Double implementation of the ratio correspondence by a market mechanism

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Abstract

To overcome deficits of the Lindahl solution concept when the economy does not exhibit constant returns to scale, Kaneko (1977a) introduced the concept of a *ratio equilibrium*. The *ratio correspondence* selects for each economy its set of ratio equilibrium allocations. In this paper we provide a simple market game that *double implements* the ratio correspondence in Nash and strong equilibria.

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1. Introduction

Given some domain of economies, a *choice correspondence* selects for each economy a set of feasible allocations. The choice correspondence may be thought of as an abstract representation of either the ideals of the society, or the preferences of a planner. The problem of *manipulation* arises when agents in a society have information about the true nature of the economy that is not known to the planner. If we were to ask the agents directly "what do you know" and then

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apply the given correspondence, some agent may strategically use this information to manipulate the allocation recommended by the correspondence.

Given the possibility of manipulation, the *(Nash) implementation problem* is to design a game such that when we allow for strategic play, the allocations that result from (Nash) equilibria of the game, are exactly the allocations that the given correspondence selects. If such a game can be found we say that *the game implements the choice correspondence on the domain.*

In this paper we consider economies with public goods, and we are interested in choice correspondences that select efficient and individually rational allocations. This literature has a long history in economics. Samuelson (1954) conjectured that any decentralized ('spontaneous') mechanism for allocating public goods efficiently would be doomed to fail, as "... it is in the selfish interest of each person to give false signals" (Samuelson, 1954, pp. 388). Samuelson's intuition can be stated formally as a theorem, proved by Hurwicz (1972), showing that there does not exist a choice correspondence which, when we ask players to directly reveal their preferences, yields Pareto-optimal and individually rational outcomes. However, subsequently, by using abstract strategy spaces, Hurwicz (1979a,b), Walker (1981), McKelvey (1989), and Tian (1989), developed games that, for constant returns to scale economies, implemented the Lindahl correspondence. Thus the Nash equilibrium outcomes of their games are both Pareto-optimal and individually rational.

In this paper we depart from the previous work in three respects: (i) we allow for more general technology than constant returns to scale (CRS); (ii) we do not exclude the possibility that some subset of the agents may communicate and conspire to manipulate the outcome, and (iii) we want the game form to resemble the operation of a 'market'. We address each of these points in turn.

Most of the public goods implementation literature has focused on the Lindahl correspondence. The Lindahl correspondence is often viewed as the public goods equivalent of the Walrasian correspondence for private goods economies. However, in the absence of CRS, Lindahl pricing generates a surplus (or deficit) that must be shared among agents to obtain an efficient outcome. The question then arises; as public goods are typically publicly provided, if the technology is jointly owned by the society, how is the share rule determined? In general the answer is not transparent. However, even if we can justify a particular share rule, the Lindahl allocations may fail to be in the core as defined by Foley (1970). Indeed the Lindahl allocations may fail to be individually rational, Kaneko (1977a), Moulin (1989). Thus if we are interested in correspondences that select efficient and individually rational outcomes, the Lindahl correspondence loses its appeal. Similarly the balanced linear cost share equilibria recently introduced by Mas-Colell and Silvestre (1989) also fails to always select individually rational allocations, Wilkie (1989). However, the *ratio correspondence*, introduced in Kaneko (1977a,b), meets our requirements and so we focus on implementing it.

Our second point concerns the possibility of coalition formation. Most of the games proposed in the literature thus far, has been concerned with strategic play by individuals, exceptions being Schmeidler (1980) and recently Peleg (1996a,b). We do not wish to exclude the possibility that agents could communicate and form coalitions, with the hope that by joint strategic play they could improve their welfare. Indeed, as we are interested in implementing core allocations, it seems inconsistent to preclude such communication. If we take this possibility seriously, then we need to use the *strong equilibrium* concept. Necessary and Sufficient conditions for implementation in strong equilibria are provided in Dutta and Sen (1991). Ideally we would like to construct a game with the property that the strong and Nash equilibrium outcomes coincide, and these coincide with the set of allocations chosen by our given correspondence, a property called *double implementation* by Maskin (1985).

Finally, we want the game form to be simple. Many games have been criticized because they have large, abstract strategy spaces and complicated outcome functions. We would like our game form to be 'similar' to the choice correspondence it is implementing. In particular, if we are implementing a choice correspondence that uses a market approach to solve allocation problems, then the game should ideally be a 'market game'. That is, the strategies are 'prices and quantities', and the outcome function should allow each agent to choose from some budget.

Our results can be stated succinctly. We provide a simple market game that double implements the ratio correspondence. Thus both the strong and Nash equilibrium outcomes are efficient and individually rational. We then show that with a simple modification to the outcome function we can double implement the ratio correspondence by a game that is both continuous and always feasible. Similar results, on a different domain are found in the recent work of Peleg (1996b), which double implements the Lindahl correspondence, on the domain of constant returns to scale economies, by a continuous and feasible mechanism.

We close this section by providing a brief outline of the paper. Section 2 presents the definitions and the solutions. Section 3 presents our main results and relates them to the literature. The modified continuous and feasible market game is introduced in Section 4. Some concluding remarks are then offered. For further properties of the ratio correspondence, the reader is referred to Moulin (1989) and Wilkie (1989), where the solutions for economies with public goods are studied from an axiomatic perspective, and to Kaneko (1977b) which links ratio allocations to the outcomes of a voting game. An generalization of ratio equilibria to a larger domain is provided in Diamantaras and Wilkie (1994).

2. Notation and definitions

We consider a domain of economies $\mathscr{E}$ where there is one private good and a set $L = \{1, 2, \ldots, k\}$ of public goods. There is a set $I = \{1, 2, \ldots, n\}$ of agents. Each agent's consumption set is $\Re_+^{k+1}$. A consumption bundle for each $i \in I$ is a pair

$(x_i, y) \in \Re_+^{k+1}$, where x_i is her consumption of the private good, and y is the vector of public good levels. For each i, u_i is any function representing her preferences defined on $\Re_+^{k+1}$. We assume that each u_i is increasing, strictly increasing in x_i, and continuous on $\Re_{++}^{k+1}$. For each i, $\omega_i > 0$, is her initial endowment of the private good. The initial level of all public goods is zero.

The technology for producing public good l is described by a function $c_l: \Re_+ \to \Re_+$, where $c_l(y_l)$ is the amount of private good required to obtain the level y_l of public good l. We assume that for each $l \in L$, c_l is continuous, strictly increasing, and satisfies $c_l(0) = 0$. Given $y = (y_1, \ldots, y_k)$ we will use $c(y)$ for the list $(c_1(y_1), \ldots, c_k(y_k))$. [1]

To summarize, an economy is a triple $e = (u, \omega, c)$, where $u = (u_1, \ldots, u_n)$, $\omega = (\omega_1, \ldots, \omega_n)$, and $c = (c_1, \ldots, c_k)$.

An *allocation* $z \in \Re_+^{n+k}$, is a list $(x_1, x_2, \ldots, x_n, y)$ where for each i, (x_i, y) is agent i's consumption.

Given $e \in \mathscr{E}$, an allocation z is *feasible for e* if $\Sigma_I x_i + \Sigma_L c_l(y_l) \leq \Sigma_I \omega_i$. Let $A(e)$ be the set of feasible allocations of e.

An allocation z is *Pareto-optimal for e* if $z \in A(e)$, and there does not exist another $z' \in A(e)$ such that $u_i(x'_i, y') \geq u_i(x_i, y)$ for all i, with strict inequality for some i. Let $P(e)$ be the set of Pareto-optimal allocations of e.

A *coalition* S is a non-empty subset of I.

The allocation $z = (x, y) \in A(e)$ can be *improved upon by S* if there exists $(x'_i)_{i \in S}$, and $y' \in \Re_+^k$ such that:

(i) $\Sigma_S x'_i + \Sigma_L c_k(y'_l) \leq \Sigma_S \omega_i$ and

(ii) $\forall i \in S$, $u_i(x'_i, y') \geq u_i(x_i, y)$, with strict inequality for some i.

An allocation $z \in A(e)$ is in the *Foley-core of e* if there does not exist a coalition S that can improve upon z.

An allocation $z \in A(e)$ is *individually-rational for e* if it cannot be improved upon by 'coalitions' of cardinality 1. Let $IR(e)$ be the set of individually-rational allocations of e.

Remark. The above definition of the core, introduced by Foley (1970), is not the only possibility. It incorporates the idea that all agents have access to the production technology, and can by some means exclude others from the public good. This definition of individual-rationality is consistent with the definition of Pareto-optimality and the core. It could be labeled "free access individual-rationality" following the Moulin (1989) terminology. Saijo (1991) calls this requirement 'autarkically individually rational'. At this level of abstraction we feel that it is a more appealing definition than the other polar case, that only by unanimity can production be undertaken.

[1] We use $\cdot$ to denote the inner product of two vectors. Thus $p \cdot c = \Sigma p_i \times c_i$ and bc denotes the scalar multiplication of c by b.

Let Δ^{n-1} denote the n-dimensional unit simplex.

Definition 2.1. A pair (r, z) where $r=(r_1,\ldots,r_n)$, each agent $r_i \in \Re^k_+$, and for each public good k, $\sum_{i \in N} r_{ik} = 1$, and $z=(x, y) \in \Re^{n+k}_+$ is a *ratio equilibrium*, and z is a ratio allocation, for $e=(u, \omega, c)$ if(i) $\forall i$, $x_i + r_i \cdot c(y) \leq \omega_i$, and(ii) $x'_i + r_i \cdot c(y') \leq \omega_i$ then $u_i(x_i, y) \geq u_i(x'_i, y')$.

Let $RE(e)$ be the set of ratio equilibria of e.

Let $\mathscr{E}_r \subset \mathscr{E}$ be the domain of economies such that a ratio equilibrium exists.

Ratio Equilibria uses a simple form of non-linear pricing. The number r_{ik} has the simple interpretation; 'the share of the cost of public good k that i is required to contribute'.

Definition 2.2. The *ratio correspondence*, R, selects for each $e \in \mathscr{E}_r$ the set of ratio allocations for e.

We close this section by presenting some simplifying notation.

Given an economy $e=(u, \omega, c)$ and an allocation $z=(x, y)$, for each i let

$$R_i(x_i, y) = \{(x'_i, y') \in \Re^{k+1}_+ | u_i(x'_i, y') \geq u_i(x_i, y)\}$$

and $P_i(x_i, y) = rel.int(R_i(x_i, y))$, where $rel.int(X)$ is the relative interior of X. We will use the conventions, $u_i(z) = u_i(x_i, y)$, $R_i(z) = R_i(x_i, y) \times \Re^{n-1}_+$ and $P_i(z) = rel.int(R_i(z))$.

Given a list of ratios $r \in \Delta^{k(n-1)}$, let

$$B_i(r, e) = \{(x'_i, y') \in \Re^{k+1}_+ | x'_i + r_i c(y') \leq \omega_i\}.$$

Let $(S_i)_{i \in N}$ be a family of sets, where S_i is the *strategy space of agent i*. A *game form*, Γ, is a pair (S, g), where S is the Cartesian product of the spaces S_i, and $g: S \times \mathscr{E} \to \Re^{n+k}$ is the outcome function.

If $g(s, e) = z = (x, y) = (x_1, \ldots, x_n, y)$, we let $g_i(s, e) = (x_i, y)$, $g_x(s) = x$, and $g_y(s, e) = y$.Note that at this stage we have not required that the outcome always be feasible, that is for some s, for some i, it may be that $g\_i(s, e) \notin \Re^{k+1}_+$. Therefore we extend the definition of each agent's utility function in such a way that given an outcome such that for some i, $g_i(s, e) \notin \Re^{k+1}_+$, then $u_i(g_i(s, e) < u_i(\omega_i, 0)$. This approach was introduced by Hurwicz (1979a).

If a mechanism $\Gamma = (S, g)$ is such that for all $e \in \varepsilon$, for all strategy profiles, the outcome of the Γ is feasible: that is

$$\forall e \in \mathscr{E}, \text{ and } \forall s \in S, \ g(s, e) \in A(e),$$

then we say Γ is a *feasible mechanism on* $\mathscr{E}$.

Given a strategy profile s, the coalition T and $(s'_i)_{i \in T}$, let the strategy profile σ be equal to s' on T and s on the compliment of T. Then define:

$$v_i(s) = u_i(g(s, e)), v'_i(s_T, s_{-T}) = u_i(g(\sigma, e)),$$

and

$$v_i(s, s'_{-i}) = u_i(g(s, s'_T, e)) \text{ when } T = N/\{i\}.$$

Given a mechanism Γ and an economy e, as the outcome function yields an allocation for each list of strategies and the utility functions of the agents then yield a utility payoff for each strategy profile, the pair (Γ, e) defines the game Γ played in e.

Definition 2.3. Given an economy $e = (u, \omega, c)$, and a mechanism $\Gamma = (S, g)$, a strategy profile s is a Nash equilibrium of Γ *played in* e if

$$\forall i \in I, \forall s'_i \in S_i, v_i(s) \geq v_i(s'_i, s_{-i}).$$

$N(\Gamma, e)$ is the set of Nash equilibria of Γ played in e. We will refer to an allocation that is the outcome of a Nash equilibrium as a *Nash equilibrium allocation*. $NA(\Gamma, e)$ is the set of Nash equilibrium allocations of Γ played in e.

Definition 2.4. Given an economy $e = (u, \omega, c)$, and a mechanism $\Gamma = (S, g)$, a strategy profile s is a *strong equilibrium of* Γ *played in* e if there does not exist a coalition T and a list of strategies, $(s'_i)_{i \in T}$, such that

$$\forall i \in T, v_i(s'_T, s_{-T}) \geq v_i(s)$$

with strict inequality for some $i \in T$.

$S(\Gamma, e)$ is the set of strong equilibria of Γ played in e. $SA(\Gamma, e)$ is the set of strong equilibrium allocations of Γ played in e.

We will consider the following sub-domains of $\mathscr{E}$: $\mathscr{E}_c \subset \mathscr{E}$, the *Classical Economies*, where each agents utility function is quasi-concave, and each cost function is convex. Let $\mathscr{E}_{ec} \subset \mathscr{E}_c$ be the *essential economies*, where all goods are *essential*, that is

$$\forall i \text{ if } (x_i, y) \in \partial \Re_+^{k+1}, (x'_i, y') \in \Re_{++}^{k+1} \text{ then } u_i(x_i, y) < u_i(x'_i, y')$$

where $\partial \Re_+^{k+1}$ denotes the boundary of $\Re_+^{k+1}$.

3. Results

3.1. A market game

Definition 3.1. The *Cost Share Game* $\Gamma_s = (S, g)$ is specified by the following components:

(i) $\forall i \in I$, $S_i = [0, 1]^k \times \Re^k$ with generic element $s_i = (r_i, y_i)$.

(ii) $g: S \to \Re^{n+k}$ where, $g(s, e) = (g_x(s, e), g_y(s, e)) = (\omega_1 - r_1 \cdot$

$c(\sum y_i), \ldots, \omega_n - r_n \cdot c(\sum y_i); \sum y_i)$ if $\forall l$ $\sum_{i=1}^n r_{il} \geq 1$ and $\sum y_i \geq 0$, and $g(s, e) = (\omega_1, \ldots, \omega_n; 0)$ otherwise.

The first component of an agent's strategy space is the proportion of the cost of providing each good she will pay, and the second component is an incremental change in the level of each public good.

Proposition 3.1. Given any $e \in \mathscr{E}_r$, any ratio equilibrium allocation for e is a Nash equilibrium outcome of the game Γ_1 played in e: $R(e) \subset NA(\Gamma_1, e)$.

Proof. Let $e \in \mathscr{E}_r$ be given and suppose $(r, z) \in RE(e)$. For all i, let $s_i = (r_i, y/n)$. Then $g(s, e) = z$. We claim that $s \in N(\Gamma_1, e)$. Consider a deviation by agent i. If she declares $r'_{il} < r_{il}$ for some l, then $\sum_{j \neq i} r_j + r'_i < 1$, and $g_i(s) = (\omega_i, 0)$, which by the individual-rationality of RE allocations is (weakly) inferior for her to (x_i, y). Furthermore, as any $y' \in \mathfrak{R}_+^{k+1}$ can be attained by playing $s'_i = (r_i, y' - \{(n-1)/n\} y)$, and u_i is strictly increasing in x_i, then if $s'_i = (r'_i, y'_i)$ where for some l, $r'_{il} > r_{il}$, s'_i cannot be a best response to s_{-i}. Thus we need only consider deviations s_i such that $r'_i = r_i$. Hence, agent i faces the problem: maximize $u_i(x'_i, y'_i + \{(n-1)/n\} y)$, where $x'_i = \omega_i - r_i \cdot c(y'_i + \{(n-1)/n\} y)$. By the definition of a ratio equilibrium, y_i is a solution to this problem. Thus $v_i(s_i, s_i) \geq v_i(s'_i, s_{-i})$ for all $s \in S_i$. The same reasoning holds for each $j \in N$. Thus $s' \in N(\Gamma_1, e)$ and $z' \in NA(\Gamma_1, e)$. □

Proposition 3.2. Given any $e \in \mathscr{E}_r$, any Nash equilibrium allocation of the game Γ_1 played in e such that the level of at least one public good is positive, is a ratio equilibrium allocation for e: $NA(\Gamma_1, e) \cap \{z \in A(e) | y \neq 0\} \subset R(e)$.

Proof. Let $e \in \mathscr{E}_r$ be given, $s \in N(\Gamma_1, e)$ and $z = g(s, e)$. Suppose that $g_y(s) \neq 0$, then by the definition of g, for all l, $\sum_{i \in I} r_{il} \geq 1$. As each u_i is strictly increasing in x_i, and $s \in N(\Gamma_1, e)$, then, for all l, $\sum_{i \in N} r_{il} = 1$. Furthermore, as $s \in N(\Gamma_1, e)$, then $z \in A(e)$ and for all i, y_i solves: max $u_i(x'_i, y'_i + \{(n-1)/n\} y)$, subject to $x_i = \omega_i - r_i \cdot c(y'_i + \{(n-1)/n\} y)$. Thus $(r, z) \in RE(e)$ and $z \in R(e)$. □

Remark. The key fact driving the above results is that the outcome function g forces each agent to act as a *ratio-taker*, where $r_i = 1 - \sum_{i \neq j} r_i$. In a CRS economy, this reduces to being a price-taker, hence agents act as if they were competitive.

Remark. Combining Propositions 3.1 and 3.2 does not exactly yield the result that Γ_1 implements R on $\mathscr{E}_r$, as the possibility remains that an equilibrium of Γ_1 for e may exist in which there is no production of the public goods. Such an allocation is possible in equilibrium only if no agent on her own would undertake some production of one public good. One way to eliminate the possibility of such an equilibrium is to modify the mechanism so that if an agent under bids, then all agent's endowment are confiscated, i.e., for all i, $x_i = 0$ if $\sum_N r_{ik} < 1$ for some k. Another cure is to assume that at least one public good is essential for one agent. Then, as each agent can choose a strategy that will yield positive production, there cannot be a Nash equilibrium with no production. Thus we obtain the following Corollary.

Corollary 3.1. Given any $e \in \mathscr{E}'_{ce}$, any Nash equilibrium allocation of the game Γ_1 played in e is a ratio equilibrium allocation for e: $NA(\Gamma_1, e) \subset R(e)$.

Another approach is to strengthen the equilibrium concept. As implementation theory is concerned with proving the equivalence of the set of equilibrium allocations of a game and the set of ϕ-optimal allocations for each economy, there is a cost and a benefit to strengthening the equilibrium concept. If for a given game $SA(\Gamma, e) = \phi(e)$, then the outcomes are very robust. No subset of the agents can conspire to upset any ϕ-optimal allocation. However, believing that only strong equilibria will played imposes a greater belief in the sophistication of the agents. Ideally we would like to find a game where the set of strong equilibria and dominant strategy equilibria coincide. However, it is impossible to construct such a game on our domain, see Groves and Ledyard (1987).

Proposition 3.3. Given any $e \in \mathscr{E}'_r$, any ratio equilibrium allocation for e is a strong equilibrium outcome of the game Γ_1 played in e: $R(e) \subset SA(\Gamma_1, e)$.

Proof. Let $e \in \mathscr{E}_r$ be given and suppose (z, r) is a ratio equilibrium for e. For all i let $s_i = (r_i, y/n)$. Then $g(s, e) = z$. We claim that $s \in S(\Gamma_1, e)$. Suppose there exists some coalition T and strategies $(s'_i)_{i \in T}\} = (r'_i, y'_i)_{i \in T}$ such that, if $z' = g_i(s'_T, s_{-T}, e)$ then $u_i(x'_i, y') \geq u_i(x_i, y)$ for all $i \in T$, with strict inequality for some $i \in T$. If $y' = 0$ then as $R(e) \subset IR(e)$, $u_i(x'_i, y') \leq u_i(x_i, y)$ for all $i \in T$, thus $y' \neq 0$. Then by the definition of g, $\sum_T r'_i \geq \sum_T r_i$ and so, $\sum_T r'_i \cdot c(y') \geq \sum_T r_i \cdot c(y')$. Suppose that for some $i \in T$, $r'_i \cdot c(y') > r_i \cdot c(y')$. Then there exist $\epsilon > 0$ such that $x'_i + \epsilon + r_i \cdot c(y') \leq \omega_i$. As by hypothesis $u_i(x_i, y) \leq u_i(x'_i, y')$, then u_i strictly increasing in x_i contradicts $z \in R(e)$. Thus for all $i \in T$, $r'_i \cdot c(y') = r_i \cdot c(y')$. Therefore as $z \in R(e)$, there cannot exist an $i \in T$ such that $v_i(s'_T, s_{-T}) > v_i(s)$. Thus T cannot improve upon z. Thus $s \in S(\Gamma_1, e)$ and $z \in SA(\Gamma_1, e)$. □

Proposition 3.4. Given any $e \in \mathscr{E}_r$, any strong equilibrium allocation for e is a ratio equilibrium allocation for e: $SA(\Gamma_1, e) \subset R(e)$.

Proof. Let $e \in \mathscr{E}_r$ be given, $s \in S(\Gamma_1, e)$, and $z = g(s, e)$. As $N(\Gamma_1, e) \subset P(\Gamma_1, e)$, if $g_y(s) \neq 0$ then Proposition 2 proves the result.

If $g_y(s) = 0$, then there are two cases:

(i) $z = (\omega_1, \ldots, \omega_n, 0) \in R(e)$, and we are done;

(ii) $z = (\omega_1, \ldots, \omega_n, 0) \notin R(e)$. Then let $(r', z') \in RE(e)$. As for all i, $(\omega_i, 0) \in B_i(r, e)$, we have that $u_i(x'_i, y') \geq u_i(\omega_i, 0)$. We claim for some i, $u_i(x'_i, y') > u_i(\omega_i, 0)$. If not then for all i, $u_i(x'_i, y') = u_i(\omega_i, 0)$ and $z \in R(e)$, contradicting hypothesis (ii). As $R(e) \neq \emptyset$ the coalition I, by playing strategies s' described in Proposition 3 obtains the ratio allocation z'. Therefore I can obtain an allocation such that $v_i(s'_i, s'_{-i}) \geq v_i(s_i, s_{-i})$ for all $i \in I$ with strict inequality for some i. Thus $z \notin PA(\Gamma_1, e)$. □

We close this section by relating the above results to the literature. First observe that we do not rule out the possibility of increasing returns. Thus our results may seem to violate the result of Calsamiglia (1977), that it is impossible to realize, let alone implement, a sub-correspondence of the Pareto correspondence with a finite dimensional strategy space when there are increasing returns. The difference is that in our implementation problem the designer has knowledge of the production technology (it is the (convex) preferences that are unknown) whereas Calsamiglia (1977) is concerned with finding the optimal allocations when preferences are known, but the (non-convex) technology is not.

Second, we note that the game Γ_1 does not ensure that the outcome is always feasible: for some disequilibrium strategy profiles, the contribution to public expenditure required of some agent may exceed her endowment. We propose a modified game that overcome this problem, at the cost of complicating the outcome function, in the next section.

4. Implementation by a feasible and continuous mechanism

The game Γ_1 is extremely simple, however it has two technically undesirable properties; the outcome function is discontinuous, and it does not guarantee individual feasibility. In this section we propose a modified game with a continuous and feasible outcome function that on $_{ce}$ double implements the ratio correspondence.

Discontinuity is considered undesirable as small 'trembles' away from an equilibrium strategy may lead to allocations very different from the equilibrium allocation. In fact it is this very type of discontinuity that drives our previous results. Recently several authors have questioned are such discontinuities necessary. Aghion (1985) and Benassy (1986) have shown the impossibility of finding a market game with a smooth outcome function and efficient Nash equilibria. Also Vega-Redondo (1985) has shown the impossibility of finding, for two person CRS economies, a smooth game form that implements the Lindahl correspondence. For CRS economies the two person case is comprehensively treated by Kwan and Nakamura (1990), who show that there is a continuous game that implements the

Lindahl correspondence. The game Γ_2 below also has a continuous outcome function, but it is not smooth at equilibrium profiles.Recall that in the previous section we extended the domain of agents utility functions to $\Re^{k+1}$ by assuming that any element in the positive orthant is preferred to any element not in it. This is an artificial construction, and the interpretation of negative consumptions remains problematic. An alternative is to require that the outcome function selects an outcome that, for every strategy profile, is feasible and in every agents consumption set. This approach has been labeled, implementation by a completely feasible mechanism, or *feasible implementation*.

In a seminal paper Maskin (1977) identified a necessary condition that a correspondence must satisfy if it can be implemented by a game that specifies such a feasible allocation for every strategy profile. Maskin's condition, called here Maskin-monotonicity, was stated for general abstract environments. [2] The application of Maskin monotonicity to our domain is presented below.

Definition 4.1. A correspondence ϕ is *Maskin-monotonic on* $\mathscr{E}$ if $\forall\, e, e' \in \mathscr{E}$ such that, $A(e) = A(e')$ and $\forall\, i$, $\omega_i' = \omega_i$, if $z \in \phi(e)$ and $\forall\, i$ $R_i'(z) \cap A(e') \subset R_i(z) \cap A(e)$, then $z \in \phi(e')$.

Hurwicz et al. (1984) prove that on $\mathscr{E}_{crs}$, L is *not* Maskin-monotonic. In particular, if an economy e admits a Lindahl equilibrium allocation z such that some agent's consumption bundle is on the boundary of her consumption set, then there is $e' \in \mathscr{E}_{crs}$, such that (e, e', z) satisfies the hypotheses of Maskin monotonicity and $z \notin L(e')$, see also Tian (1988). Therefore, as R coincides with L on $\mathscr{E}_{crs}$, it is not monotonic on $\mathscr{E}_c$. However, R does satisfy Maskin monotonicity on $\mathscr{E}_{ce}$.

Lemma 4.1. On $\mathscr{E}_{ce}$, R satisfies Maskin monotonicity.

Proof. Let $e, e' \in \mathscr{E}_{ce}$ be given, and (r, z) be a ratio equilibrium for e. Suppose that (e, e', z) satisfy the hypotheses of Maskin monotonicity and that $z \notin R(e')$. Then for some i, $R_i'(z) \cap A(e') \subset R_i(z) \cap A(e)$, while $R_i'(z) \cap B_i(r_i, e') \neq \emptyset$. Let $(x_i', y') \in P_i'(z) \cap B_i(r_i, e')$. As $e \in \mathscr{E}_{ce}$, if $z \in R(e)$, then for all j, $(x_j, y) \in \Re^{k+1}_{++}$. Therefore, as u_i is quasi-concave and continuous on $\Re^{k+1}_{++}$, for all $t \in [0, 1)$, if $(x_i^t, y^t) = t(x_i, y) + (1-t)(x_i', y')$, then $(x_i^t, y^t) \in P_i'(x_i, y)$. Furthermore, as $B_i(r_i, e)$ is convex, $(x_i^t, y^t) \in {}'B_i(r_i, e)$. However, as for all j, $(x_j, y) \in \Re^{k+1}_{++}$, for large t, $x_i^t + c(y^t) < \sum \omega_j$. Thus there exists $z' \in A(e)$ such that $(x_i^t, y^t) \in B_i(r_i, e) \cap P_i'(x_i, y)$. Thus $(x_i^t, y^t) \in B_i(r_i, e) \cap P_i(x_i, y)$, contradicting $z \in R(e)$. □

[2] For a proof of Maskin's Theorem see Saijo (1988).

Definition 4.2. The game form $\Gamma_2 = (S, g)$ is specified by the following components:

(i) $\forall\ i \in I$, $S_i = [0, 1]^k \times \Re^k$ with generic element $s_i = (r_i, y_i)$. Given $r \in \Delta^{(n-1)k}$ let

$$r'_{il} = \begin{cases} r_{il} & \text{if } \sum_N r_{il} \geq 1, \\ 1 - \sum_{j \neq i} r_{jl} + \dfrac{|1 - \sum_N r_{il}|}{n} & \text{otherwise,} \end{cases}$$

and $y(s)$ be a continuous function of s such that; for all s, for all i, $r'_i \cdot c(y(s)) \leq \omega_i$, and when for all i, $r'_i \cdot (\sum_N y_i) \leq \omega_i$, then $y(s) = \sum_N y_i$.[3]

(ii) $g: S \rightarrow \Re^{n+k}$ where $g(s, e) = (\omega_1 - r'_1 \cdot c(y(s)), \ldots, \omega_n - r'_n \cdot c(y(s)); y(s))$.

On $\mathscr{E}_{ce}$ all ratio equilibrium allocations must be interior. Suppose all agents play strategies used in the proof of Proposition 3.1. Consider a deviation by agent i. From Proposition 3.1, if i is to benefit, it must be because she moves the outcome to one that was not feasible given the declared ratios. But in Γ_2, no such unilateral deviation exists. Furthermore all Nash equilibrium allocations must be interior. Thus, by monotonicity, in an equilibrium, s, the declared ratios, r, must sum to one. Let $z = g(s, e)$ and suppose (z, r) is not a ratio equilibrium. Then for some agent, i, there must be a point, (x'_i, y'), in her budget, preferred to (x_i, y). But then as z is an interior allocation, as in the proof of Lemma 4.1, all strict convex combinations of (x, y) and (x'_i, y') are preferred to (x_i, y). Thus there is a feasible allocation that i can attain which she prefers to z, and so we the given strategies cannot be a Nash equilibrium. Similar arguments hold for strong equilibrium, and so Propositions 4.3 and 4.4 remains true if we replace and Γ_1 with Γ_2 in their statements above. Hence like Hurwicz et al. (1984) and Tian (1989), we obtain feasible implementation at the cost of restricting the domain.

We summarize with the following Proposition.

Proposition 4.5. There exists a feasible and continuous game, Γ_2, such that on $\mathscr{E}_{ce}$, Γ_2 double implements the ratio correspondence: $NA(\Gamma_2, e) = SA(\Gamma_2, e) = R(e)$.

5. Conclusion

In this paper we have examined the possibility of implementing Pareto-optimal and individually-rational allocations in public good economies where agents have

[3] For example, as each set $B_i(r, e)$ is convex for all such $r \geq 0$ and the intersection of non-empty convex sets is non-empty and convex, then we may take $y(s)$ to be the projection of $\sum_N y_i$ on $\cap_N B_i(r, e) \cap \Re_+^{k+1}$, which is single valued and continuous.

free access to the technology. We found the that the ratio correspondence, introduced by Kaneko (1977a) met our requirements. We double implemented the ratio correspondence, by means of a market like game. Our first game Γ_1 is extremely simple, and bears a close relationship to Lindahl's proposal. However the outcome function did not always select individually feasible allocations, nor was it continuous. Our second game Γ_2 was designed to overcome these problems.

Given the simplicity of the game form, and the robustness of double implementation, we think that the cost share game may be a realistic proposal for solving the free rider problem when there is a small number of players who are well informed about each other. For example it could be used to provide a set of funding rules for research joint ventures, or as a means of dividing joint costs among divisions in a multi-division firm.

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