RESIDENTIAL LAND USE
The Continous Case *

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This note presents a theorem on the existence and spatial properties of a unique competitive equilibrium of a pure exchange economy with a continuum of locations, a continuum of identical consumers and a single market place.

1. Introduction

Schweizer, Varaiya and Hartwick (1976) have given the sketch of a proof of a theorem on the existence of a compensated equilibrium for a wide class of spatial production economies, which recognize the locational indivisibility that precludes a consumer to live at more than one location. Space is treated as a discrete variable. But since fractional assignments of consumers to locations are ruled out, it is easy to construct an example with a finite number of both consumers and locations for which the set of competitive equilibria is empty. Thus, these authors note that their results are good approximations if the number of households (of each of several types) is 'large' relative to the number of available locations.

We depart from that framework for the following reasons: (1) We believe that the treatment of space as a continous variable deserves independent consideration. (2) Once one assumes a continuum of locations, it is natural to assume a continuum of consumers to handle the problem that lead to the somewhat imprecise 'largeness' assumption. (3)

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The assumption of a single market place permits to investigate some of the spatial properties of an equilibrium, which include those advanced by the early urban economists in their partial equilibrium analysis of residential land use. (4) A model with a continuum of identical consumers which includes a mobile homogeneous commodity and a locationally fixed commodity susceptible of continous differentiation, presents some peculiar features, as well as a number of difficulties absent in the discrete case, whose study is of interest in its own right.

2. The model and the main result

Think of the simplest economic environment of the Von Thunen type. Namely, a pure exchange economy with the following features. (1) There are two commodities: land uniformly distributed with density L equal to one over the set $\mathcal{K}$ of available locations; and a homogeneous and mobile commodity, called the consumption good. (2) A consumer's role is to choose both a location k, and a commodity bundle (s, x) consisting of a quantity of land, s, and a quantity of the consumption good, x. There is a continuum V of identical consumers (which is taken to be the unit interval), and each of them is described by his consumption set C, a continous utility function U representing his preferences for the two commodities, his endowment ω of the consumption good, and a function θ defined on $\mathcal{K}$ giving the land he owns at each location. (3) Consumers are assumed to make the same number of trips to the CBD where all exchange takes place. Transportation costs T in terms of the consumption good, depend only (and linearly) on distance from the CBD. Thus, $\mathcal{K}$ can be taken to be a subset $[0, \bar{k}]$ of R_+, where 0 represents the CBD and $\bar{k}$ represents a location beyond which it is not possible to live.

Definition 1. A *pure exchange spatial economy e,* is a four-tuple $\langle \mathcal{K}, L, T, (C, U, \omega, \theta)\rangle$, where

(1) $\mathcal{K} = [0, \bar{k}]$,
(2) $L: \mathcal{K} \to R_+$ such that $L(k) = 1$ for all $k \in \mathcal{K}$,
(3) $T: \mathcal{K} \to R_+$ such that $T(k) = tk$, $t > 0$, for all $k \in \mathcal{K}$,
(4) $C \subset R^2$, $U: C \to R$, $\omega \in R$, and $\theta: \mathcal{K} \to R_+$ such that $\theta(k) = 1$ for all $k \in \mathcal{K}$.

The *total endowment* (or the *mean supply*) *of the consumption good* of an economy e is equal to $\int_V \omega \, dv = \omega$.

The distribution of consumers over space can be characterized by a population density function g. Likewise, the distribution of commodity bundles over consumers can be characterized by a function (s, x) which gives the commodity bundle assigned to each location where the density is greater than zero.

Definition 2. An *allocation* of an economy e is a triple $\langle g, K, (s, x)\rangle$, where

(1) $g: \mathcal{K} \to R_+$ is an integrable function satisfying $\int_{\mathcal{K}} g(k) = 1$,
(2) $K = \{k \in \mathcal{K}: g(k) > 0\}$, and
(3) $(s, x): K \to C$.

A *normalized price system* $\langle r, 1\rangle$ consists of an integrable rent function $r: \mathcal{K} \to R_+$ which gives the price of a unit of land at each location, and the price of the consumption good which is equal to 1. The *total land rent* π is equal to $\int_{\mathcal{K}} r(k)$. The *gross wealth* of a consumer is equal to the value of his endowment of the consumption good, plus the value of the land he owns $\tau = \int_{\mathcal{K}} r(k)\theta(k) = \int_{\mathcal{K}} r(k) = \pi$. His *budget set at a location* k is the subset $B(r, 1, k) = \{(s, x) \in C: r(k)s + x \leqslant \omega - tk + \pi\}$.

In this model there is no interdependence of consumers' preferences, there are no direct preferences for distance, and each consumer is required to live at a single location. Therefore, the constraint set in a typical consumer's choice problem is the union of all the budget sets at each distance: $\cup_{k \in \mathcal{K}} B(r, p, k)$.

Definition 3. Let $\langle r, 1\rangle$ be a price system. The point $(k^*, (s^*, x^*))$ of $\mathcal{K} \times C$ is said to be an *equilibrium action relative* to $\langle r, p\rangle$ if

$$U(s^*, x^*) \geqslant U(s, x) \qquad \text{for all } (s, x) \in \bigcup_{k \in \mathcal{K}} B(r, p, k)$$

and

$$(s^*, x^*) \in B(r, p, k^*).$$

Let $A(\langle r, p\rangle)$ be the set of all equilibrium actions relative to the price system $\langle r, p\rangle$.

Definition 4. A *competitive equilibrium* of an economy e is a pair consisting of a price system $\langle \hat{r}, 1\rangle$ and an allocation $\langle \hat{g}, \hat{K}, (\hat{s}, \hat{x})\rangle$ such that

(1) $(k, (\hat{s}, \hat{x})(k)) = A(\langle \hat{r}, 1\rangle)$ for all $k \in \hat{K}$,
(2) $\hat{s}(k)\hat{g}(k) = 1$ for all $k \in \hat{K}$, and
(3) $\int_{\hat{K}} (\hat{x}(k) + tk)\hat{g}(k) = \omega$.

We now state the main theorem which provides sufficient conditions for the existence of a competitive equilibrium and lists several of its properties.

Theorem. Any economy $e = \langle \mathcal{K}, L, T, (C, U, \omega, \theta)\rangle$ *satisfying*

(A.1) there is a real number s_0 *with* $0 < s_0 < \bar{k}$ *such that* $C = \{(s, x) \in R^2_+ : s \geq s_0, x \geq 0\}$. *The zero utility level is assigned to the lower bound of the consumption set, i.e.,* $U(s_0, 0) = 0$,

(A.2) if $(s, x) \geq (s', x')$ *and if* $U(s', x') > 0$, *then* $U(s, x) > U(s', x')$,

(A.3) for any distinct (s, x) *and* (s', x') *such that* $U(s, x) = U(s', x') > 0$ *and for any* $\lambda \in (0, 1)$

$$U(\lambda s + (1-\lambda)s', \lambda x + (1-\lambda)x') > U(s', x'),$$

(A.4) indifference curves have no kinks and do not intersect the edges of the consumption set, and

(A.5) $\omega > ts_0/2$,

has a unique competitive equilibrium $\langle\langle \hat{r}, 1\rangle, \langle \hat{g}, \hat{K}, (\hat{s}, \hat{x})\rangle\rangle$. *Moreover:*

(1) the competitive equilibrium allocation is Pareto optimal,

(2) the set $\hat{K}$ *of locations occupied by consumers is an interval* $[0, \hat{y})$ *with* $\hat{y} \leq \bar{k}$,

(3) the population density function $\hat{g}$ *is strictly decreasing on* $\hat{K}$, *and hence the functions* $\hat{s}$ *and* $\hat{x}$ *are, respectively, strictly increasing and strictly decreasing on* $\hat{K}$,

(4) the function $\hat{g}$ *is continous on* $\mathcal{K}$, *and the function* $(\hat{s}, \hat{x})$ *is continous on* $\hat{K}$,

(5) the function $\hat{x}$ *is bounded, and*

(6) the rent function $\hat{r}$ *is positive, bounded above and strictly decreasing on* $\hat{K}$, *equal to zero on* $\mathcal{K} - K$, *and continous on* $\mathcal{K}$.

3. Comments

(1) The assumption (A.1) on the indispensability of a minimum of land s_0, is used to guarantee that the equilibrium population density function is bounded above. The requirement $s_0 < \bar{k}$ is a necessary condition for the existence of an allocation which yields a utility level greater than zero to a set of consumers of positive measure.

(2) The usual assumption on the strict positivity of commodity endowments, becomes here (A.5) which requires the mean endowment of the consumption good to exceed the mean transportation costs in the city of the minimum possible size. Under (A.1) and (A.2), (A.5) is a necessary and sufficient condition for the existence of an allocation which yields a utility greater than zero to a set of consumers of positive measure.

(3) The proof of the theorem proceeds as follows. Since all consumers are identical, in equilibrium they should enjoy the same utility. On the other hand, under (A.1) and (A.2) every competitive equilibrium allocation is Pareto optimal. Thus, we focus on Pareto optimal equal treatment allocations. Under (A.1), (A.2) and (A.5), there exists a maximum utility level $\hat{u} < \infty$ for which the set $\mathscr{C}_{\hat{u}}(e)$ of equal treatment feasible allocations of an economy e is non-empty. Moreover, every allocation in $\mathscr{C}_{\hat{u}}(e)$ is Pareto optimal. Under (A.3), that set turns out to be a singleton. The final step, is the construction of a unique price system which supports such allocation as a competitive equilibrium. To simplify this construction, we include here (A.4) which was not assumed in Ruiz-Castillo (1978).

(4) Next we want to single out two difficulties which are absent in the discrete case.

(A) Even though the mean supply of the consumption good is finite, a sufficiently 'small' subset of consumers might be assigned unbounded quantities of that good at an equal treatment allocation. This would pose two problems: Firstly, to establish that every allocation in $\mathscr{C}_{\hat{u}}(e)$ is Pareto optimal, one needs to show that if we start from an equal treatment allocation which yields a utility level $\tilde{u} \geqslant \hat{u}$ and involves an excess supply of the consumption good, then it is possible to reallocate such an excess supply so as to increase the utility level of every consumer up to a level $\dot{u} > \tilde{u} \geqslant \hat{u}$. To be able to do this, one needs the initial assignment of the consumption good to be bounded above. Secondly, it is impossible to support a Pareto optimal allocation by a normalized price system with a rent function bounded above, unless the demand for the consumption good is itself bounded above.

(B) Suppose that the indifference curve corresponding to $\hat{u}$ is asymtotic to the s-axis at some level $a \geqslant 0$. Then, as we approach the outer boundary $\hat{y} \leqslant \bar{k}$ of the residential zone actually occupied, consumers might be assigned commodity bundles in which the quantity of land tends to infinity. In this case, one has to ensure that, as $k \to \hat{y}$, the price of land approaches zero while the wealth net of transportation costs approaches $a \geqslant 0$.

(5) At this point, it is pertinent to compare this model with that of Ripper and Varaiya (1972). These authors establish the existence of a compensated equilibrium for a very general class of spatial production economies with a continuum of locations and a continuum of consumers of different types. They allow for some non-convexities of preferences. However, they assume that lower contour sets are compact. This means that every indifference curve intersects both edges of the consumption set. Thus, each consumer's expenditure minimization problem is guaranteed to have a solution for every price system. Moreover, this assumption rules out the two difficulties just discussed: at every equal treatment feasible allocation both the demand for land and the consumption good would be necessarily bounded.

References

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