

**ON THE GENERIC IMPOSSIBILITY OF TRUTHFUL BEHAVIOR:
A SIMPLE APPROACH***

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A B S T R A C T

We provide an elementary proof showing how in economies with an arbitrary number of agents an arbitrary number of public goods and quasi-linear utility functions, any efficient and individually rational mechanism is not strategy-proof for any economy satisfying a mild regularity requirement.

1.- INTRODUCTION

It is widely recognized that most political and economic institutions are vulnerable to their participants' strategic manipulation. In voting theory, Gibbard [1] and Satterthwaite [6] independently prove that any non-manipulable (i.e. strategy-proof) mechanism is dictatorial. In economic environments, many authors have established various results of the same nature. Hurwicz [2] shows that in two-goods, two-person, pure exchange economies, any efficient and individually rational allocation mechanism is manipulable (i.e. truth is not a dominant strategy for some agent) in some economy in the domain, provided that each agent has a positive initial endowment of at least one of the goods, and that a sufficiently wide class of convex preferences is covered (see Ledyard and Roberts [4] for the public good case). However, these results leave unanswered two questions: First, whether similar results are true under different assumptions and in cases where there are more than two agents and/or more than two goods and second, how large is the set of economies for which truth is not a dominant strategy for some agent.

The work of Saijo [5] and Zhou [7] addressed the first question. Saijo [5] studies the problem of the existence of strategy-proof and individually rational mechanisms when the Pareto efficient condition is obviated, and he proves that there is a non-constant mechanism that satisfies the above two requirements in economies with or without public goods. In the same paper Saijo proves that if the individually rational condition is strengthened, a new impossibility result appears. More precisely, in public good economies, no strategy-proof mechanism yielding participative allocations exists. An allocation is participative if every participant's bundle is no worse than the best bundle that can be achieved solely by the participant's endowment and technology, without using that of other participants. On the other hand, new proof techniques based on the identification of the geometric properties of the Pareto efficient set, enable Zhou [7] to prove fresh results. He shows that in the domain of pure exchange economies with two agents and arbitrarily many goods in which both agents' utility functions are continuous, strictly concave and increasing, any efficient and non-inversely-dictatorial allocation mechanism is manipulable for some economy in the domain.

Hurwicz and Walker [3] provide an answer to the second question. Using advanced techniques, they proved that, in economies with quasi-linear utility functions any strategy-proof mechanism defined on a convex and semi-open set of strictly concave and continuous valuations of the public goods, will not yield Pareto efficient allocations on any open and dense set of preference profiles, except by producing allocations that lie on the relative boundary of the feasible set. (Notice that a mechanism that gives the total endowment to participant one regardless of the preference announcements of other participants is strategy-proof and Pareto efficient). This result shows that, with the exception of constant mechanisms, under some restrictions on the set of admissible preferences, Pareto efficiency and strategy-proofness are two generally incompatible requirements.

In this paper we prove a result which is on the line of Hurwicz and Walker [3], namely that in economies with an arbitrary number of agents, an arbitrary number of public goods and quasi-linear utility functions, any efficient and individually rational mechanism is not strategy-proof for **any** regular economy i.e., an economy where the valuations of the public goods are strictly concave, C^2 and their Gaussian curvature are non-vanishing (see Theorem 1 below). This result is stronger and it implies that of Hurwicz and Walker [3] (see Theorem 2 below), but it differs from theirs in the assumptions -we require individual rationality, but we do not assume continuity of the mechanism- and that we use only elementary techniques. Furthermore, our Theorem 1 identifies those economies (the regular ones) for which to announce the truth is not a dominant strategy for some agent. Hurwicz and Walker did not, since they only obtain a generic result. Moreover, our approach allows for a graphical representation when there are two agents and one public good. It goes without saying that it is very easy to adapt our argument and to show that in exchange economies with an arbitrary number of agents, an arbitrary number of private goods and quasi-linear utility functions, any efficient and individually rational mechanism is not strategy-proof for any regular economy (appropriately defined).

The rest of this note goes as follows. The next Section explains the model and the main definitions and Section 3 gathers our main results.

2.- THE MODEL AND DEFINITIONS

There are n agents in the society. There is one private good (it is sometimes helpful to think of this good as "money"), and there are m public goods. Public goods are produced from the private good by means of a constant returns to scale technology represented by a linear cost function $c(\cdot)$. A consumption for agent i is a pair $(x_i, y) \in \mathbb{R}_+^{1+m}$ where $x_i \in \mathbb{R}_+$ is the private good he consumes on his own, and $y \in \mathbb{R}_+^m$ is the vector of public goods. Let $(w_i, 0) \in \mathbb{R}_+^{1+m}$ be agent i 's initial endowment (i.e., we assume that there are initially no public goods). Let

$$X = \{ (x, y) \in \mathbb{R}_+^{n+m} \mid \sum_{i=1}^n (w_i - x_i) = c(y) \}$$

be the set of feasible allocations. Each agent i has a preference relation defined over \mathbb{R}_+^{1+m} represented by a quasi-linear (on money) utility function $u_i: \mathbb{R}_+^{1+m} \rightarrow \mathbb{R}$. Let U_i be the (exogenously given) set of admissible preferences for agent i . U denotes the product space $U = U_1 \times \dots \times U_n$. A generic point $u = (u_1, \dots, u_n)$ in U is called a preference profile. Sometimes we will refer to $u \in U$ as an economy, and this will be written as (u, u_{-i}) , where $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$.

An economy $u \in U$ is said to be **regular** if for all $i = 1, \dots, n$, $v_i(\cdot)$ is C^2 and strictly concave with non-vanishing Gaussian curvature⁽¹⁾.

Let us denote by U^R the space of all regular economies. Let U^L be the space of all linear economies.

Given an economy $u \in U$, a feasible allocation (x, y) is **Pareto efficient for u** if no other feasible allocation (x', y') exists such that $u_i(x', y') \geq u_i(x, y)$ for all i with strict inequality for at least one agent.

(1) Our results remains true if we redefine a regular economy as one such that $v(\cdot)$ is concave and C^2 for all agents and it has a non-vanishing Gaussian curvature for at least two agents.

Given an economy $u \in U$, a feasible allocation (x,y) is **individually rational for u** if $u_i(x_i,y) \geq u_i(w_i,0)$ for all i .

An allocation (x^L,y^L) is a **Lindahl allocation for $u \in U$** , if it is feasible and there is a price vector $p_i \in \mathbb{R}_+^m$ one for each i such that,

- (1) $x_i^L + p_i y^L \leq w_i \quad i = 1, \dots, n$
- (2) $u_i(x_i, y) > u_i(x_i^L, y^L)$ implies $x_i + p_i y > w_i \quad i = 1, \dots, n$
- (3) $\sum_{i=1}^n p_i y^L - c(y^L) \geq \sum_{i=1}^n p_i y - c(y)$

An allocation (x^M, y^M) is called a **monopoly point for $u \in U$ when agent i behaves as a monopolist** if it is feasible and there is a price vector $p_j \in \mathbb{R}_+^m$ one for each j such that

- (1') $x_j^M + p_j y^M \leq w_j \quad , \text{ for all } j$
- (2') $u_j(x_j, y) > u_j(x_j^M, y^M)$ implies $x_j + p_j y > w_j$ for all $j \neq i$
- (3') $\sum_{j=1}^n p_j y^M - c(y^M) \geq \sum_{j=1}^n p_j y - c(y)$
- (4') $u_i(x_i^M, y^M) \geq u_i(x_i, y)$ for all $(x,y) \in X$ satisfying (1'), (2') and (3').

Since the cost function is convex, condition (3) and (3') can be replaced by $\sum_{i=1}^n p_{ik} = \frac{\partial c(y)}{\partial y_k} \quad k = 1, \dots, m$.

It is clear that for any regular economy, both the Lindahl allocation and the monopoly point exist.

A (direct) **mechanism** is a function $f: U \longrightarrow X$, which maps each preference profile into the set of feasible allocations. Let $f_i(u) = (x_i, y)$ be the consumption obtained by i if the profile u is announced.

A mechanism f is **efficient** if for any $u \in U$, $f(u)$ is a Pareto efficient allocation for u .

A mechanism f is **individually rational** if for any $u \in U$, $f(u)$ is an individually rational allocation for u .

A mechanism f is **strategy-proof** if for any agent i , any $u \in U$ and any $u'_i \in U_i$, $u_i(f_i(u_i, u_{-i})) \geq u_i(f_i(u'_i, u_{-i}))$.

Given an economy $u \in U$, a mechanism f is **strategy-proof for u** if for any agent i , and any $u'_i \in U_i$, $u_i(f_i(u_i, u_{-i})) \geq u_i(f_i(u'_i, u_{-i}))$.

Notice that the latter of these two definitions is weaker than the former and is the one which will be used in Theorem 1 below. Thus, our results are stronger, for example, than the one obtained by Ledyard and Roberts [4], because they use the former definition.

3. THE MAIN RESULTS

We start this section proving that given a regular economy, the agent who behaves as a monopolist is strictly better off in the monopoly allocation than in the Lindahl allocation (see Lemma 1 below). This Lemma will be used to prove our main result (Theorem 1). Later on Lemma 2 and Theorem 2 will prove an analogous result to that of Hurwicz-Walker [3].

Lemma 1: *Given a regular economy $u \in U$, if $(x_1^M, y^M) \gg 0$ and $y^L \neq 0$ then $u_1(x_1^L, y^L) < u_1(x_1^M, y^M)$ ⁽²⁾.*

Proof. Since (x^L, y^L) is the Lindahl allocation for u , y^L satisfies:

$$\sum_{i=1}^n \frac{\partial v_i(y)}{\partial y_j} \leq \frac{\partial c(y)}{\partial y_j} \quad j = 1, \dots, m \quad [1]$$

if (x^M, y^M) is the allocation obtained if agent i behaves as a monopolist, y^M satisfies:

$$\sum_{i=1}^n \frac{\partial v_i(y)}{\partial y_j} = \frac{\partial c(y)}{\partial y_j} - \sum_{i \neq 1} \sum_{k=1}^m \frac{\partial^2 v_i(y)}{\partial y_k \partial y_j} y_k \quad j = 1, \dots, m \quad [2]$$

Since (x^L, y^L) satisfies (1'), (2'), (3'), $u_1(x_1^L, y^L) \leq u_1(x_1^M, y^M)$, then it is enough to prove that $u_1(x_1^L, y^L) \neq u_1(x_1^M, y^M)$. Suppose that $u_1(x_1^L, y^L) = u_1(x_1^M, y^M)$. Then y^L must satisfy [2]. Since y^L satisfies [1], this implies that

$$\sum_{i \neq 1} \sum_{k=1}^m \frac{\partial^2 v_i(y^L)}{\partial y_k \partial y_j} y_k^L \geq 0 \quad j = 1, \dots, m \quad [3]$$

Let $v(y) = v_1(y) + \dots + v_{i-1} + v_{i+1} + \dots + v_n(y)$. Since v_i is strictly concave for all i , v is strictly concave. Then we can write [3] as $Hv(y^L)y^L \geq 0$, where

(2) Vector inequalities, \gg , $>$, \geq

$Hv(\cdot)$ is the Hessian matrix of the function v . Then $(y^L)^t Hv(y^L)y^L \geq 0$, but this is a contradiction since $y^L \neq 0$ and $(y^L)^t Hv(y^L)y^L < 0$ because v is strictly concave with a non-vanishing Gaussian curvature. ■

Now we are prepared to prove our main result

Theorem 1. Let $f: U \rightarrow X$ with $f(u) \gg 0 \forall u \in U$ be an efficient and individually rational mechanism with $U^L \subseteq U$. Then, f is not strategy-proof for any regular economy in U such that $(x_1^M, y^M) \gg 0$ and $y^L \neq 0$.

Proof. We first consider the case of $n = 2, m = 1$, in order to offer a graphical insight on how the proof works. A formal proof is then offered.

Given a regular economy $u \in U$ (see Figure 1) let $B-B'$ be the set of all efficient and individually rational allocations for u . Since mechanism f is efficient and individually rational, $f(u)$ must be at some point between B and B' . Suppose that it is between B and L (L is the Lindahl allocation for u). Then agent 1 can misrepresent his utility function by sending a constant marginal rate of substitution equal to his monopoly prices (the dotted line in Figure 1). He obtains an allocation in the new efficient and individually rational set $M-F$ (M is his monopoly point). For any point therein, agent 1 is better off than before. If $f(u)$ is at some point between L and B' , agent 2 would manipulate accordingly.

We now provide a formal proof of the Theorem. Let $u \in U$ be a regular economy such that $(x^L, y^L) \gg 0$, and let $(x, y) = f(u, u_{-i})$. Since (x^L, y^L) and (x, y) are Pareto efficient, there exists i such that

$$x_i + v_i(y) \leq x_i^L + v_i(y^L).$$

where $x_i + v_i(y)$ $i = 1, \dots, n$ are the true utility functions relative to the economy u . Suppose that agent i sends a utility function

$$u'_i(x_i, y) \equiv \sum_{k=1}^m p_{ik}^M y_k + x_i$$

where p_{ki}^M are the monopoly prices of the economy u with i as a monopolist. Let $(x', y') = f(u'_i, u_{-i})$, and let (x_i^M, y^M) be the consumption of agent i in the economy u when he is a monopolist. We claim that

$$x_1^M + v_1(y^M) \leq x_1' + v_1(y').$$

Suppose that

$$x_1^M + v_1(y^M) > x_1' + v_1(y').$$

Since (x^M, y^M) and (x', y') are Pareto efficient allocations for the economy (u_1, u_{-1}) , our assumptions imply that $y^M = y'$, so $x_1^M > x_1'$. At the monopoly point the budget constraint for i is satisfied, so $w_1 = \sum_{k=1}^m p_{k1}^M y_k^M + x_1^M > \sum_{k=1}^m p_{k1}^M y_k' + x_1'$, which is a contradiction because the mechanism is individually rational. This proves the claim. Therefore, and by the previous lemma

$$x_1 + v_1(y) \leq x_1^L + v_1(y^L) < x_1^M + v_1(y^M) \leq x_1' + v_1(y')$$

which shows that mechanism f is not strategy-proof for u . ■

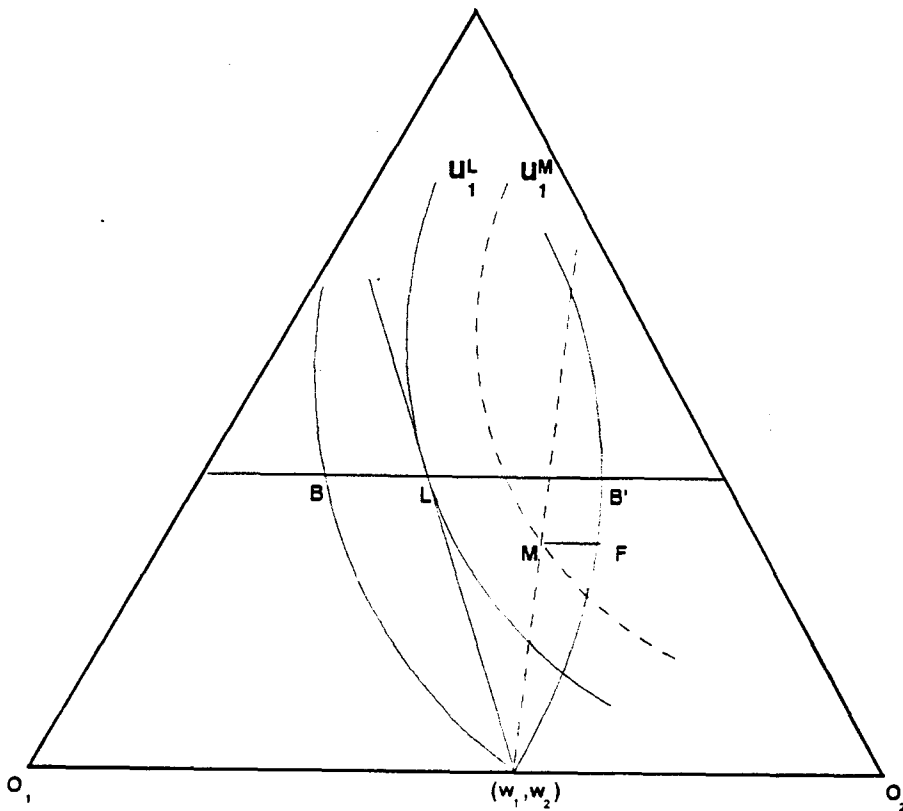


Figure 1

In Theorem 1 we have just proved the impossibility of truthful behavior on the set of regular economies when efficiency and individual rationality are imposed on the mechanism. Now we address the question of how big is the set of regular economies. We prove that the set of strictly concave functions is dense, with the Punctual Topology, in the set of concave functions. Thus, the set of regular economies is dense in the set of admissible preferences.

Lemma 2. *The set of regular economies is dense in the set of all economies with concave utility functions.*

Proof. *We will show that the set of strictly concave functions with a non-vanishing Gaussian curvature is dense in the set of concave functions.*

Let $\mathcal{F}_c = \{f: \mathbb{R}^m \rightarrow \mathbb{R} / f \text{ is concave}\}.$

Let $\mathcal{F}_{sc} = \{f: \mathbb{R}^m \rightarrow \mathbb{R} / f \text{ is strictly concave}\}.$

Claim 1. *A sequence $\{f_n: n \in \mathbb{N}\} \subseteq \mathcal{F}_{sc}$ exists such that $\{f_n: n \in \mathbb{N}\}$ converges, with the punctual convergency, to the zero function.*

For each $n \in \mathbb{N}$, let $f_n(x) = \frac{-1}{n} \sum_{i=1}^m x_i^2$. Clearly, f_n is strictly concave, with a non-vanishing Gaussian curvature and $\{f_n(x): n \in \mathbb{N}\}$ converges to zero for each $x \in \mathbb{R}^m$.

Claim 2. *For each $f \in \mathcal{F}_c$, a sequence $\{g_n: n \in \mathbb{N}\} \subseteq \mathcal{F}_{sc}$ exists such that $\{g_n: n \in \mathbb{N}\}$ converges, with the punctual convergency, to f .*

For each $n \in \mathbb{N}$, let $g_n(x) = f(x) + f_n(x)$, with $f_n(x)$ being as we described in Claim 1. Since for each $n \in \mathbb{N}$, f_n is strictly concave and f is concave, g_n is strictly concave. Thus, $g_n \in \mathcal{F}_{sc}$ for each $n \in \mathbb{N}$. By Claim 1 $\{f_n: n \in \mathbb{N}\}$ converges to the zero function, then $\{g_n: n \in \mathbb{N}\}$ converges to f . ■

An examination of the proof of Theorem 1 shows that whenever a regular economy is considered, an agent has incentives to deviate announcing an appropriate linear utility function. By Lemma 2, it is possible to carry out

the proof by taking only strictly concave (with non-vanishing Gaussian curvature) valuations of the public goods, i.e. Theorem 1 remains true if the set of admissible preferences is restricted to be the set of regular economies.

Theorem 2. *Let $f: U^R \longrightarrow X$ with $f(u) \gg 0 \forall u \in U$ be an efficient and individually rational mechanism. Then, f is not strategy-proof for any regular economy in U such that $(x_1^M, y^M) \gg 0$ and $y^L \neq 0$.*

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