The Asymptotic Nucleolus of Large Monopolistic Market Games[1]

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We study the asymptotic nucleolus of differentiable monopolistic market games in continuum economies with a finite number of traders' types, and show that, under appropriate assumptions, it is the center of symmetry of the subset of the core in which all the monopolists receive the same payoff. Thus, the nucleolus discriminates the traders in the atomless sector, whereas the competitive equilibrium does not. Moreover, if there is a single syndicated atom and a finite number of atomless sectors, the syndicate is treated more favorably under the asymptotic nucleolus than under the Shapley value associated with the pure monopolistic market. *Journal of Economic Literature* Classification Numbers: C71, D40.

1. INTRODUCTION

Aumann [2] suggested than an appropriate model of an oligopolistic economy is one in which the set of traders consists of a few large traders

[1] We thank a referee for helpful comments. Part of this work was done while Einy and Shitovitz visited the Department of Economics of Universidad Carlos III de Madrid. Einy acknowledges the financial support of the Universidad Carlos III de Madrid. Shitovitz acknowledges the support of the Spanish Ministry of Education, Grant SAB98-0059. Moreno acknowledges the support of the Spanish Ministry of Education (DGICYT), Grant PB96-0118.

and a continuum of small traders (such an economy is usually called a mixed economy). Following this suggestion, Shitovitz [25] analyzed an imperfect competition model which consists of one monopolist and a finite large number of traders each of which has a small impact on the market by associating a mixed economy with it. Since the Shitovitz [25] paper, many works on the subject have been written (for a comprehensive survey see Gabszewicz and Shitovitz [10]).

Shapley and Shubik [24] associated with every finite production economy a coalitional game with a transferable utility which is called the market game. In order to define their game, the different commodities in the economy are interpreted as production factors and the individuals are interpreted as owners of production factors. The utility functions can be viewed (under standard normalization) as production functions. The worth of a coalition in the Shapley–Shubik market game is then defined as the maximum production that its members can produce under the feasibility constraint. Aumann and Shapley extended the definition of a market game to economies with a continuum of traders (for a detailed discussion of large market games the reader is referred to Chapter VI of Aumann and Shapley [5]).

One of the fundamental concepts in economic theory and the theory of cooperative games is that of the core. As is shown in Shitovitz [25], in most monopolies, the core has a range which is large enough to contain both the competitive equilibria on the one hand and the allocations that express the exploitation of the small traders on the other. In the theory of finite cooperative games with transferable utility there are two fundamental solution concepts which assign to every game a unique outcome, the Shapley value [23] and the nucleolus [21]. In the asymptotic approach to solution concepts of games with an infinite set of players, one studies the asymptotic behavior of the solution of sequences of games with a finite set of players which approximate the original game (see, for example, Kannai [14] and Aumann and Shapley [5] in the context of the Shapley value).

Gusenerie [12] and Gardner [11] investigated the asymptotic behavior of the Shapley value in mixed markets. Legros [15] deals with the nucleolus of a bilateral market with two complementary commodities. The nucleolus of a finite coalitional game can be interpreted as the imputation which minimizes the greatest dissatisfaction of any coalition in the game and it implements in a certain respect a notion of justice due to Rawls [20]. Aumann [3] identifies the monopoly power in a mixed market with what he can prevent other coalitions from getting; that is, the strength of a monopolist bargaining power relies on the harm he can cause by refusing to trade. Thus, the monopoly power is measured by the difference of what others can get with him, and they can get without him (see Aumann [3, p. 10]). From the definition of the kernel of a game and the fact that payoff vectors in the kernel preserve the desirability relation between players (player i is at least as desirable as player j if player i's marginal contribution to every coalition which does not contain i and j is at least that of player j), it follows that the kernel may reflect the monopoly power (as interpreted above) in mixed market games. Since the nucleolus lies in the kernel, it is useful to study how it reflects the monopoly power in mixed market games.

The purpose of this work is to study the asymptotic nucleolus of mixed differentiable market games, and to compare it to the core, the competitive equilibrium, and the Shapley value. Bird [6] extends Schmeidler's definition [21] of the nucleolus to games with an infinite set of players. The problem with Bird's nucleolus is that in some cases it does not exist and also when it does exist it may yield a very large set of outcomes. In the class of games which is studied in this work Bird's nucleolus yields the core which is large, while the asymptotic nucleolus is a unique point in the core of the game.

Our mathematical model consists of a measure space of players in which the small players form a non-atomic part and the large players (i.e., the monopolists) are atoms. We assume that every atom has a corner on one of the commodities in the economy (see (4.2) in Section 4). In this model the competitive equilibrium does not reflect at all the power of discrimination of the monopolists. We show that in this model the asymptotic nucleolus is the center of symmetry of the subset of the core in which all the monopolists receive the same payoff. Since in our model each member of the core expresses some degree of exploitation of the small traders by the monopolists (that is, every payoff distribution in the core gives to a coalition of small traders a payoff which does not exceeds its payoff in the competitive distribution—this follows from Theorem 3.2 and also from Theorem A in Shitovitz [25]), this implies that the asymptotic nucleolus is the average of discrimination (of the small traders with respect to their position in the competitive distribution) in this imperfect competitive environment. Moreover, in the case of one syndicated atom and a finite number of atomless sectors, we show that the syndicate is more favored under the asymptotic nucleolus than under the asymptotic Shapley value. Similar results are have been found in finite coalitional games in which some of the players have veto power; usually in these games the nucleolus is less egalitarian than the Shapley value. In voting games with veto players, for example, the nucleolus is dominated (in the sense of Lorenz) by the Shapley value.

The paper is organized as follows: in Section 2 we define the basic notions which are relevant to our work. In Section 3 we prove some general results which will be used in Section 4 to derive our results on market games. In Section 4 we state and prove the above mentioned results on the asymptotic nucleolus of mixed market games. In Section 5 we compare the asymptotic nucleolus in our model with the asymptotic Shapley value.

2. PRELIMINARIES

In this section we define the basic notions which are relevant to our work. Let (T, Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. We refer to the member of T as *players* and to those of Σ as *coalitions*. A *coalitional game*, or simply a *game* on (T, Σ), is a function $v: \Sigma \to \Re$ with $v(\phi) = 0$. If T is finite and $\Sigma = 2^T$ is the set of all subsets of T, the game v will be called a *finite game*. A game v is *superadditive* if $v(S_1 \cup S_2) \geqslant v(S_1) + v(S_2)$ whenever S_1 and S_2 are disjoint coalitions. A *payoff measure* in a game v on (T, Σ) is a bounded finitely additive measure $\xi: \Sigma \to \Re$ which satisfies $\xi(T) \leqslant v(T)$.

We denote by $ba = ba(T, \Sigma)$ the Banach space of all bounded finitely additive measures on (T, Σ) with the variation norm. The subspace of ba which consists of all bounded countably additive measures on (T, Σ) is denoted by $ca = ca(T, \Sigma)$. If λ is a measure in ca then $ca(\lambda) = ca(T, \Sigma, \lambda)$ denotes the set of all members of ca which are absolutely continuous with respect to λ. If A is a subset of an ordered vector space we denote by A_+ the set of all non-negative members of A.

Let K be a convex subset of an Euclidean space and let $f: K \to \Re$ be a concave function. A vector p is a *supergradient* of f at $x \in K$ if $f(y) - f(x) \leqslant p \cdot (y - x)$ for all $y \in K$. The set of all supergradients of f at x will be denoted by $\partial f(x)$. It is well known that if x is an interior point of K then $\partial f(x) \neq \phi$ and f is differentiable at x iff it has a unique supergradient at x which, in this case, coincides with the *gradient* vector.

For two vectors x, y in $\Re^m$ we write $x \geqslant y$ to mean $x_i \geqslant y_i$ for all $1 \leqslant i \leqslant m$, $x > y$ to mean $x \geqslant y$ and $x \neq y$, and $x \gg y$ to mean $x_i > y_i$ for all $1 \leqslant i \leqslant m$. A function f defined on a set $A \subset \Re^m$ is called *non-decreasing* if for every x, $y \in A$ we have $x \geqslant y$ implies $f(x) \geqslant f(y)$. It is called *increasing* if, in addition, $x > y$ implies $f(x) > f(y)$.

3. THE ASYMPTOTIC BEHAVIOR OF THE KERNEL AND THE NUCLEOLUS IN MIXED GAMES

Many games that arise in economic applications can be represented as a concave function of a finite dimensional vector of measures. (Some of these applications are discussed in the introduction of Einy *et al.* [9].) In this section we characterize the asymptotic nucleolus of a class of such games.

This characterization is used in the next section in order to investigate the properties of the asymptotic nucleolus of mixed market games.

Let v be a finite game (that is, T is finite and $\Sigma = 2^T$). If $x \in \Re^{|T|}$ and $S \subset T$ we define $x(S) = \sum_{i \in S} x_i$ if $S \neq \phi$, and $x(\phi) = 0$. Denote

$$I(v) = \{x \in \Re^{|T|} \mid x_i \geqslant v(\{i\}) \text{ for every } i \in T \text{ and } x(T) = v(T)\}$$

and

$$I^*(v) = \{x \in \Re^{|T|} \mid x(T) = v(T)\}.$$

For every $i, j \in T$, $i \neq j$ and $x \in \Re^{|T|}$ define

$$s_{ij}(x) = \max\{v(S) - x(S) \mid S \subset T, i \in S \text{ and } j \notin S\}.$$

The *prekernel* of the game v is the set

$$PK(v) = \{x \in I^*(v) \mid s_{ij}(x) = s_{ji}(x)\ \forall i, j \in T, i \neq j\}.$$

The *kernel* of the game v is the set

$$K(v) = \{x \in I(v) \mid (s_{ij}(x) - s_{ji}(x))(x_j - v(\{j\})) \leqslant 0\ \forall i, j \in T, i \neq j\}.$$

It is well known that if v is a finite game which is zero monotonic (that is, $v(S \cup \{i\}) \geqslant v(S) + v(\{i\})$ for every $S \subset T$ and $i \in T \backslash S$), then $PK(v)$ and $K(v)$ coincide (see Theorem 2.7 in Maschler *et al.* [17]). For a further discussion of the kernel the reader is referred to Maschler [16].

Let v be a finite game. For every $x \in I(v)$, let $\theta(x)$ be a $2^{|T|}$-tuple whose components are the numbers $v(S) - x(S)$, $S \subset T$, arranged in non-increasing order, i.e., $\theta_i(x) \geqslant \theta_j(x)$ for $1 \leqslant i \leqslant j \leqslant 2^{|T|}$. The *nucleolus* of the game v, denoted by Nv, is the member of $I(v)$ such that $\theta(Nv)$ is the minimum in the lexicographic order of the set $\{\theta(x) \mid x \in I(v)\}$; that is, there is no $y \in I(v)$ such that $\theta(Nv)$ strictly dominates $\theta(y)$ in the lexicographic order. It is well known that the nucleolus of a finite game v always exists when $I(v) \neq \phi$ and it consists of a unique point which belongs to the kernel of v (e.g., Schmeidler [21]).

In the rest of the paper we assume that a fixed measure $\lambda \in ca_+(T, \Sigma)$ is given. We interpret λ as a *population measure*, that is, if S is a coalition, then $\lambda(S)$ is the proportion of the total population which is contained in S. We also assume that T can be represented in the form $T = T_0 \cup T_1$, where T_0 and T_1 are non-empty disjoint coalitions, the restriction of λ to (T_0, Σ_{T_0}) is non-atomic (where, here and in the sequel, if S is a coalition $\Sigma_S = \{Q \in \Sigma \mid Q \subset S\}$) and T_1 is a finite set of atoms of λ such that every subset of T_1 is in Σ.

Let v be a game on (T, Σ) and let π be a finite subfield of Σ. The set of all atoms of π is denoted by A_π. The set of all subsets of A_π is identified naturally with π, and thus a finite game with a set of players A_π is identified with a function $w\colon \pi \to \Re$ with $w(\phi)=0$. The restriction of the game v to π is denoted by v_π. An *admissible sequence* of finite fields is an increasing sequence $(\pi_n)_{n=1}^\infty$ of finite subfields of Σ such that every subset of T_1 is in π_1 and $\bigcup_{n=1}^\infty \pi_n$ generates Σ.

Let v be a superadditive game on (T, Σ). It is said that v has an asymptotic nucleolus if there exists a game ψv such that, for every admissible sequence of finite fields $(\pi_n)_{n=1}^\infty$ and every S in π_1, $\lim_{n\to\infty} Nv_{\pi_n}(S)$ exists and equals $\psi v(S)$. It follows that $\psi v \in ba$, and it is called the *asymptotic nucleolus* of the game v.

The asymptotic approach was introduced in Kannai [14] in the context of the Shapley value of non-atomic games (see also Chapter III of Aumann and Shapley [5]).

We are now ready to state and prove the main result of this section.

THEOREM 3.1. *Let m be a natural number and let $\mu=(\mu_1, ..., \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f\colon \Re^m_+ \to \Re_+$ is a non-decreasing concave function which is continuously differentiable in int $\Re^m_+$ and satisfies, $\nabla f(\mu(T)) \gg 0$ and $f(\mu(T\backslash\{a\}))=0$ for every $a \in T_1$. Then the game $v=f\circ\mu$ has an asymptotic nucleolus. Moreover, if $(\pi_n)_{n=1}^\infty$ is an admissible sequence of finite fields and $x_n \in K(v_{\pi_n})$ for every n, then for every $S \in \pi_1$ we have*

$$\lim_{n\to\infty} x_n(S) = \tfrac{1}{2}\nabla f(\mu(T))\cdot\mu(S\cap T_0) + \frac{f(\mu(T)) - \frac{1}{2}\nabla f(\mu(T))\cdot\mu(T_0)}{|T_1|}\,|S\cap T_1|.$$

Proof. Let $(\pi_n)_{n=1}^\infty$ be an admissible sequence of finite fields. We first show that if $S \in \pi_1 \cap \Sigma_{T_0}$ and $x_n \in K(v_{\pi_n})$ for every n, then $\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T))\cdot\mu(S)$. Note that since f is non-decreasing, the game v is superadditive. Therefore, for every n, the game v_{π_n} is zero-monotonic, and thus $K(v_{\pi_n}) = PK(v_{\pi_n})$ for every n. Let n be a fixed natural number and let $j \in \pi_n \cap \Sigma_{T_0}$. Assume that $x_n \in K(v_{\pi_n})$. Then for every $i \in T_1$ we have

$$s_{ji}(x_n) = \max\{v(Q) - x_n(Q) \mid Q \subset \pi_n,\, j \in Q,\, \{i\} \notin Q\} = -x_n(j)$$

and

$$s_{ij}(x_n) \geqslant v(T\backslash j) - x_n(T) + x_n(j) = f(\mu(T\backslash j)) - f(\mu(T)) + x_n(j).$$

Since $x_n \in PK(v_{\pi_n})$, we have

$$s_{ij}(x_n) = s_{ji}(x_n).$$

Therefore

$$x_n(j) \leqslant \tfrac{1}{2}(f(\mu(T)) - f(\mu(T \setminus j))).$$

Since f is concave and differentiable,

$$f(\mu(T)) \leqslant f(\mu(T \setminus j)) + \nabla f(\mu(T \setminus j)) \cdot \mu(j).$$

Thus,

$$x_n(j) \leqslant \tfrac{1}{2} \nabla f(\mu(T \setminus j)) \cdot \mu(j). \tag{3.1}$$

Let $\varepsilon > 0$. As f is continuously differentiable on int $\mathfrak{R}^m_+$, there exists $\delta > 0$ such that for every $x \in \mathfrak{R}^m_+$ we have

$$\|x - \mu(T)\| < \delta \Rightarrow \nabla f(x) \leqslant \nabla f(\mu(T)) + \varepsilon e, \tag{3.2}$$

where $e = (1, 1, ..., 1)$. Since $\mu_1, ..., \mu_m$ are absolutely continuous with respect to λ and the restriction of λ to (T_0, Σ_{T_0}) is non-atomic, there exists a natural number n_0 such that $\|\mu(j)\| < \delta$ for every $j \in \pi_{n_0} \cap \Sigma_{T_0}$. Therefore by (3.1) and (3.2), for every $n \geqslant n_0$ and $j \in \pi_n \cap \Sigma_{T_0}$ we have

$$x_n(j) \leqslant \tfrac{1}{2}(\nabla f(\mu(T)) + \varepsilon e) \cdot \mu(j).$$

Let $S \in \pi_1 \cap \Sigma_{T_0}$. Then S is the union of members of π_n for every n. Therefore for every $n \geqslant n_0$.

$$x_n(S) \leqslant \tfrac{1}{2}(\nabla f(\mu(T)) + \varepsilon e) \cdot \mu(S).$$

Since ε is arbitrary, we have

$$\overline{\lim}\, x_n(S) \leqslant \tfrac{1}{2} \nabla f(\mu(T)) \cdot \mu(S).$$

We now show that $\underline{\lim}\, x_n(S) \geqslant \frac{1}{2} \nabla f(\mu(T)) \cdot \mu(S)$. Since f is continuously differentiable on int $\mathfrak{R}^m_+$ and $\nabla f(\mu(T)) \gg 0$, there exists $\hat{\delta} > 0$ such that for every $x \in \mathfrak{R}^m_+$ we have

$$\|x - \mu(T)\| < \hat{\delta} \Rightarrow \nabla f(x) \leqslant \tfrac{3}{2} \nabla f(\mu(T)).$$

Let n_1 be a natural number such that $\|\mu(j)\| < \hat{\delta}$ for every $j \in \pi_{n_1} \cap \Sigma_{T_0}$. Then

$$\nabla f(\mu(T \setminus j)) \leqslant \tfrac{3}{2} \nabla f(\mu(T)).$$

Therefore by (3.1), for every $n \geq n_1$ and $j \in \pi_n \cap \Sigma_{T_0}$ we have

$$x_n(j) \leq \tfrac{3}{4} \nabla f(\mu(T)) \cdot \mu(j).$$

Hence,

$$x_n(S) \leq \tfrac{3}{4} \nabla f(\mu(T)) \cdot \mu(S). \tag{3.3}$$

Since f is concave, $f((1/4)\,\mu(T)) \geq (1/4)\, f(\mu(T)) > 0$, and therefore there exists a natural number $n_2 \geq n_1$ such that for every $n \geq n_2$ and $j \in \pi_n \cap \Sigma_{T_0}$ we have $\|\mu(T)\| < 4f((1/4)\,\mu(T))/3\, \|\nabla f(\mu(T))\|$; hence

$$x_n(j) \leq \frac{3}{4} \|\nabla f(\mu(T))\| \mid \|\mu(j)\| < \frac{1}{|T_1|} f\left(\frac{1}{4}\mu(T)\right). \tag{3.4}$$

Let $n \geq n_2$ be fixed and let $i \in T_1$ and $j \in \pi_n \cap \Sigma_{T_0}$. Choose $Q_n \subset \pi_n$ such that $\{i\} \in Q_n$, $j \notin Q_n$ and

$$v_{\pi_n}(Q_n) - x_n(Q_n) = \max\{v_{\pi_n}(Q) - x_n(Q) \mid Q \subset \pi_n, \{i\} \in Q, j \notin Q\}.$$

As $x_n \in K(v_{\pi_n})$, then $v_{\pi_n}(Q_n) - x_n(Q_n) = -x_n(j)$.

Let $S_n = \bigcup_{l \in Q_n} l$. We show that $S_n \supset T_1$. Assume not. Then $v(S_n) = 0$, and thus $x_n(j) = x_n(S_n) \geq x_n(\{i\})$. Since all the players in T_1 are interchangeable in the game v_{π_n} (two players in a finite game are interchangeable if they have the same marginal contribution to every coalition which does not contain them), they get the same payoff in every member of $K(v_{\pi_n})$. Hence,

$$f(\mu(T)) = x_n(T) = |T_1|\; x_n(\{i\}) + x_n(T_0).$$

By (3.3), $x_n(T_0) \leq (3/4)\, \nabla f(\mu(T)) \cdot \mu(T_0)$. Therefore,

$$x_n(\{i\}) \geq \frac{f(\mu(T)) - (3/4)\, \nabla f(\mu(T)) \cdot \mu(T_o)}{|T_1|}.$$

Since f is concave, differentiable and non-decreasing,

$$f(\mu(T)) - \tfrac{3}{4} \nabla f(\mu(T)) \cdot \mu(T_o) \geq f(\mu(T) - \tfrac{3}{4}\mu(T_o)) \geq f(\tfrac{1}{4}\mu(T)).$$

Thus, $x_n(\{i\}) \geq 1/|T_1|\; f((1/4)\,\mu(T))$. Since $x_n(j) \geq x_n(\{i\})$, this contradicts (3.4). Therefore $S \supset T_1$, and thus there exists $\hat{S}_n \in \Sigma_{T_0}$ such that $S_n = (T \setminus j) \setminus \hat{S}_n$. Hence,

$$-x_n(j) = v(S_n) - x_n(S_n) = v(S_n) - v(T) + x_n(j) + x_n(\hat{S}_n).$$

Thus

$$x_n(j) = \tfrac{1}{2}(v(T) - v(S_n) - x_n(\hat{S}_n)).$$

By (3.3)

$$x_n(\hat{S}_n) \leqslant \tfrac{3}{4}\,\nabla f(\mu(T)) \cdot \mu(\hat{S}_n).$$

Since f is concave,

$$\begin{aligned} v(T) - v(S_n) &= f(\mu(T)) - f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) \\ &\geqslant \nabla f(\mu(T)) \cdot (\mu(j) + \mu(\hat{S}_n)). \end{aligned}$$

Therefore,

$$x_n(j) \geqslant \tfrac{1}{2}\,[\nabla f(\mu(T)) \cdot \mu(j) + \tfrac{1}{4}\,\nabla f(\mu(T)) \cdot \mu(\hat{S}_n)] \geqslant \tfrac{1}{2}\,\nabla f(\mu(T)) \cdot \mu(j).$$

Hence,

$$x_n(S) \geqslant \tfrac{1}{2}\nabla f(\mu(T)) \cdot \mu(S) \qquad \text{for every} \quad n \geqslant n_2. \tag{3.5}$$

This implies that $\underline{\lim}\, x_n(S) \geqslant \frac{1}{2}\,\nabla f(\mu(T)) \cdot \mu(S)$.

Assume now that $S \in \pi_1$ is any coalition. Then for every natural number n we have

$$x_n(S) = x_n(S \cap T_0) + x_n(S \cap T_1).$$

Let t_n be the payoff which is assigned by x_n to a player in T_1. Then

$$v(T) = x_n(T) = |T_1|\; t_n + x_n(T_0) \Rightarrow \lim_{n \to \infty}\; t_n = \frac{v(T) - \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(T_0)}{|T_1|}.$$

Therefore,

$$\lim_{n \to \infty}\; x_n(S) = \frac{1}{2}\,\nabla f(\mu(T)) \cdot \mu(S \cap T_0) + \frac{v(T) - \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(T_0)}{|T_1|}\,|S \cap T_1|.$$

Q.E.D

Let v be a game on (T, Σ). The *core of* v, denoted by *Core*(v), is the set of all payoff measures $\xi \in ba$ such that $\xi(S) \geqslant v(S)$ for every $S \in \Sigma$.

We now interpret the core as a set of real-valued functions on T. A game v is *continuous at* T if for every sequence $\{T_n\}_{n=1}^{\infty}$ of coalitions such that $T_{n+1} \supset T_n$ for all n and $\bigcup_{n=1}^{\infty} T_n = T$ we have $\lim_{n \to \infty} v(T_n) = v(T)$. It is easy to see that if v is a non-negative game on (T, Σ) which is continuous at T, then *core*$(v) \subset ca_+$ (e.g., Schmeidler [22]). A coalition S_0 is *null* in a game v if for every coalition S with $S \cap S_0 = \phi$ we have $v(S \cup S_0) = v(S)$. A non-negative game v is *weakly absolutely continuous* with respect to the population measure λ if every null coalition of λ is also a null coalition of v. Now if v is non-negative, continuous at T, and weakly absolutely continuous with respect to λ, then $core(v) \subset ca_+(\lambda)$. Therefore by the Radon–Nikodym Theorem, $core(v)$ can be identified with the set of all non-negative functions g in $L_1(T, \Sigma, \lambda)$ such that $\int_T g \, d\lambda = v(T)$ and for all $S \in \Sigma$, $\int_S g \, d\lambda = v(S)$.

We want to determine the location in the core of the asymptotic nucleolus of a game which satisfies the conditions of Theorem 3.1. We first state and prove a representation theorem for the core of such games.

THEOREM 3.2. *Let $\mu = (\mu_1, \ldots, \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f\colon \Re^m_+ \to \Re_+$ is a concave function which is differentiable at $\mu(T)$ and satisfies $f(\mu(T \backslash \{a\})) = 0$ for every $a \in T_1$. Then the core of the game $v = f \circ \mu$ is given by*

$$Core(v) = \{\xi \in ca_+(\lambda) \mid \xi(T) = f(\mu(T)) \text{ and } \forall S \in \Sigma_{T_0},$$
$$\xi(S) \leqslant \nabla f(\mu(T)) \cdot \mu(S)\}.$$

Proof. Let

$$M(v) = \{\xi \in ca_+(\lambda) \mid \xi(T) = f(\mu(T)) \text{ and } \forall S \in \Sigma_{T_0},$$
$$\xi(S) \leqslant \nabla f(\mu(T)) \cdot \mu(S)\}.$$

We will show that $M(v) = Core(v)$. We first show that $M(v) \subset Core(v)$. Let $\xi \in M(v)$ and $S \in \Sigma$. Now if S does not include T_1 then $v(S) = 0$ and clearly, $\xi(S) \geqslant v(S)$. If $S \supset T_1$ then $T \backslash S \subset T_0$. As $\xi \in M(v)$,

$$\xi(T \backslash S) \leqslant \nabla f(\mu(T)) \cdot \mu(T \backslash S).$$

Therefore

$$\xi(S) = \xi(T) - \xi(T \backslash S) \geqslant \xi(T) - \nabla f(\mu(T)) \cdot \mu(T \backslash S)$$
$$= f(\mu(T)) - \nabla f(\mu(T)) \cdot \mu(T \backslash S).$$

As f is concave,

$$v(S) = f(\mu(S)) \leqslant f(\mu(T)) - \nabla f(\mu(T)) \cdot \mu(T \backslash S).$$

Hence, $\xi(S) \geqslant v(S)$, and thus $\xi \in Core(v)$.

It remains to show that $Core(v) \subset M(v)$. Let $\xi \in Core(v)$. Then for every $S \in \Sigma$ we have

$$0 \leqslant \xi(S) \leqslant \xi(T) - v(T \backslash S). \tag{3.6}$$

As f is continuous at $\mu(T)$ and $\mu_1, ..., \mu_m$ are in $ca_+(\lambda)$, the inequality in (3.6) implies that $\xi \in ca_+(\lambda)$. Since the restriction of λ to (T_0, Σ_{T_0}) is non-atomic, the restrictions of $\mu_1, ..., \mu_m$ and ξ to (T_0, Σ_{T_0}) are also non-atomic. Let $S \in \Sigma_{T_0}$. We will show that $\xi(S) \leqslant \nabla f(\mu(T)) \cdot \mu(S)$. By Lyapunov's theorem, for every $0 < \alpha < 1$ there exists a coalition $S_\alpha \in \Sigma_{T_0}$ such that $\mu(S_\alpha) = \alpha\mu(S)$ and $\xi(S_\alpha) = \alpha\xi(S)$. As f is differentiable at $\mu(T)$, for every $0 < \alpha < 1$ we have

$$f(\mu(T \backslash S_\alpha)) = f(\mu(T)) - \alpha \nabla f(\mu(T)) \cdot \mu(S) + o(\alpha).$$

As $\xi \in Core(v)$, we have

$$\xi(S_\alpha) = \xi(T) - \xi(T \backslash S_\alpha) \leqslant f(\mu(T)) - f(\mu(T \backslash S_\alpha)).$$

Hence,

$$\xi(S) \leqslant \nabla f(\mu(T)) \cdot \mu(S) + g(\alpha).$$

where $\lim_{\alpha \to 0} g(\alpha) = 0$. Therefore $\xi(S) \leqslant \nabla f(\mu(T)) \cdot \mu(S)$, and the proof is complete. Q.E.D

Note that the core of a game v in Theorem 3.2 can be identified with the following subset of $L_1(T, \Sigma, \lambda)$

$$\left\{ g \in L_1(T, \Sigma, \lambda) \mid g \geqslant 0, \int_T g \, d\lambda = f(\mu(T)), \forall t \in T_0 : g(t) \leqslant h(t) \right\},$$

where h is the Radon–Nikodym derivative (with respect to λ) of the measure $\nabla f(\mu(T)) \cdot \mu$.

Let A be a subset of a linear space. A point $x_0 \in A$ is called a *center of symmetry* of A if for every $x \in A$, the point $2x_0 - x$ also belongs to A. Note that if A is bounded, there may be at most one center of symmetry.

The following corollary is a direct consequence of Theorems 3.1 and 3.2.

COROLLARY 3.3. *Let $\mu = (\mu_1, ..., \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \mathfrak{R}^m_+ \to \mathfrak{R}_+$ is a non-decreasing concave function which is differentiable in int $\mathfrak{R}^m_{++}$ and satisfies, $\nabla f(\mu(T)) \gg 0$ and $f(\mu(T \backslash \{a\})) = 0$ for every $a \in T_1$. Then the asymptotic nucleolus of the game $v = f \circ \mu$ coincides with the center of symmetry of the subset of the core of v in which all the members of T_1 receive the same payoff.*

In this section we apply Theorem 3.1 to mixed market games.

We consider a pure exchange economy E in which the commodity space is $\Re^m_+$. The traders' space is represented by the measure space (T, Σ, λ). We assume again that $T = T_0 \cup T_1$, where T_0 and T_1 are non-empty and disjoint coalitions, T_1 is a finite set of atoms of λ such that every subset of T_1 is in Σ, and the restriction of λ to (T_0, Σ_{T_0}) is non-atomic. We will interpret the members of T_1 as monopolists or syndicates. Every trader $t \in T$ has a *utility function* u_t: $\Re^m_+ \to \Re_+$. An *assignment* in E is an integrable function $\mathbf{x}$: $T \to \Re^m_+$. There is a fixed *initial assignment* ω ($\omega(t)$ represents the *initial bundle density* of trader t). An *allocation* is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}\, d\lambda \leqslant \int_T \omega\, d\lambda$. A *transferable utility competitive equilibrium* (t.u.c.e.) of the economy E is a pair $(\mathbf{x}, p)$, where $\mathbf{x}$ is an allocation and $p \in \Re^m_+$, such that for all $t \in T$, $u_t(x) - p \cdot (x - \omega(t))$ attains its maximum (over $\Re^m_+$) at $x = \mathbf{x}(t)$. The measure $\tau(S) = \int_S [u_t(\mathbf{x}(t)) - p \cdot (\mathbf{x}(t) - \omega(t))]\, \mathrm{d}\lambda$ (when the function $u_t(\mathbf{x}(t))$ is integrable) is called the *competitive payoff distribution*, and p is the *vector of competitive prices*. We assume

$$\int_T \omega\, d\lambda \gg 0, \tag{4.1}$$

for every trader $a \in T_1$ there exists a commodity

$$1 \leqslant k_a \leqslant m \text{ such that } \omega_{k_a}(t) = 0 \text{ for every } t \in T \backslash \{a\}, \tag{4.2}$$

where ω_{k_a} denotes the k_a-component of ω.

The implication of (4.1) is that every commodity is present in the market. The interpretation of (4.2) is that every atom of λ has a corner on one of the commodities in the economy; that is, every monopolist holds a commodity that the other traders do not have. The small traders may initially own positive amounts of some commodities (the number of commodities may be greater than the number of monopolists), although (4.2) implies that their initial endowments are in the boundary of $\Re^m_+$.

Denote by U the set of all functions u: $\Re^m_+ \to \Re_+$ which are continuous and concave on $\Re^m_+$, continuously differentiable and increasing on the interior of $\Re^m_+$ and vanish on the boundary of $\Re^m_+$. Note that any differentiable neoclassical utility function is in U (see Definition 1.4.2 in Aliprantis *et al.* [1]).

Two traders in the economy E are of the *same type* if they have identical utility functions and the same initial bundle. We assume that the number of different types of traders in T_0 is n. For every $1 \leqslant i \leqslant n$, we denote by S_i the set of traders in T_0 which are of type i. We assume that

S_i is measurable (i.e., $S_i \in \Sigma$) and $\lambda(S_i) > 0$. The utility function of the traders of type i ($1 \leqslant i \leqslant n$) is denoted by u_i. We assume that the utility function of every trader in E is in U. Note that under these assumptions the economy E has a unique t.u.c.e. The Aumann–Shapley–Shubik market game associated with the economy E in this case of finite number of types is

$$v(S) = \sup \left\{ \sum_{a \in S \cap T_1} \lambda(\{a\})\, u_a(\mathbf{x}(a)) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(\mathbf{x}(t))\, d\lambda \mid \mathbf{x} \in X(S) \right\}, \tag{4.3}$$

where $X(S) = \{\mathbf{x} \mid \mathbf{x}$ is an assignment such that $\int_S \mathbf{x}\, d\lambda = \int_S \omega\, d\lambda\}$.

We first study the case in which for every $1 \leqslant i \leqslant n$ the utility function u_i is homogeneous of degree one. Define a function $f\colon \mathfrak{R}^m_+ \to \mathfrak{R}_+$ by

$$f(y) = \max \left\{ \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) \mid x_a, x_i \in \mathfrak{R}^m_+, \sum_{a \in T_1} \lambda(\{a\})\, x_a + \sum_{i=1}^{n} x_i \leqslant y \right\}. \tag{4.4}$$

Since the utility functions of the traders are continuous and concave, it is easy to see that f is well defined and concave on $\mathfrak{R}^m_+$.

LEMMA 4.1. *Let v be the market game defined in* (4.3), *and let f be the function defined in* (4.4), *then for every $S \in \Sigma$, $v(S) \leqslant f(\int_S \omega\, d\lambda)$, with equality when $\lambda(S \cap S_i) > 0$, for all $1 \leqslant i \leqslant n$.*

Proof. Let $S \in \Sigma$. We show that $v(S) \leqslant f(\int_S \omega\, d\lambda)$. Assume first that S does not include T_1. Then by (4.2), $\int_S \omega\, d\lambda$ belongs to the boundary of $\mathfrak{R}^m_+$. Since the utility functions of the traders in T vanish on the boundary of $\mathfrak{R}^m_+$, we have $v(S) = 0$ and $f(\int_S \omega\, d\lambda) = 0$. So assume that $S \supset T_1$. Let $\mathbf{x}$ be an assignment such that $\int_S \mathbf{x}\, d\lambda = \int_S \omega\, d\lambda$. For every $a \in T_1$ let $x_a = \mathbf{x}(a)$ and for every $1 \leqslant i \leqslant n$ let $x_i = \int_{S \cap S_i} \mathbf{x}\, d\lambda$. Then

$$\sum_{a \in T_1} \lambda(\{a\})\, x_a + \sum_{i=1}^{n} x_i = \int_S \mathbf{x}\, d\lambda = \int_S \omega\, d\lambda.$$

Therefore by the definition of f, we have

$$f\left(\int_S \omega\, d\lambda\right) \geqslant \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i).$$

Since the u_i are concave and homogeneous of degree one,

$$\sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(\mathbf{x}(t))\, d\lambda$$
$$\leqslant \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i).$$

As $\mathbf{x}$ was an arbitrary assignment which satisfies $\int_S \mathbf{x}\, d\lambda = \int_S \omega\, d\lambda$, we obtain that $v(S) \leqslant f(\int_S \omega\, d\lambda)$.

Assume that $S \in \Sigma$ satisfies $\lambda(S \cap S_i) > 0$ for all $1 \leqslant i \leqslant n$. Now we show that $v(S) \geqslant f(\int_S \omega\, d\lambda)$. Let $(x_a)_{a \in T_1}$ and $(x_i)_{i=1}^n$ such that

$$f\left(\int_S \omega\, d\lambda\right) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i).$$

Define an assignment $\mathbf{x}$ by $\mathbf{x}(t) = x_t$ if $t \in T_1$ and for every $t \in S_i$ $(1 \leqslant i \leqslant n)$

$$x(t) = \frac{1}{\lambda(S \cap S_i)} x_i.$$

Then

$$\int_S \mathbf{x}\, d\lambda = \sum_{a \in T_1} \lambda(\{a\})\, x_a + \sum_{i=1}^{n} x_i \leqslant \int_S \omega\, d\lambda.$$

Therefore $v(S) \geqslant \int_S u_t(\mathbf{x}(\mathbf{t}))\, d\lambda$. Since the u_i are homogeneous of degree one,

$$v(S) \leqslant \int_S u_t(\mathbf{x}(t))\, d\lambda = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) = f\left(\int_S \omega\, d\lambda\right).$$

LEMMA 4.2. *The function f defined in* (4.4) *is continuously differentiable on int $\mathfrak{R}^m_+$ and $\nabla f(\int_T \omega\, d\lambda) \gg 0$.*

Proof. We first show that f is differentiable at every point in the interior of $\mathfrak{R}^m_+$. Let $y^* \in \operatorname{int} \mathfrak{R}^m_+$. Then from the definition of f it is clear that $f(y^*) > 0$. Since f is concave on $\mathfrak{R}^m_+$, it is sufficient to show that $\partial f(y^*)$ consists of a unique point. Let $(x_a^*)_{a \in T_1}$ and $(x_i^*)_{i=1}^n$ be such that

$$f(y^*) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a^*) + \sum_{i=1}^{n} u_i(x_i^*).$$

Since the utility functions of the traders are non-decreasing, we have

$$\sum_{a \in T_1} \lambda(\{a\})\, x_a^* + \sum_{i=1}^{n} x_i^* = y^*.$$

Since $f(y^*) > 0$, the assumption that the utility functions of the traders vanish on the boundary of $\mathfrak{R}_+^m$ implies that there exists $j \in T_1 \cup \{1, ..., n\}$ such that $x_j^* \in \operatorname{int} \mathfrak{R}_+^m$. Assume first that $1 \leqslant j \leqslant n$. We will show that $\partial f(y^*) \subset \partial u_j(x_j^*)$. Let $p \in \partial f(y^*)$. Then for every $x \in \mathfrak{R}_+^m$ we have

$$\begin{aligned} u_j(x) - u_j(x_j^*) &= u_j(x) + \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a^*) + \sum_{i \neq j} u_i(x_i^*) \\ &\quad - u_j(x_j^*) - \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a^*) - \sum_{i \neq j} u_i(x_i^*) \\ &\leqslant f\left(x + \sum_{a \in T_1} \lambda(\{a\})\, x_a^* + \sum_{i \neq j} x_i^* \right) - f(y^*) \leqslant p \cdot (x - x_j^*). \end{aligned}$$

Thus $p \in \partial u_j(x_j^*)$ and $\partial f(y^*) \subset \partial u_j(x_j^*)$. Since u_j is differentiable at x_j^*, we have $\partial u_j(x_j^*) = \{\nabla u_j(x_j^*)\}$. As $\partial f(y^*) \neq \phi$, we have $\partial f(y^*) = \{\nabla u_j(x_j^*)\}$. If $j \in T_1$, for every $x \in \mathfrak{R}_+^m$ we define $\bar{u}_j(x) = \lambda(\{j\})\, u_j(x)$. Then the above argument implies that $\partial f(y^*) = \{\nabla \bar{u}_j(x_j^*)\}$. Thus, in any case $\partial f(y^*)$ consists of a unique point, and therefore f is differentiable at y^*. The assumption that the utility functions of the traders are increasing in int $\mathfrak{R}_+^m$ implies that $\nabla f(\int_T \omega \, d\lambda) \gg 0$. Now since f is concave on $\mathfrak{R}_+^m$, it is continuous on int $\mathfrak{R}_+^m$. Moreover, since the utility functions of the traders vanish on the boundary of $\mathfrak{R}_+^m$ it is easy to see that f is also continuous on the boundary of $\mathfrak{R}_+^m$. Now Proposition 39.1 of Aumann and Shapley [5] asserts that any continuous concave function on $\mathfrak{R}_+^m$ which is differentiable on int $\mathfrak{R}_+^m$ is continuously differentiable in int $\mathfrak{R}_+^m$. Therefore f is continuously differentiable on int $\mathfrak{R}_+^m$. Q.E.D

LEMMA 4.3. *Let v be the market game defined in* (4.3) *and let f be the function given in* (4.4). *Then the asymptotic nucleolus of v coincides with the asymptotic nucleolus of the game $w(S) = f(\int_S d\lambda)$ for all $S \in \Sigma$.*

Proof. We first note that by Lemma 4.2, the game w satisfies the assumption of Theorem 3.1. Now by Lemma 4.1, $v(S) = w(S)$ for every $S \in \Sigma$ satisfying $\lambda(S \cap S_i) > 0$, for all $1 \leqslant i \leqslant n$, and also $w(S) \geqslant v(S) \geqslant 0$ for all $S \in \Sigma$. Therefore the first part of the proof of Theorem 3.1 can be applied for the game v. Also the second part of this proof works for the game v by noticing that the first equality in (3.5) can be replaced by a weak inequality $(\geqslant)$. Q.E.D

We are now ready to state and prove the following theorem.

THEOREM 4.4. *Assume that the economy E satisfies* (4.1), (4.2) *and also*

(1) *There is a finite number n of traders' types in* T_0.

(2) *The utility functions* $u_1, ..., u_n$ *of the traders in* T_0 *are in U and in addition they are homogeneous of degree one on* $\Re^m_+$.

(3) *The utility functions* $\{u_a\}_{a \in T_1}$ *of the traders in* T_1 *are in U.*

Let f be the function given in (4.4). *Then the market game v defined in* (4.3) *has an asymptotic nucleolus* ψv *which is given by*

$$\psi v(S) = \frac{1}{2} \nabla f\left(\int_T \omega \, d\lambda\right) \cdot \int_{S \cap T_0} \omega \, d\lambda + \frac{f(\int_T \omega \, d\lambda) - \frac{1}{2} \nabla f(\int_T \omega \, d\lambda) \cdot \int_{T_0} \omega \, d\lambda}{|T_1|} |S \cap T_1|. \quad (4.5)$$

Moreover, if τ *is the competitive payoff distribution of the economy E, then* $\psi v(S) = \frac{1}{2}\tau(S)$ *for every* $S \in \Sigma_{T_0}$.

Proof. Equation (4.5) follows from Theorem 3.1 and Lemmata 4.1, 4.2 and 4.3. Denote $b = \int_T \omega \, d\lambda$. Let $(x_a^*)_{a \in T_1}$ and $(x_i^*)_{i=1}^n$ be such that $f(b) = \sum_{a \in T_1} \lambda(\{a\}) \, u_a(x_a^*) + \sum_{i=1}^n u_i(x_i^*)$. For every $t \in T$, let

$$\mathbf{x}^*(t) = \begin{cases} x_t^*, & t \in T_1 \\ x_i^*, & t \in S_i. \end{cases}$$

Then by a similar argument to that which was used in the proof of Lemma 4.3, we obtain that for every $t \in T$ and $x \in \Re^m_+$

$$u_t(x) \leqslant u_t(\mathbf{x}^*(t)) + \nabla f(b) \cdot (x - \mathbf{x}^*(t)). \quad (4.6)$$

Since f is non-decreasing on $\Re^m_+$, $\nabla f(b) \geqslant 0$. Let $1 \leqslant i \leqslant m$. Now if x_i^* is on the boundary of $\Re^m_+$, then $u_i(x_i^*) = 0$, and thus by (4.6), $u_i(x) - \nabla f(b) \cdot x \leqslant 0$ for every $x \in \Re^m_+$. If $x_i^* \in \text{int}\, \Re^m_+$, then $\nabla f(b) = \nabla u_i(x_i^*)$. Since u_i is homogeneous of degree one, $\nabla u_i(x_i^*) \cdot x_i^* = u_i(x_i^*)$. Therefore we again have by (4.6), $u_i(x) - \nabla f(b) \cdot x \leqslant 0$ for every $x \in \Re^m_+$ and thus

$$\max_{x \in \Re^m_+} (u_i(x) - \nabla f(b) \cdot x) = 0.$$

This implies that for every $t \in T$

$$\max_{x \in \Re^m_+} (u_i(x) - \nabla f(b) \cdot (x - \omega(t))) = \nabla f(b) \cdot \omega(t).$$

Now by (4.6), for every $a \in T_1$ and $t \in T$ we have

$$\max_{x \in \Re^m_+} (u_a(x) - \nabla f(b) \cdot (x - \omega(t))) = u_a(x_a^*) - \nabla f(b) \cdot (x_a^* - \omega(t)).$$

For every $t \in T$ let

$$g(t) = \begin{cases} u_t(\mathbf{x}^*(t)) - \nabla f(b) \cdot (\mathbf{x}^*(t) - \omega(t)), & t \in T_1 \\ \nabla f(b) \cdot \omega(t), & t \in T_0. \end{cases}$$

For every $S \in \Sigma$, define $\tau(S) = \int_S g \, d\lambda$. Then τ is the competitive payoff distribution in the economy E and for every $S \in \Sigma_{T_0}$ we have $\psi v(S) = \frac{1}{2}\tau(S)$.
Q.E.D

We now study the case in which the utility functions of the traders in the continuum (i.e., the u_i, $1 \leqslant i \leqslant n$) are not necessarily homogeneous of degree one. In this case we introduce the following assumption on the functions u_i, $(1 \leqslant i \leqslant n)$,

$$\text{for every } 1 \leqslant i \leqslant n \text{ we have } \lim_{\|x\| \to \infty} \frac{u_i(x)}{\|x\|} = 0. \tag{4.7}$$

The Assumption (4.7) is a special case of the Aumann–Perles [4] Condition. This assumption is standard in the theory on non-atomic market games (see, for example, Aumann and Shapley [5], Dubey and Neyman [7, 8], and Mertens [18]). Note that (4.7) may not be satisfied when every u_i, $(1 \leqslant i \leqslant n)$, is homogeneous of degree one; for example, when the u_i's are linear (4.7) is not satisfied.

Let $m = |T_1|$. It will be convenient to denote the member of T_1 by $a_{n+1}, ..., a_{n+m}$, their utility functions by $u_{n+1}, ..., u_{n+m}$, and their initial bundles by $\omega_{n+1}, ..., \omega_{n+m}$, respectively. Define now a function g: $\mathfrak{R}^{n+m}_+ \to \mathfrak{R}_+$ by

$$g(y, z) = \max \left\{ \sum_{i=1}^{n+m} y_i u_i(x_i) \mid x_i \in \mathfrak{R}^l_+, \text{ and } \sum_{i=1}^{n+m} y_i x_i \leqslant z \right\}. \tag{4.8}$$

Then under our assumption on the utility functions of the traders in E, by Lemma 39.9 in Aumann and Shapley [5], g is concave, non-decreasing and continuous on $\mathfrak{R}^{n+m}_+$. Moreover, by Proposition 39.13 of Aumann and Shapley [5], g is continuously differentiable in the interior of $\mathfrak{R}^{n+m}_+$.

Define now a function h: $\mathfrak{R}^n_+ \to \mathfrak{R}_+$ by

$$h(y_1, ..., y_n) = g\left(y_1, ..., y_n, \lambda(\{a_{n+1}\}), ..., \lambda(\{a_{n+m}\}), \sum_{i=1}^{n} y_i \omega_i + \sum_{i=n+1}^{n+m} \lambda(\{a_i\})\, \omega_i \right). \tag{4.9}$$

Then h is concave non-decreasing and continuous on $\Re^n_+$, and it is continuously differentiable in the interior of $\Re^n_+$. Now a similar proof to that of Lemma 4.1 (see also the proof of Lemma 39.8 in Aumann and Shapley [5]) yields that for every $S \in \Sigma$

$$v(S) = h(\lambda(S \cap S_1), ..., \lambda(S \cap S_n)),$$

where v is the market game defined in (4.3).

THEOREM 4.5. *Assume that the economy E satisfies* (4.1), (4.2) *and also*

(1) *There is a finite number n of traders' types in T_0.*

(2) *The utility functions $u_1, ..., u_n$ of the traders in T_0 are in U and in addition they satisfy condition* (4.7).

(3) *The utility functions $\{u_a\}_{a \in T_1}$ of the traders in T_1 are in U.*

Let h be the function given in (4.9). *Then the market game v defined in* (4.3) *has an asymptotic nucleolus ψv which is given for every $S \in \Sigma_{T_0}$ by*

$$\psi v(S) = \tfrac{1}{2} \nabla h(\lambda(S_1), ..., \lambda(S_n)) \cdot (\lambda(S_1 \cap S), ..., \lambda(S_n \cap S)). \qquad (4.10)$$

Moreover, if τ is the competitive payoff distribution of the economy E, then $\psi v(S) = \frac{1}{2}\tau(S)$ for every $S \in \Sigma_{T_0}$.

Proof. The fact that ψv exists and satisfies the formula (4.10) follows from Theorem 3.1. We will show that for every $S \in \Sigma_{T_0}$, $\psi v(S) = \frac{1}{2}\tau(S)$, where τ is the competitive payoff distribution of E. In order to avoid heavy notations, we assume without loss of generality that $T_0 = [0, 1]$ and $\lambda(\{a_{n+1}) = \cdots = \lambda(\{a_{n+m}\}) = 1$. Let B be the σ-field of Borel subsets of the interval $[1, m+1]$, and let F be the σ-field generated by $\Sigma_{T_0} \cup B$. Consider the measurable space $([0, m+1], F)$. Let $S \in F$. Then $S = Q_0 \cup Q_1$ where $Q_0 \in \Sigma_{T_0}$ and $Q_1 \in B$.

Define

$$\hat{v}(S) = g\Bigg(\lambda(Q_0 \cap S_1), ..., \lambda(Q_0 \cap S_n), \mu(Q_1 \cap I_1), ..., \mu(Q_1 \cap I_m),$$
$$\sum_{i=1}^{n} \lambda(Q_0 \cap S_i)\, \omega_i + \sum_{j=1}^{m} \mu(Q_1 \cap I_j)\, \omega_{n+j}\Bigg),$$

where μ is the Lebesgue measure on $[1, m+1]$ and for every $1 \leqslant j \leqslant m$, $I_j = [j, j+1]$. The game $\hat{v}$ is a non-atomic game on $([0, m+1], F)$. Moreover, by Proposition 10.17 of Aumann and Shapley [5] the game $\hat{v}$ is in the space pNA of non-atomic games (e.g., Aumann and Shapley [5]). Therefore by Theorem J and Theorem B (the diagonal formula) of Aumann and Shapley [5], we obtain that if τ is the competitive payoff distribution of the economy E, then for every $S \in \Sigma_{T_0}$

$$\tau(S) = \nabla h(\lambda(S_1), ..., \lambda(S_n)) \cdot (\lambda(S_1 \cap S), ..., \lambda(S_n \cap S)).$$

Q.E.D

5. COMPARISON WITH THE ASYMPTOTIC SHAPLEY VALUE

In this section we assume that our population measure λ has one atom a, that is $T_1 = \{a\}$. Let v be a finite game on (T, Σ). Recall that the Shapley value of a player $i \in T$ is given by

$$Sv(i) = \frac{1}{|T|!} \sum_{\sigma} (v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma)),$$

where the sum is taken over all orders σ of T and P_i^σ denotes the set of all players that precede i in the order σ. It is known (see for example, Hart [13] and Neyman [19]) that if $\mu = (\mu_1, ..., \mu_m)$ is a vector of non-trivial measures in $ca_+(\lambda)$ and $f: \mathfrak{R}_+^m \to \mathfrak{R}_+$ is a function which satisfies the assumptions of Theorem 3.1, then the game $v = f \circ \mu$ has an asymptotic Shapley value φ. Moreover,

$$\varphi v(\{a\}) = \int_0^1 [f(\mu(\{a\}) + x\mu(T_0)) - f(x\mu(T_0))] \, dx.$$

Since $f(x\mu(T_0)) = 0$ for every $0 \leqslant x \leqslant 1$, we have

$$\varphi v(\{a\}) = \int_0^1 f(\mu(\{a\}) + x\mu(T_0)) \, dx. \tag{5.1}$$

THEOREM 5.1. *Let $\mu = (\mu_1, ..., \mu_m)$ be a non-trivial measure in $ca_+(\lambda)$. Assume that $f: \mathfrak{R}_+^m \to \mathfrak{R}_+$ satisfies the assumptions of Theorem 3.1. Let φv and ψv be, respectively, the asymptotic Shapley value and the asymptotic nucleolus of the game $v = f \circ \mu$. Then*

$$\varphi v(\{a\}) \leqslant \psi v(\{a\}).$$

Proof. By Theorem 3.1 we have

$$\psi v(\{a\}) = f(\mu(T)) - \tfrac{1}{2} \nabla f(\mu(T)) \cdot \mu(T_0).$$

Since f is concave, for every $0 \leqslant x \leqslant 1$ we have

$$f(\mu(\{a\}) + x\mu(T_0)) \leqslant f(\mu(T)) + (x-1)\,\nabla f(\mu(T)) \cdot \mu(T_0).$$

By integrating both sides of (5.2) over the interval $[0, 1]$ we obtain

$$\varphi v(\{a\}) \leqslant f(\mu(T)) - \tfrac{1}{2}\nabla f(\mu(T)) \cdot \mu(T_0) = \psi v(\{a\}).$$

Q.E.D

The following corollary is a direct consequence of Theorems 5.1 and 4.5.

COROLLARY 5.2. *Let v be the market game defined in* (4.3). *If the economy E satisfies the assumptions of Theorem* 4.5, *then*

$$\varphi v(\{a\}) \leqslant \psi v(\{a\}).$$

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