \\
1 - \frac{S}{\alpha} & \text{if } S \geq \tilde{S}^{P2},
\end{cases}
\] (18)
where
\[
x_N^{P2} = \frac{1 - \alpha - c + S}{2(1 - \alpha)}. \] (19)
The threshold stock \( \tilde{x}_U^{P2} \) in (18) is obtained by substituting \( x_N^{P2} \) in \( \alpha(1 - x_N^{P2} - \tilde{x}_U^{P2}) = S \) and rearranging terms, which is then given by
\[
\tilde{x}_U^{P2} = \frac{1 - \frac{1}{2}(\frac{1-\alpha-c}{1-\alpha}) - \frac{S}{\alpha}}{1 - \frac{1}{2}(\frac{1-\alpha-c}{1-\alpha})}. \] (20)
The threshold subsidy \( S^{P2} \), which separates the cases of partial and full scrappage, is obtained from the condition \( \alpha(1 - x_N^{P2}) = S \), where \( x_N^{P2} \) is given by (19).\(^{10}\) This production rule says that if the constraints do not bind, production is increasing in the subsidy. If the subsidy is large, the full scrappage constraint binds and determines production. Instead, if the subsidy is small and so is the stock of used cars, the unconstrained choice of production does not induce partial scrappage and the level of output is given by the partial scrappage condition.\(^{11}\)

**Remark 2.** There are two observations to emphasize: (i) \( x_N^{P2} \) in (19) equals \( x_N^{P}(S) \) in the proposition, which is the optimal production with partial scrappage, and (ii) \( S^{P2} \) equals \( S_{\text{max}}(x_U) \) in the proposition for \( x_U \geq x_U \).

Finally, the problem of the firm in (P3) is to choose \( x_N \) that solves

\[
\max_{x_N} \quad (1 - x_N - c)x_N, \\
\text{s.t.} \quad \alpha(1 - x_N) \leq S.
\]

The solution is given by

\[
x_N^{P3} = \begin{cases} 
\frac{1-c}{2} & \text{if } S > \alpha \frac{1+c}{\alpha} = S^{P3}, \\
1 - \frac{S}{\alpha} & \text{otherwise},
\end{cases}
\]

where the threshold subsidy \( S^{P3} \) is obtained from the constraint \( \alpha(1 - \frac{c}{2}) = S \). What this production rule says is that, for a sufficiently large subsidy, standard profits and output are attainable for the firm, since full scrappage is attainable without modifying its level of output. Otherwise, for smaller subsidies, production is given by the constraint imposing full scrappage.

**Remark 3.** Two observations are important: (i) \( S^{P3} \) in (21) equals \( S_M \) in the proposition, which is the threshold subsidy separating full scrappage without standard profits from full scrappage with standard monopoly profits; and (ii), which is directly implied by (i), for \( S \geq S^{P3} \), production in (P3) equals \( x_N^{P} \) in the proposition and equals \( x_N^{F}(S) \) otherwise.

We now turn to the second step and derive the solution to the (overall) maximization problem in (8). The proof works by comparing profits in each of the three constrained problems above. Notice that the three problems above identify the values for \( x_N^{P}, x_N^{P2}, \) and \( x_N^{M} \) in Proposition 2. ((P3) also identifies \( S_M \), and (P2) identifies \( S^{P2} \), which is one of the subsidies defining \( S_{\text{max}}(x_U) \) in the proposition.) It remains for us to compare profits in these three problems. This comparison

---

\(^{10}\)Note that \( x_N^{P2} < 0 \) if \( S \geq S^{P2} \).

\(^{11}\)Figure 2.2 is drawn by assuming that \( x_U^{P2} < x_U^{P1} \), which we prove in Step 2.
will identify the threshold values for the subsidies and stocks of used cars in the proposition.

**Step 2. Comparison: Solution to (8)**

We first consider $S \leq S_{P2}^{P2}$, where $S_{P2}^{P2}$ is the threshold separating the cases of partial and full scrappage in (P2) and is given by (18).

**Remark 4.** Assumption 1 implies $S_{P2}^{P2} \leq S_{P3}^{P3}$, where $S_{P3}^{P3}$, which equals $S_{M}$ in the proposition and is given in (21), is the threshold subsidy that makes standard monopoly profits feasible in (P3).\(^\text{12}\)

This remark allows us to identify the actual choices of production in (16), (18), and (21) for (P1)–(P3) if $S \leq S_{P2}^{P2}$, which are given by

$$
x_{N}^{P1} = \begin{cases} 
\frac{1-c - xu}{2} - \frac{xu}{a} & \text{if } xu < x_{N}^{P1}, \\
1 - xu - \frac{s}{a} & \text{otherwise},
\end{cases}
$$

$$
x_{N}^{P2} = \begin{cases} 
1 - xu - \frac{s}{a} & \text{if } xu \leq x_{N}^{P2}, \\
x_{N}^{P2} & \text{if } xu \geq x_{N}^{P2},
\end{cases}
$$

and

$$
x_{N}^{P3} = 1 - \frac{S}{a}.
$$

A useful observation is that producing $x_{N} = 1 - \frac{S}{a}$, which is optimal in (P3), is feasible in (P2) yet not chosen by the firm. Thus, profits in (P2) are greater than in (P3), and it suffices to compare profits in (P1) and (P2).

In making this comparison, we conjecture the following production rule: there exists a threshold stock in the stock of used cars

$$
\hat{x}_{U} = \frac{1 - c - \frac{1 - \alpha - c}{\alpha}}{\sqrt{1 - \alpha}},
$$

such that the solution to (8) is given by

$$
x_{N} = \begin{cases} 
x_{N}(x_{U}) & \text{if } x_{U} \leq \hat{x}_{U}, \\
x_{N}(S) & \text{otherwise},
\end{cases}
$$

First, we verify that $\hat{x}_{U}$ in (25) equates profits in (P1) and (P2) when we use these conjectured production rules. To see this, we equate profits in both problems and obtain

$$
((1 - \alpha)(1 - x_{N}^{N}) + \alpha(1 - x_{N}^{N} - x_{U}) - c)x_{N}^{N} = ((1 - \alpha)(1 - x_{N}^{P}) + S - c)x_{N}^{P}.
$$

\(^{12}\)To see this, compute $S_{P2}^{P2} - S_{P3}^{P3} = \frac{1-c}{2} - \frac{1-c}{2} = \frac{c}{2(2-a)}$, which is non-positive by Assumption 1.
Next, we use the first-order condition to each problem to substitute the terms in parenthesis and derive

\[(x^N)^2 = (1 - \alpha)(x^P)^2.\] (28)

Then, by substituting \(x^N \) and \(x^P \) for their respective expressions in (10), we get

\[
\left(\frac{1 - c}{2} - \frac{\alpha}{2}x_U\right)^2 = (1 - \alpha)\left(\frac{1 - \alpha - c + S}{2(1 - \alpha)}\right)^2, \tag{29}
\]

which, after rearranging terms and solving for \(x_U \), yields the threshold \(\tilde{x}_U \) in (25).

Second, to prove that this production rule indeed solves (8), we proceed in two additional steps:

1. We conjectured that the production choices used when equating profits in (P1) and (P2) were \(x^N \) and \(x^P \), respectively. To verify this conjecture, we must prove that \(\tilde{x}_U \in [\tilde{x}^P_U, \tilde{x}^P_U] \) so that the choices of production are optimal in (P1) and (P2), where \(\tilde{x}^P_U \) and \(\tilde{x}^P_U \) are given by (20) and (17), respectively. To show that \(\tilde{x}_U \geq \tilde{x}^P_U \), we compute the difference \(\tilde{x}_U - \tilde{x}^P_U \) and simplify terms, to obtain

\[
\tilde{x}_U - \tilde{x}^P_U = \frac{1 - c}{\alpha} - \frac{1}{\alpha\sqrt{1 - \alpha}} - \left(\frac{1 - \alpha - c + S}{2(1 - \alpha)}\right) \tag{30}
\]

\[
= (1 - c - \alpha + S)\left(\frac{1}{\alpha - \alpha\sqrt{1 - \alpha} + \frac{1}{2(1 - \alpha)}}\right) \tag{31}
\]

\[
= \frac{1 - c - \alpha + S}{2\alpha(1 - \alpha)}\left(2 - \alpha - 2\sqrt{1 - \alpha}\right). \tag{32}
\]

The quotient in (32) is positive by Assumption 1 and the term in parenthesis is increasing in \(\alpha \). It then suffices to show that, when \(\alpha = 0 \), the term in parenthesis is non-negative, which is immediate. To show that \(\tilde{x}^P_U \geq \tilde{x}_U \), we compute the difference \(\tilde{x}^P_U - \tilde{x}_U \) and simplify terms to obtain

\[
\tilde{x}^P_U - \tilde{x}_U = \left(1 + \frac{2S}{\alpha}\right)\frac{1}{2 - \alpha} - \left(\frac{1 - c}{\alpha} - \frac{1 - \alpha - c + S}{\alpha\sqrt{1 - \alpha}}\right) \tag{33}
\]

\[
= \frac{2 - \alpha - 2\sqrt{1 - \alpha}}{\alpha(2 - \alpha)(1 - \alpha + S)}, \tag{34}
\]

where the last equality is obtained by rearranging and simplifying terms. Since \(1 - \alpha - c > 0 \) and \(2 - \alpha > 2\sqrt{1 - \alpha} \), we obtain the desired result.

2. We now show that the profit functions in (P1) and (P2) only intersect once, which implies that \(\tilde{x}_U \) in (25) is uniquely defined. To see this, notice that, as shown in Remark 1(i), the profit in (P1) is decreasing in \(x_U \). Instead, the profit in (P2) is
constant for any \( x_U \geq \bar{x}_U^{P2} \), since the price floor binds, while for \( x_U \leq \bar{x}_U^{P2} \), output is determined by the constraint, and thus the profit is increasing in \( x_U \). Together, these properties imply that the profit functions can only intersect once.

**Remark 5.** Given a subsidy \( S \leq \bar{S}^{P2} \), the threshold \( \bar{x}_U \) in \((25)\) identifies the critical stock of used cars separating the cases of no scrappage and partial scrappage. Equivalently, this expression can be rearranged to identify its counterpart, which is the critical subsidy separating the cases of no scrappage and partial scrappage for a given stock \( x_U \). We can label this subsidy \( \bar{S} \) and obtain it by solving for \( S \) in \((29)\), which implies

\[
\bar{S} = \sqrt{1 - \alpha} (1 - c - \alpha x_U) - (1 - \alpha - c).
\]

The subsidy \( \bar{S} \) equals \( S_{\min}(x_U) \) in the proposition for the case of \( x_U \in [x_U, \bar{x}_U] \). The meaning of this interval will be clear as we proceed.

**Remark 6.** The threshold \( \bar{S} \) can be negative. If this is the case, profits in \((P2)\) are greater than in \((P1)\) for any \( S \geq 0 \). To put it differently, if \( \bar{S} \) is negative, a subsidy is not necessary to induce positive scrappage, since the secondary market is in excess supply although a subsidy is not being offered (that is, the non-negativity constraint on used car prices binds). This implies \( S_{\min}(x_U) = 0 \). To see when this might be the case, we equate \( \bar{S} \) in \((35)\) to zero and solve for \( x_U \), which yields

\[
\bar{x}_U = \frac{(1-c)\sqrt{1-\alpha} - (1-\alpha-c)}{\alpha\sqrt{1-\alpha}}.
\]

Thus, as seen in the proposition, for \( x_U \geq \bar{x}_U \), \( S_{\min}(x_U) = 0 \).

Lastly, we began Step 2 with the restriction \( S \leq \bar{S}^{P2} \), yet the proposition, see \((11)\) and \((12)\), is stated in terms of the threshold stock \( x_U \). To see how one is the counterpart of the other, note that, when \( S \leq \bar{S}^{P2} \), production equals \( x_N^P(x_U) \) if \( S \leq \bar{S} \), and equals \( x_N^P(S) \) if \( S \geq \bar{S} \). This implies the following transitions for the level of effectiveness of the subsidy: if \( S \leq \bar{S}^{P2} \), scrappage moves from not inducing any scrappage to inducing partial scrappage at \( \bar{S} \), which implies that \( x_N^P(S) \) is the optimal production at \( \bar{S}^{P2} \). (It remains for us to derive what is chosen for \( S > \bar{S}^{P2} \).) Another way to write this production rule is to say that if \( S \leq \bar{S}^{P2} \), \( S_{\min}(x_U) = \bar{S} \). Otherwise, if \( \bar{S} > \bar{S}^{P2} \), producing \( x_N^P(x_U) \) is optimal for all \( S \leq \bar{S}^{P2} \), which implies \( S_{\min}(x_U) \notin [0, \bar{S}^{P2}] \). However, it remains for us to derive what the minimum subsidy equals to. All we know is that, if it exists, it falls in the region \( S > \bar{S}^{P2} \).

We can now establish the link with \( x_U \). The condition comparing \( \bar{S} \) and \( \bar{S}^{P2} \) can also be expressed in terms of the threshold stock \( \bar{x}_U \). To obtain this threshold, we
equate the two subsidies $\bar{S}$ and $\bar{S}^{P2}$, given by (35) and (18), and obtain

$$\sqrt{1-\alpha(1-c-\alpha x_U)} - (1-\alpha-c) = \alpha \frac{1-\alpha + c}{2-\alpha}. \quad (37)$$

Then, we solve for $x_U$ and derive

$$x_U = \frac{1-c}{\alpha} - \frac{1-\alpha-c+\alpha^{1-\alpha+c}}{\alpha \sqrt{1-\alpha}}, \quad (38)$$

which is the threshold identified in the proposition. Therefore:

**Remark 7.** As stated in the proposition, if $x_U \leq x_U$, which is the counterpart of $\bar{S} > \bar{S}^{P2}$, the firm maximizes profits by choosing $x_N^U(x_U)$ for all $S \leq \bar{S}^{P2}$. On the other hand, if $x_U \geq x_U$, the firm maximizes profits by producing $x_N^U$ for all $S \leq \bar{S}$, while the firm chooses $x_N^U$ for $S \in [\bar{S}, \bar{S}^{P2}]$.

It remains for us to derive the optimal production rule for $S \geq \bar{S}^{P2}$, which we relegate to the appendix.

Figure 3 depicts the production of new cars and the consumption of used cars (which is obtained from $S = \alpha(1-c-U - x_N)$) as a function of $S$. The dashed line represents production and the solid line represents used car consumption (i.e., the volume of transactions in the secondary market). The case shown corresponds to $S_{max}(x_U) > S_{min}(x_U) > 0$.

### 2.2 Effects on Primary and Secondary Markets

We are now ready to characterize the effects of a subsidy on the automobile market. We establish results relative to the case without a subsidy. Following our previous notation, $\hat{p}_N$, $\hat{p}_U$, and $\hat{x}_N$ denote without-subsidy prices and production.

**Proposition 3.** Consider $x_U$ such that $S_{max}(x_U) > S_{min}(x_U) > 0$, which is equivalent to assuming $x_U \in [x_U, \bar{x}_U]$. Then, for any $S \in [S_{min}(x_U), S_{max}(x_U)]$, the following statements hold:

1. **Sales in the primary market increase:** $x_N^P(S) > \hat{x}_N$.
2. **If $S \leq \hat{p}_U$, the price of a new car decreases:** $p_N < \hat{p}_N$.
3. **The price difference between new and used cars decreases:** $p_N - S < \hat{p}_N - \hat{p}_U$.
4. **Sales in the secondary market decrease:** $c_U < x_U$. 

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(v) The total number of cars in use increases if $S < \hat{p}_U$ and decreases if $S \geq \hat{p}_U$: $x_N^P(S) + c_U > \hat{x}_N + x_U$ and $x_N^P(S) + c_U < \hat{x}_N + x_U$, respectively.

Proof. To show (i), it suffices to note that $x_N^P(S)$ is increasing in $S$ and $x_N^P(S_{\text{min}}) > \frac{1-c}{2} - \frac{\alpha}{2} x_U = \hat{x}_N$. To prove (ii), $S \leq \hat{p}_U$ and (i) imply $p_N = (1 - \alpha)(1 - x_N^P(S)) + S < \hat{p}_N$. To prove (iii), note that $p_N - S = (1 - \alpha)(1 - x_N^P(S)) < (1 - \alpha)(1 - \hat{x}_N) = \hat{p}_N - \hat{p}_U$, where the inequality is implied by (i). To prove (iv), since a subsidy $S \geq S_{\text{min}}(x_U)$ induces scrappage, $c_U < x_U$. To show (v), note that the total of cars in use can be obtained from the used car price equation $S = \alpha(1 - x_N^P(S) - c_U)$, which implies $x_N^P(S) + c_U = 1 - S/\alpha$. If $S = \hat{p}_U$, then $x_N^P(S) + c_U = 1 - \alpha(1 - \hat{x}_N - x_U)/\alpha$, and this implies $x_N^P(S) + c_U = \hat{x}_N + x_U$. By the same argument, $x_N^P(S) + c_U > \hat{x}_N + x_U$ if $S < \hat{p}_U$, and $x_N^P(S) + c_U < \hat{x}_N + x_U$ otherwise. \hfill $\square$

If $x_U$ is such that $S_{\text{min}}(x_U) = 0$ (or, what is the same, $x_U > \bar{x}_U$), then, for any $S \in [S_{\text{min}}(x_U), S_{\text{max}}(x_U)]$, items (i), (iii), (iv), and (v) in Proposition 3 apply. That is, the primary market expands, the price difference decreases, sales in the secondary market decrease, and the fleet of cars decreases.

This proposition has implications for the choices of consumers. Figure 4 depicts their choices for the case of $S_{\text{max}}(x_U) > S_{\text{min}}(x_U) > 0$, and compares them to the
case without a subsidy, which is \( S = 0 \). Notice that the consumer types are ordered from 1 to 0. If \( S < \hat{p}_U \), the price of a used car is lower with a subsidy and the size of the primary market expands. Thus, as shown in the top line of Figure 4, some consumers switch from buying a used car to buying a new one and some others switch from not buying a car to buying a used one. An implication is that all consumers are better off with the subsidy, since all prices decrease.\(^{13}\)

![Figure 4: Consumption choices for consumers with and without a subsidy when \( S_{\text{max}} > S_{\text{min}} > 0 \)](image)

Instead, if \( S > \hat{p}_U \), used car sales decrease while new car sales increase. That is, as shown in the bottom line of Figure 4, some consumers switch from purchasing a used car to purchasing a new one, and some switch from purchasing a used car to not purchasing any car. It is clear then that those consumers who keep buying a used car or switch to not purchasing are made worse off with the subsidy, since either they pay more for their car or they derive zero surplus. For those consumers who buy a new car, the effects are ambiguous. A sufficiently large subsidy can result in a higher price for new cars, which will make these buyers worse off. However, for all other subsidies, the price of a new car decreases. To see this, suppose \( S = \hat{p}_U \). Since production expands with the subsidy, it is still the case that the price of a new car \( p_N = (1 - \alpha)(1 - x_N) + S \) is (strictly) less than the price of a new car without a subsidy, which is \( \hat{p}_N = (1 - \alpha)(1 - \hat{x}_N) + \hat{p}_U \). Thus, if the surplus of the consumers who buy a new car decreases, it will be for a subsidy \( S \gg \hat{p}_U \). If this is the case, then all consumers will be made worse off with the subsidy.

**Corollary 1.** For all \( S \leq \hat{p}_U \), the subsidy makes all consumers better off. For \( S > \hat{p}_U \), the subsidy makes those consumers who buy a used car or switch to not buying a car worse off.

\(^{13}\)Here we do not consider the income effect for those consumers who own a used car and now obtain \( S < \hat{p}_U \). See the next section for results accounting for this effect.
3 Welfare Effects

The next proposition shows that aggregate welfare is maximized at the minimum subsidy. We prove this statement in two parts. We first show that if a subsidy is effective, aggregate welfare is decreasing in the subsidy. An implication is that the (effective) subsidy that yields most welfare is the minimum subsidy. We then show that the minimum subsidy yields more welfare than not offering any subsidy. The union of these two statements implies that the minimum subsidy maximizes aggregate welfare.

Proposition 4. Consider \( x_U \in [x_L, x_U] \), where \( x_U \) and \( x_U \) are defined as in Proposition 2 and correspond to \( S_{\text{max}}(x_U) > S_{\text{min}}(x_U) > 0 \). Then,

1. Aggregate welfare is decreasing in the amount of the subsidy.
2. Aggregate welfare is greater with a minimum subsidy than without a subsidy.

Proof.

1. If \( S < S_{\text{max}}(x_U) \), aggregate welfare is given by:

\[
W(S) = \int_{1-x_N}^1 (\theta - p_N) d\theta + \int_{1-x_N-c_U}^{1-x_N} (\alpha \theta - S) d\theta + Sx_U + (p_N - c)x_N - Ss_U,
\]

where the first two terms are the total surplus for those consumers purchasing new and used cars, the third term is the resale value of used cars (the revenue for their owners), the fourth term is the profit of the firm, and the last term is the total cost of the subsidy. By simplifying terms and computing the integrals, we can express (39) as

\[
W(S) = \frac{1}{2} \left( 1 - (1 - x_N)^2 \right) + \frac{\alpha}{2} \left( (1 - x_N)^2 - (1 - x_N - c_U)^2 \right) - cx_N
\]

\[
= x_N (1 - c - \frac{3\alpha}{2}) + \alpha (1 - x_N - \frac{\alpha}{2}) c_U. \tag{40}
\]

To compute \( W'(S) \), we must calculate \( \frac{\partial x_N}{\partial S} \) and \( \frac{\partial c_U}{\partial S} \). If the subsidy is not maximum, the effect of \( S \) on output is given by \( \frac{\partial x_N}{\partial S} = \frac{1}{2(1-\alpha)} \). Similarly, the effect of \( S \) on used car consumption is obtained from the condition \( c_U = 1 - x_N - \frac{S}{\alpha} \), which implies \( \frac{\partial c_U}{\partial S} = -\frac{1}{2(1-\alpha)} - \frac{1}{\alpha} \). Then,

\[
W'(S) = (1 - c - x_N - \alpha c_U) \frac{1}{2(1-\alpha)} + \alpha (1 - x_N - c_U) \left( -\frac{1}{2(1-\alpha)} - \frac{1}{\alpha} \right), \tag{41}
\]
which simplifies into

$$W'(S) = -\frac{1}{2}(1 - x_N^P - c_U) - \frac{c}{2(1 - \alpha)}.$$  

(42)

Since $\alpha(1 - x_N^P - c_U) = S > 0$ by (3), we obtain the desired result.

If $S \in [S_{\text{max}}(x_U), S_M]$, all used cars are scrapped, $c_U = 0$, and the welfare expression in (40) simplifies into

$$W(S) = x_N^F \left(1 - c - \frac{x_N^F}{2}\right).$$

(43)

Note that $x_N^F = 1 - \frac{S}{\alpha}$, which implies $\frac{\partial x_N^F}{\partial S} = -\frac{1}{\alpha}$. Thus,

$$W'(S) = (1 - c - x_N^F) \frac{\partial x_N^F}{\partial S}$$

$$= \left(1 - c - \left(1 - \frac{S}{\alpha}\right)\right) \left(-\frac{1}{\alpha}\right)$$

$$= -\frac{S}{\alpha} - \frac{c}{\alpha},$$

(44)

(45)

(46)

which is trivially negative. Notice that, since production is continuous at $S_{\text{max}}(x_U)$, the welfare function has a kink but is continuous. Finally, for $S \geq S_M$, production is independent of $S$ and so is welfare. Again, production is continuous, and thus welfare does not increase.

2. We now show that a minimum subsidy yields more welfare than not offering a subsidy. To do so, we use (40) evaluated at partial scrappage, which corresponds to

$$W(S) = \int_{1-x_N^P}^{1} \theta d\theta + \int_{1-x_N^P-c_U}^{1-x_N^F} \alpha \theta d\theta - cx_N^P.$$

(47)

Since $x_N^P < x_N^F$ and $x_N^P + x_U < x_N^F + c_U$ at the minimum subsidy, the gain in welfare from offering a minimum subsidy is

$$W(S_{\text{min}}(x_U)) - W(0) = \int_{1-x_N^P}^{1-x_N^P} (\theta - \alpha \theta) d\theta + \int_{1-x_N^P-c_u}^{1-x_N^P-x_u} (\theta - \alpha \theta) d\theta - c(x_N^P - x_N^F).$$

(48)

The first two integrals are positive, and the first term equals

$$\int_{1-x_N^P}^{1-x_N^P} (1 - \alpha) \theta d\theta,$$

(49)
which can be written as \((1 - \alpha)(x_N^p - \hat{x}_N)(1 - (\hat{x}_N + x_N^p)/2)\) after integrating terms. Since \(\hat{x}_N < x_N^p\), we get \((1 - (\hat{x}_N + x_N^p)/2) > 1 - x_N^p\), and thus the first integral is greater than \((1 - \alpha)(x_N^p - \hat{x}_N)(1 - x_N^p)\). Since the second term in (48) is positive, we can show that \(W(S_{\text{min}}(x_U)) - W(0) > 0\) if we show that \((1 - \alpha)(1 - x_N^p)(x_N^p - \hat{x}_N) - c(x_N^p - \hat{x}_N)\) is positive. To do this, we rearrange terms in this equation and substitute \(x_N^p\) to get \(((1 - \alpha)(1 - x_N^p) - c)(x_N^p - \hat{x}_N) = ((1 - \alpha - c - S)/2)(x_N^p - \hat{x}_N)\). Then, by evaluating \(S\) at \(S_{\text{min}}\), we have \(((1 - \alpha - c - S)/2)(x_N^p - \hat{x}_N) = (1 - \alpha - c - \frac{1}{2}\sqrt{1 - \alpha(1 - c - \alpha x_U)})(x_N^p - \hat{x}_N)\), where the second term in brackets is positive and the first term is increasing in \(x_U\). Thus, it suffices for us to show that this expression is positive when evaluated at \(x_U = 0\), which is

\[
((1 - \alpha - c - S)/2)(x_N^p - \hat{x}_N) = \frac{2 - \sqrt{1 - \alpha}}{2}(1 - \alpha - c)(x_N^p - \hat{x}_N).
\]

This expression is positive by Assumption 1 and \(x_N^p > \hat{x}_N\).

\[\square\]

**Corollary 2.** If \(S_{\text{min}}(x_U) = 0\) (i.e., \(x_U \geq \hat{x}_U\), \(S = 0\) maximizes welfare.

**Proof.** Since \(S_{\text{min}}(x_U) = 0\), any subsidy \(S \geq 0\) is effective. Then, by (i) in Proposition 4, the welfare-maximizing subsidy is \(S = 0\).

\[\square\]

An implication of the previous proposition is that the increase in the profit of the firm can be less than the cost of the subsidy itself. In particular, this statement holds trivially if the subsidy is minimum: the minimum subsidy does not increase profits for the firm, but the cost of financing it is positive since used cars are scrapped.

### 4 Robustness

To complete our analysis, this section explores the generality of our main result, established in Proposition 1, by considering three extensions to our model. In each extension, we show that the argument establishing \(S_{\text{min}} < \hat{\beta}_U\) remains valid.

**Generalization 1: Imperfect Competition**

Consider \(n < \infty\) identical firms choosing quantities simultaneously. Since the product is homogenous, the demand function for new cars is still given by (4), with the difference that now \(x_N\) is aggregate output, which is given by \(x_N = \sum_{i=1}^n x_N^i\). Thus, the problem of firm \(i \in \{1, \ldots, n\}\) is to choose production \(x_N^i\) that solves

\[
\max_{x_N^i} (p_N - c)x_N^i, \tag{51}
\]
where \( p_N \) is given by (4).

Without a subsidy, \( p_N = (1-\alpha)(1-x_N) + \alpha(1-x_N-x_U) \), and the solution to (51) is \( \hat{x}_N = (1-c-\alpha x_U - \sum_{j \neq i} \hat{x}_i) / \alpha \). Now, suppose \( S = \hat{\rho}_U \), where \( \hat{\rho}_U = \alpha(1-\alpha x_N-x_U) \), since all firms are identical. Then, firm \( i \) can achieve the same profit level as it did without a subsidy by simply producing \( \hat{x}_N \). However, now the firm can do better. Since it no longer internalizes the negative effect of its own production on new car prices through the secondary market, given by the term \( -\alpha x_N \), the firm wants to expand its production. Since the without-subsidy profit remains feasible, a revealed-preference argument implies that \( S = \hat{\rho}_U \) increases the profit of the firm. Therefore, \( S_{\text{min}} < \hat{\rho}_U \). Then, Figure 1 and Proposition 1 apply as well.

**Generalization 2: Vintages**

Consider now a monopolist choosing new car production while competing with a secondary market in which one, two, up to \( J \) period old cars are transacted. Since cars may depreciate with time, we let \( \alpha_j \) denote the quality of a car that is \( j \)-periods old for \( j = 0, 1, \ldots, J \). As before, let \( \alpha_0(= \alpha_N) = 1 \). Then, the quality of used cars can be ranked by their vintage with \( \alpha_0 = 1 \geq \alpha_1 \geq \ldots \geq \alpha_J \), where the inequalities indicate weak depreciation.

The derivation of the demand function for each car vintage is straightforward. Let \( \theta_j \) denote the consumer type who is indifferent between consuming car \( j \) and \( j-1 \) and is identified from the indifference condition \( \alpha_j \theta_j - p_j = \alpha_{j-1} \theta_j - p_{j-1} \). Then, by rearranging terms in this indifference condition, we obtain \( p_j = (\alpha_j - \alpha_{j-1}) \theta_j + p_{j-1} \). That is, the price of a car of vintage \( j \) equals the price of its neighboring lower-quality car plus the gain in utility that the consumer who is indifferent between consuming \( j \) and \( j-1 \) derives from choosing car \( j \). Therefore, sales of vintage \( j \) cars, which are \( x_j \), equal \( \theta_{j+1} - \theta_j \). Since this holds for all vintages, \( \theta_j = 1 - \sum_{i=0}^j x_i \). Then, by substituting these indifferent consumers into the inverse demand function for car \( j \), we can express the inverse demand function for vintage \( j \) as

\[
p_j = (\alpha_j - \alpha_{j-1})(1 - \sum_{i=0}^j x_i) + p_{j-1}.
\]

Then, by substituting recursively all the price equations in (52) for all vintages \( j \), we obtain that the inverse demand function for new cars is given by

\[
p_0 = \sum_{j=0}^J (\alpha_j - \alpha_{j-1})(1 - \sum_{i=0}^j x_i).
\]
Now, suppose that the subsidy aims to induce the scrappage of used cars of vintage \( h \) and offers an amount equal to the without-subsidy price of an \( h \)-th vintage car. Thus, \( S = \hat{p}_h \) and \( p_0 = \sum_{j=0}^{h} (\alpha_j - \alpha_{j-1})(1 - \sum_{j=0}^{h} x_j) + S \). It is then immediate that the lowest subsidy that induces the scrappage of the \( h \)-th vintage cars is less than \( S \). With \( S \) equal to \( \hat{p}_h \), the monopolist can attain the same profit by producing \( \delta_0 \). Nevertheless, the monopolist can do better. Since the subsidy eliminates the negative externality of production through the \( h \)-th and older vintage secondary markets, the monopolist wants to expand production and Figure 1 and Proposition 1 apply as well.

**Generalization 3: Forward-looking dynamics**

Consider an infinitely-lived monopolist choosing production in every period, \( t = 1, \ldots, \infty \). The monopolist is time consistent and consumers have perfect foresight. We let \( \delta \in [0, 1] \) denote the discount factor, and \( x_{N,t} \) and \( x_{U,t} \) denote the production of new cars and the stock of used cars in period \( t \). Then, the law-of-motion for the stock of used cars is given by \( x_{U,t} = x_{N,t-1} \), which is to say that cars fully depreciate after two periods of use.\(^{15}\)

With durability, the cut-off consumers who are indifferent between purchasing a new car and a used one, and purchasing a used car and not purchasing any car are identified by the two indifference conditions

\[
\theta_{N,t} - (p_{N,t} - \delta p_{U,t+1}) = \theta_{U,t} - p_{U,t} \quad (54)
\]

and

\[
\theta_{U,t} - p_{U,t} = 0, \quad (55)
\]

where \( p_{N,t} - \delta p_{U,t+1} \) is the implicit rental price of a new car at date \( t \). (The assumptions that consumers’ preferences are quasi-linear in income and there are no transaction costs in the primary or secondary markets imply that the implicit rental price is the relevant one.) Then, the inverse demand function for new and used cars are, respectively, \( p_{N,t} = (1 - \alpha)(1 - x_{N,t}) + p_{U,t} + \delta p_{U,t+1} \), and \( p_{U,t} = \alpha(1 - x_{N,t} - x_{U,t}) \), where \( x_{U,t} = x_{N,t-1} \). The subsidy is effective if \( \alpha(1 - x_{N,t} - x_{U,t}) \leq S \).

In defining the dynamic maximization problem of the firm, production may depend on the entire history. A natural assumption is to restrict production choices to depend only on the past production that still transacts in the secondary market. This corresponds to the standard Markov assumption, which allows the choices

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\(^{14}\)See Esteban (2003) for the complete derivation and characterization of the equilibrium dynamics with imperfect competition, yet without a subsidy.

\(^{15}\)Partial depreciation can be accommodated with a minor modification.
of the firm to depend only on the payoff-relevant variables. Given the assumed
depreciation schedule, these correspond to \(x_{N,t-1}\). Then, the problem of the time-
consistent monopolist can be written as the dynamic-programming problem

\[
V(x_U) = \max_{x_N} \left( (1 - \alpha)(1 - x_N) + \alpha(1 - x_N - x_U) + \delta(1 - g(x_N) - x_N) - c \right) x_N
+ \delta V(x_N),
\]

(56)

where \(V(\cdot)\) is the value function and \(g(\cdot)\) is the equilibrium decision rule.

It is easy to see that, if the equilibrium decision rule in (56) is linear in \(x_U\) and the
value function is quadratic, the maximization problem of the monopolist is linear-
quadratic in the control and stock. Thus, the choice of production solving (56) is
a linear function \(x_N = g(x_U) = a_0 + a_1 x_U\), where \(a_0\) and \(a_1\) are its coefficients.\(^\text{16}\)
Since \(a_0\) equals the optimal production if the stock is zero, we have \(a_0 > 0\). Since
new and used cars are substitutes in consumption, we have \(a_1 < 0\). (We can also
show that \(a_1 > -1\), which says that production slowly converges back to its steady
state level once the subsidy is discontinued.\(^\text{17}\) This implies that, although a subsidy
changes the distribution of vehicle holdings, its effects are transitory.)

Now, consider a one-time unanticipated scrappage subsidy. Suppose that the
subsidy equals \(\tilde{\rho}_U\), which is the without-subsidy price. As before, the firm can
achieve the same profit level as it did without a subsidy by choosing \(\hat{x}_N\), which
solves (56) and has new car price \(p_N = (1 - \alpha)(1 - \hat{x}_N) + S + \delta \alpha(1 - \hat{x}_N - (a_0 +
a_1 \hat{x}_N))\).\(^\text{18}\) That is, with \(\hat{x}_N\), prices are the same in all markets and the profit of the
firm remains unchanged. However, the firm can do better by expanding production

\(^{16}\)The expressions for \(a_0\) and \(a_1\) can be obtained by computing the first-order condition to the
firm's dynamic programming problem in (56), which is

\[
(1 - \alpha)(1 - 2x_N) + \alpha(1 - 2x_N - x_U) + \delta \alpha(1 - a_0 - 2a_1 x_N - 2x_N) - c + \delta V'(x_N) = 0,
\]

(57)

where, by the envelope theorem,

\[
V'(x_N) = -\alpha(a_0 + a_1 x_N).
\]

(58)

Solving for \(x_N\) in these expressions, we obtain

\[
x_N = \frac{1 + \alpha \delta - c - 2 \alpha \delta a_0}{2 + 2 \alpha \delta + 3 \alpha \delta a_1} x_U,
\]

(59)

\(^{17}\)For a full derivation, see Esteban (2003).

\(^{18}\)The quantitative benefit of expanding production might be limited in a dynamic model since
the used car market in \(t + 1\) can be driven into excess supply. That is, the inverse demand function
for new cars is given by \(p_N = (1 - \alpha)(1 - \hat{x}_N) + S + \delta \max\{\alpha(1 - \hat{x}_N - (a_0 + a_1 \hat{x}_N)),0\}\), where the
since it no longer internalizes the negative effect of production on new car prices via the secondary market, which is given by the term $-\alpha x_{N,t}$. Then, the feasibility of the without-subsidy profit level completes the argument and Figure 1 and Proposition 1 also apply.

5 Conclusion

We have shown that market power has non-trivial effects on determining the effectiveness of a scrappage subsidy and its resulting implications on car markets. One key result shows that a subsidy can induce scrappage even if it pays less for a used car than its without-subsidy price. This result is robust to a number of generalizations. We also show that the smallest subsidy that is effective in inducing scrappage is also the one that maximizes welfare. Our results contrast with the predictions derived from a model with perfect competition.

There are relevant extensions to this work. On the one hand, the results can be compared to a framework in which subsidies tie the monetary incentive to the replacement vehicle being new. On the other hand, they can be compared to other forms of regulation that also aim at controlling emissions such as different taxation schemes for new and used automobiles.

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$max$ operator is the price of a used car in $t+1$, which must be non-negative. The non-negativity of prices imposes a bound on the benefits of expanding production if its choice of production makes the non-negativity constraint on $t+1$’s used car prices bind. Nonetheless, the incentive to expand production persists.
A Appendix

Proof of Proposition 2

What follows completes the proof of Proposition 2 by considering \( S \geq S^{P2} \). As previously shown, \( S^{P2} \) is a cutoff in determining when partial scrappage is feasible in (P2).

In the main proof, we showed that \( S^{P2} \leq S^M \), where \( S_M \) equals \( S^{P3} \), which is the threshold subsidy determining whether standard monopoly profits are feasible for the firm. Thus, if \( S \geq S_M \), it is straightforward to see that (P3) results in higher profit than (P1) and (P2), since it achieves standard profits. Therefore, the following is immediate.

Remark 8. For \( S \geq S_M \), \( x_N^M \) in the proposition solves the overall maximization problem of the firm as seen in the proposition.

Then, what remains is to consider \( S \in [S^{P2}, S_M] \). Recall first that if \( \bar{S} \leq S^{P2} \) (or \( x_U \geq x_U \)), the monopolist’s choice of production is given by \( x_N^S \) for \( S \leq \bar{S} \) and \( x_N^P \) for \( S \in [\bar{S}, S^{P2}] \). Thus, \( S \) is the threshold subsidy separating the cases of partial and no scrappage. A critical observation is that the firm’s profit is increasing in the amount of the subsidy. Therefore, if \( x_N^P \) is preferred to \( x_N^S \) for \( S \in [\bar{S}, S^{P2}] \), the same ranking between \( x_N^P \) and \( x_N^S \) must be preserved for \( S \in [S^{P2}, S^M] \), which establishes the choice between (P1) and (P2). Further, for \( S \in [S^{P2}, S^M] \), the solutions to (P2) and (P3) coincide for this range of subsidies, with the firm choosing \( x_N^P \). Then, \( x_N^P \) is the optimal choice of production in the overall problem if \( \bar{S} \leq S^{P2} \), which is the same as \( x_U \geq x_U \).

Remark 9. In summary, if \( x_U \in [x_U, x_U] \), production equals \( x_N^S \) for \( S \leq \bar{S} \), equals \( x_N^P \) for \( S \in [\bar{S}, S^{P2}] \), equals \( x_N^F \) for \( S \in [S^{P2}, S^M] \), and equals \( x_N^M \) otherwise. Thus, \( S_{min}(x_U) = \bar{S} \), \( S_{max}(x_U) = S^{P2} \), and \( S_{min}(x_U) < S_{max}(x_U) \). It can also be the case that \( x_U \geq x_U \). As previously derived, \( S_{min}(x_U) = 0 \) for this case and, as just shown, \( S_{max}(x_U) = S^{P2} \).

The proof in the main text also shows that, when \( x_U \leq x_U \) or \( S > S^{P2} \), production equals \( x_N^S(x_U) \) for all \( S \leq S^{P2} \). It remains for us to derive the firm’s choice when \( S \in [S^{P2}, S^M] \). For this range of subsidies, the solutions to (P1)–(P3) are given by

\[
\begin{align*}
x_N^{P1} &= \begin{cases} 
\frac{1-c}{2} - \frac{a}{2} x_U & \text{if } S \leq S^{P1}, \\
1 - x_U - \frac{S}{\alpha} & \text{otherwise},
\end{cases} \\
x_N^{P2} &= \frac{S}{\alpha}
\end{align*}
\]
and

\[ x_{N}^{P3} = 1 - \frac{S}{\alpha} \]  \hspace{1cm} (62)

Notice that the choices in (P2) and (P3) are identical, and thus the profit that the firm earns is the same. An important observation is that producing \( 1 - x_{U} - \frac{S}{\alpha} \) is feasible in (P2), yet the monopolist chooses \( 1 - \frac{S}{\alpha} \). Therefore, the choices of the monopolist are either \( x_{N}^{P1} \) in (P1) or \( x_{N}^{P3} \) in (P3), which equals \( x_{N}^{P} \) in the proposition.

We conjecture that there exists a threshold subsidy, which we denote by \( \hat{S} \), such that \( \hat{S} \in [\hat{S}^{P2}, \hat{S}^{M}] \) and this subsidy equates profits in (P1) and (P3), with production given by \( \frac{1-c}{2} - \frac{x_{U}}{2} \) (or \( x_{N}^{P} \)) and \( 1 - \frac{S}{\alpha} \), respectively, and is obtained from

\[
((1 - \alpha) (1 - x_{N}^{P1}) + \alpha (1 - x_{N}^{P1} - x_{U}) - c) x_{N}^{P1} = \left(1 - \left(1 - \frac{\hat{S}}{\alpha}\right) - c\right) \frac{\hat{S}}{\alpha}. \hspace{1cm} (63)
\]

To solve for \( \hat{S} \), we substitute on the left-hand side of the equation the first-order condition to (P1), substitute \( x_{N}^{P1} \) for its expression, and rearrange terms, to derive

\[
\left(\frac{1-c}{2} - \frac{\alpha}{2} x_{U}\right)^{2} = \left(\frac{\hat{S}}{\alpha} - c\right) \left(1 - \frac{\hat{S}}{\alpha}\right). \hspace{1cm} (64)
\]

Then, by solving for \( \hat{S} \) in the equation above, we obtain

\[
\hat{S} = \frac{\alpha}{2} + \frac{1+c}{2} - \alpha \sqrt{\left(\frac{1-c}{2}\right)^{2} - \left(\frac{1-c}{2} - \frac{\alpha}{2} x_{U}\right)^{2}}. \hspace{1cm} (65)
\]

which equals \( S_{\min}(x_{U}) \) in the proposition for the case of \( x_{U} \leq x_{L} \).

It remains to be shown that \( \hat{S} \in [\hat{S}^{P2}, \hat{S}^{M}] \) for all \( x_{U} \leq x_{L} \). Notice that (65) is continuous and decreases in \( x_{U} \). With some simple mathematical computations, we can show that \( \hat{S} = \hat{S}^{P2} \) if \( x_{U} = x_{L} \). Further, if \( x_{U} \to 0 \), we get \( \hat{S} \to \hat{S}^{M} \). Together, these statements establish our claim that \( \hat{S} \in [\hat{S}^{P2}, \hat{S}^{M}] \). Lastly, we may want to gauge some intuition for why \( \hat{S} \to \hat{S}^{M} \) if \( x_{U} \to 0 \). Notice that, if \( x_{U} \to 0 \), the firm can achieve standard profits by not inducing any scrappage. Therefore, the only way in which a firm will induce scrappage is if the profit accrued by doing so is the standard monopoly profit, which corresponds to offering a subsidy \( S \to \hat{S}^{M} \).

**Remark 10.** For \( \hat{S} \geq \hat{S}^{P2} \) (or \( x_{U} \leq x_{L} \)), production equals \( x_{N}^{P} \) for \( S \leq \hat{S} \), equals \( x_{N}^{P} \) for \( S \in [\hat{S}, \hat{S}^{M}] \), and equals \( x_{M}^{P} \) for \( S \geq \hat{S}^{M} \). Thus, \( \hat{S} = S_{\min}(x_{U}) = S_{\max}(x_{U}) \) if \( x_{U} < x_{L} \), as stated in the proposition.
References


