

# On the Limit Points of Discrete Selection Dynamics

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This paper provides an analog to the aggregate monotonicity condition introduced by Samuelson and Zhang [*J. Econ. Theory*, 1992] in a study of continuous dynamics. Our condition guarantees that limit points of discrete selection dynamics are rationalizable strategies. We show that the condition will be satisfied by the discrete replicator dynamic if the population does not change rapidly. These results reconcile the Samuelson–Zhang theorem, which implies that limit points of continuous replicator dynamics must be rationalizable, with an example of Dekel and Scotchmer [*J. Econ. Theory*, 1992], which shows that limit points of the discrete replicator dynamic may place positive probability on strictly dominated strategies. *Journal of Economic Literature* Classification Numbers: C72, C73. © 1992 Academic Press, Inc.

## 1. INTRODUCTION

This paper presents a condition on discrete game dynamics which guarantees that limiting outcomes must survive iterated deletion of strictly dominated strategies. We are motivated by two recent studies. Samuelson and Zhang [9] have given conditions under which the limit points of continuous game dynamics must be rationalizable. To prove their result, they introduce a monotonicity condition that is satisfied by the replicator dynamic of evolutionary game theory. In contrast, Dekel and Scotchmer [2] have shown by example that the discrete replicator dynamic need not eliminate a strategy that is strictly dominated by a mixture of the other strategies. We wish to reconcile these results by providing conditions under which discrete dynamics rule out dominated strategies and relating those conditions to the discrete replicator dynamic.

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We obtain the following results. First, we provide an analog to the Samuelson–Zhang condition; our assumption guarantees that limit points of discrete selection dynamics are rationalizable strategies. Second, we show that the condition will be satisfied by the discrete replicator dynamics if the population does not change rapidly. These results help to relate the Samuelson–Zhang theorem to the Dekel–Scotchmer example.

Section 2 of the paper introduces the basic model. Section 3 examines the Dekel–Scotchmer example. Section 4 discusses the monotonicity assumptions needed in the continuous and discrete models and explains the relationship to each other and to the replicator dynamic. Section 5 proves a result that gives conditions under which limit points of discrete selection dynamics must be rationalizable.

Our framework and main result simply translate the Samuelson and Zhang result to a discrete setting. We use their notation whenever possible.

## 2. PRELIMINARIES

We deal with a finite two-player game.<sup>1</sup> We let  $I$  be the set of  $n_1$  pure strategies of player 1;  $J$  denotes player 2's set of  $n_2$  pure strategies. Player  $k$ 's payoff function is denoted by  $\pi_k(i, j)$  for  $(i, j) \in I \times J$ . Let  $S^n$  denote the standard  $(n - 1)$ -dimensional simplex. We extend  $\pi_k(\cdot)$  to the space of mixed strategies  $S^{n_1} \times S^{n_2}$  in the usual fashion by linearity. Hence, for  $(x, y) \in S^{n_1} \times S^{n_2}$ ,  $\pi_k(x, y) \equiv \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \pi_k(i, j)x_i y_j$ ; we will identify the pure strategy  $i$  with the degenerate distribution that places probability one on the  $i$ th pure strategy. In keeping with the usual framework for evolutionary game theory, it is useful to imagine that there are infinite populations of players who take on the roles of player 1 and 2; that players from each population are randomly and anonymously paired; and that all individuals play pure strategies so that an element of  $S^{n_i}$  represents the fraction of members of the population that play each pure strategy.

Samuelson and Zhang consider continuous dynamics determined by functions  $(f, g)$ , where

$$\dot{x}_i = f_i(x, y) \quad i = 1, \dots, n_1 \quad \text{and} \quad \dot{y}_j = g_j(x, y) \quad j = 1, \dots, n_2 \quad (1)$$

that satisfy, for all  $(x, y) \in S^{n_1} \times S^{n_2}$ ,

(C1)  $f(\cdot)$  and  $g(\cdot)$  are Lipschitz continuous,

(C2)  $\sum_{i=1}^{n_1} f_i(x, y) = 0 = \sum_{j=1}^{n_2} g_j(x, y)$ ,

<sup>1</sup>We choose to work with two-player games because that is the standard setting for evolutionary dynamics and is the context for the Dekel–Scotchmer and Samuelson–Zhang papers. Our results generalize in a straightforward manner to  $n$ -player games.

- (C3) for all  $x \in S^{n_1}$ , if  $x_i = 0$ , then  $f_i(x, y) \geq 0$ ,  
(C4) for all  $y \in S^{n_2}$ , if  $y_j = 0$ , then  $g_j(x, y) \geq 0$ , and  
(CR)  $\lim_{x_i \rightarrow 0} [f_i(x, y)/x_i]$  and  $\lim_{y_j \rightarrow 0} [g_j(x, y)/y_j]$  exist and are finite.

The first condition is the standard regularity condition needed to guarantee the existence and uniqueness of solutions to the dynamic system (1) for any initial condition. Conditions (C2)–(C4) guarantee that the path of solutions stays within the strategy simplices. (CR) is a regularity condition which implies that growth rates  $f_i(x, y)/x_i$  and  $g_j(x, y)/y_j$  can be extended continuously to the boundary of  $S^{n_1} \times S^{n_2}$ ; we shall denote these extensions, given by the limits in (CR), by  $f_i(0, y)/0$  and  $g_j(x, 0)/0$ . This condition prevents strategies that are present in the population from being eliminated in finite time. Samuelson and Zhang also impose another condition that is crucial for their result on rationalizability. We discuss this condition in Section 4.

We will instead look at discrete dynamics defined by continuous functions  $(F, G)$ , where  $F: S^{n_1} \times S^{n_2} \rightarrow S^{n_1}$  and  $G: S^{n_1} \times S^{n_2} \rightarrow S^{n_2}$  and

$$\begin{aligned} x_i(t+1) &= F_i(x(t), y(t)) \quad i = 1, \dots, n_1 \quad \text{and} \\ y_j(t+1) &= G_j(x(t), y(t)) \quad j = 1, \dots, n_2. \end{aligned} \tag{2}$$

(D1) If  $x_i > 0$ , then  $F_i(x, y) > 0$  and if  $y_j > 0$ , then  $G_j(x, y) > 0$ .

(DR)  $\lim_{x_i \rightarrow 0} [F_i(x, y)/x_i]$  and  $\lim_{y_j \rightarrow 0} [G_j(x, y)/y_j]$  exist and are positive and finite.

(DR) plays the same role as (CR).

We wish to know whether the set of limit points of a solution  $(x(t), y(t))$  to (1) or (2) places positive probability on dominated strategies. First we make precise our definitions.

Strategy  $x' \in S^{n_1}$  is *strictly dominated* in  $M_1 \subset S^{n_1}$  relative to  $M_2 \subset S^{n_2}$  if there exists  $x \in M_1$  such that  $\pi_1(x, y) > \pi_1(x', y)$  for all  $y \in M_2$ . Let  $\bar{D}_1(M_1, M_2)$  be the set of mixed strategies in  $M_1$  that are not strictly dominated in  $M_1$  relative to  $M_2$ . The strategy  $x \in S^{n_1}$  survives *strict iterated admissibility* (SIA) if there exist sequences of the form  $S^{n_1} = M_{10}, M_{11}, \dots, M_{1T}$  and  $S^{n_2} = M_{20}, M_{21}, \dots, M_{2T}$  where  $M_{1n+1} = \bar{D}_1(M_{1n}, M_{2n})$  and  $M_{2n+1} = \bar{D}_2(M_{1n}, M_{2n})$  for  $n = 1, \dots, T-1$ , with  $M_{1T} = \bar{D}_1(M_{1T}, M_{2T})$  and  $M_{2T} = \bar{D}_2(M_{1T}, M_{2T})$  and with  $x \in M_{1T}$ . Similar definitions apply for player 2.

$(x^*, y^*)$  is said to be a *limit point* of a solution  $(x(t), y(t))$  to (1) or (2) if there exists an increasing sequence of times,  $\{t_n\}_{n=1}^{\infty}$ , such that  $\lim_{n \rightarrow \infty} (x(t_n), y(t_n)) = (x^*, y^*)$ . According to this definition a limit point is the limit of a sequence of points generated by the dynamics.

Pearce [8] shows that in two-player games the pure strategies that survive SIA coincide with the set of pure strategies that are rationalizable (Bernheim [1] and Pearce [8]) in the set of all mixed strategies. We will say that a solution  $(x(t), y(t))$  to (1) or (2) avoids nonrationalizable strategies if all limit points survive SIA.

### 3. DISCUSSION OF THE DEKEL–SCOTCHMER EXAMPLE

Dekel and Scotchmer [2] present an example in which for almost all initial conditions, limit points of paths generated by the discrete replicator dynamic place positive probability on a strategy that is strictly dominated by a mixture of the other strategies. Since the theorem of Samuelson and Zhang guarantees that this cannot occur for the continuous replicator dynamic, the example is puzzling. We briefly discuss the example in this section.

The example is given in Fig. 1.<sup>2</sup> For both players an equal mixture of the first three strategies dominates the fourth strategy. To discuss stability, we define the replicator dynamic. The continuous replicator dynamic  $(f^*, g^*)$  takes the form

$$f_i^*(x, y) = [\pi_1(i, y) - \pi_1(x, y)] x_i$$

and

$$g_j^*(x, y) = [\pi_2(x, j) - \pi_2(x, y)] y_j;$$

in the discrete replicator dynamic  $(F^*, G^*)$ ,  $F^*(x, y) \equiv x + f^*(x, y)/\pi_1(x, y)$  and  $G^*(x, y) \equiv y + g^*(x, y)/\pi_2(x, y)$  so that, for example,  $x_i(t+1) - x_i(t) = f_i^*(x, y)/\pi_1(x, y)$ . Friedman [3], Hofbauer and Sigmund [5], and Nachbar [7] discuss the replicator dynamic.

Dekel and Scotchmer show that for any dynamic path  $x(t)$  that satisfies the discrete replicator dynamic with  $x(0)$  completely mixed,  $x_4(t)$  converges to zero if and only if  $x_1(0) = x_2(0) = x_3(0)$ . Hence, for almost all initial

<sup>2</sup> Note that the game is symmetric. Dekel and Scotchmer analyze the game as if there is only one population of players, and analyze symmetric dynamics. While Samuelson–Zhang and we explicitly allow asymmetric dynamics, this difference cannot account for the different results: Strategies evolving according to asymmetric replicator dynamics of a symmetric game starting from a symmetric initial condition are identical to symmetric replicator dynamics. This observation follows because the path of the asymmetric replicator dynamic of a symmetric game starting from a symmetric initial condition is symmetric for all time. We would expect stochastic dynamics to differ depending on whether there were one or two populations of players. This difference is not captured by the deterministic replicator dynamic, but may be captured in part by the difference between the static ESS stability conditions for symmetric and asymmetric contests.

	ROCK	SCISSORS	PAPER	DUMB
ROCK	1,1	2.35,0	0,2.35	0.1,1.1
SCISSORS	0,2.35	1,1	2.35,0	0.1,1.1
PAPER	2.35,0	0,2.35	1,1	0.1,1.1
DUMB	1.1,0.1	1.1,0.1	1.1,0.1	0,0

FIGURE 1

conditions, the dominated strategy is not eliminated in the limit.<sup>3</sup> Their argument is subtle, but it hinges on two observations. First, the discrete replicator dynamic restricted to the game determined by the upper three-by-three submatrix of Fig. 1 (the rock–scissors–paper game) must approach the boundary of  $\Sigma_0 = \{x \in S^4 : x_4 = 0\}$ . Second, if a dynamic path has a limit point on  $\Sigma_0$ , then  $x_4$  does not converge to zero. The first observation follows standard analyses of the rock–scissors–paper game (see, for example, Hofbauer and Sigmund [5, p. 134] and Weissing [11]). The second observation follows (loosely) by noting that if a dynamic path does approach the boundary of  $\Sigma_0$ , then  $x_4$  will grow because, while it is not a best response to the population strategy, it does better than the population average.

Samuelson and Zhang’s result implies that all limit points of the continuous replicator dynamic (that start from the interior of the strategy simplex) are contained in  $\Sigma_0$ . What accounts for the difference? Here we offer brief comments. The next two sections of the paper provide a general treatment. First, continuous replicator dynamics applied to the rock–scissors–paper game determined by the upper three-by-three submatrix in Fig. 1 do not approach the boundary.<sup>4</sup> For this reason, after a finite interval, the population does not return to states in which the dumb fourth strategy is better than average.

<sup>3</sup> Dekel and Scotchmer modify the replicator dynamic to allow inheritance of mixed strategies. For this specification they prove that limit points of the discrete replicator dynamic cannot be dominated.

<sup>4</sup> Continuous replicator dynamics in the rock–scissors–paper game approach the boundary if (all other numbers held constant) the payoffs that are equal to 2.35 in Fig. 1 are less than 2 (see Hofbauer and Sigmund [5, p. 161]). In this case, however, the fourth strategy is not dominated.

Second, we note that if a constant is added to each of the entries in Fig. 1, then continuous replicator dynamic does not change, but the discrete dynamic does change. Indeed, if  $C > 20/7$  is added to all of the entries in Fig. 1, then  $(1/3, 1/3, 1/3)$  is a globally asymptotically stable strategy of the rock–scissors–paper game under the discrete replicator dynamic; both discrete and continuous replicator dynamics converge to  $(1/3, 1/3, 1/3, 0)$  in this case. Discrete replicator dynamics need not behave like continuous replicator dynamics when payoffs (fitnesses) are low; they have qualitatively similar properties if a sufficiently large constant is added to all payoffs.<sup>5</sup>

#### 4. MONOTONICITY FOR DISCRETE AND CONTINUOUS DYNAMICS

In this section we introduce the monotonicity condition needed for our result and relate it to the Samuelson–Zhang monotonicity condition. We provide a sense in which the conditions are equivalent when the time between periods in a discrete dynamic converges to zero. We discuss the special case of the replicator dynamic. Finally, we point out that there is a correspondence between the conditions in the following sense: Any regular continuous dynamic that satisfies the Samuelson–Zhang condition maps to a regular discrete dynamic that satisfies our condition, and conversely.

Samuelson and Zhang introduce a condition, which they call aggregate monotonicity, that combined with (C1) through (C4) and (CR) implies that continuous selection dynamics avoid nonrationalizable strategies.<sup>6</sup> The regular selection dynamic  $(f, g)$  satisfies *aggregate monotonicity* if

(AM)  $\pi_1(p, y) > \pi_1(p', y)$  implies  $\sum_{i=1}^{n_1} (p_i - p'_i)[f_i(x, y)/x_i] > 0$  and  $\pi_2(x, q) > \pi_2(x, q')$  implies  $\sum_{j=1}^{n_2} (q_j - q'_j)[g_j(x, y)/y_j] > 0$ .

<sup>5</sup> The fact that adding a positive constant to payoffs changes the qualitative behavior of the discrete replicator dynamic is not troubling for biological applications; relative reproduction rates should not be invariant with respect to adding constants. It is disturbing for economic applications if players are expected utility maximizers. We see no reason, however, why adaptive behavior of economic agents must obey invariance properties derived from the assumption of full rationality. On the other hand, as Eddie Dekel-Tabak has pointed out to us, some of the stories that can be used to motivate studying the replicator dynamic in economic contexts suggest that dynamics should be invariant with respect to all increasing monotonic transformations. It is unlikely that strictly dominated strategies are assigned probability zero in all limits of increasing monotonic transformations of the continuous replicator dynamic.

<sup>6</sup> Samuelson and Zhang introduce another, easier to satisfy, monotonicity condition, the requirement that  $\pi_1(i, y) > \pi_1(k, y)$  implies that  $f_i(x, y)/x_i > f_k(x, y)/x_k$  and show that this condition implies that regular selection dynamics avoid strategies that are dominated by pure strategies. The condition and result carry over with no changes to the discrete case. Friedman [3] and Nachbar [7] impose the same condition.

Similarly, we say that the regular discrete selection dynamic  $(F, G)$  satisfies *aggregate log monotonicity* if

(LM)  $\pi_1(p, y) > \pi_1(p', y)$  implies  $\sum_{i=1}^{n_1} (p_i - p'_i) \log[F_i(x, y)/x_i] > 0$  and  $\pi_2(x, q) > \pi_2(x, q')$  implies  $\sum_{j=1}^{n_2} (q_j - q'_j) \log[G_j(x, y)/y_j] > 0$ .

(LM) is the assumption that we need in order to prove our main result. While it does not appear to be a direct translation of (AM) to the discrete dynamic, (AM) and (LM) serve identical purposes. Consider the function  $E(x, p) \equiv \sum_{i=1}^{n_1} p_i \log x_i$ . It is straightforward to check that the first condition of (AM) holds along a solution to (1) if and only if

$$dE(x, p)/dt > dE(x, p')/dt \quad \text{if } \pi_1(p, y) > \pi_1(p', y), \quad (3)$$

and the first condition of (LM) holds if and only if

$$\begin{aligned} E(x(t+1), p) - E(x(t), p) &> E(x(t+1), p') - E(x(t), p') \\ \text{if } \pi_1(p, y) &> \pi_1(p', y). \end{aligned} \quad (4)$$

Conditions (3) and (4) assert that on a solution path in both the continuous and discrete dynamics,  $E(\cdot)$  grows faster for better strategies.

In order to see how discrete dynamics change as the interval between adjustments shrink, begin with a system of the form

$$\dot{z}(t) = h(z). \quad (5)$$

Let

$$H(z; \Delta) \equiv z + h(z)\Delta. \quad (6)$$

The difference equation  $z(t+\Delta) = H(z(t); \Delta)$  corresponds to (5) since  $[z(t+\Delta) - z(t)]/\Delta = h(z)$  for all  $\Delta$ . Assume (by analogy to (DR)) that for each  $i$ ,  $\lim_{z_i \rightarrow 0} [H_i(z; \Delta)/z_i]$  exists and is finite so that we can extend  $H_i(z; \Delta)/z_i$  to the boundary of the mixed-strategy simplex by continuity. Since  $\log(1+x)$  is approximately equal to  $x$  when  $x$  is close to zero, if  $\sum_{i=1}^{n_1} (p_i - p'_i) [h_i(z)/z_i] > 0$  then there exists  $\Delta_0 > 0$  such that for all  $\Delta \in (0, \Delta_0)$ ,  $\sum_{i=1}^{n_1} (p_i - p'_i) \log[H_i(z; \Delta)/z_i] > 0$ . Moreover,  $\Delta_0$  can be taken to be independent of  $z$ . It follows that aggregate monotonicity for the continuous dynamic implies aggregate log monotonicity of the discrete dynamic for sufficiently small period length.

Since our primary objective is to reconcile results relating to the discrete and continuous replicator dynamic, we will study that case in a bit more detail. It is straightforward to verify that the continuous replicator dynamic satisfies (AM); the Dekel–Scotchmer example, combined with

Proposition 2 (which we state and prove in the next section) demonstrates that the discrete replicator dynamic need not satisfy (LM).<sup>7</sup>

Now imagine that the period length shrinks. A natural interpretation is that if the period length is  $\Delta$ , then only the fraction  $\Delta$  of the population reproduces in proportion to its fitness; the remainder of the population lives on in its past composition. In this case, the relationship between the population distribution of strategies at consecutive time periods is

$$x_i(t + \Delta) - x_i(t) = (1 - \Delta)x_i(t) + \Delta[\pi_1(i, y(t)) - \pi_1(x(t), y(t))]x_i(t)/\pi_1(x(t), y(t)), \quad (7)$$

where the first term on the right hand side of (7) reflects the assumption that the fraction  $(1 - \Delta)$  of the population does not change, while the second term on the right hand side reflects the assumption that the remaining fraction of the population changes according to the replicator equation. Dividing both sides of Eq. (7) by  $x_i(t)$  we obtain

$$[x_i(t + \Delta) - x_i(t)]/x_i(t) = \Delta[\pi_1(i, y(t))/\pi_1(x(t), y(t)) - 1],$$

which is a special case of the transformations given by (6) that we used before (for player 1 let  $h_i(x, y) = \{\pi_1(i, y)/\pi_1(x, y) - 1\}x_i$ ). Hence, since the continuous replicator dynamic always satisfies (AM), the discrete replicator dynamic will satisfy (LM) when the time between periods is sufficiently short.

For the replicator dynamic there is another way to see the relationship between continuous and discrete dynamics. Add a large positive constant  $C$  to all payoffs. Doing so does not change  $f^*(\cdot)$ , but it increases  $\pi_1(\cdot)$  by  $C$ . It follows that the ratio  $f^*(x, y)/\pi_1(x, y)$  can be made arbitrarily close to zero with the addition of a large enough constant. That is, one can find a constant  $C$  large enough so that if  $C$  is added to all payoffs, then (AM) implies (LM); hence (LM) is satisfied for the replicator dynamics if we add a sufficiently large constant to all payoffs. Adding a constant to all of the payoffs has a natural interpretation in the biological context. Here the payoffs represent fitnesses. If the strategies chosen in the game make only a small contribution to relative fitness, then different strategies will not grow rapidly in a time period. In this case the discrete dynamic behaves like the continuous dynamic.<sup>8</sup>

<sup>7</sup> Proposition 2 assumes (DR), which does not hold for the discrete replicator dynamic in the Dekel-Scotchmer example of Fig. 1. Adding a small positive constant to all of the payoffs in Fig. 1 allows the discrete replicator dynamic to satisfy (DR) for the game without destroying the qualitative properties of limiting behavior in the example. Hence it is the failure of aggregate log monotonicity, not the failure of regularity, that leads to the possibility that dominated strategies are limits of the discrete replicator dynamic.

<sup>8</sup> Hofbauer and Sigmund [5, pp. 273–274] make the same observation.



There is a one-to-one relationship between discrete dynamics that satisfy aggregate log monotonicity and continuous dynamics that satisfy aggregate monotonicity. Given any continuous dynamic  $(f, g)$  define the *log-associated discrete dynamic*  $(F, G)$  by  $F_i(x, y) = \omega(x, y)x_i e^{f_i(x, y)/x_i}$  and  $G_j(x, y) = \mu(x, y)y_j e^{g_j(x, y)/y_j}$ , where  $\omega(x, y)$  and  $\mu(x, y)$  are normalizations;  $\omega(x, y) \equiv [\sum_{i=1}^{n_1} x_i e^{f_i(x, y)/x_i}]^{-1}$  and  $\mu(x, y) \equiv [\sum_{j=1}^{n_2} y_j e^{g_j(x, y)/y_j}]^{-1}$ . It is clear that  $(f, g)$  satisfy (CR) if and only if the associated  $(F, G)$  satisfy (DR). Also, the following proposition is an immediate consequence of the definitions.

**PROPOSITION 1.** *The continuous dynamic  $(f, g)$  satisfies aggregate monotonicity if and only if the log-associated discrete dynamic  $(F, G)$  satisfies aggregate log monotonicity.*

Let  $(f^*, g^*)$  denote the continuous replicator dynamics. Theorem 3 in Samuelson and Zhang shows that  $(f, g)$  is a regular, aggregate monotone continuous dynamic if and only if  $f_i(x, y) = \lambda(x, y)f_i^*(x, y)$  and  $g_j(x, y) = \beta(x, y)g_j^*(x, y)$  for  $\lambda(\cdot)$  and  $\beta(\cdot)$  continuous and positive.<sup>9</sup> Combined with Proposition 1, we have a characterization of the set of regular, aggregate log monotone discrete dynamics: They are simply the discrete dynamics log-associated with multiples of continuous replicator dynamics. As we use this characterization in the next section, it is useful to state it separately.

**COROLLARY.** *The discrete dynamic  $(F, G)$  satisfies log aggregate monotonicity if and only if there exist positive and continuous functions  $\omega(\cdot)$ ,  $\mu(\cdot)$ ,  $\lambda(\cdot)$  and  $\beta(\cdot)$  on  $S^{n_1} \times S^{n_2}$  such that  $\log[F_i(x, y)/x_i] = \lambda(x, y)[\pi_1(i, y) - \pi_1(x, y)] + \log \omega(x, y)$  and  $\log[G_j(x, y)/y_j] = \beta(x, y)[\pi_2(x, y) - \pi_2(x, y)] + \log \mu(x, y)$*

In the corollary  $\omega(\cdot)$  and  $\mu(\cdot)$  are normalizations that guarantee that the dynamics satisfy  $F: S^{n_1} \times S^{n_2} \rightarrow S^{n_1}$  and  $G: S^{n_1} \times S^{n_2} \rightarrow S^{n_2}$ . If  $\lambda(x, y) \equiv \beta(x, y) \equiv 1$ , then  $F(\cdot)$  and  $G(\cdot)$  are the log-associated dynamics derived from the continuous replicator dynamic.

<sup>9</sup> Samuelson and Zhang's Theorem 3 only claims that if  $(f, g)$  is a regular, aggregate monotone continuous dynamic then  $f_i(x, y) = \lambda(x, y)f_i^*(x, y)$  and  $g_j(x, y) = \beta(x, y)g_j^*(x, y)$  for  $\lambda(\cdot)$  and  $\beta(\cdot)$  positive. Continuity of  $\lambda(\cdot)$  and  $\beta(\cdot)$  follow from the continuity and regularity of  $(f, g)$  and  $(f^*, g^*)$ . The converse implication follows immediately from the definitions.

5. AGGREGATE LOG MONOTONE DISCRETE DYNAMICS  
AVOID DOMINATED STRATEGIES

Proposition 2 states that every limit point of a regular, aggregate log monotone dynamic must be an optimal response to some mixed strategy of the other player. Moreover, the mixed strategy can be taken to be a limit of averages of strategies played in the past. In particular, if a pure strategy is strictly dominated then it must receive zero weight in the limit of any regular, aggregate log monotonic selection dynamic that starts from an interior point.

The carrier of a mixed strategy  $u$ , denoted  $C(u)$ , is the set of all pure strategies it gives positive probability.

**PROPOSITION 2.** *Let  $(F, G)$  be a regular, aggregate log monotonic selection dynamic. If  $(x(t), y(t))$  is an evolutionary path with  $(x(0), y(0))$  completely mixed and  $(x^*, y^*)$  is a limit point of  $(x(t), y(t))$ , then there exist subsequences  $\{s_n\}$  and  $\{t_n\}$  of the positive integers; a constant  $K$  and positive weights  $\alpha(s; s_n)$  and  $\beta(t; t_n)$  for  $s=0, 1, \dots, s_n$  and  $t=0, 1, \dots, t_n$ , such that for all  $n$ ,  $\sum_{s=0}^{s_n-1} \alpha(s; s_n) = \sum_{t=0}^{t_n-1} \beta(t; t_n) = 1$ ,  $\alpha(s; s_n) \leq K/s_n$  and  $\beta(t; t_n) \leq K/t_n$ ; and  $(\bar{x}, \bar{y})$  such that  $\bar{x} = \lim_{n \rightarrow \infty} \sum_{s=0}^{s_n-1} \alpha(s; s_n) x(t)$ ,  $\bar{y} = \lim_{n \rightarrow \infty} \sum_{t=0}^{t_n-1} \beta(t; t_n) y(t)$ , and*

$$\pi_1(i, \bar{y}) \geq \pi_1(x, \bar{y}) \quad \text{for all } x \in S^{n_1} \text{ and } i \in C(x^*) \quad (8)$$

and

$$\pi_2(\bar{x}, j) \geq \pi_2(\bar{x}, y) \quad \text{for all } y \in S^{n_2} \text{ and } j \in C(y^*). \quad (9)$$

*Proof.* We prove (8). Since  $x_i(T)/x_i(0) = \prod_{t=0}^{T-1} x_i(t+1)/x_i(t) = \prod_{t=0}^{T-1} F_i(x(t), y(t))/x_i(t)$ ,

$$\begin{aligned} & \log[x_i(T)/x_i(0)] - \log[x_{i'}(T)/x_{i'}(0)] \\ &= \sum_{t=0}^{T-1} \log\{[F_i(x(t), y(t))/x_i(t)]/[F_{i'}(x(t), y(t))/x_{i'}(t)]\}. \end{aligned} \quad (10)$$

By (LM) and the corollary, there exists  $\lambda(\cdot)$  such that the right-hand side of (10) is equal to  $\sum_{t=0}^{T-1} \lambda(x(t), y(t))[\pi_1(i, y(t)) - \pi_1(i', y(t))]$ . Hence (10) implies, for all  $i$  and  $i'$ ,

$$\begin{aligned} & \log[x_i(T)/x_i(0)] - \log[x_{i'}(T)/x_{i'}(0)] \\ &= \sum_{t=0}^{T-1} \lambda(x(t), y(t))[\pi_1(i, y(t)) - \pi_1(i', y(t))]. \end{aligned} \quad (11)$$

Let  $A(T) \equiv \sum_{t=0}^{T-1} \lambda(x(t), y(t))/T$ . Since  $\lambda(\cdot)$  is continuous and positive on  $S^{n_1} \times S^{n_2}$  it attains its positive maximum and minimum values  $M$  and  $m$  so that

$$M \geq A(T) \geq m > 0. \quad (12)$$

Define  $\beta(t; T) \equiv \lambda(x(t), y(t))/(TA(T))$ ; by (12) and the definition of  $A(\cdot)$ ,  $\beta(\cdot)$  is positive,  $\beta(t; T) \leq M/[mT]$ , and  $\sum_{t=0}^{T-1} \beta(t; T) = 1$ . Define  $\bar{y}(T) \equiv \sum_{t=0}^{T-1} \beta(t; T) y(t)$ ; using linearity of  $\pi_1(\cdot)$ , we can rewrite (11) as

$$\begin{aligned} & \{\log[x_i(T)/x_{i'}(T)] - \log[x_i(0)/x_{i'}(0)]\}/T \\ & = A(T)[\pi_1(i, \bar{y}(T)) - \pi_1(i', \bar{y}(T))]. \end{aligned} \quad (13)$$

By the definition of  $x^*$ , there exists  $\varepsilon > 0$  such that it is possible to find a subsequence of  $T = 1, 2, \dots$ , denoted by  $\{t_n\}$ , for which  $x_i(t_n) > \varepsilon$  for all  $i \in C(x^*)$ . By compactness, it is possible to select this subsequence so that  $\{\bar{y}(t_n)\}$  and  $\{A(t_n)\}$  converge as  $n$  approaches infinity. Denote these limits by  $\bar{y}$  and  $A^*$ . Taking limits in (13) (using  $x_i(t) \in [0, 1]$  and  $x_i(0) > 0$ ) it follows that if  $i \in C(x^*)$ , then  $0 \leq A^*[\pi_1(i, \bar{y}) - \pi_1(i', \bar{y})]$  with equality if  $i' \in C(x^*)$ . Since  $A^* > 0$  by (12), (8) holds. Similar logic establishes (9).

The proof of Proposition 2 uses techniques found in Schuster, Sigmund, Hofbauer, and Wolff [10] who prove that time averages of periodic solutions to the replicator equation are Nash equilibria. Note that the proof does not use the assumption that there are only two players.

While (LM) provides a sufficient condition for a dynamic to avoid dominated strategies, it is not a necessary condition. In particular, dynamics that only increase the probability on strategies that respond optimally to the opponent's recent actions need not satisfy (AM) or (LM); dynamics in this class must avoid strictly dominated strategies, however. Gul [4] and Milgrom and Roberts [6] show that limit points of adjustment processes in this class must be rationalizable.

We can use Proposition 2 to show that selection dynamics are consistent with adaptive learning in the sense of Milgrom and Roberts [6]. Given a set of strategies  $R$ , let  $U_n^\varepsilon(R)$  be the set of  $\varepsilon$ -undominated strategies for player  $n$  given that opponents are using strategies in  $R$ . A sequence of strategies  $\{x_n(t)\}$  is *consistent with  $\varepsilon$ -adaptive learning* by player  $n$  if for all  $\hat{t}$  there exists  $T$  such that for all  $t \geq T$ ,  $x_n(t) \in U_n^\varepsilon(\{x(s) : \hat{t} \leq s < t\})$ . It is *consistent with adaptive learning* if it is consistent with  $\varepsilon$ -adaptive learning for all  $\varepsilon > 0$ . When  $\{x(t)\}$  is a path derived from a selection dynamic that satisfies (LM) that begins from a totally mixed initial condition, for any  $\varepsilon > 0$  there exists a  $T$  sufficiently large so that if  $t > T$ , then  $x(t)$  is an  $\varepsilon$ -best

response to an average of past population strategies. Hence the sequence  $\{x(t)\}$  is consistent with adaptive learning.<sup>10</sup>

In view of Proposition 2, it is a simple matter to prove the discrete analog of Samuelson and Zhang's Theorem 2.

**PROPOSITION 3.** *Let  $(F, G)$  be a regular, aggregate log monotonic selection dynamic. Let  $x' \in S^{n_1}$  fail strict iterated admissibility. If  $(x(t), y(t))$  is an evolutionary path with  $(x(0), y(0))$  completely mixed, there exists a function  $\varepsilon(t)$  with  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$  such that for every  $t$ , there exists a pure strategy  $i(t)$  in the  $C(x')$  such that  $x_{i(t)} \leq \varepsilon(t)$ . A similar statement holds for a strategy of player 2 that fails strict iterated admissibility.*

Proposition 3 differs from Samuelson and Zhang's Theorem 2 in only two respects: It treats discrete rather than continuous dynamics; and it assumes (LM) rather than (AM). Samuelson–Zhang's proof can be adapted to the discrete context with only small changes. We choose to present a proof based on Proposition 2, rather than the direct proof of Samuelson and Zhang in order to establish that limit points of the dynamic must be best responses to averages of past strategies.

*Proof.* Let  $(x(t), y(t))$  be an evolutionary path with  $(x(0), y(0))$  completely mixed. Let  $P$  be the set of limit points of  $\{x(t)\}$  and  $Q$  be the set of limit points of  $\{y(t)\}$ . Define sets  $A_1$  and  $A_2$  by

$$A_1 = \{u \in S^{n_1} : u \text{ fails SIA and there exists } x^* \in P \text{ such that } C(u) \subset C(x^*)\}$$

and

$$A_2 = \{v \in S^{n_2} : v \text{ fails SIA and there exists } y^* \in Q \text{ such that } C(v) \subset C(y^*)\}.$$

It suffices to show that  $A_1 \cup A_2$  is empty. We will assume instead that  $A_1 \cup A_2$  is not empty and argue to a contradiction. For  $a' \in A_1 \cup A_2$ , let  $K(a')$  be such that  $a' \in M_{1K(a')} \setminus M_{1K(a')+1}$  or  $a' \in M_{2K(a')} \setminus M_{2K(a')+1}$ , depending on whether  $a' \in S^{n_1}$  or  $S^{n_2}$ . Let  $a$  be a minimizer of  $K(\cdot)$  on

<sup>10</sup>  $x(t)$  may place positive probability on pure strategies that fail to be  $\varepsilon$ -best responses to population strategies. Consequently discrete selection dynamics that satisfy (LM) are consistent with adaptive learning if we take our strategy set to be the entire simplex of mixed strategies. They need not be consistent with adaptive learning if we limit attention to pure strategies. As we think of population strategies  $x(t)$  as arising from individuals playing pure strategies, some of the individuals may never be consistent with adaptive learning since they could be playing pure strategies that are not  $\varepsilon$ -best responses to the population's strategy. However, the proportion of players that are not using  $\varepsilon$ -best responses goes to zero as time goes to infinity.

$A_1 \cup A_2$ , and let  $k = K(a)$ . Without loss of generality, assume  $a \in A_1$ . Since  $a \in M_{1k} \setminus M_{1k+1}$ , there exists  $b \in M_{1k}$  such that

$$\pi_1(a, y) - \pi_1(b, y) < 0 \quad \text{for all } y \in M_{2k} \quad (14)$$

or, in words,  $b$  strictly dominates  $a$  relative to  $M_{2k}$ . Let  $Y$  consist of all those  $y \in S^{m_2}$  such that  $y_j > 0$  only if  $j \in M_{2k}$ . It follows from (14) that

$$\pi_1(a, y) - \pi_1(b, y) < 0 \quad \text{for all } y \in Y. \quad (15)$$

Since  $k$  is a minimum of  $K(\cdot)$ ,

$$\lim_{t \rightarrow \infty} y_j(t) = 0 \quad \text{for } j \notin M_{2k}. \quad (16)$$

By Proposition 2 and the definition of  $a$  we know that there exists a subsequence  $\{t_n\}$  such that  $a$  responds optimally to  $\bar{y}$ , where  $\bar{y} = \lim_{n \rightarrow \infty} \sum_{t=0}^{t_n-1} \beta(t; t_n) y(t)$  for positive weights  $\beta(t; t_n)$ ,  $t=0, 1, \dots, t_n$ ,  $\beta(t; t_n) \leq K/t_n$ , and  $\sum_{t=0}^{t_n-1} \beta(t; t_n) = 1$ . From (16) and  $\beta(t; t_n) \leq K/t_n$  for  $t=0, 1, \dots, t_n$  it follows that  $\bar{y}_j = 0$  for  $j \notin M_{2k}$ . Therefore,  $\bar{y} \in Y$ . Hence (15) implies that  $a$  cannot be an optimal response to  $\bar{y}$ ; the contradiction completes the proof.

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