



Working Paper : #09-35 (13)  
Statistics and Econometrics Series  
June 2009

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## **INEQUALITIES FOR THE RUIN PROBABILITY IN A CONTROLLED DISCRETE- TIME RISK PROCESS**

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**Abstract:** Ruin probabilities in a controlled discrete-time risk process with a Markov chain interest are studied. To reduce the risk there is a possibility to reinsure a part or the whole reserve. Recursive and integral equations for ruin probabilities are given. Generalized Lundberg inequalities for the ruin probabilities are derived given a constant stationary policy. The relationships between these inequalities are discussed. To illustrate these results some numerical examples are included.

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# INEQUALITIES FOR THE RUIN PROBABILITY IN A CONTROLLED DISCRETE-TIME RISK PROCESS

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AMS 2000 Subject Classification: Primary: 91B30.

Secondary: 60J05;60K10.

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October 8, 2008

## Abstract

Ruin probabilities in a controlled discrete-time risk process with a Markov chain interest are studied. To reduce the risk of ruin there is a possibility to reinsure a part or the whole reserve. Recursive and integral equations for ruin probabilities are given. Generalized Lundberg inequalities for the ruin probabilities are derived given a constant stationary policy. The relationships between these inequalities are discussed. To illustrate these results some numerical examples are included.

## 1 Introduction

This paper studies an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. Controlling a risk process is a very active area of research, particularly in the last decade; see [5, 6, 8, 9], for instance. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. Hence, an alternative method commonly used in ruin theory is to derive *inequalities* for ruin probabilities (see Asmussen [1], Grandell [4], Schmidli [9], and Willmot

and Lin [10]). Following Cai [2] and Cai and Dickson [3], we model the interest rate process as a denumerable state Markov chain. This model can be in fact a discrete counterpart of the most frequently occurring effect observed in continuous interest rate process, e.g., mean-reverting effect. Stochastic inequalities for the ruin probabilities are derived by martingales and inductive techniques. The inequalities can be used to obtain upper bounds for the ruin probabilities. For the sake of simplicity, we restrict ourselves to use stationary control policies. Explicit condition are obtained for the optimality of employing no reinsurance.

The outline of the paper is as follows. In Section 2 the risk model is formulated. Some important special cases of this model are briefly discussed. In Section 3 we derive recursive equations for finite-horizon ruin probabilities and integral equations for the ultimate ruin probability. In Section 4 we obtain upper bounds for the ultimate probability of ruin. An analysis of the new bounds and a comparison with the Lundberg's inequality is also included. Finally, in section 5 we illustrate our results on the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process. We conclude in Section 6 with some general comments and some some further research.

## 2 The model

We consider a discrete-time insurance risk process in which the surplus  $X_n$  varies according to the equation

$$X_n = X_{n-1} (1 + I_n) + \mathcal{C}(b_{n-1}) \cdot Z_n - h(b_{n-1}, Y_n), \text{ for } n \geq 1 \quad (1)$$

with  $X_0 = x \geq 0$ . Following Schmidli [9] p. 21, we introduce an absorbing (cemetery) state  $\varkappa$ , such that if  $X_n < 0$  or  $X_n = \varkappa$ , then  $X_{n+1} = \varkappa$ . We denote the state space by  $\mathbb{X} = \mathbb{R} \cup \varkappa$ . Let  $Y_n$  be the  $n$ -th claim payment, which we assume to form a sequence of i.i.d. random variables with common probability distribution function (p.d.f.)  $F$ . The random variable  $Z_n$  stands for the *length of the  $n$ -th period*, that is, the time between the occurrence of the claims  $Y_{n-1}$  and  $Y_n$ . We assume that  $\{Z_n\}$  is a sequence of i.i.d. random variables with p.d.f.  $G$ . This case includes a controlled version of the Cramér-Lundberg model if we assume that the claims occur as a Poisson process. Of course, we can also think of the case where  $Z_n = 1$  is deterministic. In addition, we suppose that  $\{Y_n\}_{n \geq 1}$  and  $\{Z_n\}_{n \geq 1}$  are independent.

The process can be controlled by reinsurance, that is, by choosing the *retention level* (or proportionality factor or risk exposure)  $b \in \mathcal{B}$  of a reinsurance contract for one period, where  $\mathcal{B} := [b_{\min}, 1]$ , and  $b_{\min} \in (0, 1]$  will be introduced below. Let  $\{I_n\}_{n \geq 0}$  be the *interest rate process*; we suppose that  $I_n$  evolves as a Markov chain with a denumerable (possibly finite) state space  $\mathbb{I}$  consisting of nonnegative integers.

The function  $h(b, y)$  with values in  $[0, y]$  specifies the fraction of the claim  $y$  paid by the insurer, and it also depends on the retention level  $b$  at the beginning of the period. Hence  $y - h(b, y)$  is the part paid by the reinsurer. The retention level  $b = 1$  stands for the control

action *no reinsurance*. In this article, we consider the case of *proportional reinsurance*, which means that

$$h(b, y) := b \cdot y, \text{ with retention level } b \in \mathcal{B}. \quad (2)$$

The premium (income) rate  $c$  is fixed. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level  $b$ , we denote by  $\mathcal{C}(b)$  the premium left for the insurer if the retention level  $b$  is chosen, where

$$0 \leq \mathcal{C}(b) \leq c, \quad b \in \mathcal{B}.$$

We define  $b_{\min} := \min\{b \in (0, 1] | \mathcal{C}(b) \geq 0\}$ . Moreover,  $\mathcal{C}(b)$  is an increasing function that we will calculate according to the *expected value principle* with added safety loading  $\theta$  from the reinsurer:

$$\mathcal{C}(b) = c - (1 + \theta)(1 - b) \frac{E[Y]}{E[Z]}, \quad (3)$$

where  $Y$  and  $Z$  are generic random variables with p.d.f.  $F$  and  $G$ , respectively.

We consider Markovian control policies  $\pi = \{a_n\}_{n \geq 1}$ , which at each time  $n$  depend only on the current state, that is,  $a_n(X_n) := b_n$  for  $n \geq 0$ . Abusing notation, we will identify functions  $a : \mathbb{X} \rightarrow \mathcal{B}$  with stationary strategies, where  $\mathcal{B} = [b_{\min}, 1]$ , the decision space. Consider an arbitrary initial state  $X_0 = x \geq 0$  (note that the initial value is not stochastic) and a control policy  $\pi = \{a_n\}_{n \geq 1}$ . Then, by iteration of (1) and assuming (2), and (3), it follows that for  $n \geq 1$ ,  $X_n$  satisfies

$$X_n = x \prod_{l=1}^n (1 + I_l) + \sum_{l=1}^n \left( \mathcal{C}(b_{l-1}) Z_l - b_{l-1} \cdot Y_l \prod_{m=l+1}^n (1 + I_m) \right). \quad (4)$$

Let  $(p_{ij})$  be the matrix of transition probabilities of  $\{I_n\}$ , i.e.,

$$p_{ij} := P(I_{n+1} = j | I_n = i), \quad (5)$$

where  $p_{ij} \geq 0$  and  $\sum_j p_{ij} = 1$  for all  $i, j \in \mathbb{I}$ . The ruin probability when using the policy  $\pi$ , given the initial surplus  $x$ , and the initial interest rate  $I_0 = i$  is defined as

$$\psi^\pi(x, i) := P^\pi \left( \bigcup_{k=1}^{\infty} \{X_k < 0\} | X_0 = x, I_0 = i \right), \quad (6)$$

which we can also express as

$$\psi^\pi(x, i) = P^\pi (X_k < 0 \text{ for some } k \geq 1 | X_0 = x, I_0 = i). \quad (7)$$

Similarly, the ruin probabilities in the finite horizon case are given by

$$\psi_n^\pi(x, i) := P^\pi \left( \bigcup_{k=1}^n \{X_k < 0\} | X_0 = x, I_0 = i \right). \quad (8)$$

Thus,

$$\psi_1^\pi(x, i) \leq \psi_2^\pi(x, i) \leq \cdots \leq \psi_n^\pi(x, i) \leq \cdots,$$

and

$$\lim_{n \rightarrow \infty} \psi_n^\pi(x, i) = \psi^\pi(x, i).$$

The following lemma is used below to simplify some calculations.

**Lemma 1.** *For any given policy  $\pi$ , there is a function  $\psi^\pi(x)$  such that*

$$\psi^\pi(x, i) \leq \psi^\pi(x)$$

for every initial state  $x > 0$  and initial interest rate  $I_0 = i$ .

*Proof.* By (1) and (2), the risk model is given by

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n.$$

Since  $I_n \geq 0$ , we have

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n \geq X_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n. \quad (9)$$

Define recursively

$$\tilde{X}_n := \tilde{X}_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n, \quad (10)$$

with  $X_0 = \tilde{X}_0 = x$ . Hence,  $X_n \geq \tilde{X}_n$  for all  $n \in \mathbb{N}$ . Clearly, if  $X_n < 0$ , then  $\tilde{X}_n < 0$ .

Let

$$\mathcal{E}_1 := \left\{ \omega \in \Omega \mid \bigcup_{n=1}^{\infty} \{X_n(\omega) < 0\} \right\} \text{ and } \mathcal{E}_2 := \left\{ \omega \in \Omega \mid \bigcup_{n=1}^{\infty} \{\tilde{X}_n(\omega) < 0\} \right\},$$

and note that  $\mathcal{E}_1 \subset \mathcal{E}_2$ . Therefore,

$$P^\pi \left( \bigcup_{n=1}^{\infty} \{X_n < 0\} \mid I_0 = i \right) \leq P^\pi \left( \bigcup_{n=1}^{\infty} \{\tilde{X}_n < 0\} \mid I_0 = i \right),$$

and since the  $\tilde{X}_n$  do not depend on  $I_n$ , we obtain from (6)

$$\psi^\pi(x, i) = P^\pi \left( \bigcup_{n=1}^{\infty} \{X_n < 0\} \mid X_0 = x, I_0 = i \right) \leq P^\pi \left( \bigcup_{n=1}^{\infty} \{\tilde{X}_n < 0\} \mid X_0 = x \right) =: \psi^\pi(x).$$

□

We denote by  $\Pi$  the policy space. A control policy  $\pi^*$  is said to be optimal if for any initial values  $(X_0, I_0) = (x, i)$ , we have

$$\psi^{\pi^*}(x, i) \leq \psi^\pi(x, i)$$

for all  $\pi \in \Pi$ . Schmidli [9] and Schäl [8] show the existence of an optimal control policy for some special cases of the model risk (1). However, even in these special cases it is extremely difficult to obtain closed expressions for  $\psi^{\pi^*}(x, i)$ . We are thus led to consider bounds for the ruin probabilities, which we do in sections 3, 4, and 5, below. First, we note that (1) includes some interesting ruin models.

**Special cases.** To conclude this section we note the following subcases of the risk model (1).

1. If  $I_n = 0$  for all  $n \geq 1$ , then the risk model (4) reduces to the discrete-time risk model with proportional reinsurance:

$$X_n = x - \sum_{t=1}^n (b_{t-1}Y_t - \mathcal{C}(b_{t-1})Z_t). \quad (11)$$

or equivalently,

$$X_n = X_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n.$$

The corresponding ruin probability is

$$\psi^\pi(x) := P^\pi \left( \bigcup_{n=1}^{\infty} \{X_n < 0\} \mid X_0 = x \right).$$

Assuming, constant stationary strategies, say  $b_n = b_0$ , then, by (11),

$$\psi^\pi(x) = P^\pi \left( \bigcup_{n=1}^{\infty} \{ \sum_{t=1}^n [b_0Y_t - \mathcal{C}(b_0)Z_t] > x \} \mid X_0 = x \right).$$

Moreover, if we assume that  $b_0E[Y] < \mathcal{C}(b_0)E[Z]$ , then there exists a constant  $R_0 \equiv R_0(b_0) > 0$  satisfying

$$E \left[ e^{-R_0(\mathcal{C}(b_0)Z - b_0Y)} \right] = 1. \quad (12)$$

Therefore, by the classical Lundberg inequality for ruin probabilities (see [1, 4, 10]),

$$\psi^\pi(x) \leq e^{-R_0x} \text{ for } x \geq 0. \quad (13)$$

Since determining ruin probabilities is essentially an infinite-horizon problem, it suffices to consider stationary strategies.

**Remark 1.** *It is enough to consider constant stationary strategies in this paper, i.e.,  $b_n = b$  for all  $n \geq 1$  and we will to argue: first, we assume that  $P(bY > \mathcal{C}(b)Z) > 0$  for all  $b \in \mathcal{B}$ . Because, if there is some  $b_e \in \mathcal{B}$  such that  $P(b_eY > \mathcal{C}(b_e)Z) = 0$ , the ruin can be prevented by retention level  $b_e$  and the risk process considered in this case becomes trivial. Secondly, we assume the net profit condition  $E^\pi[\mathcal{C}(b)Z - bY] > 0$  for some  $\pi \in \Pi$ . Otherwise, ruin cannot be prevented because the surplus would be decreasing in time for all reinsurance treaties. Therefore, using the law of large numbers, we have*

$$\frac{1}{n} \sum_{i=1}^n [\mathcal{C}(b)Z_i - bY_i] \rightarrow E^\pi[\mathcal{C}(b)Z - bY],$$

this implies that for the stationary strategy  $b_n = b$  the process  $X_n$  tends to infinity (in particular,  $\inf_n X_n > -\infty$ ). Hence, there is an initial capital  $X_0 = x$  such that  $P(\inf_n X_n \geq 0 | X_0 = x) > 0$ . Because there is a strictly positive probability that from initial capital zero the set  $[x, \infty)$  is reached before the set  $(-\infty, 0)$ , we get also that  $P(\inf_n X_n \geq 0 | X_0 = 0) > 0$ . Finally, we have a stationary strategy for which ruin is not certain.

2. **Exponentially distributed length periods.** Our process (1) a controlled version of Cramér-Lundberg model if the claims occur as a Poisson process, in which case the  $Z_n$  are exponentially distributed, say  $Z_n \simeq \text{Exp}(\lambda)$ . Suppose that, in addition,  $I_n = 0$  for all  $n \geq 1$ , and the that single claims have expectation  $\mu$  and moment generating function  $M_Y(s)$ . Thus,  $Y_n$  has a compound distribution with expectation  $\lambda\mu$  and moment generation function  $e^{\lambda(M_Y(s)-1)}$ . Let  $M_Y(b; r) := \int_0^\infty e^{bry} dF(y)$  be the moment-generating function of the part of the claim the insurer has to pay if the retention level  $b$  is chosen.

We assume constant stationary strategies, say  $b_n = b_0$  for all  $n \geq 1$ . Moreover, we assume that  $\mathcal{C}(b_0) > b_0\lambda\mu$  and  $M_Y(b_0; r) < \infty$  for some  $r > 0$  and  $b_0 \in \mathcal{B}$ . It is clear that the risk process  $X_n - x = \sum_{k=1}^n (\mathcal{C}(b)Z_n - bY_k)$  satisfies all the hypotheses of theorem 14 in [4] p.10 [4]. Then

$$E^\pi [e^{-R_0[\mathcal{C}(b)-bY_n]}] = e^{-R_0\mathcal{C}(b_0)} \cdot e^{\lambda(M_Y(b_0; R_0)-1)}.$$

Then, by (12), we have that the adjustment coefficient  $R_0 = R_0(b_0)$  fulfils

$$-R_0\mathcal{C}(b_0) + \lambda(M_Y(b_0; R_0) - 1) = 0. \quad (14)$$

By Lemma 4.1 Schmidli [9],  $R_0$  is unimodal and it attains its maximum value at a point  $b_0^* \in \mathcal{B}$ . Then, it is easy to see that it is optimal to have no reinsurance ( $b_0^* = 1$ ) if and only if the safety loading  $\theta$  is too high in the sense that

$$1 + \theta \geq \frac{M_Y'(1, R_0)}{\mu}. \quad (15)$$

3. Let  $d_n$  be the constant, short-term dividend rate in the  $n$ -th period (the dividends are payments made by a corporation to its shareholder members). Then the discrete-time risk model with stochastic interest rate and dividends is given by

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1})Z_n - h(b_{n-1}, Y_n) - d_n X_n,$$

with  $h(b, y)$  as in (2). Thus, rearranging terms,

$$X_n = X_{n-1} \left( \frac{1 + I_n}{1 + d_n} \right) + \frac{\mathcal{C}(b_{n-1})}{(1 + d_n)} Z_n - \frac{h(b_{n-1}, Y_n)}{(1 + d_n)}.$$

Let  $Y'_n := \frac{Y_n}{(1+d_n)}$  and  $I'_n := \frac{I_n - d_n}{(1+d_n)}$ . Since  $\{I_n\}$  and  $\{Y_n\}$  are independent, then so are  $\{I'_n\}$  and  $\{Y'_n\}$ . Let  $\mathcal{C}'(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1+d_n)}$ . Then the model becomes

$$X_n = X_{n-1}(1 + I'_n) + \mathcal{C}'(b_{n-1})Z_n - h(b_{n-1}, Y'_n),$$

which from a statistical viewpoint is essentially the same as the model without dividends (1) and can be analyzed in a similar way.

4. As an extension of the latter case, some companies have dividend reinvestment plans (or DRIPs). These plans allow shareholders to use dividends to systematically buy small amounts of stock. Let  $\tilde{d}_n$  be the short term dividend reinvestment rate in the  $n$ -th period,  $\tilde{d}_n \in [0, 1)$ . Then, the discrete-time risk model with stochastic interest rate and dividends reinvestment is given by

$$X_n = X_{n-1} (1 + I_n) + \mathcal{C}(b_{n-1})Z_n - h(b_{n-1}, Y_n) + \tilde{d}_n X_n.$$

Hence, rearranging terms, we obtain

$$X_n = X_{n-1} \left( \frac{1 + I_n}{1 - \tilde{d}_n} \right) + \frac{\mathcal{C}(b_{n-1})}{(1 - \tilde{d}_n)} Z_n - \frac{h(b_{n-1}, Y_n)}{(1 - \tilde{d}_n)}.$$

Let  $Y_n'' := \frac{Y_n}{(1 - \tilde{d}_n)}$ ,  $I_n'' := \frac{I_n - \tilde{d}_n}{(1 - \tilde{d}_n)}$ , and  $\mathcal{C}''(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1 - \tilde{d}_n)}$ . It follows that

$$X_n = X_{n-1} (1 + I_n'') + \mathcal{C}''(b_{n-1})Z_n - h(b_{n-1}, Y_n''),$$

which, again, is essentially the same as the model (1).

Let us go back to the original risk model (1). In the next section, we will derive recursive equations for the ruin probabilities and integral equations for the ultimate ruin probability associated to the model (1).

**Remark 2.** Given a p.d.f.  $G$ , we denote the tail of  $G$  by  $\bar{G}$ , that is,  $\bar{G}(x) := 1 - G(x)$ .

### 3 Recursive and integral equations for ruin probabilities

In this section, we first derive a recursive equation for  $\psi_n^\pi(x, i)$ . Secondly, we give an integral equation for  $\psi^\pi(x, i)$ . Finally, we obtain an equation for the ruin probability with horizon  $n = 1$  given  $I_0 = i$ ,  $X_0 = x$  and a stationary policy  $\pi$ . These results, which are valid for any initial interest rate, are summarized in the following lemma.

**Lemma 2.** Let  $u(y, z) := b_0 y - \mathcal{C}(b_0)z$ , where  $b_0$  is the initial retention level. Let  $\tau_j(z) := (x(1 + j) + \mathcal{C}(b_0)z) / b_0$ ,  $X_0 = x \geq 0$ , and  $p_{ij}$  as in (5). Then

$$\psi_1^\pi(x, i) = \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \bar{F}(\tau_j(z)) dG(z), \quad (16)$$

and for  $n = 1, 2, \dots$

$$\begin{aligned} \psi_{n+1}^\pi(x, i) &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \int_0^{\tau_j(z)} \psi_n^\pi(x(1 + j) - u(y, z), j) dF(y) dG(z) \\ &+ \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \bar{F}(\tau_j(z)) dG(z). \end{aligned} \quad (17)$$



Moreover,

$$\begin{aligned}\psi^\pi(x, i) &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \int_0^{\tau_j(z)} \psi^\pi(x(1+j) - u(y, z), j) dF(y) dG(z) \\ &+ \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \bar{F}(\tau_j(z)) dG(z).\end{aligned}\tag{18}$$

*Proof.* Let  $U_k := u(Y_k, Z_k) = b_0 Y_k - \mathcal{C}(b_0) Z_k$ . Given  $Y_1 = y, Z_1 = z$ , the control strategy  $\pi$ , and  $I_1 = j$ , from (4) we have  $U_1 = u(y, z)$ . Therefore,

$$X_1 = x(1 + I_1) - U_1 = h_1 - u(y, z), \text{ where } h_1 = x(1 + j)$$

Thus, if  $u(y, z) > h_1$  then

$$P^\pi(X_1 < 0 | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i) = 1.$$

This implies that for  $u(y, z) > h_1$

$$P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i \right) = 1,$$

while if  $0 \leq u(y, z) \leq h_1$ , then

$$P^\pi(X_1 < 0 | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i) = 0.\tag{19}$$

Let  $\{\tilde{Y}_n\}_{n \geq 1}$ ,  $\{\tilde{Z}_n\}_{n \geq 1}$ , and  $\{\tilde{I}_n\}_{n \geq 0}$  be independent copies of  $\{Y_n\}_{n \geq 1}$ ,  $\{Z_n\}_{n \geq 1}$ , and  $\{I_n\}_{n \geq 0}$ , respectively.

Let  $\tilde{U}_k := b_0 \tilde{Y}_k - \mathcal{C}(b_0) \tilde{Z}_k$ . Thus, (19) and (4) yield that for  $0 \leq u(y, z) \leq h_1$ ,

$$\begin{aligned}& P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i \right) \\ &= P^\pi \left( \bigcup_{k=2}^{n+1} \{X_k < 0\} | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i \right) \\ &= P^\pi \left( \bigcup_{k=2}^{n+1} \left\{ (h_1 - u(y, z)) \prod_{l=1}^k (1 + I_l) - \sum_{l=1}^k U_l \prod_{m=l+1}^k (1 + I_m) < 0 \right\} | X_0 = x, I_1 = j \right) \\ &= P^\pi \left( \bigcup_{k=1}^n \left\{ (h_1 - u(y, z)) \prod_{l=1}^k (1 + \tilde{I}_l) - \sum_{l=1}^k \tilde{U}_l \prod_{m=l+1}^k (1 + \tilde{I}_m) < 0 \right\} | X_0 = x, \tilde{I}_0 = j \right) \\ &= \psi_n^\pi(h_1 - u(y, z), j) = \psi_n^\pi(x(1 + j) - u(y, z), j)\end{aligned}$$

where the second equality follows from the Markov property of  $\{I_n\}_{n \geq 0}$ , and the independence of  $\{Y_n\}_{n \geq 1}$ ,  $\{Z_n\}_{n \geq 1}$  and  $\{I_n\}_{n \geq 0}$ .

Let us now consider the event  $\mathcal{A} = \{Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i\}$ , and recall that  $F(y) = P(Y \leq y)$  and  $G(z) = P(Z \leq z)$ . From (8) and (4) we obtain

$$\begin{aligned}\psi_{n+1}^\pi(x, i) &= P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} \mid X_0 = x, I_0 = i \right) \\ &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \int_0^\infty P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} \mid \mathcal{A} \right) dF(y) dG(z).\end{aligned}$$

Then, recalling that  $\tau_j(z) = \frac{x(1+j) + \mathcal{C}(b_0)z}{b_0}$ ,

$$\begin{aligned}\psi_{n+1}^\pi(x, i) &= \sum_{j \in \mathbb{I}} p_{ij} \left\{ \int_0^\infty \int_0^{\tau_j(z)} P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} \mid \mathcal{A} \right) dF(y) dG(z) \right. \\ &\quad \left. + \int_0^\infty \int_{\tau_j(z)}^\infty P^\pi \left( \bigcup_{k=1}^{n+1} \{X_k < 0\} \mid \mathcal{A} \right) dF(y) dG(z) \right\} \\ &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \left\{ \int_0^{\tau_j(z)} \psi_n^\pi(x(1+j) - u(y, z), j) dF(y) dG(z) + \int_0^\infty \int_{\tau_j(z)}^\infty dF(y) dG(z) \right\} \\ &= \sum_{j \in \mathbb{I}} p_{ij} \left\{ \int_0^\infty \int_0^{\tau_j(z)} \psi_n^\pi(x(1+j) - u(y, z), j) dF(y) dG(z) + \int_0^\infty \bar{F}(\tau_j(z)) dG(z) \right\}.\end{aligned}\tag{20}$$

This gives (17). In particular,

$$\psi_1^\pi(x, i) = \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \bar{F}(\tau_j(z)) dG(z).$$

Finally, letting  $n \rightarrow \infty$  in (20) and using dominated convergence we obtain  $\lim_{n \rightarrow \infty} \psi_{n+1}^\pi(x, i) = \psi^\pi(x, i)$ , and (18) follows.  $\square$

**Remark 3.** *If we consider the risk model without reinsurance, that is,  $b = 1$ , we obtain similar results to those in Cai and Dickson [3].*

## 4 Bounds for ruin probabilities

We will use the results obtained in Section 3 to find upper bounds for the ruin probabilities with infinite horizon taking into account the information contributed by the Markov chain of the interest rate process. We derive a functional for the ultimate ruin probability in terms of the new worse than used in convex (NWUC) ordering; see Remark 4, below. This idea was first introduced by Willmot and Lin [10] and has been generalized by other authors.

We will present two upper bounds for the ruin probabilities. The first bound is obtained by an inductive approach, and the second by a martingale approach. These bounds are discussed in Remark 5, at the end of this section.

## 4.1 Bounds obtained by the inductive approach

**Theorem 1.** *Let  $R_0 > 0$  be the constant satisfying (12). Then, for all  $x \geq 0$  and  $i \in \mathbb{I}$ ,*

$$\begin{aligned}\psi^\pi(x, i) &\leq \beta \sum_{j \in \mathbb{I}} p_{ij} E^\pi [e^{-R_0 x(1+j)}] \\ &= \beta E^\pi [e^{-R_0 [x(1+I_1)]} | I_0 = i],\end{aligned}\tag{21}$$

where  $\beta \equiv \beta(b_0)$  and is given by

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 t} \bar{F}(t)}.$$

*Proof.* It suffices to show that the rightmost term in (21) is an upper bound for  $\psi_n^\pi(x, i)$ , for all  $n \geq 1$ . We will prove this by induction. First note that

$$\begin{aligned}\bar{F}(\vartheta) &= \left( \frac{\int_\vartheta^\infty e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 \vartheta} \bar{F}(\vartheta)} \right)^{-1} e^{-R_0 b_0 \vartheta} \int_\vartheta^\infty e^{R_0 b_0 y} dF(y) \\ &\leq \beta e^{-R_0 b_0 \vartheta} \int_\vartheta^\infty e^{R_0 b_0 y} dF(y) \leq \beta e^{-R_0 b_0 \vartheta} E^\pi [e^{R_0 b Y_1}]\end{aligned}\tag{22}$$

for any  $\vartheta \geq 0$ . This implies that for every  $x \geq 0$ ,  $i \geq 0$ , and  $b_0 \in \mathcal{B}$ , by (16) and (22) we have

$$\begin{aligned}\psi_1^\pi(x, i) &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \bar{F}(\tau_j(z)) dG(z) \\ &\leq \sum_{j \in \mathbb{I}} p_{ij} \left( \beta E^\pi [e^{R_0 b Y_1}] \cdot \int_0^\infty e^{-R_0 b_0 \left( \frac{x(1+j) + \mathcal{C}(b_0)z}{b_0} \right)} dG(z) \right) \\ &= \beta E^\pi [e^{R_0 b Y_1}] \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty e^{-R_0 [x(1+j) + \mathcal{C}(b_0)z]} dG(z) \\ &= \beta E^\pi [e^{R_0 b Y_1}] \sum_{j \in \mathbb{I}} p_{ij} E^\pi [e^{-R_0 [x(1+j) + \mathcal{C}(b_0)Z_1]} | I_0 = i] \\ &= \beta E^\pi [e^{R_0 b Y_1}] \cdot E^\pi [e^{-R_0 [x(1+I_1) + \mathcal{C}(b)Z_1]} | I_0 = i] \\ &= \beta E^\pi [e^{R_0 b Y_1}] E^\pi [e^{-R_0 \mathcal{C}(b)Z_1}] E^\pi [e^{-R_0 x(1+I_1)} | I_0 = i] \\ &= \beta E^\pi [e^{-R_0 [\mathcal{C}(b)Z_1 - bY_1]}] \cdot E^\pi [e^{-R_0 x(1+I_1)} | I_0 = i] \\ &= \beta E^\pi [e^{-R_0 x(1+I_1)} | I_0 = i] \quad (\text{by (12)}).\end{aligned}$$

This shows that the desired result holds for  $n = 1$ . To prove the result for general  $n \geq 1$ , the induction hypothesis is that, for some  $n \geq 1$ , and every  $x \geq 0$  and  $i \in \mathbb{I}$ ,

$$\psi_n^\pi(x, i) \leq \beta E^\pi [e^{-R_0 x(1+I_1)} | I_0 = i].\tag{23}$$

Now, let  $0 \leq y \leq \tau_j(z)$ , with  $\tau_j(z)$  as in Lemma 2. Further, in (23) replace  $x$  and  $i$  by  $x(1+j) + \mathcal{C}(b_0)z - b_0y$  and  $j$ , respectively, to obtain

$$\begin{aligned} \psi_n^\pi(x(1+j) + \mathcal{C}(b_0)z - b_0y, j) &\leq \beta E^\pi[e^{-R_0[x(1+j)+\mathcal{C}(b_0)z-b_0y](1+I_1)} | I_0 = j] \\ &\leq \beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)z-b_0y]}. \end{aligned} \quad (24)$$

Therefore, replacing (24) in (17), we get

$$\begin{aligned} \psi_{n+1}^\pi(x, i) &\leq \sum_{j \in \mathbb{I}} p_{ij} \left( \beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_{\tau_j(z)}^\infty e^{R_0b_0y} dF(y) dG(z) \right) \\ &\quad + \sum_{j \in \mathbb{I}} p_{ij} \left( \beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_0^{\tau_j(z)} e^{R_0b_0y} dF(y) dG(z) \right) \\ &= \sum_{j \in \mathbb{I}} p_{ij} \left( \beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_0^\infty e^{R_0b_0y} dF(y) dG(z) \right) \\ &= \beta E^\pi[e^{R_0bY_1}] \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} dG(z) \\ &= \beta E^\pi[e^{R_0bY_1}] \cdot E^\pi[e^{-R_0\mathcal{C}(b)Z_1}] \cdot E^\pi[e^{-R_0x(1+I_1)} | I_0 = i] \\ &= \beta E^\pi[e^{-R_0x(1+I_1)} | I_0 = i]. \end{aligned}$$

Hence, (23) holds for any  $n = 1, 2, \dots$ . Finally, letting  $n \rightarrow \infty$  in (23) we obtain (21).  $\square$

As an application of Theorem 1, we next consider the special case in which the claim distribution is in the class of NWUC distributions [10] p. 25, which are defined as follows.

**Remark 4.** A distribution  $F$  concentrated on  $(0, \infty)$  is said to be **new worse than used in convex (NWUC) ordering** if, for all  $x, y \geq 0$

$$\int_{x+y}^\infty \bar{F}(z) dz \geq \bar{F}(y) \int_x^\infty \bar{F}(z) dz.$$

For example, let  $F$  a phase-type distribution with parameters  $(\alpha, T)$  (see [1] pp. 215–222). Then  $F$  is NWUC if and only if  $T^{-1}$  and  $T^{-1}e^{Ty}(\mathcal{I} - \vec{\mathbf{1}}\alpha)$  are both non-negative or non-positive definite simultaneously for all  $y \geq 0$  (where  $\mathcal{I}$  represent the identity matrix and  $\vec{\mathbf{1}}$  is the column vector of ones).

**Corollary 1.** Under the hypotheses of Theorem 1, and assuming that  $E^\pi[e^{R_0bY_1}] < \infty$  for all  $b \in \mathcal{B}$ , and that, in addition,  $F$  is a NWUC distribution, we have

$$\psi^\pi(x, i) \leq (E^\pi[e^{R_0bY_1}])^{-1} E^\pi[e^{-R_0x(1+I_1)} | I_0 = i]. \quad (25)$$

*Proof.* Following Willmot and Lin [10] pp. 96–97, let  $r := R_0b > 0$ . Therefore

$$\beta^{-1} := \inf_{t \geq 0} \frac{\int_t^\infty e^{ry} dF(y)}{e^{rt} \bar{F}(t)} = \int_0^\infty e^{ry} dF(y),$$

that is,  $\beta^{-1} = E^\pi[e^{R_0bY_1}]$ . Finally, replacing this equality in (21), we obtain (25).  $\square$

## 4.2 Bounds by means of the martingale approach

Another way for deriving upper bounds for ruin probabilities is the martingale approach. To this end, let  $V_n := X_n \prod_{l=1}^n (1 + I_l)^{-1}$  with  $n \geq 1$ , be the so-called discounted risk process. The ruin probabilities  $\psi_n^\pi$  in (8) associated to the process  $\{V_n, n = 1, 2, \dots\}$  are

$$\psi_n^\pi(x, i) = P^\pi \left( \bigcup_{k=1}^n (V_k < 0) \mid X_0 = x, I_0 = i \right).$$

In the classical risk model,  $\{e^{-R_0 X_n}\}_{n \geq 1}$  is a martingale. However, for our model (4), there is no constant  $r > 0$  such that  $\{e^{-r X_n}\}_{n \geq 1}$  is a martingale. Still, there exists a constant  $r > 0$  such that  $\{e^{-r V_n}\}_{n \geq 1}$  is a *supermartingale*, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

**Proposition 1.** *Assume that for each  $i \in \mathbb{I}$ , there exists  $\rho_i > 0$  satisfying that*

$$E^\pi \left[ e^{-\rho_i [\mathcal{C}(b)Z_1 - bY_1](1+I_1)^{-1}} \mid I_0 = i \right] = 1. \quad (26)$$

Then

$$R_1 := \min_{i \in \mathbb{I}} \rho_i \geq R_0 \quad (27)$$

and, furthermore, for all  $i \in \mathbb{I}$

$$E^\pi \left[ e^{-R_1 [\mathcal{C}(b)Z_1 - bY_1](1+I_1)^{-1}} \mid I_0 = i \right] \leq 1. \quad (28)$$

*Proof.* For each  $i \in \mathbb{I}$ , let

$$l_i(r) := E^\pi \left[ e^{-r [\mathcal{C}(b)Z - bY](1+I_1)^{-1}} \mid I_0 = i \right] - 1, \quad \text{for } r > 0.$$

Then the first derivative of  $l_i(r)$  at  $r = 0$  is

$$l_i'(0) = E^\pi [-\mathcal{C}(b)Z - bY] \cdot E \left[ (1 + I_1)^{-1} \mid I_0 = i \right] < 0 \quad (\text{by independence}),$$

and the second derivative is

$$l_i''(r) = E^\pi \left[ ((\mathcal{C}(b)Z - bY)(1 + I_1)^{-1})^2 \cdot e^{-r [\mathcal{C}(b)Z - bY](1+I_1)^{-1}} \mid I_0 = i \right] > 0.$$

This shows that  $l_i(r)$  is a convex function. Let  $\rho_i$  be the unique positive root of the equation  $l_i(r) = 0$  on  $(0, \infty)$ . Further, if  $0 < \rho \leq \rho_i$ , then  $l_i(\rho) \leq 0$ . However,

$$E^\pi \left[ e^{-R_0 [\mathcal{C}(b)Z - bY](1+I_1)^{-1}} \mid I_0 = i \right] = \sum_{j \in \mathbb{I}} p_{ij} E \left[ e^{-R_0 [\mathcal{C}(b_0)Z - b_0 Y](1+j)^{-1}} \right]$$

$$(\text{by Jensen's inequality}) \leq \sum_{j \in \mathbb{I}} p_{ij} E \left[ e^{-R_0 [\mathcal{C}(b_0)Z_1 - b_0 Y_1]} \right]^{(1+j)^{-1}}.$$

Consequently, by (12), we have  $E \left[ e^{-R_0[\mathcal{C}(b_0)Z_1 - b_0Y_1]} \right] = 1$ . Hence, since  $\sum_{j \in \mathbb{I}} p_{ij} = 1$ ,

$$E^\pi \left[ e^{-R_0[\mathcal{C}(b)Z - bY](1+I_1)^{-1}} \mid I_0 = i \right] \leq 1.$$

This implies that  $l_i(R_0) \leq 0$ . Moreover,  $R_0 \leq \rho_i$  for all  $i$ , and so

$$R_1 := \min_{i \in \mathbb{I}} \rho_i \geq R_0.$$

Thus, (27) holds. In addition  $R_1 \leq \rho_i$  for all  $i \in \mathbb{I}$ , which implies that  $l_i(R_1) \leq 0$ . This yields (28).  $\square$

**Theorem 2.** *Under the hypotheses of Proposition 1, for all  $i \in \mathbb{I}$  and  $x \geq 0$ ,*

$$\psi^\pi(x, i) \leq e^{-R_1 x}. \quad (29)$$

*Proof.* By (4), the discounted risk process  $V_k := X_k \prod_{l=1}^k (1 + I_l)^{-1}$  satisfies that

$$V_k := x + \sum_{l=1}^k \left( (\mathcal{C}(b_0)Z_l - b_0Y_l) \prod_{t=1}^l (1 + I_t)^{-1} \right). \quad (30)$$

Let  $S_n = e^{-R_1 V_n}$ . Then

$$S_{n+1} = S_n e^{-R_1(\mathcal{C}(b_0)Z_{n+1} - b_0Y_{n+1}) \prod_{t=1}^{n+1} (1+I_t)^{-1}}.$$

Thus, for any  $n \geq 1$ ,

$$\begin{aligned} & E^\pi[S_{n+1} \mid Y_1, \dots, Y_n, Z_1, \dots, Z_n, I_1, \dots, I_n] \\ &= S_n E \left[ e^{-R_1(\mathcal{C}(b_0)Z_{n+1} - b_0Y_{n+1}) \prod_{t=1}^{n+1} (1+I_t)^{-1}} \mid Y_1, \dots, Y_n, Z_1, \dots, Z_n, I_1, \dots, I_n \right] \\ &= S_n E \left[ e^{-R_1(\mathcal{C}(b_0)Z_{n+1} - b_0Y_{n+1})(1+I_{n+1})^{-1} \prod_{t=1}^n (1+I_t)^{-1}} \mid I_1, \dots, I_n \right] \\ &\leq S_n E \left( \left[ e^{-R_1(\mathcal{C}(b_0)Z_{n+1} - b_0Y_{n+1})(1+I_{n+1})^{-1}} \mid I_1, \dots, I_n \right] \right)^{\prod_{t=1}^n (1+I_t)^{-1}} \\ &\leq S_n. \end{aligned}$$

This implies that  $\{S_n\}_{n \geq 1}$  is a supermartingale.

Let  $T_i = \min\{n : V_n < 0 \mid I_0 = i\}$ , where  $V_n$  is given by (30). Then  $T_i$  is a stopping time and  $n \wedge T_i := \min\{n, T_i\}$  is a finite stopping time. Thus, by the optional stopping theorem for martingales, we get

$$E^\pi(S_{n \wedge T_i}) \leq E^\pi(S_0) = e^{-R_1 x}.$$

Hence,

$$\begin{aligned}
e^{-R_1 x} &\geq E^\pi(S_{n \wedge T_i}) \geq E^\pi((S_{n \wedge T_i})I_{(T_i \leq n)}) \geq E^\pi((S_{T_i})I_{(T_i \leq n)}) \\
&= E^\pi(e^{-R_1 V_{T_i}} I_{(T_i \leq n)}) \geq E^\pi(I_{(T_i \leq n)}) = \psi_n^\pi(x, i),
\end{aligned} \tag{31}$$

where (31) follows because  $V_{T_i} < 0$ . Thus, by letting  $n \rightarrow \infty$  in (31) we obtain (29).  $\square$

**Remark 5.** *Summarizing, we have three upper bounds for the ruin probabilities with infinite horizon. First, the Lundberg bound, which only depends on  $R_0$ , the Lundberg exponential in (12), (13). Second, the inductive bound (21) which depends on  $R_0$  and also on the interest rate process. Third, the martingale bound in (29), which depends on  $R_1$ . Note that the last two bounds are sharper than the Lundberg bound. Observe also that the number of operations to get  $R_1$  in (29) is higher than that to get  $R_0$  in (21).*

In the next section we present some numerical results.

## 5 Numerical results

To illustrate the bounds given by Theorems 1 and 2 we present two numerical examples that use Matlab and Maple implementations. Without loss of generality we can work in monetary units equal to  $E[Y]$  in all examples.

### 5.1 Exponentially distributed claims

Let consider the special case 2 in section 2, in which  $Z_n$  and  $Y_n$  are exponentially distributed with parameters  $\lambda$  and  $1/\mu$ , respectively. In addition, we will consider an interest model with three possible interest rates:

$$\mathbb{I} = \{6\%, 8\%, 10\%\}.$$

The transition matrix (see (5)) is given by

$$\begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.15 & 0.7 & 0.15 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

Thus, our interest rate model incorporates mean reversion to a level of 8%. If  $\theta$  is too high, in the sense that

$$1 + \theta \geq (1 - \mu R_0)^{-2},$$

then the optimal policy is given by  $\pi^* = \{a_n^*\}_{n \geq 1}$  with  $a_n^* = 1$  for all  $n$ . If we assume that  $c > \lambda\mu$ , then we have that the ruin probability for the Cramér-Lundberg model is

$$\psi^{\pi^*}(x) = \left(\frac{\lambda\mu}{c}\right) e^{-x\left(\frac{1}{\mu} - \frac{\lambda}{c}\right)}.$$

Recalling (14), the Cramér-Lundberg exponent  $R_0$  is the solution of equation

$$\lambda + cR_0 = \lambda(1 - \mu R_0)^{-1}.$$

By Lemma 1 and (13), we have that  $\psi^{\pi^*}(x, i) \leq \psi^{\pi^*}(x) \leq e^{-R_0 x}$ . In the case that  $Y$  has **NWUC** distribution, then the inductive bound is given by (25). The martingale bound can be obtained from Theorem 2. The Table 1 shows the numerical results when  $\lambda = 1$ ,  $\mu = 2$ ,  $\theta = 3$ ,  $c = 4$ , and  $x = 1$ . Note that

Lundberg	$\psi^{\pi^*}(x)$	Inductive	Martingale	$R_0$	$R_1$
0.7788	0.3894	0.3817	0.4366109286	0.25	0.8287128040

Table 1: *Table of upper bounds for ruin probabilities, with  $x = 1$  and  $i = 8\%$*

$$\psi^{\pi^*}(x, i) \leq 0.3817 < \psi^{\pi^*}(x).$$

The numerical results in Table 1 show that the upper bound in (21) can be tighter than that in (29). This suggests that the upper bounds derived by the inductive approach are tighter than the upper bounds obtained by supermartingales. In addition, the upper bounds derived by the inductive approach are tighter than the ruin probability without interest rate. Moreover, Table 1 shows that the upper bounds derived in this article are sharper than the Lundberg upper bound.

## 5.2 Claims with phase-type distribution

We consider claim distributions of the phase-type because they and their moments can be written in a closed form, various quantities of interest can be evaluated with relative ease, and furthermore, the set of phase-type distributions is dense in the set of all distributions with support in  $[0, \infty)$ .

Suppose that the claim size  $Y$  has a phase-type density with parameters  $(\alpha, T)$  where

$$T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \text{ and } \alpha = (1/2, 1/2).$$

Let

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{1} = (1, 1), \quad \text{and} \quad t = -T \cdot \vec{1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

In this case,

$$M_Y(s) = E[e^{s \cdot Y}] = \alpha(-s\mathcal{I} - T)^{-1} t.$$



Thus,  $E[Y] = \frac{d}{ds}M_Y(s) |_{s=0} = \alpha(T)^{-2}t = 0.75$ , and  $Y$  has NWUC distribution. Let  $Z \simeq \text{Exp}(1)$ ,  $E[Z] = 1$ , and  $M_Z(s) = E[e^{sZ}] = (1 - s)^{-1}$ .

We consider an interest model with three possible interest rates:  $\mathbb{I} = \{6\%, 8\%, 10\%\}$ . We would like to have an idea of the dependence of our bounds on the transition probability matrix of the interest rate process. To this end, we consider two transition probability matrices, namely,

$$P_1 = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0 \\ 0.9 & 0.1 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.2 & 0.8 \\ 0 & 0.1 & 0.9 \end{pmatrix}.$$

We fix the premium income rate  $c = 0.975$  and the safety loading  $\theta = 0.1$  of the reinsurer. In addition,  $\mathcal{B} = (0, 1]$ . In this case (15) is not satisfied.

**The Lundberg bound:** In this example we can guarantee that the Lundberg bound (13) holds for each  $b \in \mathcal{B}$ . Then there exists a constant  $R_0$  such that (12) is achieved. Moreover, solving

$$E^\pi[e^{R_0 b Y_1}] \cdot E^\pi[e^{-R_0 \mathcal{C}(b) Z_1}] = 1,$$

is equivalent to find the Cramér-Lundberg adjustment coefficient such that

$$1 + \mathcal{C}(b)R_0 = \alpha(-bR_0\mathcal{I} - T)^{-1}t.$$

Then the Lundberg bound for the ruin probability is

$$\psi^b(x) \leq e^{-R_0 x}, \text{ for } x \geq 0.$$

Figure 1 shows the relation between  $R_0$  and  $b$  in this inequality is inversely proportional. Table 2 presents numerical values of the bounds obtained for several admissible decision policies.

**The Induction bound:** Here, the claim distribution is a NWUC (see [10], page 24) and such that  $E^\pi[e^{R_0 b Y_1}] = M_Y(R_0 b) < \infty$  for each  $b \in \mathcal{B}$ . Then Corollary 1 applies and for each  $i \in \mathbb{I}$  and  $x \geq 0$ , we have

$$\begin{aligned} \psi^\pi(x, i) &\leq (E^\pi[e^{R_0 b Y_1}])^{-1} E^\pi[e^{-R_0 x(1+I_1)} | I_0 = i] \\ &\leq [\alpha(-bR_0\mathcal{I} - T)^{-1} t]^{-1} \sum_{k \in \mathbb{I}} p_{ik} e^{-R_0 x(1+k)}. \end{aligned}$$

See Table 2 for numerical values of this bound obtained for several admissible decision policies. As it is to be expected we get induction bounds smaller than the Lundberg bounds for the same decision policies.

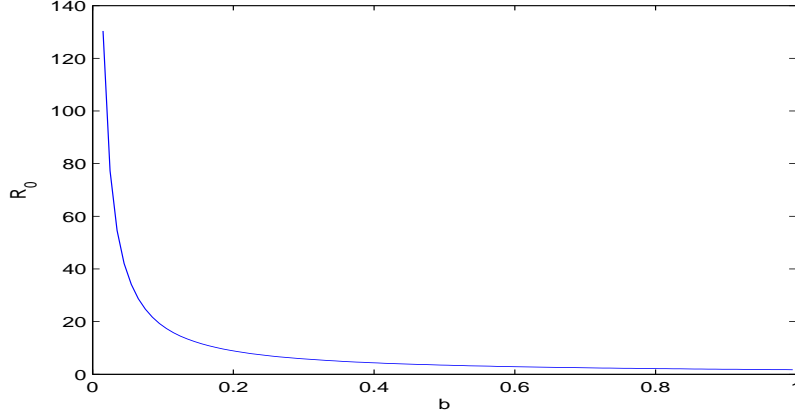


Figure 1: Relation between  $R_0$  and  $b$ .

**The Martingale bound:** By the condition (26) of Proposition 1 and Theorem 2, we get the martingale bound (29). Observe that

$$E \left[ e^{-\rho_i(\mathcal{C}(b)Z_1 - bY_1)(1+I_1)^{-1}} | I_0 = i \right] = 1$$

which is equivalent to the following condition for each  $i \in \mathbb{I}$ :

$$\sum_{k \in \mathbb{I}} p_{ik} e^{\rho_i(1+k)^{-1}} M_Y \left( \frac{b\rho_i}{1+k} \right) M_Z \left( -\frac{\mathcal{C}(b)\rho_i}{1+k} \right) = 1.$$

In our example we solve  $R_1 = \min_{i \in \mathbb{I}} \rho_i \geq R_0$ , and then we obtain  $\psi^\pi(x, i_1) \leq e^{-R_1 x}$  for  $x \geq 0$ . Numerical results of this bound are reported in Table 2. It is obvious that this martingale bound improves the results of the induction bound.

We run numerical experiments to compare, for a fixed retention level  $b$ , the ruin probability bounds that could be achieved. The Figure 2 shows the upper bounds of ruin probability

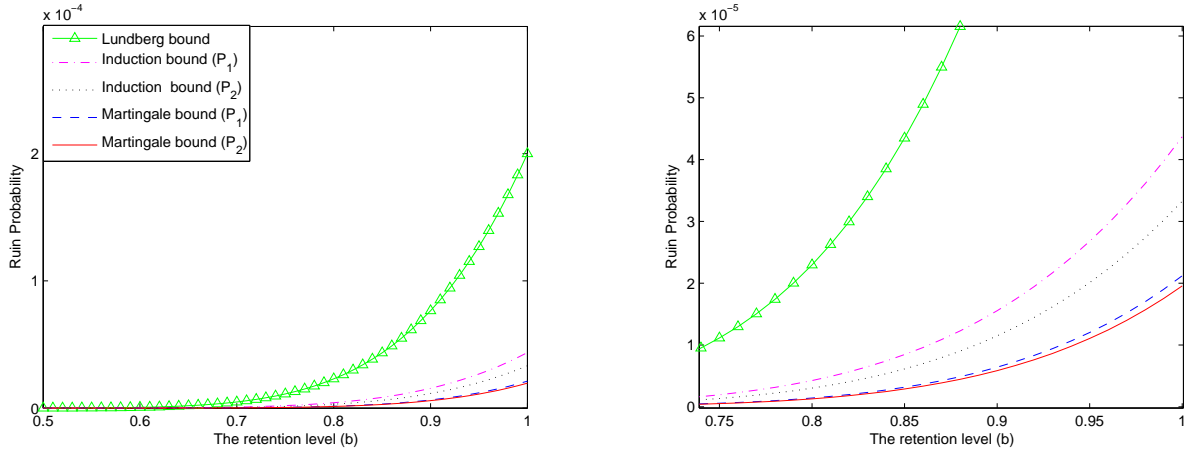


Figure 2: *Bounds for the ruin probabilities. Left panel:  $b \in [0.5, 1]$ . Right panel:  $b \in [0.75, 1]$ .*

from different approaches with the initial state  $x = 5$  and  $i = 8\%$ .

Finally, we find of special interest the case of small reinsurers for which the retention level could be restricted by economic considerations. Thus the Table 2 shows the numerical values of the bounds from different values of  $b$  when  $b$  is increasing towards 1. Recall, that  $b = 1$  stands for the control action *no reinsurance*. Clearly, the best results are in the case where the transition when the interest rate matrix is  $P_2$ .

The numerical results in Table 2 show that the upper bound in (29) can be tighter than

$P_\kappa$	$b$	<i>Lundberg</i>	<i>Induction</i>	<i>Martingale</i>	$R_0$	$R_1$
$P_1$	0.5	$0.323e - 7$	$0.369e - 8$	$0.448e - 9$	3.4491	4.3048
$P_1$	0.75	$0.111e - 4$	$0.196e - 5$	$0.586e - 6$	2.2810	2.8697
$P_1$	0.85	$0.434e - 4$	$0.846e - 5$	$0.317e - 5$	2.0086	2.5321
$P_1$	0.95	$0.126e - 3$	$0.268e - 4$	$0.120e - 4$	1.7943	2.2655
$P_1$	1	$0.2e - 3$	$0.436e - 4$	$0.212e - 4$	1.7034	2.1522
$P_2$	0.5	$0.323e - 7$	$0.213e - 8$	$0.382e - 9$	3.4491	4.3368
$P_2$	0.75	$0.111e - 4$	$0.136e - 5$	$0.527e - 6$	2.2810	2.8911
$P_2$	0.85	$0.434e - 4$	$0.614e - 5$	$0.288e - 5$	2.0086	2.5509
$P_2$	0.95	$0.126e - 3$	$0.201e - 4$	$0.110e - 4$	1.7943	2.2824
$P_2$	1	$0.2e - 3$	$0.333e - 4$	$0.195e - 4$	1.7034	2.1683

Table 2: Numerical bounds of ruin probability.

that in (21). This suggests that the upper bounds derived by the martingale approach are tighter than the upper bounds obtained by induction. In addition, the table also shows that the upper bounds derived in this article are sharper than the Lundberg upper bound.

## 6 Concluding remarks

Our main results in this paper, Theorem 1 and Theorem 2, give upper bounds for the probability of ruin of a certain risk process, which (as shown in Section 2) includes as special cases several relevant models. To obtain Theorem 1 and Theorem 2, first, we obtain an important preliminary result, Lemma 2, which gives recursive equations for finite-horizon ruin probabilities and an integral equation for the ultimate ruin probability. We illustrate our results with an application to the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process. This application suggests that the upper bounds derived by inductive approach are tighter than the ruin probability without interest rate (the function considered in Lemma 1). In addition, the upper bounds derived in this article are sharper than the Lundberg upper bound.

Our paper leaves, of course, many open issues. For instance:

- (a) Is it possible to obtain bounds tighter than those in Theorems 1 and 2 ?.

(b) Actually, what do we need to obtain the ruin probabilities in closed form ?.

(c) Let  $\tau := \inf \{k \geq 1 | X_k < 0\}$  be the *time of ruin*.

Can we calculate or estimate quantities such as  $E[\tau]$ , or  $P(\tau \leq T)$  for given  $T > 0$ ?

These are just a few of the many questions that we can ask ourselves. But two immediate queries are:

- (i) Since  $\{I_n\}$  in (1) is supposed to be a Markov chain, can we rewrite the minimization of the ruin probability as a *Markov decision problem*? ([5, 6, 9], for instance).
- (ii) Suppose that in (1) we include an investment process. What can we say about these models?

Further research in some of these directions is in progress.

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