ASYMMETRIC PRICE-BENEFIT AUCTIONS

M. Angeles de Frutos*

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Keywords: Efficient dissolution; Auction; Asymmetry; Bayesian Equilibrium.

Journal of Economic Literature Classification Number: C72, D39, D44, D52, D74, D82.

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Asymmetric Price-Benefit Auctions

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This paper studies the performance of three different auction procedures for allocating the assets of a dissolving partnership when the partners have valuations for the assets that are independent but asymmetrically distributed, with one partner reputed to be more interested in the assets to be divided. I provide results on existence, and uniqueness of the equilibrium induced by these auctions. I also show some properties of these equilibria, in particular that with positive probability the ex post outcome is inefficient. The results contrast with those for the case of symmetric distributions.

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1 Introduction

Consider two-person partnerships in which each partner is endowed with an equal share of a good to be traded such that it is inefficient to sell it in the market and split the proceeds. Divorce settlements and the dissolution of joint venture agreements might fall into this class of problems. If the partners are concerned with achieving an efficient dissolution but they want to restrict themselves to the use of simple mechanisms1, what procedure should be used at termination?

As another application, consider the competition between two raider firms with identical initial toeholds seeking to take over a target firm. What simple mechanism will ensure that the target is worth more if run by the winner?

From Cramton, Gibbons and Klemperer (1987; CGK, for short) and McAfee (1992) it is known that the \textit{k+1-price auctions} achieve efficient and individually-rational dissolution when the valuations for the asset are drawn independently from a common probability distribution. In this kind of auctions the players submit sealed bids and the good is transferred to the highest bidder who pays each of the others a price that is a given combination of the highest and the second highest bid.

In this paper I consider that the agents' valuations are independently but asymmetrically distributed on some common real interval. I also assume that one partner is reputed to be more interested in the asset so that this agent's beliefs are higher in the sense of first order stochastic dominance. I will model the asymmetry on beliefs but, clearly, a wide variety of asymmetries in preferences can be interpreted as an asymmetry in beliefs. Asymmetries are important in contract bidding as well as in standard auctions. The asymmetry may be due to different budget constraints, as is the case in art auctions, or to the presence of collusive behavior. In a takeover case it may due to synergies or to different managerial skills. Another source of asymmetry might be the existence of

1 For an argument in favor of the use of simple mechanisms in the context of a dissolution see McAfee (1992).
heterogeneous risk preferences among the partners. Relaxing the assumption of symmetry makes the study of efficient dissolution more realistic, but it also results in major analytical complications, similar to those faced in the analysis of asymmetric auctions.

The auction procedures that I consider here are the Winner's Bid Auction (WBA) in which the winning bidder pays half of his bid to the losing bidder, the Loser's Bid Auction (LBA) in which the winning bidder pays half of the losing bidder's bid to the losing bidder and the Splitting the Difference Auction (SDA) in which the winning bidder pays to the losing bidder half the average of the winning and the losing bids. All of these procedures are particular cases of the \( k+1 \)-price auctions. I ask which of these auctions will achieve an ex-post efficient allocation—one for which the ownership of the asset is always transferred to the party with the highest valuation—and I study properties of the equilibrium induced by them.

With symmetric independent private values, the WBA, the LBA and the SDA are ex-post efficient. Under these auctions bids are closer to valuations than in standard first and second price auctions since bidders have countervailing incentives; they are tempted to exaggerate their true valuations if they are to sell but will want to understate if they are to purchase.

Under asymmetric distributions the bidding equilibria of these auctions consist of continuously differentiable and strictly monotonically increasing strategies which are characterized as solutions of systems of differential equations with two boundary conditions. I provide results on existence and uniqueness of Nash equilibrium under some regularity assumptions on the probability distributions. I also show that the partner that is reputed to be more interested in the asset will bid more conservatively, irrespective of the auction in use. Besides, her equilibrium bids distribution will stochastically dominates the one of the other partner. In contrast to the symmetric case, all of them fail to achieve allocation efficiency.

Under the WBA, I show that the bidders always shade their bids whereas under the LBA they always submit bids in excess of their valuation.\(^2\) Hence, it is impossible to get

\(^2\)Overbidding is also the optimal strategy in a takeover when bidders have partial ownership of the item and the private valuations of the bidders are drawn from a common distributions. This overbidding
a mutually beneficial resale under this auction since the purchase price exceeds the loser's value of the asset. Under the SDA, for larger values, both partners shade their bids, and for smaller values, they mark up their bids. A numerical resolution of these auctions for particular valuation distributions show that the SDA outperforms the other two since it generates the largest total expected gain from trade. This example also shows that under the LBA and the SDA both partners may overbid by enough such that either winner loses money. In the takeover context, this result is consistent with the observation that the stock price of a successful raider sometimes falls dramatically.3

The paper is organized as follows. Section 2 presents the model and assumptions. In section 3, I study the $k+1$-price auctions and focus on the WBA, the LBA, and the SDA. Section 3 contains the results on existence, uniqueness and properties of the equilibrium induced by these mechanisms. Section 4 provides a comparison among them based on a numerical resolution of the equilibrium bids induced by these auctions. Section 5 concludes. Finally, most of the proofs are included in the appendix.

2 The Model

There are two partners ($i = 1, 2$) who want to dissolve their partnership. It is assumed that each partner has an equal share of the asset to be traded and is risk neutral concerning the division of the asset. The partnership will be dissolved efficiently if the partner with the highest value obtains the asset. Partner $i$ has a valuation for the entire object of $v_i$ which is only known to him. Valuations for the asset are independent but asymmetrically distributed. It is common knowledge that partner 1's beliefs about the value that partner 2 places in the good are drawn from the c.d.f. $F_2(v)$, and that partner 2's beliefs about the valuations of partner 1 are summarized by the c.d.f. $F_1(v)$. These distributions are common knowledge.

In order to compare the properties of the different mechanisms that are proposed in this paper, I will consider the following assumptions related to the players' beliefs:

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1. This fact has been pointed out by Bernhardt (1995) for the LBA equilibrium bids of two raider firms with symmetric valuations for the target firm but with different toeholds.

2. May lead to inefficient outcomes if the initial toeholds are not equal. For an argument, see Burkart (1995).
(A1) Both cumulative distribution functions are absolutely continuous over their common support \([0, v]\). Their density functions \(f_1, f_2\) are continuous, bounded away from zero and locally Lipschitz over \((0, v]\).

(A2) Partner 2's valuations are higher in the sense of first order stochastic dominance, that is, \(F_2(v) \leq F_1(v)\) for all \(v\).

(A3) \(F_2(v)/F_1(v)\) is increasing in \(v\).

(A4) Partner 2's valuations are higher in the sense of hazard rates dominance, that is,

\[
H_1(v) = \frac{f_1(v)}{1 - F_1(v)} \geq H_2(v) = \frac{f_2(v)}{1 - F_2(v)}
\]

for all \(v\).

(A2) is an assumption of stochastic dominance of \(F_1\). Under this assumption the probability distribution \(F_1\) gives less weight to high values of \(v\) than \(F_2\) does. This implies that bidder 2 is reputed to be more likely interested in the asset than bidder 1. (A3) is an assumption of reverse hazard rate dominance of bidder 2's valuations, which is a stronger version of stochastic dominance. This assumption has been used by Maskin and Riley (1992) to ensure that in a first price auction the more optimistic buyer is the less "aggressive bidder" (we are using their terminology here). Finally, (A4) is an assumption of hazard rates dominance which is also a stronger version of stochastic dominance. It states that bidder 1 has a higher probability of having a low valuation conditional on valuations being above \(v\). A sufficient condition for (A3) and (A4) simultaneously hold, and therefore (A2), is that bidder 2's valuations be higher in the sense of monotone likelihood ratio dominance (for an argument see Shaked and Shanthikumar, 1994).

3 \(k + 1\) Price Auctions

This section is devoted to the study of three auction procedures that are special cases of the \(k + 1\) price auctions. In these auctions each player submits a sealed bid. The bidder with the highest bid, \(b_w\), gets the object and has to pay the other \(0.5(kb_1 + (1 - k)b_w)\).
where $b_i$ denotes the loser’s bid. In particular I am going to consider the following three members of this class of auctions: the Winner’s Bid Auction (WBA) that corresponds to the case $k = 0$, the Loser’s Bid Auction (LBA) that corresponds to $k = 1$ and the Split-the-Difference auction (SDA), which results from taking $k = 1/2$.4

Suppose that the object has a true value $v_1$ for bidder 1 and that bids $b_1$ and $b_2$ are submitted by bidders 1 and 2. Then bidder 1’s payoff is5

$$u_1(v_1, b_1, b_2) = \begin{cases} v_1 - 0.5(kb_2 + (1 - k)b_1), & \text{if } b_1 > b_2, \\ 0.5(kb_1 + (1 - k)b_2), & \text{if } b_1 < b_2. \end{cases}$$

A strategy of bidder $i = 1, 2$ is a (Lebesgue-measurable) function $B_i : [0, v] \rightarrow [0, \infty)$, specifying for each possible value of $v_i$ a bid $b_i = B_i(v_i)$. A pair $(B_1, B_2)$ constitutes an equilibrium if for all $i$ and $v_i \in [0, v]$, $U_i(v_i, b_i, B_j) \geq U_i(v_i, b_i', B_j)$ for all $b_i'$, where

$$U_i(v_i, b_i, B_j) = \int_{B_j(v_j < b_i)} [v_i - 0.5(kB_j(v_j) + (1 - k)b_i)]dF_j(v_j)$$

$$+ \int_{B_j(v_j > b_i)} [0.5(kb_i + (1 - k)B_j(v_j))]dF_j(v_j),$$

is partner $i$’s expected payoff if she conjectures that her opponent will use the strategy $B_j$.

### 3.1 The Winner’s Bid Auction

The bidder with the highest bid wins the auction and pays half of her bid to the other bidder. Ties are resolved by the flip of a coin.

I will show that there exists a unique equilibrium for this auction and that this equilibrium is such that the bidder that is reputed more interested in the asset bids less aggressively. To prove this result I begin by showing that equilibrium bid strategies $(B_1(v_1), B_2(v_2))$ are strictly increasing with differentiable inverse bid functions. Denote these inverse functions $y_i(b) \equiv B_i^{-1}(b)$, for $i = 1, 2$; where $y_i(b)$ is the maximum value of $v$ such that $B_i(v) \leq b$.

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4The WBA and LBA have been studied for the symmetric case by McAfee (1992).

5We are ignoring here the event of identical bids. We will show that in equilibrium the probability both firms make the same bid is zero.
Since strategies are increasing, if bidder \( i \) bids \( b \) and bidder \( j \) follows the strategy \( B_j \) then her expected payoff is:

\[
U_i(v_i, b, E_j) = (v_i - \frac{b}{2})F_j(B_j^{-1}(b)) + \int_{B_j^{-1}(b)} 0.5B_j(v_j)dF_j(v_j).
\]

At an interior solution partner \( i \)'s bid solves the first order condition

\[(v_i - b)f_j(B_j^{-1}(b)) = 0.5B_j'(b)F_j(B_j^{-1}(b)).\]

The right hand side is the marginal cost of bidding more when \( i \) wins. The left hand side represents the marginal revenue for increasing the bid and winning. Notice that this first order condition implies that the inverse bid functions \((y_1, y_2), y_i(b) = B_i^{-1}(b)\), must be solutions of the system of differential equations,

\[(1) \quad 2(y_i - b)f_j(y_j(b))y_j'(b) - F_j(y_j(b)) = 0.\]

To see that equilibrium strategies, \((B_1, B_2)\), have to be monotone increasing on \([0, v]\) let \( b' = B_i(v'_i) \) and \( b'' = B_i(v''_i) \) with \( v'_i \geq v''_i \). Equilibrium requires the following two conditions to be satisfied: \(U_i(v'_i, v', B_j) \geq U_i(v''_i, v'', B_j)\) and \(U_i(v''_i, v'', B_j) \geq U_i(v'_i, v', B_j)\), i.e.,

\[
(2v'_i - b')F_j(y_j(b')) + \int_{y_j(b')} B_j(x)dF_j(x) \geq (2v''_i - b'')F_j(y_j(b'')) + \int_{y_j(b'')} B_j(x)dF_j(x), \text{ and}
\]

\[
(2v''_i - b'')F_j(y_j(b'')) + \int_{y_j(b'')} B_j(x)dF_j(x) \geq (2v'_i - b')F_j(y_j(b')) + \int_{y_j(b')} B_j(x)dF_j(x).
\]

Subtracting the right side of the second inequality from the left hand side of the first, and subtracting the left hand side of the second inequality from the right hand side of the first, yields: \((v''_i - v'_i) [F_j(y_j(b'')) - F_j(y_j(b'))] \geq 0\). This implies \( b'' \leq b' \). Moreover, \( B_i(\cdot) \) must be gapless: if there is a gap \([b', b)\] in \( B_i \), then there must be a gap \((b', b)\) in \( B_j \), because for any \( v_j \) it would be better to bid \( b' \) than any other bid in that interval (since with this bid bidder \( j \) lowers the price if winning without affecting the probability of winning). But then the type of bidder \( i \) planning to bid \( b \) would be better off bidding \((b' + b)/2\). Furthermore, \( B_i \) must be atomless. Suppose not, then there is \( b \) and \( j \) such that \( P(B_j(u_j) = b) > 0 \). Then there exists \( \varepsilon > 0 \) such that bidder \( i \) will assign zero probability to \((b, b + \varepsilon)\) creating a gap. Equilibrium strategies are strictly increasing since they are
atomless and weakly monotonically increasing; and because they are also gapless then equilibrium has to be in pure strategies. Continuity and monotonicity of $B_i(\cdot)$ imply that $B_i(\cdot)$ is differentiable almost everywhere on its domain.

Consider now the boundary conditions. Let $B_1, B_2$ be equilibrium strategies, and assume, w.l.o.g., that $B_1(0) < B_2(0)$. Because of continuity, there exists $\tilde{v}$ such that $B_1(v) < B_2(0)$ for all $v \in [0, \tilde{v}]$. Since 2's bid is always greater than $B_1(\tilde{v})$, then partner 1's probability of winning with the bid $B_1(\tilde{v})$ is zero and bidder 2's probability of winning with her equilibrium strategy is nonzero for all $v_2 \geq 0$.

In equilibrium, for all $\tilde{v} \in [0, \tilde{v}]$, it is satisfied that $U_2(\tilde{v}, B_2(\tilde{v}), B_1) \geq U_2(\tilde{v}, B_2(\tilde{v}), B_1) > U_2(\tilde{v}, B_2(\tilde{v}), B_1) \geq 0$, where the first inequality holds because $B_2(\tilde{v})$ is an equilibrium strategy, the second one is due to the fact that $U$ is strictly increasing in $v$, and the last one holds because there exists a bid that has a zero probability of winning and bids are non-negative. Therefore $U_2(\tilde{v}, B_2(\tilde{v}), B_2(\tilde{v})) > 0$, which implies $\tilde{v} > B_2(\tilde{v})$. Consider now bidder 1. If she plays $B_2(\tilde{v})$, she has a nonzero probability of winning, hence, $\tilde{v} > B_2(\tilde{v})$ implies $U_1(\tilde{v}, B_2(\tilde{v}), B_2) > U_1(\tilde{v}, B_1(\tilde{v}), B_2)$ since

$$(\tilde{v} - 0.5B_2(\tilde{v}))F_2(\tilde{v}) + \int_{\tilde{v}}^{\tilde{v}} 0.5B_2(x)dF_2(x) > \int_{0}^{\tilde{v}} 0.5B_2(x)dF_2(x).$$

This contradicts that $B_1(\cdot)$ is an equilibrium bidding strategy. Assume now that $B_1(0) = B_2(0) = a > 0$. Continuity implies that there exists $\tilde{v}$ such that $B_i(\tilde{v}) > \tilde{v}$ with $U_i(\tilde{v}, B_i(\tilde{v}), B_j) = (\tilde{v} - 0.5B_i(\tilde{v}))F_j(B_j^{-1}(B_i(\tilde{v}))) + \int_{B_j^{-1}(B_i(\tilde{v}))}^{\tilde{v}} 0.5B_j(x)dF_j(x) = \int_{B_j^{-1}(B_i(\tilde{v}))}^{\tilde{v}} 0.5B_j(x)dF_j(x).$

Since $B_i(\tilde{v}) > \tilde{v}$, we obtain

$$U_i(\tilde{v}, B_i(\tilde{v}), B_j) \leq 0.5\tilde{v}F_j(B_j^{-1}(\tilde{v})) + \int_{B_j^{-1}(\tilde{v})}^{\tilde{v}} 0.5B_j(x)dF_j(x) = U_i(\tilde{v}, \tilde{v}, B_j).$$

The contradiction proves that the lower support for bidder i's bid distribution cannot be $B_i(0) > 0$. Therefore $B_1(0) = B_2(0) = 0$. The argument above also implies that equilibrium strategies have to satisfy $B_i(v) < v$, for all $v > 0$. Consider now the upper support of the bids distribution. Since $B_1(\cdot)$ is strictly increasing, bidder 2 wins with probability 1 by bidding $B_1(\tilde{v})$. Hence $B_2(\tilde{v}) \leq B_1(\tilde{v})$. The same argument holds for bidder 1; thus, we also have $B_1(\tilde{v}) \leq B_2(\tilde{v})$, and hence $B_1(\tilde{v}) = B_2(\tilde{v}) = \tilde{b} < \tilde{v}$.

System (1) and the boundary conditions define equilibrium strategies if these conditions define a best decision for each bidder. We check this additional condition as part of the proof in the following theorem which characterizes the equilibrium of the WBA.
**Theorem 1** A pair of strategies \((B_1, B_2)\) is an equilibrium for the WBA if and only if the strategies are pure, the bid functions are strictly increasing and differentiable, with \(B_i(v) < v\) for all \(v \in (0, \bar{v}]\), and there exists \(\bar{b}, 0 < \bar{b} < \bar{v}\), such that the inverse bid functions \((y_1, y_2)\), \(y_i(b) = B_i^{-1}(b)\), are solutions over the interval \([0, \bar{b}]\) of the system of differential equations,

\[
2(y_i - b)f_i(y_j(b))y'_j(b) - F_i(y_i(b)) = 0
\]

satisfying the boundary condition \(y_i(0) = 0, y_i(\bar{b}) = \bar{v}\) for \(i = 1, 2\).

**Proof:** See Appendix.

I will now show that such an equilibrium exists and that under (A3) it is unique.

**Theorem 2** If (A1) holds then there exists a Bayesian equilibrium for the WBA. If, in addition, (A3) holds, then the equilibrium is unique.

**Proof:** See Appendix.

The following proposition shows some of the properties of the equilibrium induced by the winner bid auction.

**Proposition 1** Under (A3) if \((B_1, B_2)\) is an equilibrium strategy profile to a winner-price auction then, for all \(i = 1, 2\),

i) \(B_1(v) \geq B_2(v)\), for all \(v \in (0, \bar{v}]\),

ii) The distribution of bidder 2's bids stochastically dominates the distribution of bidder 1's bid.

iii) Bidder 2 has a higher probability of winning.

**Proof:** i). For all \(b \in [0, \bar{b}]\) if \(y_2(b) = y_1(b)\) then \(\frac{F_2(y_2(b))}{F_2(y_1(b))} \frac{f_1(y_1(b))}{f_2(y_2(b))} = \frac{y_2(b)}{y_1(b)}\). Assumption (A3) implies that the left hand side of this equality is not larger than one and hence that \(y_2(b) < y_1(b)\), or equivalently \(B_2(b) > B_1(b)\), whenever \(y_2(b) = y_1(b)\).
I now show, by way of contradiction, that it cannot exist $v \in (0, \theta)$ such that $B_2(v) > B_1(v)$. If such a $v \in (0, \theta)$ would exist then there exists $m \in [v, \theta]$ such that $B_2(m) = B_1(m)$ and $B'_2(m) \leq B'_1(m)$, which can never be the case. To see this, let $m = \min\{k \in (v, \theta) \mid B_2(k) = B_1(k)\}$. The existence of $m$ is guaranteed by $B_2(\theta) = B_1(\theta)$. By continuity, for all $n \in [v, m)$, $B_2(n) > B_1(n)$. This implies

$$\frac{B_2(m) - B_2(n)}{m - n} < \frac{B_1(m) - B_1(n)}{m - n}$$

which yields to $B'_2(m) < B'_1(m)$.

Hence, it has to be the case that $B_2(v) \leq B_1(v)$ for all $v \in [0, \theta]$.

ii). Part (i) of proposition implies $y_1(b) \geq y_2(b)$ and therefore (1) yields

$$\frac{F_2(y_2(b))}{F_2(y_1(b))} \frac{1}{y_2(b)} \leq \frac{F_1(y_2(b))}{F_1(y_1(b))} \frac{1}{y_1(b)},$$

or equivalently,

$$\frac{f_2(y_2(b))}{f_2(y_1(b))} \cdot \frac{y'_2(b)}{y'_1(b)} = \frac{d}{db} \ln[F_2(y_2(b))] \geq \frac{f_1(y_1(b))}{F_1(y_1(b))} \cdot \frac{y'_1(b)}{y'_1(b)} = \frac{d}{db} \ln[F_1(y_1(b))].$$

Since $\ln(F_2(y_2(b))) = \ln(F_1(y_1(b)))$, then $\ln[F_2(y_2(b))] \leq \ln[F_1(y_1(b))]$ for all $b \in [0, \theta]$, and hence it follows that $F_2(y_2(b)) \leq F_1(y_1(b))$.

iii). Let $P_i$ denote bidder $i$'s probability of winning. Through a transformation of variables, it is easy to see that $P_2 = \int_0^\theta F_1(b) f_2(b) db$. Now (A3) implies $P_2 = \int_0^\theta F_1(b) f_2(b) db > \int_0^\theta F_2(b) f_1(b) db = P_1$.

In the WBA the reputed more interested bidder will bid “less aggressively”. Since bidder 1 faces a fiercer competition, the one from 2 that is reputed to be very interested in the asset to be sold, it is natural that she bids “more aggressively”. She is in a worse position than bidder 2 because she faces a bid probability distribution that gives high “weights” to high values, consequently one would expect her payoffs to be smaller.

3.2 The Loser’s Bid Auction

The bidder with the highest bid wins the auction and pays half the loser bid to the other bidder. Ties are resolved by the flip of a coin.

I will show that there exists a unique equilibrium for this auction that it is such that the bidder who is reputed more interested in the asset bids less aggressively. To prove this
result I first show that equilibrium bid strategies \((B_1(v_1), B_2(v_2))\) are strictly increasing with differentiable inverse bid functions. Given that \(B_i\) is strictly increasing, we can write bidder \(i\)'s expected gain when bidder \(j\) follows strategy \(B_j\) as

\[
U_i(v_i, b, B_j) = \int_0^{B_j^{-1}(b)} \left( v_i - \frac{B_j(v_j)}{2} \right) dF_j(v_j) + \frac{b}{2} \left( 1 - F_j(B_j^{-1}(b)) \right).
\]

At an interior solution, \(i\)'s bid \(b = B_i(v_i)\) satisfies

\[
(b - v_i) f_j(B_j^{-1}(b)) \frac{d}{db} f_j(b) = B_j'(b) 0.5 \left( 1 - F_j(B_j^{-1}(b)) \right).
\]

Thus, in equilibrium, the marginal cost from overbidding, \(b - v_i\), has to be equal to the marginal increase in the payment that partner \(i\) gets when she loses. This equation can be derived by differentiating \(U_i(v_i, b, B_j)\) with respect to \(b\) and setting that derivative to zero.

To obtain the appropriate boundary conditions assume first, w.l.o.g., \(B_1(0) < B_2(0)\). Because equilibrium bidding strategies are continuous, there exists \(\theta\) such that \(B_1(v) < B_2(v)\) for all \(v \in [0, \theta]\) and thus bidder \(1\)'s probability of winning the auction with the bid \(B_1(\theta)\) is zero.

This implies \(U_1(\theta, B_1(\theta), B_2) = 0.5 B_1(\theta) < U_1(\theta, B_2(0), B_2) = 0.5 B_2(0)\). But this contradicts the assumption that \(B_1(\cdot)\) is an equilibrium bidding strategy. Therefore \(B_1(0) = B_2(0)\). Since bidder \(1\)'s expected payoff when her valuation is zero is half her bid (she always loses), she will bid \(a > 0\) when her draw is zero. Consider now the terminal boundary condition. Assume, for contradiction, that \(B_1(\bar{v}) = b > \bar{v}\). Because of continuity, there exists \(\theta\) such that \(B_1(v) > v\) for all \(v \in [\theta, \bar{v}]\). Over this interval, partner 2 has a zero probability of winning when bidding her valuation. It is straightforward to prove that, to drive up the price that 1 has to pay for her shares, she will bid in equilibrium, \(B_2(v) = \lim_{\varepsilon \to 0} B_1(v) - \varepsilon\) for all \(v \in [\theta, \bar{v}]\). This implies \(U_1(\bar{v}, b, B_2) < U_1(\bar{v}, \bar{v}, B_2)\). To see this, note that

\[
\int_0^{\bar{v}} \frac{B_2(v_2)}{2} dF_2(v_2) < \int_0^{B_2^{-1}(\bar{v})} \frac{B_2(v_2)}{2} dF_2(v_2) + \int_{B_2^{-1}(\bar{v})}^{\theta} 0.5 \varepsilon dF_2(v_2).
\]

Thus \(B_1(\bar{v}) \leq \bar{v}\). The same argument applies to partner 2, thus \(B_2(\bar{v}) \leq \bar{v}\). Now \(B_i(\bar{v}) = \bar{v}\) for \(i = 1, 2\), follows from standard auction theory since each partner will bid at least her
valuation if there is a positive probability that the other one will bid at least that high. These results are summarized in next theorem.

I now proceed to state the main results for this mechanism.

**Theorem 3** A pair of strategies \((B_1, B_2)\) is an equilibrium if and only if the strategies are pure; the bid functions are strictly increasing and differentiable, with \(B_i(v) > v\) for all \(i\), and \(v \in [0, \bar{v}]\) and their inverse functions \((y_1(b), y_2(b))\) are such that they are solutions to the system of differential equations,

\[
2(y_1(b) - b)f_j(y_j(b))y_j'(b) + (1 - F_j(y_j(b))) = 0,
\]

with boundary conditions \(y_1(a) = y_2(a) = 0\), and \(y_1(\bar{v}) = y_2(\bar{v}) = \bar{v}\).

**Proof:** See Appendix.

**Theorem 4** Suppose assumption \((A1)\) holds. Then there exists one and only one Bayesian equilibrium for the LBA.

**Proof:** See Appendix.

The following proposition shows some of the properties of the equilibrium induced by the loser bid auction.

**Proposition 2** Under \((A4)\) if \((B_1, B_2)\) is an equilibrium strategy profile to a loser-price auction then for all \(i = 1, 2\),

i). \(B_1(v) \geq B_2(v)\), for all \(v \in [0, \bar{v}]\),

ii). The distribution of bidder 2's bids stochastically dominates the distribution of bidder 1's bid.

iii). Bidder 2 has a higher probability of winning the auction.
Proof: i). For all \( b \in [0, \theta) \) if \( y_2(b) = y_1(b) \) then \[
\frac{H_2(y_2(b))}{H_1(y_1(b))} = \frac{y_1'(b)}{y_2'(b)} \geq 1, \]
where the inequality is due to assumption (A4) and hence that \( y_2'(b) < y_1'(b) \).
Since at \( a, y_2(a) = y_1(a), \) it must be the case that \( y_2(b) \geq y_1(b) \) and therefore \( B_1(v) \geq B_2(v) \) for all \( v \in (0, \theta) \).

ii). Part (i) implies \( y_2(b) \geq y_1(b) \) which yields \( H_2(y_2(b))y_2'(b) \leq H_1(y_1(b))y_1'(b) \) or equivalently \(-\frac{d}{db} \ln[1 - F_2(y_2(b))] \leq -\frac{d}{db} \ln[1 - F_1(y_1(b))].\)
Because \( F_2(y_2(a)) = F_1(y_1(a)) \) then \( \ln[1 - F_2(y_2(b))] \geq \ln[1 - F_1(y_1(b))] \) for all \( b \in (a, \theta), \) and hence \( F_2(y_2(b)) \leq F_1(y_1(b)). \)

iii). Let \( P_i \) denote bidder \( i \)'s probability of winning. Part ii) of this proposition and (A3) imply
\[
P_2 = \int_0^\theta \left( \int_0^{\theta} f_1(v) dv \right) f_2(v) dv = \int_0^\theta F_1(y_1(b)) f_2(v) dv \geq \int_0^\theta F_2(y_2(b)) f_1(v) dv = P_1. \]

In the LBA, as in the WBA, the reputedly more interested bidder bids less aggressively. Since in a LBA bidders overbid in equilibrium whereas in a WBA they shade their bids the former will lead to higher expected equilibrium prices than the latter.6 In both auctions bidder 2 has a higher probability of winning. Recall that bidder 1 faces a bid probability distribution which gives high weight to high values, consequently, she is more likely to lose the auction. Thus bidder 2 is better off in a WBA whereas bidder 1 prefers a LBA. Moreover, it seems reasonable to expect bidder 1 to have higher interim expected payoffs in a LBA than bidder 1. Notice that when she wins she has to pay a lower price for the assets while when losing she gets a higher price. Actually, \( U_1(v) = \int_0^\theta (v - 0.5B_1(v_1)) f_2(v_2) dv_2 > \)
\[
\int_0^\theta (v - 0.5B_1(v_1)) f_1(v_1) dv_1 = U_2(v) \quad \text{and} \quad U_1(0) = U_2(0) = 0.5a. \]
Therefore either \( U_1(v) \geq U_2(v) \) for all \( v, \) or at least this is the case for high valuations. Note that the loser does not regret ex-post to lose since she gets a price for her shares that is higher than her valuation for the asset. The winner, on the other hand, regrets ex-post that she overbid so much

6These auctions have also been studied by Engelbrecht-Wiggans (1994) and Bulow, Huang and Kempeler (1996) for the common-values case. Engelbrecht-Wiggans assumes equal shares on the assets and focus on the question of low giving bidders a share in the proceeds affects equilibrium bidding. In particular, He shows that the expected sale price is higher in a LBA than in a WBA. Bulow, Huang and Kempeler consider a two-bidders takeover model where each bidder has a different toehold in the takeover target. They show that with asymmetric toeholds the expected sale price is higher in a WBA than in a LBA if the toeholds are sufficiently small.
but this overbidding is optimal ex-ante.

3.3 The Splitting-the-difference Auction

Under this mechanism the parties submit sealed bids, and the highest bidder obtains the asset at a price that “splits the difference” between the bids. Ties are resolved by the flip of a coin. If bidder $j$ uses a strategy $B_j$ then the expected payoff to bidder $i$ with valuation $v$ if she submits a bid $b = B_i(v)$ will be

$$U_i(v, b, B_j) = \int_{B_j(v_j) < b} (v - 0.5(\frac{b + B_j(v_j)}{2}))dF_j(v_j) + \int_{B_j(v_j) > b} 0.5(\frac{b + B_j(v_j)}{2})dF_j(v_j).$$

Assume that equilibrium bid strategies $B_i(v_i)$ are strictly increasing with differentiable inverse bid functions. If this the case, we can write bidder $i$’s expected gain as

$$U_i(v, b, B_j) = \int_0^{B_j^{-1}(v)} (v - \frac{b + B_j(v_j)}{4})dF_j(v_j) + \int_{B_j^{-1}(b)}^\infty (\frac{b + B_j(v_j)}{4})dF_j(v_j).$$

At an interior solution, $i$’s bid $b = B_i(v)$ satisfy

$$(b - v_i)f_j(B_j^{-1}(b)) = 0.5B_j^{-1}(b)(0.5 - F_j(B_j^{-1}(b))).$$

In equilibrium the marginal revenue from bidding more, $(v_i - b)f_j(B_j^{-1}(b)) + 0.5 \times 0.5 \times (1 - F_j(B_j^{-1}(b)))$, where the last term represents the marginal increase in payment for $i$’s shares from bidding more when she loses, has to be equal to the marginal cost due to increase in payment when winning, $0.25F_j(B_j^{-1}(b))$.

From McAfee (1992) it is known that if valuations are independently and symmetrically distributed then the WBA and the LBA produce the same expected utilities, so that there is no basis to choose one over the other.\footnote{Nevertheless, if there exists an outside option that can be exercised then WBA is the only member of the $k+1$-price auctions that is efficient. See McAfee (1992).} In order to compare the SDA with the previous auctions let me start by analysis its performance in the symmetric case.
3.3.1 The SDA: The Symmetric Case

Along this subsection I assume that partners' valuations are independent and identically distributed from a cumulative distribution function \( F \) with support \([0, \theta]\) and positive continuous density function \( f \). Under these assumptions it is easy to see that the equilibrium bid for the SDA is as follows:

\[
B_i(v_i) = v_i - \int_{z=F^{-1}(0.5)}^{v_i} \frac{(F(z) - 0.5)^2}{f(v_i) - 0.5^2} \, dz
\]

for \( i = 1, 2 \).

Note that a partner bids her valuation if and only if her valuation is the median of the distribution function. For larger values she shades her bid \((b(v) < v)\), and for smaller she marks up her bid \((b(v) > v)\). This is a very intuitive behavior. For \( v > v_{med} \) the partner is more likely to obtain the good. Thus, as the potential buyer, she has incentives to shade her bid and lowers the price. For \( v < v_{med} \) the partner is a potential seller hence she has incentives to increase her bid and thus the price she will receive for the asset. This bidding scheme is clearly ex-post efficient since the equilibrium bidding strategies are increasing in the valuations.

In the symmetric case the SDA generates the same expected utility to the parties than the WBA or the LBA. Besides in the presence of an outside option the SDA only loses efficiency if that is valued less than the median of the valuations distribution. It seems an appealing mechanism to undertake a dissolution by competitive bidding. Let us now study its performance in the asymmetric case.

3.3.2 The SDA: the Asymmetric Case.

To analyze the performance of the SDA in the asymmetric case I first characterize the equilibrium. Then I will show existence of an equilibrium, and finally I will study the properties of the equilibrium induced by this auction.

**Theorem 5** A pair of strategies \((B_1, B_2)\) constitute a Bayesian equilibrium for the SDA if and only if the strategies are pure, the bid functions are strictly increasing and differentiable, with \( B_i(v) = v \) for \( v_{med} \), and they have differentiable inverse bid
functions \((y_1, y_2), y_i(b) = B_i^{-1}(b),\) which are solution over the interval \([a, b]\) of the system of differential equations.

\[
(b - y_i)f_i(y_i(b))y_i'(b) = 0.5(0.5 - F_j(y_j(b)),
\]
with boundary condition \(y_1(c) = y_2(c) = 0, y_1(d) = y_2(d) = v,\) where \(0 < c < d < v.\)

I omit the formal proof since it is very similar to the ones I have already provided for the WBA and the LBA. Let me just mention the intuition of this result. Equilibrium strategies have to be weakly increasing in valuation since the payoff to winning is increasing in a partner's valuation while the payoff to losing is independent of her valuation. They have to be continuous because if there were a jump in a partner's bid, let's say \(i,\) then \(j\) will never bid in the interval of the jump but then \(i\) will find more profitable to bid in the interior of that interval. By similar reasoning it is easy to conclude that in equilibrium no bidder will make the same bid for a range of values. Each bidder equilibrium bid is characterized by the first order condition that leads to bids that are monotonically increasing in valuations. Finally, the boundary conditions can be obtained by using equilibrium arguments and by examining the first order conditions.

**Theorem 6** If \((A1)\) holds then there exists an equilibrium for the SDA.

Again, I will omit the formal proof. Notice that, in order to show existence, it is not enough to show that a solution to the differential equation exists but that solution must also satisfy the boundary conditions. From (3) we can see that \(b_1 = B_1(v_1) = v_1 at v_1 = m, m \in (v_1^{med}, v_2^{med}) ,\) and \(b_2 = B_2(v_2) = v_2 at v_2 = n, n \in (v_1^{med}, v_2^{med}) , n < m.\) Therefore we can consider system (3) with boundary conditions \(y_1(c) = y_2(c) = 0, y_1(b_1) = m and y_2(b_2) = n,\) for which a solution exists because of lipschitz continuity of the trajectories on \([c, n - \varepsilon]\) and \([c, m - \varepsilon]\) and then extend \(y_1\) by \(m\) at \(m\) and \(y_2\) by \(n\) at \(n.\) Consider next system (3) with boundary conditions \(y_1(b_1) = m and y_2(b_2) = n and y_1(d) = y_2(d) = v.\) A solution to this problem exists because of lipschitz continuity of the trajectories on \((n + \varepsilon, d)\) and \((m + \varepsilon, d)\). By extending \(y_1\) by \(m\) at \(m\) and \(y_2\) by \(n\) at \(n,\) we can show existence.

Let me now focus on the properties of the equilibrium induced by the SDA. Next proposition shows that the equilibrium, under the assumptions that I have considered,
is not always efficient.

**Proposition 3** Under (A2) if \((B_1, B_2)\) is an equilibrium strategy profile to a splitting-the-difference auction then:

i) \(B_2(v) \leq B_1(v)\), for all \(v \in (0, \bar{v})\),

ii) The distribution of bidder 2’s bids stochastically dominates the distribution of bidder 1’s bids.

iii) Bidder 2 has a higher probability of winning the auction.

**Proof:** i). The boundary conditions for the SDA imply that there exist \(v_1, v_2\) such that \(B_1(v_1) = v_1\) and \(B_2(v_2) = v_2\). Let \(v_1 = m\) and \(v_2 = n\). I now show that \(m, n \in [v_1^{med}, v_2^{med}]\) with \(m > n\). By inspection of (3) it is clear that \(B_1(m) = m\) when \(B_2(v_2^{med}) = m\) and \(B_2(n) = n\) when \(B_1(v_1^{med}) = n\). Besides (A2) implies \(v_1^{med} < v_2^{med}\).

Suppose \(m < n\). Then \(B_2(n) = n\) and \(B_2(v_2^{med}) = m < n\) with \(v_2^{med} < n\), otherwise partner 2’s bids would not be strictly increasing in valuations. This implies \(B_1(m) = m < n\) and \(B_1(v_1^{med}) = n > v_2^{med} > v_1^{med}\). Hence at \(b = n\) the right hand side of (3) is positive \(n - v_1^{med}\) but the left hand side is negative since \(0.5 < F_2(n)\), a contradiction. The contradiction proves the claimed \(m > n\).

Assume now that \(m < v_1^{med}\). This yields to \(B_1(m) = m\) and \(B_1(v_1^{med}) = n < m\) a contradiction with bids increasing in valuations.

Finally, let me show that \(m > v_2^{med}\) can not be the case. If \(m > v_2^{med}\) then \(B_2(n) = n\) and \(B_2(v_2^{med}) = m > v_2^{med}\) but this yields to a contradiction with system (3) since after bids cross the 45° line they have to satisfy that \(B_i(v) < v_i\) for all \(i = 1, 2\). The only values of \(m, n\) that are consistent with system (3) and with bids increasing in valuations are \(m, n \in [v_1^{med}, v_2^{med}]\) with \(m > n\). This also implies that there is no \(b \in (c, d)\) such that \(y_2(b) = y_1(b)\) and therefore that \(B_2(v) \leq B_1(v)\), for all \(v \in (0, \bar{v})\).

ii). Part (i) implies \(y_2(b) \geq y_1(b)\).

Consider first the case \(b < y_1(b) \leq y_2(b)\). If this is the case (3) yields

\[
\frac{d}{db} \ln[F_2(y_2(b)) - 0.5] \geq \frac{d}{db} \ln[F_1(y_1(b)) - 0.5].
\]
Because $F_2(y_2(d)) = F_1(y_1(d))$ then $\ln[F_2(y_2(b))-0.5] \leq \ln[F_1(y_1(b))-0.5]$ for all $b \in (c,d)$, and hence $F_2(y_2(b)) \leq F_1(y_1(b))$.

Consider now the case $y_1(b) \leq y_2(b) < b$ which implies

$$- \frac{d}{db} \ln[0.5 - F_2(y_2(b))] \leq - \frac{d}{db} \ln[0.5 - F_1(y_1(b))].$$

Since $F_2(y_2(c)) = F_1(y_1(c)) = 0$ then $\ln[0.5 - F_2(y_2(b))] \geq \ln[0.5 - F_1(y_1(b))]$ for all $b \in (c,d)$, and hence $F_2(y_2(b)) \leq F_1(y_1(b))$.

iii). Let $P_i$ denotes bidder $i$'s probability of winning. Part ii) of this proposition and (A3) imply

$$P_2 = \int_0^b \left( \int_0^{B_2^{-1}(b)} f_1(v)dv \right) f_2(v)dv = \int_0^b F_1(y_1(b))f_2(v)dv \geq \int_0^b F_2(y_2(b))f_1(v)dv = P_1.$$

4 Efficiency: A Numerical Example

The results in this paper show that no member of the $K+1-$ auctions generates an efficient dissolution if the partners have valuations that are independent but asymmetrically distributed. Furthermore, I do not think it is possible to rank these auctions in terms of the interim expected utility they generate without relying on particular numerical examples. When ex-post efficiency is unattainable it is natural to seek for the auction among the ones I have discussed that maximizes expected total gains from dissolution.

The expected total gains derived from the implementation of any of these auctions is

$$\int_0^b v_1 P(B_1(v_1) \geq B_2(v_2)) f_1(v_1)dv_1 + \int_0^b v_2 P(B_1(v_1) < B_2(v_2)) f_2(v_2)dv_2 =$$

$$\int_0^b v_1 f_1(v_1)dv_1 + \int_0^b \int_0^{v_1} (v_2 - v_1)m(v_1, v_2)f_1(v_1)f_2(v_2)dv_1dv_2,$$

If to guarantee ex-post efficiency is the only goal of the partners then the following bidding procedure can be used. Each partner submits a bid and has to pay the expected externality associated with her bid; the asset is awarded to the partner making the highest bid. In our set-up partner $i$ submits $b_i$ and pays $p_i = \int_0^{b_i} v_i f_1(v)dv$ to the other partner. By internalizing the expected externality, this bidding scheme induces each partner to make an equilibrium bid $b_i = v_i$, and therefore ensures an ex post-efficient outcome. Note that this procedure has an important caveat since it imposes different prices on the partners. If both submit the same bid, partner 1 will have to pay a higher price to get the asset than 2 does.
where \( m(v_1, v_2) = P(B_1(v_1) < B_2(v_2)) \).

Since an ex post efficient mechanism maximizes the expected total gains from trade, and since in such a mechanism

\[
m(v_1, v_2) = \begin{cases} 
1 & \text{if } v_1 < v_2, \\
0 & \text{if } v_1 > v_2,
\end{cases}
\]

then we have to seek for the auction that minimizes \( \int_0^1 (y_2(b) - y_1(b)) \text{d}b \).

The purpose of this section is to compare the auctions I have considered from an efficient viewpoint. To do so I will rely on a particular numerical example. This example will also illustrate some features of the equilibrium induced by the WBA, the LBA and the SDA.

Assume that partner1’s beliefs about the value that partner 2 places in the asset are summarized by the c.d.f. \( F_2(v) = v \), whereas those of partner 2 are summarized by the c.d.f. \( F_1(v) = \frac{1 - \exp(-v)}{1 - \exp(-1)} \). Note that \( F_1(v), F_2(v) \) satisfy all the assumptions in the \( 1 - \exp -1 \) model. I now show the equilibrium induced by the auctions studied here.

The Winner Bid Auction

The equilibrium bid function are the solutions of the following system of differential equations:

\[
2(y_1(b) - b)y_1'(b) = y_2(b),
\]

\[
2(y_2(b) - b)(\exp(-y_1(b)))y_1'(b) = 1 - \exp(-y_1(b)),
\]

\[
y_2(0.621) = 1,
\]

\[
y_1(0.621) = 1.
\]

The proof of Theorem 2 suggests a simple algorithm for finding a solution numerically. Pick any \( b \in (0, v) \) and compute the associated solution in system (1). If this solution intersects the 45 degree line below 0, then increase the starting value. Otherwise if \( y_i(0) > 0 \), then reduce the starting value.\(^9\)

The results for this mechanism are summarized in the following table:

\(^9\)For an study on how to solve numerically differential equations as those in (1), see Marshall, Meurer, Richard and Strompequatist (1994).
\[ y_1(b) = \frac{156}{b} \quad y_2(b) = \frac{163}{b} \]

\[
\begin{array}{cccccccc}
    b & .1 & .2 & .3 & .4 & .5 & .55 & .6 \\
    y_1(b) & .156 & .297 & .443 & .595 & .760 & .852 & .953 \\
    y_2(b) & .163 & .320 & .488 & .659 & .823 & .900 & .971 \\
    \text{Difference} & .0068 & .0230 & .0453 & .0639 & .0627 & .0477 & .018 \\
\end{array}
\]

The Loser Bid Auction

The equilibrium bid functions are the solutions of the following system of differential equations:

\[
2(y_1(b) - b)y_1'(b) = y_2(b) - 1 \\
2(y_2(b) - b)(\exp(-y_1(b)))y_2'(b) = \exp(-1) - \exp(-y_1(b)) \\
y_2(0.296) = 0 \\
y_1(0.296) = 0
\]

The algorithm that I have used here for finding a solution numerically is based on Theorem 4. Pick any \( a \in (0, \bar{v}) \) and compute the associated solution in system (2). If this solution intersects the 45 degree line below \( \bar{v} \), then decrease the starting value. Otherwise if \( y_i(\bar{v}) > \bar{v} \), then increase the starting value.

The results for this mechanism are summarized in the following table:

\[
\begin{array}{cccccccc}
    b & .3 & .4 & .5 & .55 & .6 & .7 & .8 \\
    y_1(b) & .004 & .120 & .251 & .321 & .393 & .543 & .695 \\
    y_2(b) & .007 & .164 & .308 & .377 & .444 & .579 & .714 \\
    \text{Difference} & .0024 & .0441 & .0567 & .0555 & .0509 & .0358 & .0185 \\
\end{array}
\]

The Splitting-the Difference Bid Auction

The equilibrium bid functions are the solutions of the following two systems of differential equations:

\[
2(y_1(b) - b)y_1'(b) = y_2(b) - 0.5 \\
2(y_2(b) - b)(\exp(-y_1(b)))y_2'(b) = 0.5(\exp(-1) + 1) - \exp(-y_1(b)) \\
y_2(0.1406) = 0 \\
y_1(0.1406) = 0 \\
y_2(0.79637) = 1 \\
y_1(0.79637) = 1
\]

20
To solve this case I have followed the arguments in Theorem 6.

The results for this mechanism are summarized in the following tables:

<table>
<thead>
<tr>
<th>b</th>
<th>.2</th>
<th>.25</th>
<th>.3</th>
<th>.35</th>
<th>.410</th>
<th>.45</th>
<th>.474</th>
<th>.56</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1(b)$</td>
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<td>.140</td>
<td>.213</td>
<td>.288</td>
<td>.379</td>
<td>.445</td>
<td>.474</td>
<td>.543</td>
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<td>.175</td>
<td>.248</td>
<td>.321</td>
<td>.410</td>
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<table>
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<td>.741</td>
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<td>.0444</td>
<td>.0481</td>
<td>.0452</td>
<td>.0304</td>
</tr>
</tbody>
</table>

RESULTS:

- The numerical results confirm that partner 2 bids more conservatively in all the auctions. We have $y_2(b) > y_1(b)$, or equivalently, $B_1(v) > B_2(v)$, for all $v \in (0, v)$.

- The domain of equilibrium bids is as follows: $B_{WBA}^W \in [0, 0.621]$, $B_{SDA}^S \in [0.1406, 0.796]$ and $B_{LBA}^L \in [0.296, 1]$, with $B_i^{WBA}(v) < B_i^{SDA}(v) < B_i^{LBA}(v)$ for all $v$, and all $i = 1, 2$.

- Under the LBA we obtain that both partners may overbid so that either winner loses money. If $v_1, v_2 \in (0.3, 0.85)$ either winner will suffer a loss. In the takeover case this is consistent with the observation that the stock price of a successful raider sometimes falls dramatically.

- Under the SDA we obtain $B_1(v) > v$ for all $v \in (0, 0.474)$, and $B_2(v) > v$ for all $v \in (0, 0.410)$ with $B_1(0.474) = 0.474 = B_2(v_1^{med})$ and $B_1(v_2^{med}) = 0.410 = B_2(0.410)$. Note that under the SDA it is also true that both partners may overbid so that either winner loses money. If $v_1, v_2 \in (0.15, 0.45)$ either winner will suffer a loss.

- The LBA generates the highest expected selling price.

- The numerical results suggest that the SDA is more efficient than the WBA or the LBA.
5 Conclusions

It is well-known that private-value auctions with equal ownership of the item yield efficient outcomes when the valuations of the bidders are drawn from a common distributions. This result, as shown in this paper, does not carry out to the case of asymmetric distributions. The inefficiency is due to the fact that the bidder who is reputed to be more interested in the asset to be traded will bid less aggressively in equilibrium. As a result of this, the bidder with the lower valuation buys the target with positive probability.

Results in this paper have a wide range of applications including the sharing of profits in bidding rings or creditors' bidding in Bankruptcy auctions. However, a limitation of the analysis in this paper is that attention is restricted to the special case of equal shares in the partnership. One should study the range of partnerships that can be dissolved efficiently with these auctions once it is known that the equal-shares partnerships may not be in this set. It would also be interesting to consider the general n-partners case. This is left for future research.

6 Appendix

Proof of theorem 1.

Let $B_i(·)$ denote an equilibrium strategy for bidder $i$. The fact that $B_i(·)$ is strictly increasing and differentiable almost everywhere on $[0, \bar{b}]$ implies that it has an inverse function $y_i$ defined on $[0, \bar{b}]$ which is increasing and differentiable almost everywhere and satisfies that $y_i(0) = 0$ and $y_i(\bar{b}) = \bar{v}$ for all $i = 1, 2$. Thus for all $i = 1, 2$, $U_i(v_i, B_i, B_j)$ is differentiable in its second argument. Hence $B_i(·)$ must satisfy the first order condition for a maximum and therefore its inverse will satisfy system (1).

I will now show that the necessary conditions are sufficient. Assume $y_1, y_2$ verify (1), and satisfy the boundary conditions with $0 < y_i(b) < \bar{v}$ for all $i = 1, 2$ and $b \in (0, \bar{b}]$. Note that $y'_i(b) > 0$ for all $i$ and $b \in (0, \bar{b}]$. If there exists $i$ and $\theta \in (0, \bar{b}]$ such that $y'_i(\theta) = 0$, then system (1) implies that $y_i(\theta) = \infty$ in contradiction with $0 < y_i(b) < \bar{v}$, for $b \in (0, \bar{b}]$. Similarly, if there exists $\pi \in (0, \bar{b}]$, such that $y'_i(\pi) < 0$, then the boundary
condition \( y_1(0) = 0 \) and the fact that \( y_i(b) > 0 \) for \( b > 0 \) imply that some \( \xi \in (0, \pi) \) satisfying \( y_i'(\xi) = 0 \) would exists. But this yields a contradiction with \( 0 < y_i(b) < \bar{b} \) for \( b \in (0, \bar{b}] \). Therefore \( y_1, y_2 \) are continuous, differentiable over \((0, \bar{b}]\) with \( y_i(b) > b \), for all \( b \in (0, \bar{b}] \) and all \( i = 1, 2 \). I have to show that \( B_1 = y_1^{-1} \) and \( B_2 = y_2^{-1} \) is an equilibrium. To do so, let us assume that partner 2 follows the strategy \( B_2 = y_2^{-1} \). Clearly, a bid larger than \( \bar{b} \) is never a best response since for all \( b \geq \bar{b} \) we have \( U_1(v_1, b, B_2) = v_1 - 0.5b \) and \( \frac{\partial}{\partial b} U_1 = -0.5 \).

If \( v_1 = 0 \), then the best response is \( B_1(0) = 0 \) because \( b = 0 \) maximizes \( U_1 \) for \( v_1 = 0 \).

Recall that
\[
U_1(0, b, B_2) = \begin{cases} \int_{0}^{b} 0.5B_2(v_2)dF_2(v_2) & \text{if } b = 0, \\ -bF_2(B_2^{-1}(b)) + \int_{B_2^{-1}(b)}^{\bar{b}} 0.5B_2(v_2)dF_2(v_2) & \text{if } b > 0. \end{cases}
\]

If \( v_1 \in (0, \bar{b}) \) then we have
\[
\frac{\partial U_1(v_1, b, B_2)}{\partial b} = \begin{cases} \frac{1}{2}F_2(y_2(b)) + (v_1 - b)F_2(y_2(b))y_2'(b). \end{cases}
\]

Since \( y_2'(b) \) satisfies (1) then
\[
\frac{\partial U_1(v_1, b, B_2)}{\partial b} = -\frac{1}{2}F_2(y_2(b)) + \frac{v_1 - b}{y_1(b)} - b \cdot \frac{1}{2}F_2(y_2(b)).
\]

Because \( y_1(b) > b \), \( \frac{\partial U_1}{\partial b} \) is positive if \( v_1 > y_1(b) \) and negative if \( v_1 < y_1(b) \). For \( v_1 \in (0, \bar{b}) \), the continuity of \( U_1 \) as a function of \( b \) and the above results imply that \( U_1 \) is monotonically nondecreasing on \([0, B_1(v_1))\) and monotonically nonincreasing on \((B_1(v_1), \infty)\).

Therefore bidder 1's expected payoff is maximized if \( v_1 = y_1(b) \) that is if \( b_1 = B_1(v_1) \). Thus \( B_1(v_1) \) is a best response if bidder 2 bids according with \( B_2(v_2) \). Since a symmetric argument applies for bidder 2, any strictly increasing and differentiable \( y_1(b), y_2(b) \) satisfying system (1) and the boundary conditions define an equilibrium.

**Proof of Theorem 2**

From the characterization theorem we know that equilibrium inverse bidding functions are solutions to the system:

\[
y_i'(b) = \frac{F_i(y_i(b))}{f_i(y_i(b))} \cdot \frac{1}{2(\nu_i(b) - b)},
\]

with boundary conditions \( y_i(0) = 0 \) and \( y_i(\bar{b}) = \bar{b} \), where \( \nu_i = \nu_i(b) \) for \( i = 1, 2 \).
The system presents a singularity at \( y_1(0) = y_2(0) = 0 \), which is the only singularity since \( y_i(b) > b \) for all \( b \in (0, \tilde{b}) \). Thus I cannot apply the results on existence of the theory of differential equations. Nevertheless, system (1), with the initial condition \( y_i(\tilde{b}) = \bar{v} \), for \( i = 1, 2 \), defines a free - boundary problem, for which those results hold. Note that for all \( v \in (e, \tilde{v}] \), \( e > 0 \), the right-hand side of (1) is Lipschitz continuous. Therefore to show existence is equivalent to show that there exists \( \tilde{b}, 0 < \tilde{b} < \bar{v} \), \( y_i(\tilde{b}) = \bar{v} \), for which the solution of (1) consists of strictly increasing functions in the domain \([0, \bar{v}]\) and such that \( y_i(0) = 0 \) for all \( i \).

For any \( b \in (0, \tilde{b}] \) let us define the integral curves \( \Psi_i(b, \tilde{b}) \) as follows: \( \Psi_i(b, \tilde{b}) = \bar{v} \) for \( i = 1, 2 \) and \( \Psi_1(b, \bar{b}) \) and \( \Psi_2(b, \bar{b}) \) determined by integrating (1) backwards from \( \bar{b} \). As \( \tilde{b} \) decreases, \( \Psi_i(0, \bar{b}) \) decreases as long as it remains well defined. It is not well defined when \( \Psi_i(0, \bar{b}) = 0 \). Appealing to Lemma 2 in his appendix, there exists \( i, \) and \( \bar{b} \) such that \( \lim_{b \to \bar{b}} \Psi_i(0, b) = 0 \). Let assume w.l.o.g. \( i = 1 \), so that \( \lim_{b \to \bar{b}} \Psi_1(0, b) = 0 \). I now show that this implies \( \lim_{b \to \bar{b}} \Psi_2(0, b) = 0 \) as well.

Assume the contrary, that is, assume \( \lim_{b \to \bar{b}} \Psi_2(0, b) = a > 0 \). Extend \( \Psi_1(b, \bar{b}) \) and \( \Psi_2(b, \bar{b}) \) by zero and \( a \) at zero. Then \( \Psi_2(0, \bar{b}) > 0 \). From (1) the slope of \( \Psi_1(0, \bar{b}) \) at 0 is finite so that \( \Psi_1(b, \bar{b}) = \theta_1 b \) for some \( \theta_1 > 0 \). This implies

\[
\Psi_2'(b, \bar{b}) = \frac{F_2(\Psi_2(b, \bar{b}))}{f_2(\Psi_2(b, \bar{b}))} \cdot \frac{1}{2(\Psi_1(b, \bar{b}) - \bar{b})} \approx \frac{\theta_2}{2(\theta_1 b - b)} \approx \frac{\theta_3}{\bar{b}}.
\]

It is clear from the above expression that the integral of \( \Psi_2(b, \bar{b}) \) goes to infinite as \( b \) goes to 0, and therefore it does not converge to \( a \). Thus \( \lim_{b \to \bar{b}} \Psi_2(0, b) = 0 \).

Thus \( \Psi_1(b, \bar{b}), \Psi_2(b, \bar{b}) \) extended by zero at zero satisfy system (1) and the boundary conditions. It is an equilibrium, and the proof on existence is complete.

To show uniqueness, suppose, by way of contradiction, that there exists another equilibrium \( (\tilde{z}_1, \tilde{z}_2) \). This implies that there exists \( \tilde{b} \) such that \((\tilde{z}_1, \tilde{z}_2)\) are solution to (1) with initial conditions \( z_1(0) = z_2(0) = 0 \), and \( z_1(\tilde{b}) = z_2(\tilde{b}) = \bar{v} \). The uniqueness of the solution to the free - boundary problem implies \( \tilde{b} \neq \bar{b} \). Assume w.l.o.g. \( \tilde{b} < \bar{b} \).

\[\text{Recall that the density functions are assumed locally Lipschitz over } (0, \bar{v}], \text{ thus by the theory of ordinary differential equations it is known that there exists one and only one solution passing through any interior point of that domain (See Hurewicz (1958)).}\]
I first show that if this is the case then for all \( b \in (0, \tilde{b}) \), \( (z_1(b) - y_1(b))(z_2(b) - y_2(b)) < 0 \). Assume first \( z_1(b) = y_1(b) \) and \( z_1(\theta) < y_1(\theta) \) for all \( \theta \in (0, b) \). Because of Lipschitz continuity \( z_2(b) \neq y_2(b) \), thus assume w.l.o.g. that \( z_2(b) > y_2(b) \). Then (1) implies \( \beta_1'(b) < y_1'(b) \) and therefore that, for \( m \) slightly smaller than \( b \), \( z_1(m) > y_1(m) \), a contradiction. Therefore \( z_1(\theta) > y_1(\theta) \) for all \( \theta \in (0, b) \). In order to show that \( z_2(b) < y_2(b) \) holds for all \( b \in (0, \tilde{b}) \), suppose, by way of contradiction, that this is not the case. Fix \( \lambda \in (0, b) \) and define the following function \( \beta_1 \) on \( [0, \lambda] \):

\[
\beta_1(\lambda) = y_1(\lambda),
\]

\[
\beta_1'(\theta) = \frac{F_1(\beta_1(\theta))}{f_1(\beta_1(\theta))} \cdot \frac{1}{2(z_1(\theta) - \theta)}.
\]

Note that for all \( \theta \in (0, \lambda) \) \( \beta_1'(\theta) : \frac{f_1(\beta_1(\theta))}{F_1(\beta_1(\theta))} = y_1'(\theta) \cdot \frac{f_1(z_1(\theta))}{F_1(z_1(\theta))} \) and for all \( \theta \in (0, \lambda) \) \( z_1(\theta) > \beta_1(\theta) > y_1(\theta) \). Since the slopes are proportional at \( \lambda \) but \( z_2(\theta) - \theta > y_2(\theta) - \theta \) then \( \beta_1'(\lambda) < y_1'(\lambda) \) and therefore \( z_1(m) \leq \beta_1(m) \) for some \( m \in (0, \lambda) \), a contradiction. The contradiction proves that for all \( b \in (0, \tilde{b}) \), \( (z_1(b) - y_1(b))(z_2(b) - y_2(b)) < 0 \).

Let us assume w.l.o.g. \( \beta_1(v) < B_1(v) \). By the argument above, we have that \( \beta_2(v) > B_2(v) \). This implies \( \beta_2(\tilde{v}) = \tilde{b} > B_2(\tilde{v}) = \tilde{b} \). This contradicts \( \tilde{b} > \tilde{b} \). The contradiction proves the theorem.

**Proof of theorem 3**

Let \( B_i(\cdot) \) denote an equilibrium strategy for bidder \( i \). I first show that \( B_i(\cdot) \) has to be weakly monotonically increasing on \( [0, \tilde{v}] \). To see this let \( v' = B_i(v') \) and \( v'' = B_i(v'') \) with \( v' \geq v'' \). Equilibrium requires the following two conditions to be satisfied:

\[
U_i(v', b', B_j) = \int_0^{v'(b')}(2v' - B_j(x))dF_j(x) + b'(1 - F_j(y_j(b'))) \geq \int_0^{v''(b'')}(2v'' - B_j(x))dF_j(x) + b''(1 - F_j(y_j(b''))) = U_i(v'', b'', B_j) \]

and

\[
U_i(v'', b'', B_j) = \int_0^{v''(b'')}(2v'' - B_j(x))dF_j(x) + b''(1 - F_j(y_j(b''))) \geq \]

11This property of the equilibria is reminiscent of the results for the war of attrition with incomplete information where the equilibria can be indexed by the relative toughness of the two players. See Fudenberg and Tirole (1986).
\[ \int_0^{u''(b')} (2v'' - B_j(x))dF_j(x) + u'(1 - F_j(y_j(b'))) = U_i(v'', b', B_j). \]

Subtracting the right side of the second inequality from the left hand side of the first, and subtracting the left hand side of the second inequality from the right hand side of the first, yields: \( (u'' - u') [F_j(B_j^{-1}(b'')) - F_j(B_j^{-1}(b'))] \geq 0. \) This implies \( b'' \leq b'. \) Now I show that \( B_i(\cdot) \) is strictly increasing and differentiable. Note that \( B_i(\cdot) \) must be gapless: if there is a gap \([b', b]\) in \( B_i \) then there must be a gap \((b', b)\) in \( B_j \) because for any \( v_j \) it would be better to bid \( b \) than any other bid in that interval (the probability of winning is unchanged but it increases her payoff when losing). But then the type of bidder \( i \) planning to bid \( b' \) would be better off bidding \((b' + b)/2\), a contradiction. Furthermore, \( B_i \) must be atomless. Suppose it is not, then there is \( b \) and \( j \) such that \( P(B_j(v_j) = b) > 0 \). But then, there exists \( \varepsilon > 0 \), such that bidder \( i \) will never bid on \([b, b + \varepsilon]\), since \( i \) does better by bidding \( \lim_{\delta \to 0} b - \delta \).

An upper bound in the decrease in profit from getting a lower payment when she loses is \( 0.5e(1 - \lim_{\delta \to 0} F_j(B_j^{-1}(b + \delta))) \), which is, for \( \varepsilon \) small enough, less than a lower bound on the increase in profit when winning, \( \lim_{\delta \to 0} [F_j(B_j^{-1}(b + \delta)) - F_j(B_j^{-1}(b - \delta))]|(b - v_i) \). By symmetry, in equilibrium, partner \( j \) cannot bid \( b \) for a range of valuations. Strategies are pure since they are continuous and strictly increasing. Because the distribution of bids must be both gapless and atomless, continuity and monotonicity leads to differentiability almost everywhere on its domain.

In equilibrium \( B_i(v) > v \) for all \( v \in [0, \bar{v}) \). To see this note that when bidding \( B_i(v) \) instead of \( v \), an upper bound in the loss for winning more often and paying more for the assets is \( 0.5|F_j(B_j^{-1}(B_i(v)) - F_j(B_j^{-1}(v))|B_i(v) - v) \). For \( B_i(v) \) close enough to \( v \), this upper bound is smaller than the increase in profit from getting a higher payoff when losing \( 0.5(B_i(v) - v)(1 - F_j(B_j^{-1}(B_i(v)))) \).

Because the bidding strategies are increasing and differentiable, their inverses \( y_1, y_2 \) are increasing and differentiable. The differentiability of \( U_i(v_i, b_i, B_j) \) for all \( i = 1, 2 \), in its second argument allows to conclude that \( y_1, y_2 \) must satisfy system (2).

Finally, I now show that the necessary conditions are also sufficient. Let \( y_1, y_2 \) be solutions of (2) with \( 0 < y_i(b) < \bar{v} \) for all \( i = 1, 2 \) and \( b \in (a, \bar{v}) \), which satisfy the boundary conditions. From (2) it is clear that \( y'_i(b) > 0 \) for all \( i \). If there exists \( i \) and \( \theta \in (a, \bar{v}) \) such that \( y'_i(\theta) = 0 \) then (2) implies that \( y_j(\theta) = -\infty \) in contradiction with
$y_j(b) > 0$. Analogously, if there exists $\pi \in (a, \bar{v})$ such that $y_j(\pi) < 0$ then the boundary conditions imply that there exists some $\xi \in (\pi, \bar{v})$ such that $y_j(\xi) = 0$. This implies $y_j(\xi) = -\infty$. But this yields a contradiction with $0 < y_j(b)$ for $b \in (0, \bar{v})$. Therefore $y_1$, $y_2$ are continuous, differentiable over $(0, \bar{v})$, and such that $y_i(b) < b$ for all $b \in (0, \bar{v})$ and all $i = 1, 2$. To show that $B_1 = y_1^{-1}$ and $B_2 = y_2^{-1}$ is an equilibrium let us assume that partner 2 follows the strategy $B_2 = y_2^{-1}$. Due to the definition of $B_2$, and its strict monotonicity we have

$$U_1(v, b, B_2) = \begin{cases} 0.5b & \text{if } b \leq a, \\ \int_0^{B_2^{-1}(b)} v - 0.5B_2(v_2)dF_2(v_2) + 0.5b(1 - F_2(B_2^{-1}(b))) & \text{if } b \in (b, \bar{v}), \\ \int_0^{B_2^{-1}(b)} v - 0.5B_2(v_2)dF_2(v_2) & \text{if } b \geq \bar{v}. \end{cases}$$

If $v_1 = 0$, then the best response is $B_1(0) = a$ because $b = a$ maximizes $U_1$ for $v_1 = 0$. Note that $\frac{\partial U_1}{\partial b} < 0$ for $b > a$ and $v = 0$. If $v_1 \in (0, \bar{v})$ then we have

$$\frac{\partial U_1}{\partial b} = 0.5(1 - F_2(y_2(b))) + (v_1 - b) f_2(y_2(b))y_2'(b).$$

Since $y_2(b)$ satisfies (2) then

$$\frac{\partial U_1}{\partial b} = 0.5(1 - F_2(y_2(b))) + \frac{v_1 - b}{b - y_1(b)} \times 0.5(1 - F_2(y_2(b))).$$

Because $y_1(b) < b$, then $\frac{\partial U_1}{\partial b}$ is positive if $v_1 < y_1(b)$ and negative if $v_1 > y_1(b)$. For $v_1 \in (0, \bar{v})$ the continuity of $U_1$ as a function of $b$ and the above results imply that $U_1$ is monotonically non-decreasing on $[0, B_1(v_1))$ and monotonically non-increasing on $(B_1(v_1), \infty)$.

Therefore bidder 1’s expected payoff is maximized if $v_1 = y_1(b)$ that is if $b_1 = B_1(v_1)$. Thus $B_1(v_1)$ is a best response if bidder 2 bids according with $B_2(v_2)$. A symmetric argument applies to bidder 2. Thus, any strictly increasing and differentiable $y_1(b)$, $y_2(b)$ satisfying system (2) and the boundary conditions define an equilibrium.

**Proof of Theorem 4**

From the characterization theorem we know that equilibrium inverse bidding functions are solution to system (2) with boundary conditions $y_2(a) = y_1(a) = 0$, and $y_1(\bar{v}) = \bar{v}$. 

27
\( y_0(\vartheta) = 0. \)

We can rewrite system (2) in the following way:

\[
(2') \quad \frac{d}{db} \ln(1 - F_j(y_j(b))) = \frac{1}{(y_i - b)}.
\]

For any starting value \( a \), \( a \in (0, \bar{v}) \), the system is Lipschitz continuous on \([a, \bar{v} - \varepsilon)\) since by assumption \( H_i(y) \) (the hazard rate function) is bounded away from zero. Thus, by the theory of differential equations, it is known that the solution is unique on \([a, \bar{v} - \varepsilon)\). I denote the solution by \( y(x) \). At \( \bar{v} \) the system is not well defined and it is not Lipschitz continuous. Hence, there may exist a second solution \( \hat{y}(x) \) with initial condition \( \hat{a} = a + \varepsilon \) such that \( \hat{y}_i(\bar{v}) = \bar{v} \) for all \( i \). I will show that this can not be the case by proving that for all \( \varepsilon > 0 \) it is satisfied that \( \hat{y}(x + \varepsilon) \leq y(x) \) which implies \( \hat{y}(\bar{v}) \leq y(\bar{v} - \varepsilon) < \bar{v} \).

At the initial conditions \( \hat{y}_i(a + \varepsilon) = 0, y_i(a) = 0, \) for all \( i = 1, 2 \) the left hand side of the system is smaller for \( \hat{y} \). Furthermore, we have

\[
\frac{\hat{y}_i'(a + \varepsilon)}{\hat{y}_i'(a + \varepsilon)} = \frac{H_i(\hat{y}_i(a + \varepsilon))}{H_i(y_i(a))} = \frac{H_i(y_i(a))}{H_i(y_i(a))} = y_i'(a).
\]

At the starting value the slopes are proportional but the left hand side of the system is smaller for \( \hat{y} \), which implies \( \hat{y}'(a + \varepsilon) < y'(a) \), and thus \( \hat{y}(a + \varepsilon) \leq y(a) \). Therefore at the starting value the statement is true.

Let us assume that \( i \) and \( \hat{x} \) exist such that \( \hat{y}(\hat{x} + \varepsilon) > y(\hat{x}) \). This would imply that there exists \( x \) such that \( \hat{y}(x + \varepsilon) = y(x) \). Assume w.l.o.g. \( \hat{y}_1(x + \varepsilon) = y_2(x), \hat{y}_1(x + \varepsilon) > y_2(x) \) with \( \hat{y}_1(x + \varepsilon) \leq y_1(x) \). From (2) we have

\[
\text{Proof:} \quad \frac{1}{x + \varepsilon - \hat{y}_1(x + \varepsilon)} > \frac{1}{x - y_1(x)}. \quad \text{This implies} \quad \hat{y}_1(x + \varepsilon) > y_1(x), \quad \text{a contradiction.} \]

I will now show that an equilibrium exists. To do so, let us denote with \( \phi_i(a, b) \) the trajectory solving (2) and that \( \phi_i(a, a) = 0 \). To prove that an equilibrium exists is equivalent to showing that there exists \( \hat{a} \in (0, \bar{v}) \) such that \( \phi_i(a, \hat{a}) = \bar{v} \), for \( i = 1, 2 \). Note that \( \phi_i(a, b) \) is a continuous function on \([a, \bar{v} - \varepsilon)\) because of the continuity of trajectories with respect to the initial conditions due to the Lipschitz continuity of the system.

Pick an arbitrary \( a, 0 < a < \bar{v} \), and assume that for this \( a \) the maximal solution (i.e. the solution that cannot be defined over a bigger interval and still be a solution) has domain \([a, \eta]\) with \( \phi_1(c, \eta) = \bar{v} \). By Lemma 3 in this appendix we know that \( \phi_2(a, \eta) = \bar{v} \) and by Lemma 5 we can see that \( \bar{v} - \eta + a = 0.5 \int_0^\bar{v} 1 - F_1(B_2^{-1}(B_1(v)))dv. \) Since \( \eta \) is increasing

28
in a, appealing to continuity and Lemma 4 there exists \( \tilde{a} > a \) for which \( \eta = \delta \). Therefore \((\phi_1(\tilde{a}, b), \phi_2(\tilde{a}, b))\) is a solution to (2) that satisfies the boundary conditions; hence, it is an equilibrium, and the proof is completed.

**Lemma 1** Let \( 0 < \tilde{b} < \tilde{b} < \tilde{b} \). Then for all \( b \) belonging to the domains of definition of \( \Psi_i(\cdot, \tilde{b}) \) and \( \Psi_i(\cdot, \tilde{b}) \), \( \Psi_i(\cdot, \tilde{b}) < \Psi_i(\tilde{b}, \tilde{b}) \).

**Proof:** Since \((y_1, y_2)\) are strictly increasing then \( \Psi_i(\tilde{b}, \tilde{b}) = \bar{v} < \Psi_i(\tilde{b}, \tilde{b}) \). Let \( m \) be the largest bid less than \( \tilde{b} \) such that for some \( i \) \( \Psi_i(b, \tilde{b}) = \Psi_i(\tilde{b}, \tilde{b}) \). This implies \( \Psi_i(m, \tilde{b}) < \Psi_i(m, \tilde{b}) \) (otherwise the solutions would be equal over their common definition support contradicting Lipschitz continuity) and \( \Psi_i(m, \tilde{b}) \leq \Psi_i(m, \tilde{b}) \). But from (1) we have \( \frac{d\Psi_i(b)}{db} \) is a decreasing function of \( y_j \), and this implies \( \Psi_i(m, \tilde{b}) > \Psi_i(m, \tilde{b}) \), a contradiction.

**Lemma 2** There exists \( i, \tilde{b} \) such that:

(a) for all \( b < \tilde{b} \), \( \Psi_i(0, b) < 0 \) and \( \Psi_i(0, b) < 0 \) and

(b) \( \lim_{b \to \tilde{b}} \Psi_i(0, b) = 0 \).

**Proof:** The result follows from the continuity of \( \Psi_1(0, b) \) and \( \Psi_2(0, b) \) and from their monotonicity with respect to the terminal boundary value as shown in the previous lemma.

**Lemma 3** Let \((\phi_1, \phi_2)\) be a solution of (2) over the interval \([a, m] \), with initial boundary condition \( \phi_i(a, a) = 0 \). Then \( \lim_{b \to m} \phi_i(a, b) = m \) implies \( \lim_{b \to m} \phi_j(a, b) = m \) as well.

**Proof:** Assume w.l.o.g. \( \phi_1(a, m) = m \) and \( \phi_2(a, m) = r < m \). From (2') we can deduce that \( \lim_{b \to m} \frac{d}{db} \ln(1-F_1(y_1(b))) = \frac{1}{(y_1(m) - m)} \).

Since \( \frac{d}{db} \ln(1-F_1(y_1(b))) \) and \( \frac{d}{db} \ln(1-F_2(y_2(b))) \) are bounded as \( b \to m \), then \( \frac{1}{(y_1(b) - b)} \frac{1}{(y_2(b) - b)} \) is also bounded. This implies that \( \lim_{b \to m} \phi_2(a, b) = m \), that is \( \phi_2(a, m) = m \).
Lemma 4 Let \((\phi_1, \phi_2)\) be a solution of (2) over the interval \([a, m)\), with initial boundary condition \(\phi_i(a, a) = 0\). If \(a > 0\) then \(\phi_i(a, b) > \hat{\phi}_i(a, b)\) for all \(b\) belonging to the domain of definition of \(\phi\) and \(\hat{\phi}\).

Proof: Because \((\phi_1, \phi_2)\) are strictly increasing we have \(\phi_i(a, a) = \hat{\phi}_i(a, a) < \phi_i(a, a)\), for all \(i\). Thus at the initial condition the claim holds.
Assume now that there exists \(i\) and \(b \in (a, m)\) such that \(\hat{\phi}_i(a, b) > \phi_i(a, b)\). By conti­

nuity there exists \(b' \in (a, m)\) for which \(\hat{\phi}_i(a, b') = \phi_i(a, b')\) with \(\hat{\phi}_i(a, b') > \phi_i(a, b')\) and \(\hat{\phi}_j(a, b') < \phi_j(a, b')\) (remember that if \(\phi_j(a, b') = \phi_j(a, b')\) then the two solutions would be equal over their common definition support in contradiction with Lipschitz continuity).

From (2') it is clear that \(\frac{d}{db} \phi_i\) is increasing in \(\phi_i\). This fact, along with \(\hat{\phi}_j(a, b') < \phi_j(a, b')\), implies \(\hat{\phi}_i(a, b') \leq \phi_i(a, b')\), a contradiction.

Lemma 5 Let \((\phi_1, \phi_2)\) be a solution of (2) over the interval \([a, m)\), with \(m < 0\). Then, for all \(v \in [0, \phi^{-1}(m)]\) it is satisfied that \(\frac{d}{dv} \{(v - B_j(v)) \ln (1 - F_i(B_i^{-1}(B_j(v))))\} = 0.5 \ln (1 - F_i(B_i^{-1}(B_j(v))))\).

Proof: Since \((\phi_1, \phi_2), (B_1, B_2)\) and \((B_i^{-1}(B_2), B_i^{-1}(B_1))\) are differentiable we have

\[
\frac{d}{dv} \ln (1 - F_i(B_i^{-1}(B_j(v)))) = \left. \frac{d}{db} \ln (1 - F_i(b)) \right|_{b = B_j(v)} \cdot \frac{d}{dv} B_j(v).
\]

System (2') with \(b = B_j(v)\) can be rewritten as:

\[
2 \frac{d}{dv} \ln (1 - F_i(B_i^{-1}(B_j(v)))) = \frac{d}{dv} B_j(v) \cdot \frac{1}{v - B_j(v)}.
\]

Rearranging, and after some computations, we get

\[
\frac{d}{dv} \{(v - B_j(v)) \ln (1 - F_i(B_i^{-1}(B_j(v))))\} = 0.5 \ln (1 - F_i(B_i^{-1}(B_j(v))))\).
\]


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