Comparative statics for aggregative games
The strong concavity case

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Abstract

In this paper we study the effects of a change in some exogenous variable (the number of players or a parameter in the payoff functions) on the strategies played and payoffs obtained in a Nash equilibrium in the framework of an Aggregative Game (a generalization of the Cournot model). We assume a strong concavity condition which implies that the best reply function of any player is decreasing in the sum of the strategies of the remaining players (i.e. strategic substitution). Our results generalize and unify those known in the Cournot model.

Keywords: Comparative statics; Games

1. Introduction

In this paper we study the effects of changes in the number of players and shifts in their payoff function on the strategies played and the payoffs obtained in a Nash equilibrium. We will assume, in the class of games under consideration, that the payoff function of each player fulfills the following:

(1) It can be written as a function of a player's own strategy (assumed to be one dimensional) and the sum of the strategies of all players. This assumption has been called the 'Aggregation Axiom' by Dubey et al. (1980), p. 346) and the corresponding games are called aggregative games. According to Shubik (1984) p. 325): 'Games with the above property clearly have much more structure than a

game selected at random. How this structure influences the equilibrium points has not yet been explored at depth'.

(2) It satisfies a strong concavity condition slightly stronger than the Strategic Substitutes case studied by Bulow et al. (1985). The latter implies that the best reply function of each player (i.e. the mapping selecting the best strategy for a player, given the strategies of the remaining players) is decreasing in the strategies of the other players. This strong concavity condition is a generalization of the so-called Hahn stability conditions (see Hahn, 1962).

Notice that the class of games satisfying (1) and (2) is large and include:

(a) Models of strategic interaction among firms like competition in quantities (i.e. the Cournot model and certain models with heterogeneous goods), competition under rationing schemes (see Romano, 1988), technological competition (see Loury, 1979), the problem of the commons (see Dasgupta and Heal, 1979, pp. 55–78), and pollution games.

(b) Models of group interaction, i.e. rent-seeking, public goods, etc. (see Olson, 1971, esp. pp. 23–43).

(c) Models focusing on the internal organizaion of firms or the like as contribution and revelation games and principal–many agents models.

In all the above cases uncertainty, taxes and payoff functions different from profit functions (i.e. sales) are allowed. The latter is specially important in imperfectly competitive markets since the classical hypothesis of profit maximization lacks a convincing foundation in those markets (see, for example, Baumol, 1959, and Vickers, 1985).

We first prove that the best reply functions of a game satisfying the aggregation axiom and the strategic substitution condition do not have any structural property beyond that they depend on the sum of strategies of the remaining players and that they are decresing (Proposition 0). This result can be used to motivate the need of our strong concavity assumption. Assuming the latter we show that:

(1) An increase in the number of players (a) decreases the value of the strategy of any incumbent, and increases the sum of all strategies (Proposition 1); (b) decreases the payoff of incumbents (Propositions 2 and 3).[1]

(2) A shift raising the marginal payoff curve of a player, say i, (a) increses the sum of strategies and the strategy of Player i, and decreases the strategy of any other player (Proposition 4); (b) increases the payoff of player i and decreases the payoff of any other player (Proposition 5).

(3) A shift raising the marginal payoff curve of *all* players increases the sum of all strategies (Proposition 6).

[1] It should be remarked that the fact that the utility of incumbents decreases with the number of agents plays a crucial part in Olson's theory of collective action (see Olson, 1971). According to this author, large groups are unlikely to form since the private benefit to joining such a coalition may be negative. On the contrary, small groups usually provide high benefits to its members and thus the incentive to stay is preserved.

(4) We provide counterexamples to all propositions when the strong concavity assumption is not fulfilled. Also two more examples are used to show that in the case considered in (3) above nothing can be said about individual strategies and utilities. Some of these examples are taken from previous work and are included here for the sake of completeness.

Summing up, (1), (2) and (3) above show that under our assumptions, the effects of an increase in the number of players or a shift in their payoff function agrees with our a priori intuition. (1) above has been studied in the Cournot case by McManus (1962, 1964), Frank (1965), Ruffin (1971), Okuguchi (1973), Seade (1980) and Szidarovsky and Yakowitz (1982). It must be noticed that our approach not only generalizes these results but allows for simpler proofs and does not require that the number of players can be treated as a continuous variable. Parts (2)–(3) above have been studied in the Cournot case by Dixit (1986) and Quirmbach (1988).

Our results can be compared with those obtained under the (polar) assumption of supermodularity. Roughly speaking, a game is supermodular when for each player her strategy set is the product of compact intervals and the marginal profitability of any action increases with any other action of any player (see Topkis, 1979), for a more general definition). When strategy sets are one-dimensional the above definition reduces to that of a game with strategic complementarities (see Bulow et al., 1985). It can be shown that if the marginal profitability of any action is increasing in a parameter, say τ (this is identical to our Assumption 4 below), the largest and smallest Nash equilibria are increasing functions of τ, so if the Nash equilibrium is unique, it is increasing in τ (see Lippman et al., 1987; Milgrom and Roberts, 1990; and Milgrom and Shannon, 1992).[2] This is analogous to our Propositions 4 and 6 (but in our case individual strategies are not always increasing in τ, see Example 6). Notice that the distinction between idiosyncratic and generalized shocks does not play any role in supermodular games. To the best of my knowledge there are no results in this literature on the effect of entry (Propositions 1–3 below) or the effect of a change in τ on payoffs (Proposition 5 and Example 6).

The rest of the paper is organized as follows. Section 2 explains the basic model and the main assumptions. Section 3 studies the effect of an increase in the number of players and Section 4 focuses on shifts of the marginal payoff curve. Finally Section 5 gathers our final comments.

[2] Other properties of supermodular games are that (1) the existence of a Nash equilibrium does not require quasi-concavity of the payoff functions, and (2) under certain cirumstances, if there are several Nash equilibria, they can be Pareto-ranked. Applications of supermodular games include Bayesian games and oligopolistic competition (see Vives, 1990), stability and learning (see Lippman et al., 1987; Milgrom and Roberts, 1990; and Krishna, 1992 and coordination problems in a macroeconomic framework (see Silvestre, 1993, for a survey of this literature). For general surveys on supermodular games, see Fudenberg and Tirole (1991) and Vives (1993).

2. The model

In this section we explain the main concepts which will be used in the rest of the paper.

Definition 1. An aggregative game $(U_i(\cdot), S_i)_{i \in I}$ consists of
(a) a set of players (also called agents) $I = 1, 2, \ldots, n$;
(b) a collection of strategy sets $S_i = R_+$;
(c) a collection of payoff functions U_i: $\times_{i \in I} S_i \rightarrow R$ of the form $U_i(x_i, x)$, where $x_i \in S_i$ and $x = \sum_{j \in I} x_j$.

In words, in an aggregative game, the so-called 'Aggregation Axiom' holds (see Dubey et al., 1980, p. 346), so the (one-dimensional) strategies of the players can be aggregated in an additive way. We remark that all the propositions below can be proved if $x = f(x_1, \ldots, x_n)$ ($f(\cdot)$ strictly increasing), introducing suitable concavity assumptions. An aggregative game can be thought of as a generalization of the well-known Cournot model. In this case $U_i = p(x)x_i - C_i(x_i)$, x_i being the output of firm i, x total output, $p(x)$ the inverse demand function and $C_i(x_i)$ the cost function of firm i. This case will be used in most examples below. We remark that our approach can deal with (a) payoff functions different from profit (i.e. welfare-maximizing publicly owned firms; see Fershtman, 1990; (b) symmetric uncertainty (for the Cournot case see Horowitz, 1987); (c) taxes (for the Cournot case see Dierickx et al., 1988); and (d) in some cases, heterogeneous product (using the trick of Yarrow, 1985, p. 517). Other examples of aggregative games (technological competition, the problem of the commons, preference revelation, contribution games, pollution and wage-setting trade unions) are explained in Table 1. Olson (1971) provides several applications to political science.

Now we state our solution concept.

Definition 2. Given an aggregative game $(U_i(\cdot), S_i)_{i \in I}$, $(x_i^*, x^*)_{i \in I}$ with $x^* = \sum_{i \in I} x_i^*$, $x_i^* \in S_i$, $\forall i \in I$, is said to be a Nash equilibrium (N.E.) if $\forall i \in I$

$$U_i(x_i^*, x^*) \geqslant U_i(x_i, x^* - x_i^* + x_i)\,, \quad \forall x_i \in S_i\,.$$

Now we state and discuss our main assumptions. Let $\mathscr{C}^0$ be the class of continuous functions and $\mathscr{C}^s$ be the class of functions which are s times continuously differentiable.

Assumption 1. $U_i(\cdot) \in \mathscr{C}^1$, $\forall i \in I$.

Notice that under Assumption 1 (A.1 in what follows) if $x_i^* \in \text{int. } S_i$ the necessary condition of a N.E. reads as follows:

Table 1

	x_i	x	$U_i(x_i, x)$	$x = f(x_1, \ldots, x_n)$
Trade unions	Percent increase in wage rate	Inflation rate	Utility function of trade union i	Inflation rate as a function of wage rate increases
Pollution	Output of firm i	Amount of pollution	Profit function	Production of pollution
Contribution games (public goods, principal agents)	Prive inputs offered by i	Quantity of the public good/reward	Utility function of agent i	Production function of the public good/reward function
Preference revelation	Preference parameters to be revealed	Social state	Utility function	Social rule
Problem of the commons	Inputs used by firm i	An environmental variable	Profit	Environment as a function of inputs
Oligopoly	Output of firm i	Price	Profit function	Inverse demand function
Technological competition	Input needed to produce the technology used by firm i	Technological level	Profit function	Technology as a function

$$\frac{\partial U_i(x_i^*, x^*)}{\partial x_i} + \frac{\partial U_i(x_i^*, x^*)}{\partial x} = 0\,, \quad \forall i \in I\,.$$

Let us define

$$T_i = T_i(x_i, x) \equiv \frac{\partial U_i(x_i, x)}{\partial x_i} + \frac{\partial U_i(x_i, x)}{\partial x}\,, \quad \forall i \in I\,.$$

Let N be the set of active agents (i.e. those for whom $x_i^* \in \text{int.}\ S_i$ in a N.E. with n players). $N+1$ is defined accordingly. We will assume that $N \cap N+1 \neq \emptyset$, i.e. at least one player is active in a N.E. with n and $n+1$ agents, respectively.

Assumption 2. $T_i(x_i, x)$ is strictly decreasing on x_i and x, $\forall i \in I$.

A.2 is what we have called in the Introduction the strong concavity condition. A sufficient condition for A.2 to hold is that $U_i(\cdot)$ be strictly concave on x and x_i and (if $U_i \in \mathscr{C}^2$) that $\partial^2 U_i()/\partial x,\ x_i < 0$.

Implicit differentiation of $T_i(\cdot)$ shows that A.2 implies that the best reply function is strictly decreasing, i.e. the assumption of Strategic Substitution in the terminology of Bulow et al. (1985).

In the homogeneous oligopoly case A.2 is equivalent to a much used condition in the literature on Cournot equilibrium (see, for example, Friedman, 1982, p. 496, Assumption 3, and the references therein), namely

$$\frac{\partial^2 p(\cdot)}{\partial x^2} x_i + \frac{\partial p(\cdot)}{\partial x} < 0 \quad \text{and} \quad \frac{\partial p(\cdot)}{\partial x} - \frac{\partial^2 C_i(\cdot)}{\partial x_i^2} < 0 .$$

This assumption was first used by Hahn (1962) in connection with the dynamic stability of the Cournot equilibrium.

Finally, we state our third assumption.

Assumption 3. $U_i(\cdot)$ is strictly decreasing on x, $\forall i \in I$.

This assumption will be only used in Propositions 2, 3 and 5. In the Cournot case A.3 requires a strictly decreasing inverse demand curve.

Notice that A.1 and A.2, plus a compactness requirement, imply the existence of an unique N.E. and that under A.1 and A.3 any interior N.E. can be shown to be inefficient, i.e. there is a strategy vector for which all players are better off (for proofs of these facts see Friedman, 1977, pp. 25–26 and 169–171). Obviously these conditions are far from necessary; see, for example, Kukushkin (1994)).

The reader may wonder if, under the aggregation axiom, strategic substitution alone may be sufficient to yield well-defined answers to our comparative statics questions. The following auxiliary proposition looks for structural properties of best reply functions under these two assumptions and finds a negative result. First, let us define $x_{-i} \equiv \Sigma_{j \neq i} x_j$.

Proposition 0. *Let $x_i = f_i(x_{-i})$, $i = 1, \ldots, n$, be a collection of $\mathscr{C}^0$ functions with $x_i \in S_i$ (S_i compact) and such that $f_i(\cdot)$ is strictly decreasing $\forall i$, then*

(a) $\forall i$, $\exists U_i(x_i, x)$, $U_i(\cdot) \in \mathscr{C}^1$, *concave on x_i such that*

$$f_i(x_{-i}) \equiv \arg\max_{a \in S_i} U_i(a, a + x_{-i}) , \quad \forall x_{-i}$$

Moreover, $U_i(\cdot)$ can be taken to be decreasing on x (i.e. fulfilling A.3).

(b) $\forall i$, $\exists a \mathscr{C}^1$ *cost function $C_i(x_i)$ and a linear inverse demand function $p = A - x$ such that*

$$f_i(x_{-i}) \equiv \arg\operatorname{mag}_{b \in S_i} (A - b - x_{-i})b - C_i(b) , \quad \forall x_{-i} .$$

Proof. (a) First notice that $f_i(\cdot)$ is invertible. Also, $f_i^{-1}(\cdot)$ is integrable since $f_i^{-1}(\cdot) \in \mathscr{C}^0$ (by the continuity of $f_i(\cdot)$; see Bartle, 1976, p. 156), and it is bounded (see Bartle, 1976, p. 427). Let $q_i(x_i)$ be the primitive of $f_i^{-1}(x_i)$. Define $U_i \equiv q_i(x_i) + x_i^2 - x_i x$. Notice that U_i is decreasing on x. Then we have that

$$\frac{\partial U_i}{\partial x_i} = f_i^{-1}(x_i) + 2x_i - x_i - x \equiv f_i^{-1}(x_i) - x_{-i} = 0 \,.$$

And since $f_i^{-1}(\cdot)$ is strictly decreasing, U_i is concave on x_i, so the second-order condition of payoff maximization is satisfied, and thus (a) holds.

(b) Let $p(x) \equiv A - x$ and $C_i(x_i) \equiv Ax_i - x_i^2 - q_i(x_i) + B$, where $q_i(\cdot)$ is as defined in part (a) above. Since x_i is defined on a compact set, B can be taken large enough such that $C(x_i) \geq 0$, $\forall x_i$. Also, taking A large enough, the marginal cost is positive. Then,

$$\begin{aligned} U_i &\equiv p(x)x_i - C(x_i) = (A - x)x_i - Ax_i + x_i^2 + q_i(x_i) - B \\ &= q_i(x_i) + x_i^2 - xx_i - B \,, \end{aligned}$$

which is identical to the utility function constructed in part (a) above.

The main consequence of Proposition 0 is that in games in which both the aggregation axiom and the strategic substitution assumption hold, the best reply functions depend on the sum of strategies of the other players and that they are decreasing exhaust all the properties of best reply functions. Thus, they are, to some extent, arbitrary (this result may be regarded as analogous to the lack of structural properties of excess demand functions in General Equilibrium; see Shafer and Sonnenschein, 1982, but in our case the root of the problem is not on the aggregation side). Even if payoff functions are restricted to be profit functions, no structural property beyond those quoted above can be found!

As an easy corollary of Proposition 0 we have (a) the equilibrium set of strategies is arbitrary and (b) comparative statics will not yield definitive answers. Both points can be easily seen in the case of two players by constructing best reply mappings that intersect at any given se of points and by considering shifts of these curves and comparing non-adjacent equilibria. Thus, we are led to conclude that in general we need additional properties to those quoted before in order to tackle comparative statics. As we will see, our A.2 will be sufficient for this job.

3. The effects of entry

In this section we study the effects of an increase in the number of players (see Bresnahan and Reiss, 1991, and the references therein for the empirical evidence in oligopolistic markets). In order to save notation let $y \equiv x_{n+1}(n+1)$. Also, let us denote by $x(n)$, $x_i(n)$ and $U_i(n)$ the equilibrium values of x, x_i and U_i in a game with n players. We remind the reader that not all agents in N are active.

Proposition 1. *Under A.1 and A.2 we have that*

(a) $x(n) \leqslant x(n+1)$, $x_i(n) \geqslant x_i(n+1)$, $\forall i \in N$, *and*

(b) *if* $y > 0$ *the above inequalities are strict.*

Proof. We first notice that if $x(n) \geqslant x(n+1)$ and $x_i(n) > 0$, $x_i(n+1) = 0$ is impossible since $T_i(x_i(n), x(n)) = 0$ and $T_i(0, x(n+1)) \leqslant 0$, so $T_i(x_i(n), x(n)) \geqslant T_i(0, x(n+1)) \geqslant T_i(0, x(n))$, which contradicts that $T_i(\cdot)$ is strictly decreasing on x_i[3]. Take any $i \in N \cap N+1$ (if $i \not\in N+1$, by definition $x_i(n) > x_i(n+1) = 0$). In both N.E. first-order conditions hold so

$$T_i(x_i(n), x(n)) = T_i(x_i(n+1), x(n+1)) \,. \tag{1}$$

Therefore because of A.2 we have only two possiblities:

(I) $x(n+1) \leqslant x(n)$ and $x_i(n+1) \geqslant x_i(n)$, with a strict inequality, or

(II) $x(n+1) \geqslant x(n)$ and $x_i(n+1) \leqslant x_i(n)$.

If *(I)* holds, since all active players at n are active at $n+1$ and $x = \Sigma\, x_i$, we have a contradiction. Therefore part (a) is proved. Part (b) is proved noticing that (1) implies that if $x(n) = x(n+1)$, then $x_i(n) = x_i(n+1)$, $\forall i \in N \cap N+1$. But since all active players at n will be active at $n+1$ and $y > 0$, we reach a contradiction. Therefore $x(n) < x(n+1)$, A.2 plus (1) show that $x_i(n) > x_i(n+1)$, $\forall i \in N \cap N+1$. Finally, if $i \not\in N+1$ but $i \in N$, $x_i(n) > x_i(n+1) = 0$. □

If A.2 does not hold, Proposition 1 fails as the following examples – which refer to the Cournot model – show.

Example 1. (Seade, 1980). Let $p = x^{-0.8}$, $C_i = x_i$. Using the first-order conditions of profit maximization, it is easily seen that $x_i(1) < x_i(2)$.

Example 2. $p = a - bx$, $C_i = cx_i + d/2x_i^2$, with $a > c$, $d < 0$, $d + 2b > 0$ and $d + b < 0$. (Total costs will be negative for x_i large enough, but this problem does not arise if $(a-c)/(2b+d) < -c/d$). Then $x = (a-c)n/(b+d+nb)$ so x is decreasing on x if $b + d < 0$. On the other hand, second-order conditions are fulfilled if $d + 2b > 0$. A graphical argument similar to this example can be found in McManus (1964).

We now turn to study how payoffs change with entry.

Proposition 2. *Under A.1, A.2 and A.3*

(a) $U_i(n) \geqslant U_i(n+1)$, $\forall i \in N$, and

(b) If $y > 0$, the above inequalities are strict.

[3] A similar argument shows that if $x(n) \leqslant x(n+1)$ and $x(n) = 0$, then $x(n+1) = 0$, so the second inequality in (a) in Proposition 1 holds $\forall i \in I$.

Proof. In order to save notation let us write $x_i(n)$ as the strategies of all players except i in a N.E. with n players, i.e. $x_{-i}(n) = x(n) - x_i(n)$. Also define $V_i(\cdot) \equiv U_i(x_i, x_{-i} + x_i) \equiv V_i(x_i, x_{-i})$. Then, if Proposition 2(a) were not true, $V_i(x_i(n+1), x_{-i}(n+1)) > V_i(x_i(n), x_{-i}(n)) \geqslant V_i(x_i(n+1), x_{-i}(n))$. Thus, $x_{-i}(n) > x_{-i}(n+1)$, which contradicts that $x_{-i}(n)$ is non-decreasing in n by Proposition 1(a). In order to show (b), let us assume that $U_i(n) = U_i(n+1)$. Then, reasoning as above we get $x_{-i}(n) \geqslant x_{-i}(n+1)$, contradicting that if $y > 0$, $x_{-i}(n)$ is strictly increasing in n (by Proposition 1(b)). □

If A.2 holds but $U_i(\cdot)$ is increasing in x, we have the reverse conclusion, i.e. that entry increases the payoff of incumbents. The following example shows that if A.2 does not hold, Propostition 2 may fail.

Example 3. Let us assume two agents with identical payoff functions (see Fig. 1). Because of A.3, payoffs increase in the direction of the arrows. Point A is a symmetrical N.E. with two players since any player can only change unilaterally x and x_i on the 45° line (x and x_i change by the same amount since the strategies of the other players are given). By the same token, B is a symmetrical N.E. with three players and such that the payoffs of 1 and 2 are now greater (notice that if $n = 1$ $A'A$ and OA were identical, the example does not work).

Notice than in Example 3 we have that $n > 1$. If this is not the case, i.e. there is a unique incumbent player, the entry of a new player will always decrease the payoff of the incumbent, i.e. her payoff is bigger under monopoly than under duopoly, as shown by the next proposition.

Proposition 3. *Under A.3 we have that*

(a) $u_1(1) \geqslant U_1(2)$ *and*

(b) *if* $x_2(2) > 0$, *then the above inequality is strict.*

Proof. Suppose it is not. Defining $V_i(\cdot)$ as before we have that

$$V_1(x_1(2), x_2(2)) \geqslant V_1(x_1(1), 0) \geqslant V_1(x_1(2), 0)\,.$$

And since $V_i(\cdot)$ is decreasing on x_{-i}, we get a contradiction. □

Notice that A.1 and A.2 are not required for the proposition to hold. As in the case of the previous proposition, if $U(\cdot)$ is increasing in x, it is easy to show that entry increases the payoff of the incumbent.

4. The effects of shocks

In this section we study the effect of an exongenous shift in the payoff function on the relevant variables. We will assume that the payoff function of player i can

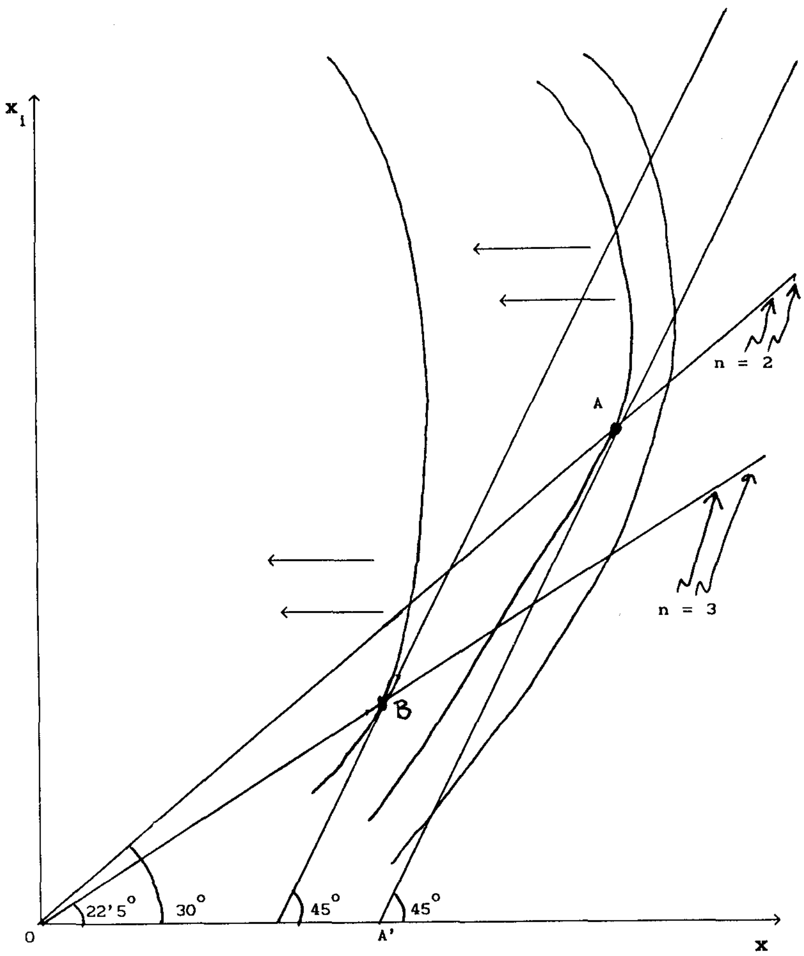

Fig. 1.

be written as $U_i = U_i(x_i, x, t_i)$, where t_i is a one-dimensional parameter which is possibly different for different players (in the Cournot model t_i may represent either the factors behind the demand side or the cost side or, as in Farrell and Shapiro, 1990, the quanity of capital owned by firm i). In this section, in order to simplify the proofs, we assume that Nash equilibria are interior. Then, the first-order condition reads $T_i(x_i, x, t_i) = 0$. Finally, the values of the strategies and payoffs in a Nash equilibrium will be denoted by x_i^*, x^*, and U_i^*.

Assumption 4. $T_i(\cdot)$ is strictly increasing in t_i.

This assumption allows us to interpret increases in t_i as shifts to the right of the marginal payoff curve, i.e. t_i can be regarded as a measure of the impact of a shock on the marginal payoff of player i.

We distinguish two types of shocks: idiosyncratic and generalized. In the first we study the impact on the market of a variation in a single t_i (i.e. an increase in the price of the factors or the taxes paid by player i). In the second we consider a simultaneous variation in all t_i, $i = 1, \ldots, n$. This corresponds, for instance, to a shift in the common demand function or the price of a facto used by all players in the industry. In this case, without loss of generality we will write the first-order condition as a function of a single t, i.e. $T_i(x_i, x, t) = 0$.

Intuition suggests that in the case of an idiosyncratic shock an increase in t_i will increase the strategy of player i and it will decrease the strategies of her competitors. This intuition is formalized in the next proposition:

Proposition 4. *Under A.1, A.2 and A.4 an increase in* t_i, (a) *increases the sum of strategies*, (b) *increases the strategy of player i and* (c) *decreases the strategy of any other player in the market.*

Proof. Since the proof is fairly analogous to the proof of Proposition 1, we will indicate only the guidelines. First it is proven that the sum of strategies cannot be constant. Second, if the sum of strategies decreses, the strategy of all players must increase in order to maintain first-order conditions, and this is a contradiction. Thus, the sum of strategies increases. Again the first-order conditions of all players except i imply that the stretegies of these players must fall. Therefore the strategy of i must increase. □

Of course if the inequality in A.4 is reversed so are the conclusions of Proposition 4. An implication of this proposition is – in contrast with supermodular games – the absence of multiplier effects, i.e. $\mathrm{d}x/\mathrm{d}t_i < \mathrm{d}x_i/\mathrm{d}t_i$ (see Fudenberg and Tirole, 1991, p. 498). The next example – which again refers to the Cournot model – will show that A.2 is needed for the result to hold.

Example 4. Suppose that there are three firms and that in a (sufficiently large) neighborhood of a N.E. the relevant functions read $p = a' - x$, $C_1 = cx_1 - d/2x_1^2 - t_1x_1$ with $a' > c$, $d > 0$, $d - 2 < 0$ (so the second-order condition holds), $d - 1 > 0$, and $C_i = c'x_i$, with $a' > c'$, $i = 2, 3$. Let $\tilde{a} \equiv a' - c$ and let $a \equiv a' - c'$. Profit maximization implies that $x_1 = (x - \tilde{a} - t_1)/)(d - 1)$ and $x_i = a - x$, $i = 2, 3$. Solving the system we get $x = (2a(d - 1) - \tilde{a} - t_1)/(3(d - 1) - 1)$. If, for instance, $a = 10$, $d = 1.5$, $t_1 = 5$ and $\tilde{a} = 1$, we have that $x^* = 8$, $x_i^* = 4$ and $x_i^* = 2$. But if $t_1 = 5.5$, $x^* = 7$, $x_1^* = 1$ and $x_i^* = 3$.

For the next proposition we need an additional assumption. This assumption plus A.4 implies that a variation in t_i affects both marginal and total payoff in the same direction.

Assumption 5. $U_i(\cdot)$ is increase on t_i.

Proposition 5. *If all payoff functions are $\mathscr{C}^2$ and A.2–A.5 hold, an increase in t_i (a) increases the payoff of i and (b) decreases the payoff of any other player.*

Proof. First, it is easy to show that A.2 implies that the Jacobian matrix of $T_i(\cdot)$ has a non-vanishing determinant. Thus the implicit function theorem implies that all the relevant variables are continuously differentiable functions of t_i in a neighborhood of equilibrium. Then, taking into account the first-order conditions for player $j \neq i$, we have that

$$\mathrm{d}U_j/\mathrm{d}t_i = \partial U_j(\cdot)/\partial x \circ (\mathrm{d}x/\mathrm{d}t_i - \mathrm{d}x_j/\mathrm{d}t_i)$$

and Proposition 4 and A.3 imply (b) above. In the case of player i we have that

$$\mathrm{d}U_i/t_i = \partial U_i(\cdot)/\partial x \circ (\mathrm{d}x/\mathrm{d}t_1 - \mathrm{d}x_i/\mathrm{d}t_i) + \partial U_i(\cdot)/\partial t\ ,$$

and since the strategy of all competitors has decreased and because of A.5 we obtain (a) above. □

The next example shows the necessity of A.2 for Proposition 5 to hold

Example 5. Suppose that the market is as in Example 4. Then it is easily calculated that if $t_1 = 5$, $U_1^* = 4$ and $U_i^* = 4$, $i = 2, 3$. But if $t_1 = 5.5$, $U_i^* = 0$ and $U_i^* = 9$, $i = 2, 3$.

We end this section by studying the effects of a generalized shock.

Proposition 6. *Under A.1, A.2 and A.4 an increase in t increases x.*

Proof. First, by analogous reasoning to Proposition 1, it can be shown that x cannot be constant. And if x decreases all x_i must increase. A contradiction. □

The effect of t on individual strategies and payoffs in equilibrium depends on how payoff functions are affected (see Dixit, 1986, and Quirmbach, 1988). This means that, in the Cournot model, a technological improvement in costs might decrease the output and profits *of the most efficient firm* (see Example 6). Finally, without A.2 Proposition 6 does not hold (see Example 7).

Example 6. Let $p = a - x$, $n = 2$, $C_1 = c_1 x_1$ and $C_2 = \alpha c_1 x_2$. Take $t = -c_1$, so A.2 and A.4 hold. It can be easily shown that in a N.E. $x_1^* = (a + \alpha c_1 - 2c_1)/3$ and

$U_1^* = ((a - t_1(\alpha - 2))/3)^2$. Thus if $\alpha > 2$, the output and profits of firm 1 (which is the most efficient firm) decreases with t.

Example 7. Let $p = x + t - A$, $C_i = 2.5\ x_i^2/2$ and $n = 2$, with $A > t$ (this implies that for x small p is negative, but since p is positive in equilibrium the inverse demand function can be substituted by $p = \max(0, x + t - A)$). Thus, $T_i = x + t - A - 1.5x_1$ so A.4 and the second-order condition are fulfilled. Then, $x = 4(A - t)$, i.e. x is decreasing on t.

5. Conclusions

In this paper we have tried to integrate several models – some of them often used in Industrial Organization and Welfare Economics – and to show that the qualitative properties of comparative statics of these models conform with our intuition as long as (i) the game is an aggregative game and (ii) a strong concavity condition, which implies strategic substitution, is met.

It would be interesting to know if the qualitative propeties of models of strategic substitutes and strategic complements are similar. However, the latter case presents greater difficulties and might require different methods. First, an additional assumption is needed in order to guarantee that the equilibrium is unique (see, for example, Friedman, 1982, p. 504, Assumption 6, that implies that the best reply function of any player is a contraction, or the dominant diagonal assumption used by Dierker and Grodal, 1994). And second, unless additional assumptions are made, the game is not an aggregative game so it is not clear how to model the strategy of a player who is not in the market. Given all that, it is scarcely surprising that, as we mentioned in the introduction, results in this area are restricted to the study of the effect of a change in an exogenous parameter on the equilibrium strategies. Further results in this direction are likely to be greatly welcomed.

Acknowledgments

This is a revised version of Chapter 1 of my Ph.D. dissertation submitted to London University. My supervisor, O. Hart, and M. González-Maestre, C. Herrero, N. Kukushkin, C. Mazón, J. Moore, H. Moulin, J. Segura, D. Ulph, A. Villar, X. Vives and an anonymous referee provided useful comments. The usual caveat applies. The Fundacion del I.N.I. and D.G.I.C.Y.T., under projects PB88-0289 and PB91-0756, provided financial support in an early stage of this work. A later version of the paper was written while visiting the Indian Statistical Institute (Delhi) with support from the Subdireccion General de Cooperacion Internacional. I thank all these persons and institutions for their help. This paper is dedicated to James Friedman.

References

Bartle, R.G., The Elements of Real Analysis, 2nd edn. (Wiley, New York, 1976).

W.J. Baumol, Business Behavior, Value and Growth (Macmillan, New York, 1959).

T.F. Bresnahan and P.C. Reiss, Entry and competition in concentrated markets, J. Polit. Econ. 99 (5) (1991) 977–1009.

J. Bulow, J. Geanakoplos and P. Klemperer, Multimarket oligopoly: Strategic substitutes and complements, J. Polit. Econ. 93 (1985) 488–511.

P. Dasgupta and G. Heal, Economic Theory and Exhaustible Resources, Cambridge Economics Handbooks (Cambridge University Press, 1979).

0. Dierickx, C. Matutes and D. Neven, Indirect taxation and Cournot equilibrium, Int. J. of Indust. Org. 6 (1988) 385–399.

E. Dierker and B. Grodal, Profit maximization and utility maximization, mimeo., Vienna University, 1994.

A. Dixit, Comparative statics for oligopoly, Int. Econom. Rev. 27 (1986) 107–122.

P. Dubey, A. Mas-Colell and M. Shubik, Efficiency properties of strategic market games, J. Econom. Theory 22 (1980) 339–362.

J. Farrell and C. Shapiro, Asset ownership and market structure in oligopoly, Rand J. Econom. 21 (1990) 275–292.

C. Fershtman, The interdependence between ownership status and market structure: The case of privatization, Economica 57 (1990) 319–328.

C.R. Frank, Entry in a Cournot Market, Rev. Econom. Studies 32 (1965) 245–250.

J. Friedman, Oligopoly and the Theory of Games (North-Holland, Amsterdam, New York, Oxford, 1977).

J. Friedman, Oligopoly theory, in: K. Arrow and M. Intrilligator, eds., Handbook of Mathematical Economics (North-Holland, Amsterdam, 1982) ch. 11.

D. Fudenberg and J. Tirole, Game Theory (MIT Press, Cambridge, MA, 1991).

F.H. Hahn, The Stability of the Cournot Oligopoly Solution, Rev. Econom. Studies 29 (1962) 929–931.

I. Horowitz, Regression-estimated market demand and quasi-Cournot behavior, Int. J. Ind. Org. 5 (1987) 247–253.

V. Krishna, Learning in games with strategic complementarities, Working Paper 92–073, Harvard Business School, Cambridge, MA (1992).

N.S. Kukushkin, A condition for the existence of a Nash equilibrium with public and private objectives, Games and Economic Behavior (1994) forthcoming.

S.A. Lippman, J.W. Mamer and K.F. McCardle, Comparative statics in non-cooperative games via transinfinitely iterated play, J. Econom. Theory 41 (1987) 288–303.

G. Loury, Market structure and innovation: A reformulation, Quart. J. Econom. 93 (3) (1979) 395–409.

M. McManus, Numbers and size in Cournot oligopoly, Yorkshire Bull. 14 (1962) 14–22.

M. McManus, Equilibrium, numbers and size in Cournot oligopoly, Yorkshire Bull. 16 (1964) 68–75.

P. Milgrom and J. Roberts, Rationalizability, learning and equilibrium in games with strategic complementarities, Econometrica 58 (6) (1990) 1255–1277.

P. Milgrom and C. Shannon, Monotone comparative statics, mimeo., Stanford University, 1992.

K. Okuguchi, Quasi-competitiveness and Cournot oligopoly, Rev. Econom. Studies 40 (1973) 145–148.

M. Olson, The Logic of Collective Action (Harvard University Press, 1971).

H. Quirmbach, Comparative statics for oligopoly. Demand shift effects, Int. Econom. Rev. 29 (1988) 451–459.

R. Romano, Oligopolistic competition for market share via voluntary excess supply, Int. J. Ind. Org. 6 (1988) 447–468.

R.J. Ruffin, Cournot oligopoly and competitive behavior, Rev. Econom. Studies 38 (1971) 493–502.

J. Seade, On the effects of entry, Econometrica 48 (2) (1980) 256–278.

W. Shafer and H. Sonnenschein, Market demand and excess demand functions, in: K. Arrow and M. Intrilligator, eds., Handbook of Mathematical Economics (North-Holland, Amsterdam, 1982) ch. 14.

M. Shubik, Game Theory in the Social Sciences: Concepts and Solutions (MIT Press, Cambridge, MA, 1984).

J. Silvestre, Market power and unemployment, J. Econom. Lit. 31 (1993) 105–141.

F. Szidarovsky and S. Yakowitz, Contributions to Cournot oligopoly theory, J. Econom. Theory 28 (1982) 51–70.

D. Topkis, Equilibrium points in nonzero-sum n-person submodular games, SIAM J. Control Opt. 17 (1979) 773–787.

J. Vickers, Delegation and the theory of the firm, Econom. J. Suppl. 95 (1985) 138–147.

X. Vives, Nash equilibrium with strategic complementarities, J. Math. Econom. 19 (1990) 305–321.

X. Vives, Edgeworth and modern oligopoly theory, Europ. Econom. Rev. 37 (1993) 463–476.

G.K. Yarrow, Welfare losses in oligopoly and monopolistic competition, J. Indust. Econom. 33 (4) (1985) 515–529.