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SOME DISCRETE APPROACHES TO CONTINUUM ECONOMIES

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Keywords: Continuum economy; Core; Coalitions; Preference Relations; Cantor Sets.

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Some discrete approaches to continuum economies

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Abstract. Given the preferences of the agents of a continuum economy, we define the average and unanimous preference. This allow us to consider several sequences of economies, in which only a finite number of different agents' characteristics can be distinguished. We obtain approximation results for the core of these economies.

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1 Introduction

In the classical n-agents model of Arrow-Debreu-McKenzie, it is stated the hypothesis that the agents behave as price-takers. However, mathematically, it is not possible to assume tha the influence of each agent is negligible in economies with a finite set of agents.

Later, Aumann (1964) proposed the study of economies with a continuum of agents (also called atomles, continuum or perfectly competitive economies). The reason for this last name is that, in these models, the influence of each agent, or of a set of measure zero, is null because the integral does not change if the behaviour of such a set of agents is modified. The mathematical elegance of this approach may not be immune to the criticism that, often enough, economic reality only allow us to distinguish a finite number of participants.

A first attempt to solve this kind of criticism is made by García and Hervés (1993). They define what we can call a continuum n-types economy, that is, a continuum economy which is observed by the market as an economy with n types of agents. They prove that this later economy can be interpreted as a classical n-agents economy, and vice versa.

In this paper, we consider a perfectly competitive economy, focusing on different discrete approaches that may be adopted, analyzing some implications on the veto mechanism. We postulate that the market (or the observer) only distinguishes a finite number of different characteristics, that is, endowments and preference relations. So, the agents included in a same group or type are seen as all the same. It seems reasonable to consider the average endowment for all the agents belonging to the same type, and so we do. Meanwhile, it is not clear what preference is perceived by the observer in a set of agents that he considers of the same type. For example, the observer can look at a set of agents, that he detects as the same one, assigning them a preference defined by the mean of the whole agent set. Or else, the observer can estimate that a consumption bundle is prefered to another one, by all the agents he assesses as equal, if it is unanimously prefered.

In order to formalize these ideas, to each continuun economy $\mathcal{E}_c$ we associate (for each n) a continuum economy $\mathcal{E}_c^n$, in which only a finite number, namely 2^n, of different agent characteristics can be distinguished. To each economy $\mathcal{E}_c^n$ we associate a discrete economy $\mathcal{E}_n$ with 2^n agents. In this way, we define different discrete approaches to continuum economies, by means of what we call average and unanimous preferences. We study some properties of all these preferences and we analyze relations between the initial continuum economy and its discrete approaches. That is, we obtain results about allocations belonging to the core of the continuum economies in terms of the corresponding allocations in the core of the associated discrete economies (or in the core of the continuum economies in which only a finite number of different agent characteristics can be distinguished by the observer or the market). Precisely, we obtain how the core of a continuum economy can be approximated, in some sense, by the sequence of the cores of the associated discrete economies.

The paper is organized as follows. Section 2 details the model. In section 3 we study a first discrete approach to continuum economies, introducing the average preference. Specifically, we define the average preference, analyzing some of its properties, we prove the main results and we present some examples. In section 4 we follow a similar set up for the case of the unanimous preference.

2 The model

Let us consider a pure exchange economy $\mathcal{E}_c = ((I, \mathcal{A}, \mu), \omega(t), \preceq_t, t \in I)$, having $\mathbb{R}^\ell$ as commodity space. $(I, \mathcal{A}, \mu)$ is an atomless positive, bounded measure space which represents the space of agents. For simplicity, we assume that I is the real interval $[0, 1]$, $\mathcal{A}$ is the Lebesgue σ-algebra of subsets of I, and μ is the Lebesgue measure. The consumption set of each agent $t \in I$ is $X_t = \mathbb{R}^\ell_+$, his initial endowment is $\omega(t) \in \mathbb{R}^\ell_+$, and his preference relation is $\preceq_t$ represented by a continuous utility function $U(t, \cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}$.

Following Aumann (1964), we suppose that the map $\omega : I \to \mathbb{R}^\ell_+$, that associates to each agent his initial endowment is integrable, and that the function $\preceq$, that associates to each agent $t \in I$ his preference relation $\preceq_t$, is measurable, in the sense that if $x, y : I \to \mathbb{R}^\ell$ are feasible allocations in the economy $\mathcal{E}_c$, then the set $\{t \in I | x(t) \succ_t y(t)\}$ is measurable.

As we have noticed in the introduction, often enough, economic reality only allows us to distinguish a finite number of different agent characteristics. So, our aim is to consider an economy with a continuum of agents, introducing several discrete approaches of this economy in order to analyze the implications that can be obtained from this simplification in relation to the veto mechanism.

In this way, we are interested in the core allocations of the initial economy $\mathcal{E}_c$, as limit core allocations of the discrete economies, depending on the discrete approach considered.

For this, for each positive integer n, we define the continuum economy $\mathcal{E}_c^n$ in which only a finite number of different agents can be distinguished. Specifically and for technical reasons, let us consider that the set of agents I is divided into 2^n pairwise disjoint subintervals, each of them representing a type of agent. That is, $I = \bigcup_{i=1}^{2^n} I_i^n$, where $I_i^n = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, if $i \neq 2^n$, $I_{2^n}^n = \left[\frac{2^n - 1}{2^n}, 1\right]$. Each consumer $t \in I$ is characterized in the economy $\mathcal{E}_c^n$ by his consumption set $\mathbb{R}^\ell_+$, his initial endowment $\omega^n(t) = \frac{1}{\mu(I_i^n)} \int_{I_i^n} \omega(t) d\mu(t)$, for all $t \in I_i^n$, and his preference relation $\succeq_t^n = \succeq_i^n$ for all $t \in I_i^n$. We will refer to I_i^n as the set of agents of type i in the economy $\mathcal{E}_c^n$.

Note that if f is a feasible allocation in $\mathcal{E}_c$, then f^n is a feasible allocation in $\mathcal{E}_c^n$, where $f^n(t) = \frac{1}{\mu(I_i^n)}\int_{I_i^n} f(t)d\mu(t)$, for all $t \in I_i^n$. Moreover, as $\int_I \omega(t)d\mu(t) = \int_I \omega^n(t)d\mu(t)$, one has that f is a feasible allocation in $\mathcal{E}_c$ iff f^n is a feasible allocation in $\mathcal{E}_c^n$, for all n.

Let us consider the discrete economy $\mathcal{E}_n$ associated with the continuum economy $\mathcal{E}_c^n$. That is, $\mathcal{E}_n$ is an economy with 2^n agents, where each agent $i \in \{1, \ldots, 2^n\}$ is characterized by $\omega_i^n = \omega^n(t)$, and $U_i^n = U^n(t, \cdot)$, with $t \in I_i^n$.

Observe that one allocation f in our economy $\mathcal{E}_c$, can be interpreted either as an allocation f^n in $\mathcal{E}_c^n$, or as an allocation $x^n = (x_1, \ldots, x_{2^n})$ in $\mathcal{E}_n$, where $x_i^n = \frac{1}{\mu(I_i^n)}\int_{I_i^n} f(t)d\mu(t)$, equivalently $x_i^n = f^n(t)$, con $t \in I_i^n$. Reciprocally, an allocation x in $\mathcal{E}_n$ can be interpreted as an allocation f in $\mathcal{E}_c^n$, where f is the step function defined by $f(t) = x_i$, if $t \in I_i^n$.

Remark. We have assumed that the real interval $[0, 1]$ is divided into 2^n subintervals of equal length. Obviously, this kind of partition is made for technical reasons. What it is important to note is that if the initial economy $\mathcal{E}_c$ has a finite number of different characteristics, then it may be transformed in order to garantee that, for n big enough, the sequence $\mathcal{E}_c^n$ of economies with 2^n types is constant and equal to $\mathcal{E}_c$.

3 Average Preference

3.1 Definition and some properties.

Let us consider that for each n the preference relation of each agent $t \in I$, $\succeq_t^n$, is represented by the utility function $U^n(t,x) = \frac{1}{\mu(I_i^n)}\int_{I_i^n} U(t,x)d\mu(t)$, whoever $t \in I_i^n$ may be. In this case, we will call $\succeq_t^n$ as average preference. All the agents of $\mathcal{E}_c^n$ belonging to I_i^n are observed as equal, with endowments and preferences given by the average of all the agents of I_i^n.

Next we show some properties of this average preference.

1. Observe that if $U(t, \cdot)$ is continuous in x for almost all $t \in I$, then by the dominated convergence theorem we can obtain that $U^n(t, \cdot)$ is also continuous in x, for all $t \in I$ and whatever n may be.

2. Observe that if $U(t, \cdot)$ is a monotone (resp. strictly monotone) function for almost all $t \in I$, $U^n(t, \cdot)$ is a monotone (resp. strictly monotone) function for all $t \in I$ and all n.

3. Note that if $U(t,\cdot)$ is a concave function for almost all $t \in I$, then $U^n(t,\cdot)$ is a concave function for all $t \in I$ and all n. However, the quasi-concavity of $U(t,\cdot)$ for almost all $t \in I$ does not imply the quasi-concavity of the average preferences $U^n(t,\cdot)$.

4. By other way, Lebesgue differentiation theorem allows us to conclude that $\omega^n(t)$ converges to $\omega(t)$ pointwisely, and given $x \in \mathbb{R}^\ell_+$ it is verified that $U^n(t,x)$ converges to $U(t,x)$, for almost all $t \in I$. Therefore, using Egoroff theorem, one has that ω^n (resp. $U^n(\cdot,x)$) converges to $\omega(t)$ (resp. to $U(\cdot,x)$) almost uniformly.

5. Observe that for each n we have that $\int_{I_i^n} \omega^n(t)d\mu(t) = \int_{I_i^n} \omega(t)d\mu(t)$, whatever $i \in \{1,\ldots,2^n\}$ may be.

6. Having into account the results in García and Hervés (1993), you can deduce that if the utility function $U(t,\cdot)$ is concave for almost all $t \in I$, then $((x_1,\ldots,x_{2^n}),p)$ is a walrasian equilibrium for the economy $\mathcal{E}_n$ with 2^n agents iff (f,p) is a walrasian equilibrium for the continuum economy $\mathcal{E}_c^n$. You can also conclude that $(x_1,\ldots,x_{2^n})$ is an Edgeworth equilibrium for the economy $\mathcal{E}_n$ iff f is a core allocation in the economy $\mathcal{E}_c^n$.

3.2 Main results

As we have pointed out, we are interested in studying the relationship between $\mathcal{E}_c$ and $\mathcal{E}_n$ related to the veto mechanism. Specifically, in this section we prove the following: given an allocation f in the economy $\mathcal{E}_c$, it is verified that if the corresponding allocation $x^n \in Core(\mathcal{E}_n)$ for all $n \geq n_0$, then $f \in Core(\mathcal{E}_c)$, where $x_i^n = \frac{1}{\mu(I_i^n)}\int_{I_i^n} f(t)d\mu(t)$. In order to obtain this result, we state some previous facts.

Lemma 3.1 *Let $g^n, g : I \to \mathbb{R}^\ell$ be integrable functions, such that $g^n(t)$ converges to $g(t)$ almost everywhere. Then, for each $\varepsilon > 0$ there exist $k(\varepsilon) > 0$, $n(\varepsilon)$, and $J_\varepsilon \subset I$, with $\mu(J_\varepsilon) < \varepsilon$, such that $\|g^n(t)\|, \|g(t)\| < k(\varepsilon)$, for all $t \notin J_\varepsilon$, and for all $n \geq n(\varepsilon)$.*

Proof. By Chebysev inequality, there exists k such that $\mu(\{t \in I | \|g(t)\| > k\}) < \frac{\varepsilon}{2}$. On the other hand, by Egoroff theorem there exists $J \subset I$, with $\mu(J) < \frac{\varepsilon}{2}$, such that g^n converges to g uniformly on $I \setminus J$. Now it is enough to take $J_\varepsilon = J \bigcup \{t \in I | \|g(t)\| > k\}$, and $k(\varepsilon) > k$.

Q.E.D.

Let us consider now a feasible allocation $f : I \to \mathbb{R}^\ell_+$ in the economy $\mathcal{E}_c$. For each n, let $f^n : I \to \mathbb{R}^\ell_+$ be the feasible allocation in the economy $\mathcal{E}_c^n$, given by

$f^n(t) = \frac{1}{\mu(I_i^n)}\int_{I_i^n} f(t)d\mu(t)$, for each agent $t \in I_i^n$.

Given $z \in \mathbb{Q}^\ell$ we define $\Gamma(z) = \{t \in I | U(t, z + \omega(t)) > U(t, f(t))\}$, and $\Gamma^n(z) = \{t \in I | U^n(t, z + \omega^n(t)) > U^n(t, f^n(t))\}$, for each $n \in \mathbb{N}$. Let $\mathcal{Z} = \{z \in \mathbb{Q}^\ell | \mu(\Gamma(z)) = 0\}$. So, $\mu(\bigcup_{z\in\mathcal{Z}} \Gamma(z)) = 0$. Finally, for each agent $t \in I$, let $\psi(t) = \{z \in \mathbb{R}^\ell | U(t, z + \omega(t)) > U(t, f(t))\}$.

Lemma 3.2 *Let S be a coalition of agents blocking the allocation f in the economy $\mathcal{E}_c$. Then, for each $i \in \{1, \ldots, \ell+1\}$, there exist $\alpha_i \in \mathbb{Q}_+$, $z_i \in \mathbb{Q}^\ell$, and $t_i \in \hat{S} = S \setminus \bigcup_{z\in Z} \Gamma(z)$, such that $\sum_{i=1}^{\ell+1} \alpha_i = 1$, $\sum_{i=1}^{\ell+1} \alpha_i z_i = 0$, and $z_i \in \psi(t_i)$.*

Proof. As S blocks f, there exists $g : S \to \mathbb{R}_+^\ell$, such that $\int_S g(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$ and $g(t) \succ_t f(t)$, for almost all $t \in S$. Because of $\mu(\bigcup_{z\in Z} \Gamma(z)) = 0$, we can deduce that $\hat{S}$ is a coalition which also blocks f by the same allocation g. On the other hand, it is verified that

$$\frac{1}{\mu(\hat{S})}\int_{\hat{S}} (g(t) - \omega(t))d\mu(t) \in co\left((g - \omega)(\hat{S})\right)$$

Therefore, $0 \in co\left(\bigcup_{t\in\hat{S}} \psi(t)\right)$. By Caratheodory's theorem, one obtains that there exist $\alpha_i \geq 0$, and $z_i \in \psi(t_i)$, with $t_i \in \hat{S}$, $i = 1, \ldots, \ell+1$, such that $0 = \sum_{i=1}^{\ell+1} \alpha_i z_i$.

Let us show that we can take $\alpha_i \in \mathbb{Q}_+$. In fact, for each $k \in \mathbb{N}$, let $\alpha_i^k = E[k\alpha_i + 1]$, where $E[t]$ denotes the entire part of the real number t. Let $z_i^k = \frac{k\alpha_i}{\alpha_i^k} z_i \in \mathbb{R}^\ell$. As $\lim_{k\to\infty} \frac{k\alpha_i}{\alpha_i^k} = 1$, we obtain that: $\lim_{k\to\infty} z_i^k = z_i$. Then, by the continuity of the preferences, there exists k_0 such that $z_i^k \in \psi(t_i)$, for all $i \in \{1, \ldots, \ell+1\}$, and for all $k \geq k_0$. Moreover, it is verified that $\sum_{i=1}^{\ell+1} \alpha_i^k z_i^k = \sum_{i=1}^{\ell+1} k\alpha_i z_i = 0$, where α_i^k are integer numbers, for all $i \in \{1, \ldots, \ell+1\}$.

Finally, let us show that we can take $z_i \in \mathbb{Q}^\ell$. For each i, let (z_i^n) be a sequence which converges to z_i, with $z_i^n \in \mathbb{Q}^\ell$, and $z_i^n \geq z_i$. By the continuty of the preferences, there exists n_0 such that $z_i^n \in \psi(t_i)$, for all $i \in \{1, \ldots, \ell+1\}$, and for all $n \geq n_0$. By construction, we have that $\sum_{i=1}^{\ell+1} \alpha_i z_i^n = -r^n$, with $r^n \geq 0$. The monotonicity of the preferences implies that $z_i^n + r^n \in \psi(t_i)$, for all $i \in \{1, \ldots, \ell\}$, and all $n \geq n_0$.

Q.E.D.

Let us denote by $\mathcal{C}(\mathbb{R}_+^\ell)$ the set of all real continuous functions defined on $\mathbb{R}_+^\ell$. We consider on $\mathcal{C}(\mathbb{R}_+^\ell)$ the compact-open topology. This is a metric space. In fact, η is a metric of this space, being η defined by

$$\eta(f,g) = \sum_{n=1}^{\infty} 2^{-n}\eta_n(f,g), \quad \text{where } \eta_n(f,g) = \sup_{\|x\|\leq n} \frac{| f(x) - g(x) |}{1+ | f(x) - g(x) |}$$

Let $U : I \to \mathcal{C}(\mathbb{R}_+^\ell)$ be the function that associates to each agent $t \in I$ his utility function $U(t, \cdot)$. Unless we state the contrary, in the rest of the paper we assume that U is continuous.

Lemma 3.3 *For almost all $t \in I$, it is verified that $U^n(t,\cdot)$ converges to $U(t,\cdot)$ uniformly on compact subsets of $\mathbb{R}_+^\ell$.*

Proof. Given $x \in \mathbb{R}_+^\ell$, by Lebesgue differentiation theorem, it is verified that $U^n(t,x)$ converges to $U(t,x)$ for almost all $t \in I$. In particular, for each $x \in \mathbb{Q}_+^\ell$ there exists $J(x) \subset I$, with $\mu(J(x)) = 1$, such that $U^n(t,x)$ converges to $U(t,x)$ for all $t \in J(x)$. Let $J = \bigcap_{x\in\mathbb{Q}_+^\ell} J(x)$. Then $\mu(J) = 1$ and $U^n(t,x)$ converges to $U(t,x)$ for all $t \in J$, whatever $x \in \mathbb{Q}_+^\ell$ may be.

Let us see that $U^n(t,x)$ converges to $U(t,x)$ for all $t \in J$, for any $x \in \mathbb{R}_+^\ell$. For this, given $x \in \mathbb{R}_+^\ell$ let $(x^k) \subset \mathbb{Q}_+^\ell$ be a sequence such that $x^k \to x$. By the continuity of the functions $U^n(t,\cdot)$ we can deduce that $\lim_{k\to\infty} U^n(t,x^k) = U^n(t,x)$ for all n and for all $t \in I$. It is also verified that $\lim_{n\to\infty} U^n(t,x^k) = U(t,x^k)$ for all k and for all $t \in J$. Moreover, this convergence is uniform on k. In fact, let $K = \{U(\cdot,x_k), k \in \mathbb{N}\}$. As U is continuous, K is a equicontinuous set. Therefore, by Ascoli-Arzela's theorem, K is a relatively compact subset of $\mathcal{C}(I)$. As the inclusion of $\mathcal{C}(I)$ in $L^\infty(I)$ is continuous, K is a relatively compact subset of $L^\infty(I)$. This implies (see Dunford-Schwartz, IV.8.18) that $U^n(t,x^k)$ converges to $U(t,x^k)$, uniformly on k . Applying Moore's lemma (see Dunford-Schwartz, I.7.6), $\lim_{n\to\infty} \lim_{k\to\infty} U^n(t,x^k) = \lim_{k\to\infty} \lim_{n\to\infty} U^n(t,x^k) = \lim_{k\to\infty} U(t,x^k) = U(t,x)$. So, $\lim_{n\to\infty} U^n(t,x) = \lim_{n\to\infty} \lim_{k\to\infty} U^n(t,x^k) = U(t,x)$. Therefore $U^n(t,x)$ converges to $U(t,x)$ for almost all $t \in I$ and all $x \in \mathbb{R}_+^\ell$.

Finally, let $x \in \mathbb{R}_+^\ell$ and $(x^m) \subset \mathbb{R}_+^\ell$, such that $x^m \to x$. Reasoning as before, we obtain that $\lim_{(n,m)} U^n(t,x^m) = U(t,x)$, for all $t \in J$. In particular, if $n = m$, it is verified that $\lim_{n\to\infty} U^n(t,x^n) = U(t,x)$, for all $t \in J$. That is, $U^n(t,\cdot)$ converges continuously to $U(t,\cdot)$ for each $x \in \mathbb{R}_+^\ell$. Equivalently, $U^n(t,\cdot)$ converges to $U(t,\cdot)$ uniformly on compact subsets of $\mathbb{R}_+^\ell$. (See Royden, problem 9.40).

Q.E.D.

Lemma 3.4 *Let K be a compact subset of $\mathbb{R}_+^\ell$. It is verified that $U^n(\cdot,\cdot)$ converges to $U(\cdot,\cdot)$ almost uniformly on I and uniformly on K. That is, for each*

$\varepsilon > 0$ *there exists* $J_\varepsilon \subset I$, *with* $\mu(J_\varepsilon) > 1 - \varepsilon$, *such that* $U^n(\cdot,\cdot)$ *converges to* $U(\cdot,\cdot)$ *uniformly on* $J_\varepsilon \times K$..

Proof. By lemma 3.3, there exists $J \subset I$, with $\mu(J) = 1$, such that for all $t \in J$ it is verified that $U^n(t,x)$ converges to $U(t,x)$ for all $x \in \mathbb{R}^\ell_+$, and this convergence is uniform on K. Given k, m positive integers, we define

$$J_{k,m} = \left\{ t \in J \;\middle|\; | U^n(t,x) - U(t,x) | < \frac{1}{m}, \text{ for all } n \geq k,\ x \in K \right\}$$

Then $J_{k,m} \subset J_{k+1,m}$ for all m and k. Moreover, by the uniform convergence on K for all $t \in J$, we have that $J = \bigcup_{k=1}^\infty J_{k,m}$ for all m. So, for each $\varepsilon > 0$ and for each m, there exists $k(m)$ such that $\mu(J \setminus J_{k(m),m}) < \varepsilon 2^{-m}$. Let $J_\varepsilon = \bigcap_{m=1}^\infty J_{k(m),m}$. Then, $\mu(J \setminus J_\varepsilon) < \varepsilon$ and besides $| U^n(t,x) - U(t,x) | < m^{-1}$, for all $n \geq k(m)$ and for all $t \in J_\varepsilon$. Therefore, $U^n(\cdot,\cdot)$ converges to $U(\cdot,\cdot)$ uniformly on $J_\varepsilon \times K$.

Q.E.D.

Next we state one of the main results in this section.

Theorem 3.1 *Let f be a feasible allocation in the economy $\mathcal{E}_c$. For each $n \in \mathbb{N}$, let us consider the discrete economy $\mathcal{E}_n$ and the allocation x^n, defined by* $x_i^n = \dfrac{1}{\mu(I_i^n)} \displaystyle\int_{I_i^n} f(t) d\mu(t)$. *If $x^n \in Core(\mathcal{E}_n)$ for all $n \geq n_0$, then $f \in Core(\mathcal{E}_c)$.*

Proof. Let us suppose that $f \notin Core(\mathcal{E}_c)$. Then, there exists a coalition S blocking f. By lemma 3.2, there exist $\alpha_i \in \mathbb{Q}_+$ and $z_i \in \psi(t_i)$, $i = 1, \ldots, \ell+1$, with $t_i \in \hat{S} = S \setminus \bigcup_{z \in Z} \Gamma(z)$ and $\sum_{i=1}^{\ell+1} \alpha_i = 1$, such that $0 = \sum_{i=1}^{\ell+1} \alpha_i z_i$.

By the definition of $\hat{S}$, as $z_i \in \psi(t_i)$, we have that $\mu(\Gamma(z_i)) > 0$. So, there exists $\alpha > 0$, such that $\mu(\Gamma(z_i)) \geq \alpha$ for all $i \in \{1, \ldots, \ell+1\}$. It is also verified that $z_i \in \psi(t)$ for all $t \in \Gamma(z_i)$, that is, $U(t, z_i + \omega(t)) - U(t, f(t)) > 0$ for all $t \in \Gamma(z_i)$. Therefore, there exists $B_i \subset \Gamma(z_i)$, with $\mu(B_i) < \frac{\alpha}{8}$, and there exists $\delta > 0$, such that $U(t, z_i + \omega(t)) - U(t, f(t)) \geq \delta$ for all $t \in \Gamma(z_i) \setminus B_i$.

By lemma 3.1, there exist $A \subset I$, n_0 and a compact $K \subset \mathbb{R}^\ell$, such that $\mu(A) < \frac{\alpha}{8}$, $f(t), f^n(t), z_i + \omega^n(t) \in K$, for all $t \in A$ and for all $n \geq n_0$. Let us recall that $f^n(t) = x_i^n$, if $t \in I_i^n$. By lemma 3.4, there exists a set of agents $B \subset I$, with $\mu(B) < \frac{\alpha}{8}$, such that $f^n(t) \to f(t)$, $\omega^n(t) \to \omega(t)$ and $U^n(t,x) \to U(t,x)$ uniformly, for all $t \notin B$ and for all $x \in K$.

For each $\varepsilon > 0$, let us consider the map $\varphi(\cdot, \varepsilon) : I \to \mathbb{R}_+$, given by

$$\varphi(t,\varepsilon) = \sup_{\substack{x \in K \\ \|y\| \leq \varepsilon}} | U(t, x+y) - U(t,x) |$$

Note that $\varphi(t,\varepsilon) > 0$ for all $\varepsilon > 0$ and $\varphi(t,\varepsilon)$ converges to 0 whenever $\varepsilon \to 0$. Besides, for each ε, the map $\varphi(\cdot,\varepsilon)$ is measurable. Let $I_\varepsilon = \{t \in I | \varphi(t,\varepsilon) < \frac{\delta}{4}\}$.

So, if $\varepsilon < \varepsilon'$ then $I_{\varepsilon'} \subset I_\varepsilon$. As $I = \bigcup_\varepsilon I_\varepsilon$, there exists ε_0 such that $\mu(I \setminus I_{\varepsilon_0}) < \frac{\alpha}{8}$ and $\varphi(t, \varepsilon_0) < \frac{\delta}{4}$ for all $t \in I_{\varepsilon_0}$.

Let $\Gamma'(z_i) = (\Gamma(z_i) \bigcap I_{\varepsilon_0}) \setminus (A \bigcup B \bigcup B_i)$. Then $\mu(\Gamma'(z_i)) > \frac{\alpha}{2}$. Moreover, f^n (resp. ω^n) converges to f (resp. to ω) uniformly on $\Gamma'(z_i)$. So, there exists n_1 such that $\|f^n(t) - f(t)\|, \|\omega^n(t) - \omega(t)\| < \varepsilon_0$ for all $n \geq n_1$ and $t \in \Gamma'(z_i)$. We obtain that if $n \geq \max\{n_0, n_1\}$ then $U(t, z_i + \omega^n(t)) - U(t, f^n(t)) > \frac{\delta}{2}$ for all $t \in \Gamma'(z_i)$. By the uniform convergence of $U^n(t,x)$ with $t \in \Gamma'(z_i)$ and $x \in K$, there exists n_2 such that $| U^n(t,x) - U(t,x) | < \frac{\delta}{4}$, for all $x \in K$, $t \in \Gamma'(z_i)$ and $n \geq n_2$.

Let $\bar{n} = \max\{n_0, n_1, n_2\}$. If $n \geq \bar{n}$, it is satisfied that $U^n(t, z_i + \omega^n(t)) - U^n(t, f^n(t)) > 0$, for all $t \in \Gamma'(z_i)$. This implies that there exists $\hat{n}$ such that $\Gamma'(z_i) \subset \Gamma^n(z_i)$, for all $i \in \{1, \ldots, \ell+1\}$ and $n \geq \hat{n}$. Moreover, $\mu(\Gamma^n(z_i)) > \frac{\alpha}{2}$.

On the other hand, for each i we can write $\alpha_i = \frac{\beta_i}{\beta}$, with $\beta_i, \beta \in \mathbb{N}$ and $\beta_i \leq \beta$. By the definition of Γ^n, it is verified that for each n and i there exists a type subset $T_i^n \subset \{1, \ldots, 2^n\}$, such that $\Gamma^n(z_i) = \bigcup_{j \in T_i^n} I_j^n$. As $\mu(I_i^n)$ converges to zero when n goes to ∞, but simultaneously $\mu(\Gamma^n(z_i)) > \frac{\alpha}{2} > 0$, we obtain that there exists n^* such that $Card(T_i^n) > \beta$, for all i and $n \geq n^*$. Let us consider $J_i^n \subset T_i^n$ with $Card(J_i^n) = \beta_i$. Given $n \geq n^*$, let y^n be an allocation that associates to each agent $j \in J_i^n$ the consumption vector $y_j^n = z_i + \omega^n(t_{j,i})$, with $t_{j,i} \in I_j^n \subset \Gamma^n(z_i)$, $1 \leq j \leq \beta_i$. Let us show that the coalition $J^n = \bigcup_{i=1}^{\ell+1} J_i^n$ can obtain the allocation y^n. Precisely,

$$\begin{aligned}
\sum_{i=1}^{\ell+1} \sum_{j=1}^{\beta_i} (z_i + \omega^n(t_{j,i})) &= \sum_{i=1}^{\ell+1} \beta_i z_i + \sum_{i=1}^{\ell+1} \sum_{j=1}^{\beta_i} \omega^n(t_{j,i}) \\
&= \beta \sum_{i=1}^{\ell+1} \alpha_i z_i + \sum_{i=1}^{\ell+1} \sum_{j=1}^{\beta_i} \omega^n(t_{j,i}) \\
&= \sum_{i=1}^{\ell+1} \sum_{j=1}^{\beta_i} \omega^n(t_{j,i})
\end{aligned}$$

Therefore, we conclude that for $n \geq n^*$ the coalition J^n blocks the allocation x^n in the economy $\mathcal{E}_n$.

Q.E.D.

Let $\hat{\mathcal{S}} \subset \mathcal{A}$. As in Hervés and Moreno (1996), we say that an allocation belongs to the $\hat{\mathcal{S}}$-Core of the economy $\mathcal{E}_c$ if it is not blocked by any coalition $S \in \hat{\mathcal{S}}$. Let $\mathcal{S}^n$ denote the σ-algebra that is generated by the subintervals I_i^n, $i = 1, \ldots, 2^n$. Let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}^n$.

Corollary 3.1 *Let f be a feasible allocation in the economy $\mathcal{E}_c$. Given $n \in \mathbb{N}$, we consider in the 2^n types continuum economy $\mathcal{E}_c^n$ the allocation f^n, defined by $f^n(t) = x_i^n$ if $t \in I_i^n$. If $f^n \in \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$ for all $n \geq n_0$, then $f \in Core(\mathcal{E}_c)$. In particular, if $f^n \in Core(\mathcal{E}_c^n)$ for all $n \geq n_0$, then $f \in Core(\mathcal{E}_c)$.*

Proof. It is enough to notice that $x^n \in Core(\mathcal{E}_n)$ iff $f^n \in S^n\text{-}Core(\mathcal{E}_c^n)$, and that $Core(\mathcal{E}_c^n) \subset S^n\text{-}Core(\mathcal{E}_c^n)$.

Q.E.D.

Remark. Note that all the previous results remain true if U is piecewise continuous. We have assume U to be continuous only for simplicity.

We shall show in subsection 3.3 that the converse of theorem 3.1 and corollary 3.1 do not hold. In spite of this, we can obtain some weaker converse results. This is now our aim.

Following Kannai (1970), we denote by $\mathcal{P}$ the set of all preferences which are derived from complete, reflexive, transitive, strictly monotone and continuous preorders, defined on $\mathbb{R}_+^\ell$. Given $\succ \in \mathcal{P}$, it can be represented by a continuous utility function as follows. For each $x \in \mathbb{R}_+^\ell$ there exists an unique vector y on the principal diagonal of $\mathbb{R}_+^\ell$, such that $x \sim y$. Let $U(x) = \|y\|$, where $\|\cdot\|$ is the euclidean norm. U is continuous and it is characterized by $U(x) = \|x\|$ on the principal diagonal. Moreover, the constant $k = \sqrt{\ell}$ satisfies $0 \leq U(x) \leq k\|x\|$, for all $x \in \mathbb{R}_+^\ell$.

Let us denote by $\mathcal{U}$ the set of all the utility functions obtained as above. It is verified that the minimal topology on $\mathcal{P}$ which makes the set $\{(x, y, \succ) | x \succ y\}$ open in the product space $\mathbb{R}_+^\ell \times \mathbb{R}_+^\ell \times \mathcal{P}$, is induced by the metric on $\mathcal{P}$

$$\rho(\succ_1, \succ_2) = \max_{x \in \mathbf{R}_+^\ell} \frac{| U_1(x) - U_2(x) |}{1 + \|x\|^2}$$

being U_i belonging to $\mathcal{U}$ and representing $\succ_i$. This topology has a countable basis. Moreover, the map $\succ : I \to \mathcal{P}$ is measurable iff it is measurable in the Aumann sense, which we have assumed in section 2.

Theorem 3.2 *Let $(\succeq_n) \subset \mathcal{P}$ be a sequence of preferences and let $\succeq \in \mathcal{P}$. Then, $\rho(\succeq_n, \succeq)$ converges to zero iff U_n converges to U uniformly on compact subsets of $\mathbb{R}_+^\ell$, being U_n, U the utility functions belonging to $\mathcal{U}$ which represent $\succeq_n$ and $\succeq$, respectively.*

Proof. First, we prove the necessary condition. Let us suppose that $\rho(\succeq_n, \succeq)$ converges to zero. Let $\varepsilon > 0$ and K a compact subset of $\mathbb{R}_+^\ell$. Then, there exists r such that $K \subset B(0, r) = \{x \in \mathbb{R}_+^\ell ; \|x\| < r\}$. We take $\hat{\varepsilon} = \dfrac{\varepsilon}{1 + r^2}$. As $\succeq_n \xrightarrow{\rho} \succeq$, there exists n_0 such that $\rho(\succeq_n, \succeq) \leq \hat{\varepsilon}$, for all $n \geq n_0$.

On the other hand, it is verified that

$$\begin{aligned} \max_{x \in K} | U_n(x) - U(x) | &\leq \max_{x \in K} \frac{| U_n(x) - U(x) |}{1 + \|x\|^2} \max_{x \in K} (1 + \|x\|^2) \\ &\leq \max_{x \in \mathbf{R}_+^\ell} \frac{| U_n(x) - U(x) |}{1 + \|x\|^2} (1 + r^2) \end{aligned}$$

Therefore, $\max_{x\in K} \mid U_n(x) - U(x) \mid \leq \varepsilon$, for all $n \geq n_0$. This allows us to conclude that U_n converges to U uniformly on compact subsets of $\mathbb{R}^\ell_+$.

We prove now the sufficient condition. Let us suppose that U_n converges to U uniformly on compact subsets of $\mathbb{R}^\ell_+$. As $U_n \in \mathcal{U}$ for all n, and $U \in \mathcal{U}$, there exists a constant k, which only depends on ℓ, such that $U_n(x) \leq k\|x\|$ and $U(x) \leq k\|x\|$, for all n and $x \in \mathbb{R}^\ell_+$. Let $\varepsilon > 0$ and $r > 1$ such that $\dfrac{2kr}{1+r^2} < \varepsilon$. As U_n converges to U uniformly on $K = \{x \in \mathbb{R}^\ell_+; \|x\| \leq r\}$, there exists n_0 such that, $\max_{x\in K} \mid U_n(x) - U(x) \mid \leq \varepsilon$, for all $n \geq n_0$.

By the definition of K, we obtain that $\max_{x\in K} \dfrac{\mid U_n(x) - U(x) \mid}{1+\|x\|^2} \leq \varepsilon$, for all $n \geq n_0$. And, if $x \notin K$, as $\|x\| > r > 1$, it is verified that

$$\frac{\mid U_n(x) - U(x) \mid}{1+\|x\|^2} \leq \frac{\mid U_n(x) \mid + \mid U(x) \mid}{1+\|x\|^2} \leq \frac{2k\|x\|}{1+\|x\|^2} \leq \frac{2kr}{1+r^2}$$

This allows us to conclude that $\max_{x\in \mathbb{R}^\ell_+} \dfrac{\mid U_n(x) - U(x) \mid}{1+\|x\|^2} \leq \varepsilon$, for all $n \geq n_0$. Therefore, $\rho(\succeq_n, \succeq)$ converges to zero.

Q.E.D.

Remark. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing and continuous function. Then, in the theorem above, U_n and U can be replaced by $\phi \circ U_n$ and $\phi \circ U$, respectively. Even more, if $\phi_n : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a sequence of increasing and continuous functions, such that ϕ_n converges to ϕ uniformly on compact subsets, then U_n and U can also be replaced by $\phi_n \circ U_n$ and $\phi \circ U$, respectively.

Corollary 3.2 *For each agent $t \in I$, let $\succeq_t$ (resp. $\succeq^n_t$) be his preference relation in the economy $\mathcal{E}_c$ (resp. $\mathcal{E}^n_c$) , represented by the utility function $U(t,\cdot)$ (resp. $U^n(t,\cdot)$). Let us suppose that $U(t,\cdot) \in \mathcal{U}$ for almost all $t \in I$.*

Then $\rho(\succeq^n_t, \succeq_t)$ converges to zero, for almost all $t \in I$.

Proof. As $U(t,\cdot) \in \mathcal{U}$ for almost all $t \in I$, we have that $U(t,x) = \|x\|$, for all x in the principal diagonal of $\mathbb{R}^\ell_+$, for almost all $t \in I$. So, it is verified that $U^n(t,\cdot) \in \mathcal{U}$, for all $t \in I$ and n, because $U^n(t,x) = \|x\|$, for all x in the principal diagonal of $\mathbb{R}^\ell_+$.

On the other hand, by lemma 3.3, $U^n(t,\cdot)$ converges to $U(t,\cdot)$ uniformly on compact subsets of $\mathbb{R}^\ell_+$, for almost all $t \in I$. Applying theorem 3.2, we conclude that $\rho(\succeq^n_t, \succeq_t)$ converges to zero, for almost all $t \in I$.

Q.E.D.

It is important to notice that the hypothesis of the strict monotony of the preferences can not be omited, as we will see in subsection 3.3.

On the other hand, this result allows us to obtain a weaker converse version of corollary 3.1 in terms of the ε-Core concept stated in Kannai (1970). For this, given $a, b \in \mathbb{R}^\ell$ let $a \ominus b$ be the vector in $\mathbb{R}^\ell_+$ whose k-th coordinate is $\max\{a_k - b_k, 0\}$. Given $\varepsilon > 0$, it is said that an allocation f belongs to the ε-Core of the economy $\mathcal{E}_c$, and we denote $f \in \varepsilon\text{-}Core(\mathcal{E}_c)$, if $g(t) \succ_t f(t)$ for almost all $t \in S$ implies that the inequality $\int_S g(t)d\mu(t) < \int_S \omega(t)d\mu(t) \ominus \varepsilon$ does not hold.

Corollary 3.3 *Let $U(t,\cdot) \in \mathcal{U}$ for almost all $t \in I$. Let $f : I \to \mathbb{R}^\ell_+$ be a feasible allocation in the economy $\mathcal{E}_c$. For each n let $f^n : I \to \mathbb{R}^\ell_+$ be defined by $f^n(t) = \frac{1}{\mu(I_i^n)}\int_{I_i^n} f(t)d\mu(t)$, for each agent $t \in I_i^n$.*

If $f \in Core(\mathcal{E}_c)$, then for each $\varepsilon > 0$ there exists n_0, such that for all $n \geq n_0$ it is verified that f^n belongs to the ε-Core of $\mathcal{E}_c^n$. So, $f^n \in \varepsilon - S^n\text{-}Core(\mathcal{E}_c^n)$, for all $n \geq n_0$.

Proof. We recall that f^n is a feasible allocation in $\mathcal{E}_c^n$ and $f^n(t)$ converges to $f(t)$ for almost all $t \in I$. On the other hand, $\{\omega, \omega^n : n \in \mathbb{N}\}$ is a weakly sequentially compact subset of $L^1(I)$, because ω^n converges weakly to ω. So, corollary 3.2 and theorem D in Kannai (1970) prove our statement.

Q.E.D.

Remarks. Note that corollary 3.2 and other results in Kannai (1970) allow us to conclude, under the hypothesis stated in corollary 3.3, the following propositions:

- Let $f \in \hat{\varepsilon}\text{-}Core(\mathcal{E}_c)$. Then for all $\varepsilon > 0$ there exists n_0, such that for all $n \geq n_0$ it is verified that $f^n \in (\varepsilon + \hat{\varepsilon})\text{-}Core(\mathcal{E}_c^n)$.

  Therefore, $f^n \in (\varepsilon + \hat{\varepsilon}) - S^n\text{-}Core(\mathcal{E}_c^n)$, for all $n \geq n_0$.

- If $f^n \in \hat{\varepsilon}\text{-}Core(\mathcal{E}_c^n)$ for all n, then for all $\varepsilon > 0$ it is verified that

$$f \in (\varepsilon + \hat{\varepsilon}) - Core(\mathcal{E}_c)$$

Observe that these results are weaker versions of the reciprocals of corollary 3.3 and theorem 3.1, respectively.

3.3 Some counterexamples

Example 3.1. Our first example shows that the converse results of theorem 3.1 or else corollary 3.1 are not true. That is to say, the fact that f belongs to $Core(\mathcal{E}_c)$ does not garantee that f^n belongs to $S^n\text{-}Core(\mathcal{E}_c^n)$ for n large enough. To prove our point, let α and β satisfy $0 < \alpha < \beta < 1$, and such that for infinitely many n, that is, for a sequence $n = n_k$, and for all $i = i(n), j = j(n) \in \{1, \ldots, 2^n\}$, with $i \neq j$, such that $\alpha \in I_i^n$, and $\beta \in I_j^n$, it is verified that $\mu(\{t \in I_i^n | t < \alpha\}) > \mu(\{t \in I_i^n | t > \alpha\})$, and $\mu\left(\left\{t \in I_j^n | t < \beta\right\}\right) > \mu\left(\left\{t \in I_j^n | t > \beta\right\}\right)$. By Cantor's nested intervals theorem we can take α and β as above.

Let us consider the economy $\mathcal{E}_c$ with the commodity space $\mathbb{R}^2$. Each agent $t \in [0,1]$ is characterized by his initial endowment $\omega(t)$ and his preference relation represented by the utility function $U(t,(x,y))$, defined as follows

$$\omega(t) = \begin{cases} (1,0) & \text{if } t < \alpha \text{ or } t > \beta \\ (0,1) & \text{if } \alpha < t < \beta \end{cases}$$

$$U(t,(x,y)) = \begin{cases} 2x + y & \text{if } t < \alpha \text{ or } t > \beta \\ x + 2y & \text{if } \alpha < t < \beta \end{cases}$$

It is easy to prove that the allocation f, given by $f(t) = \omega(t)$, belongs to $Core(\mathcal{E}_c)$. However, the subsequence (n_k) above, verifies that $f^{n_k} \notin \mathcal{S}^{n_k}$-$Core(\mathcal{E}_c^{n_k})$ whatever n_k may be. In fact, for each n_k the coalition $S_{n_k} = I_i^{n_k} \bigcup I_j^{n_k}$ blocks the allocation f^{n_k} via g_{n_k} in the economy $\mathcal{E}_c^{n_k}$, being

$$g_{n_k}(t) = \begin{cases} f^{n_k}(t) + (\varepsilon_{n_k}, -\varepsilon_{n_k}) & \text{if } t \in I_i^{n_k} \\ f^{n_k}(t) + (-\varepsilon_{n_k}, \varepsilon_{n_k}) & \text{if } t \in I_j^{n_k} \end{cases}$$

with $\varepsilon_{n_k} > 0$, ε_{n_k} little enough.

Example 3.2. Note that, in example 3.1, we can get that $f^n \notin \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$ for all even number n, or $f^n \notin \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$ for all odd number n, but not both of them simultaneously. Now, following a similar idea, we define an economy $\mathcal{E}_c$, such that there exists $f \in Core(\mathcal{E}_c)$, but $f^n \notin \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$ for all n.

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$, with $0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2$, such that:

For all odd n and for each $i = i(n), j = j(n) \in \{1, \ldots, 2^n\}$, with $i \neq j$, such that $\alpha_1 \in I_i^n$, and $\alpha_2 \in I_j^n$, it is verified that $\mu(\{t \in I_i^n | t < \alpha_1\}) > \mu(\{t \in I_i^n | t > \alpha_1\})$, and $\mu\left(\left\{t \in I_j^n | t < \alpha_2\right\}\right) < \mu\left(\left\{t \in I_j^n | t > \alpha_2\right\}\right)$.

For all even n and for each $h = h(n), k = k(n) \in \{1, \ldots, 2^n\}$, with $h \neq k$, such that $\beta_1 \in I_h^n$, and $\beta_2 \in I_k^n$, it is verified that $\mu(\{t \in I_h^n | t < \beta_1\}) < \mu(\{t \in I_h^n | t > \beta_1\})$, and $\mu(\{t \in I_k^n | t < \beta_2\}) > \mu(\{t \in I_k^n | t > \beta_2\})$.

Let the economy $\mathcal{E}_c$ with $\mathbb{R}^2$ as commodity space. Each agent $t \in [0,1]$ is characterized by his initial endowment $\omega(t)$ and his utility function $U(t,(x,y))$, given by the following formulae

$$\omega(t) = \begin{cases} (1,0) & \text{if } t \in [0,\alpha_1) \bigcup [\beta_1,\alpha_2) \bigcup [\beta_2,1] \\ (0,1) & \text{if } t \in [\alpha_1,\beta_1) \bigcup [\alpha_2,\beta_2) \end{cases}$$

$$U(t,(x,y)) = \begin{cases} 2x+y & \text{if } t \in [0,\alpha_1) \bigcup [\beta_1,\alpha_2) \bigcup [\beta_2,1] \\ x+2y & \text{if } t \in [\alpha_1,\beta_1) \bigcup [\alpha_2,\beta_2) \end{cases}$$

In the economy $\mathcal{E}_c^{2n+1}$, while all the agents of type i_{2n+1} prefer the first commodity, all the agents of type j_{2n+1} prefer the second one. On the other hand, in the economy $\mathcal{E}_c^{2n}$ while all the agents of type h_{2n} prefer the first commodity, all the agents of type k_{2n} prefer the second one.

It is not hard to show that the allocation f, given by $f(t) = \omega(t)$, belongs to $Core(\mathcal{E}_c)$. However, $f^n \notin S^n\text{-}Core(\mathcal{E}_c^n)$ for all n. In fact, for each n the coalition $S_{2n+1} = I_i^{2n+1} \bigcup I_j^{2n+1}$ blocks the allocation f^{2n+1} via g_{2n+1} in the economy $\mathcal{E}_c^{2n+1}$, and the la coalition $S_{2n} = I_h^{2n} \bigcup I_k^{2n}$ blocks f^{2n} via g_{2n} in the economy $\mathcal{E}_c^{2n+1}$, where g_n is defined as follows

$$g_{2n+1}(t) = \begin{cases} f^{2n+1}(t) + (\varepsilon_{2n+1}, -\varepsilon_{2n+1}) & \text{if } t \in I_{i_{2n+1}}^{2n+1} \\ f^{2n+1}(t) + (-\varepsilon_{2n+1}, \varepsilon_{2n+1}) & \text{if } t \in I_{j_{2n+1}}^{2n+1} \end{cases}$$

$$g_{2n}(t) = \begin{cases} f^{2n}(t) + (\varepsilon_{2n}, -\varepsilon_{2n}) & \text{if } t \in I_{h_{2n}}^{2n} \\ f^{2n}(t) + (-\varepsilon_{2n}, \varepsilon_{2n}) & \text{if } t \in I_{k_{2n+1}}^{2n} \end{cases}$$

being ε_n any positive real number verifying $g_n(t) > 0$ for all n.

Observe that, in both examples above, U and ω are piecewise continuous functions. However, it is not difficult to show that U and ω can be taken continuous functions, leading us to the same claim.

Example 3.3. The last example shows that, in general, the strict monotony assumed on preferences $\succeq_t$ can not be deleted. In particular, our next example proves that for all agent t in a positive measure subset of I, the sequence of average preferences $(\succeq_t^n)$ converges, in the Kannai sense of the metric ρ, to a preference $\succeq_t' \neq \succeq_t$.

For this, let us consider an economy $\mathcal{E}_c$ with a single commodity and let K be a Cantor subset of I, with $\mu(K) > 0$. For each agent $t \in I$, let $U(t,\cdot)$ be his utility function, given by

$$U(t,x) = \begin{cases} x & \text{if } x \leq 1 \\ 1 & \text{if } x \geq 1 \text{ and } t \in K \\ 1+\alpha_t(x-1) & \text{if } x \geq 1 \text{ and } t \in I \setminus K \end{cases}$$

being $\alpha_t = \min\{t - a_k, b_k - t\}$ if $t \in J_k = (a_k, b_k)$, where $I \setminus K = \bigcup_{k=1}^{\infty} J_k$.

It can be verified the following claims:

Claim 1. If we consider on $\mathcal{C}(\mathbb{R}_+)$ the compact-open topology, then the function $U : I \longrightarrow \mathcal{C}(\mathbb{R}_+)$, which associates to each agent $t \in I$ his utility function $U(t, \cdot)$, is a continuous function.

Claim 2. As $I \setminus K$ is an open and dense subset of I, then $\mu\left(I_i^n \bigcap (I \setminus K)\right) > 0$, for all n and i. On the other hand, $U(t, \cdot)$ is a non decreasing function for all $t \in I$, and an strictly increasing function for all $t \in I \setminus K$. Now, we recall that if $t_0 \in I_i^n$, then $U^n(t_0, x) = \frac{1}{\mu(I_i^n)} \int_{I_i^n} U(t, x) d\mu(t)$.

Therefore, $U^n(t, \cdot)$ is an strictly increasing utility function for all n and all agent $t \in I$, due to the fact that $I \setminus K$ is an open and dense subset of I.

Claim 3. Let $\succeq_t^n$ be the average preference relation represented by $U^n(t, \cdot)$. Then, the utility function $\hat{U}^n(t, \cdot) \in \mathcal{U}$, which represents $\succeq_t^n$ is defined by

$$\hat{U}^n(t, x) = x$$

Claim 4. For all agent $t \in K$ it is verified that $\rho(\succeq_t^n, \succeq_t') = 0$, for all n, where $\succeq_t'$ is represented by $U'(t, x) = x$. Obviously $\succeq_t \neq \succeq_t'$.

Claim 5. The measure of K can be chosen taking any value less than 1.

4 Unanimous Preference

4.1 Definition and some properties.

Let us consider now that for each n the preference relations $\succeq^n$ in the economy $\mathcal{E}_c^n$ are defined as follows

$$x \succeq_{t_0}^n y \;\Leftrightarrow\; x \succeq_t y \text{ for almost all } t \in I_{i(t_0)}^n$$

being $I_{i(t_0)}^n$ the real subinterval, such that $t_0 \in I_{i(t_0)}^n$. In this case, we will refer to $\succeq_t^n$ as unanimous preference. This preference states that a consumption vector is prefered to another one if it is unanimously prefered by all the corresponding set of agents.

Let $\succeq_i^n = \succeq_t^n$, with $t \in I_i^n$.

Next we show some properties of this average preference.

1. If x is unanimously prefered to y, then x is prefered to y with the average preference which we have defined in section 3.

2. The unanimous preference relation $\succeq_t^n$ is not complete. In fact, if we have $\mu(\{t \in I_i^n | x \succ_t y\}) > 0$ and $\mu(\{t \in I_i^n | x \prec_t y\}) > 0$, then x can not be compared to y in the economy $\mathcal{E}_c^n$ for the agents of type i.

3. $\succeq_t^n$ and $\succ_t^n$ are transitive for all n and $t \in I$.

4. Let $t \in I_i^n$. The strict preference relation $\succ_t^n$ and the indifference relation $\sim_t^n$ are given by

$$x \succ_t^n y \quad \Leftrightarrow \quad x \succeq_t y \text{ for almost all } t \in I_i^n \text{ and } \mu(\{t \in I_i^n | x \succ_t y\}) > 0$$

$$x \sim_t^n y \quad \Leftrightarrow \quad x \sim_t y \text{ for almost all } t \in I_i^n$$

5. Let $t_0 \in I_i^{n_0}$. It is verified that $x \succeq_{t_0}^{n_0} y$ iff $x \succeq_t^n y$ for all $t \in I_i^{n_0}$ and for all $n \geq n_0$. Note that, although the equivalence remains true if $\succeq_t^n$ is replaced by $\sim_t^n$, this equivalence is no longer true if $\succeq_t^n$ is substituted by the strict preference $\succ_t^n$.

6. If $\succeq_t$ is continuous for almost all $t \in I$, then $\succeq_t^n$ is continuous for all $t \in I$ and for all n. That is, the sets $\{y \in \mathbb{R}_+^\ell | y \succeq_t^n x\}$, $\{y | y \preceq_t^n x\}$, and $\{y | y \sim_t^n x\}$ are closed, whatever $x \in \mathbb{R}_+^\ell$ may be.

7. If $\succeq_t$ is convex for almost all $t \in I$, then $\succeq_t^n$ is also convex for all $t \in I$ and for all n.

8. If $\succeq_t$ is monotone (resp. strictly monotone) for almost all $t \in I$, then $\succeq_t^n$ is monotone (resp. strictly monotone) for all $t \in I$ and all n.

4.2 Main results

In subsection 3.2, we have obtained several results concerning to sufficient conditions (theorem 3.1 and corollary 3.1) and necessary conditions (see the remarks at the end of the subsection) for an allocation f to belong to the core of the economy $\mathcal{E}_c$, in terms of the core of discrete economies, for the average preference.

In this subsection, our aim is to follow a similar path for the unanimous preference. For this we first state sufficient conditions for an allocation f to belong to the core of the economy $\mathcal{E}_c$, in terms of the core of discrete economies.

Theorem 4.1 *Let $f : I \to \mathbb{R}_+^\ell$ be a continuous function that is a feasible allocation in the economy $\mathcal{E}_c$. If $f^n \in Core(\mathcal{E}_c^n)$ for all $n \geq n_0$, then $f \in Core(\mathcal{E}_c)$.*

Proof. Let us assume that $f \notin Core(\mathcal{E}_c)$. Then, there exist a coalition S, and $g : S \to \mathbb{R}^{\ell}_{+}$, such that $\int_S g(t)d\mu(t) \ll \int_S \omega(t)d\mu(t)$, and $U(t, g(t)) > U(t, f(t))$, for all $t \in S$. As ω^n converges weakly to ω in $L^1(I)$, there exists n_0, such that $\int_S g(t)d\mu(t) \leq \int_S \omega^n(t)d\mu(t)$, for all $n \geq n_0$. Moreover, by Lusin's theorem we can choose S to be compact and g to be continuous on S (see Hildebrand (1974), page 140).

By the continuity properties of $\mathcal{U}$, f and g, there exists $\varepsilon > 0$ such that, if $\hat{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^{\ell}_{+}$, then we have

$$U(t, g(t)) - \frac{\varepsilon}{2} \; > \; U(t, f(t) + \hat{\varepsilon}) + \frac{\varepsilon}{2}, \;\; \text{for all} \;\; t \in S$$

Noticing that f^n converges to f uniformly on S, there exists K a compact subset of $\mathbb{R}^{\ell}_{+}$, such that $f(S), g(S), f^n(S) \subset K$, and there exists $n_1 = n_1(\varepsilon)$, such that $f(t) + \hat{\varepsilon} \geq f^n(t)$, for all $t \in S$ and $n \geq n_1$. So, by monotonicity of preferences we obtain that

$$U(t, g(t)) - \frac{\varepsilon}{2} \; > \; U(t, f^n(t)) + \frac{\varepsilon}{2}, \;\; \text{for all} \;\; t \in S$$

On the other hand, as $\mathcal{U}$ is continuous, there exists $\delta = \delta(\varepsilon, K)$ such that, if $| t - t' | < \delta$, and $x \in K$, then $| U(t', x) - U(t, x) | < \frac{\varepsilon}{2}$. Let n_2, such that $2^{-n_2} < \delta$, and let $\bar{n} = \max\{n_0, n_1, n_2\}$. Then, if $n \geq \bar{n}$, and $| t - t' | < \delta$, it is verified that

$$U(t', g(t)) > U(t, g(t)) - \frac{\varepsilon}{2} \;\; \text{and} \;\; U(t, f^n(t)) + \frac{\varepsilon}{2} > U(t', f^n(t))$$

Note that if $n \geq \bar{n}$, then $| t - t' | < \delta$ if $t, t' \in I^n_i$, whatever i may be.

Therefore, $U(t', g(t)) > U(t', f^n(t))$ for all $n \geq \bar{n}$, and $t, t' \in I^n_i$. So, we can conclude that the coalition S blocks f^n via g in the economy $\mathcal{E}^n_c$.

Q.E.D.

Remark. We have actually proved that if an allocation f is blocked by the coalition S via g in the economy $\mathcal{E}_c$, then f^n is also blocked by the the same coalition S and via the same allocation g in the economy $\mathcal{E}^n_c$.

This theorem 4.1 is a weaker version of theorem 3.1, in the case of unanimous preference. As we shall see in subsection 4.3, neither the continuity of f nor the continuity of $\mathcal{U}$ can be dropped.

In order to find necessary conditions for an allocation f to belong to the core of the economy $\mathcal{E}_c$, we need to state first some lemmas.

Lemma 4.1 *Let $S \subset I^n_i$, with $\mu(S) > 0$. Let $g : S \to \mathbb{R}^{\ell}_{+}$ be a μ-integrable function and let $x \in \mathbb{R}^{\ell}_{+}$, such that $g(t) \succ^n_i x$ for all $t \in S$. Then $\frac{1}{\mu(S)}\int_S g(t)d\mu(t) \succ^n_i x$.*

Proof. Let $A = \{y \in \mathbb{R}^\ell_+ | y \succ^n_i x\}$. First, let us show that A is a convex set. For this, let $y_1, y_2 \in A$. Then $y_1 \succeq_t x$, $y_2 \succeq_t x$ for almost all $t \in I^n_i$ and there exist $S_1, S_2 \subset I^n_i$, with $\mu(S_1) > 0$ and $\mu(S_2) > 0$, such that $y_1 \succ_t x$, for all $t \in S_1$, $y_2 \succ_t x$ for all $t \in S_2$. Let $y_\lambda = \lambda y_1 + (1-\lambda)y_2$ with $0 < \lambda < 1$. By convexity of the preferences, there exist $S'_1 \subset S_1$, $S'_2 \subset S_2$, with $\mu(S'_1) = \mu(S_1)$ and $\mu(S'_2) = \mu(S_2)$, such that $y_\lambda \succ_t x$, for all $t \in S' = S'_1 \bigcup S'_2$. So, $y_\lambda \in A$. Therefore A is a convex set. The convexity theorem of Hüsseinov (1987) allow us to conclude that

$$\frac{1}{\mu(S)} \int_S g(t) d\mu(t) \in co(g(S))$$

As $g(S) \subset A$, we have that $\frac{1}{\mu(S)} \int_S g(t) d\mu(t) \succ^n_i x$.

Q.E.D.

Lemma 4.2 *Let f be a feasible allocation in the economy $\mathcal{E}^n_c$, such that $f(t) = f_i$ for all $t \in I^n_i$, and for all i. If $f \in \mathcal{S}^{n+1}$-$Core(\mathcal{E}^{n+1}_c)$, then $f \in \mathcal{S}^n$-$Core(\mathcal{E}^n_c)$.*

Proof. Let us suppose that $f \notin \mathcal{S}^n$-$Core(\mathcal{E}^n_c)$. In this case, there exist $S \in \mathcal{S}^n$ and $g : S \to \mathbb{R}^\ell_+$, such that $\int_S g(t) d\mu(t) \leq \int_S \omega^n(t) d\mu(t)$ and $g(t) \succ^n_t f(t)$ for almost all $t \in S$. By the definition of $\mathcal{S}^n$ we have that there exists $T_S \subset \{1, \ldots, 2^n\}$ such that $S = \bigcup_{i \in T_S} I^n_i$. That is, T_S is the set of types which form the coalition S. So, for each $i \in T_S$ it is verified that $g(t) \succ^n_i f_i$ for all $t \in I^n_i$. For each $i \in T_S$, let $g_i = \frac{1}{\mu(I^n_i)} \int_{I^n_i} g(t) d\mu(t)$. Let us consider $\tilde{g} : S \to \mathbb{R}^\ell_+$, given by $\tilde{g}(t) = g_i$ if $t \in I^n_i$. Note that $\int_S \tilde{g}(t) d\mu(t) = \sum_{i \in T_S} \mu(I^n_i) g_i = \int_S g(t) d\mu(t)$. By lemma 4.1, $g_i \succ^n_i f_i$ for all $i \in T_S$. This means that, for each $i \in T_S$, it is verified that $g_i \succeq_t f_i$ for almost all $t \in I^n_i$, and besides there exists $S_i \subset I^n_i$, with $\mu(S_i) > 0$, such that $g_i \succ_t f_i$ for all $t \in S_i$. On the other hand, $I^n_i = I^{n+1}_{2i-1} \bigcup I^{n+1}_{2i}$.

Let the sets T_1 and T_2 be defined as follows

$$T_1 = \{i \in T_S | \mu(S_i \bigcap I^{n+1}_{2i-1}) > 0\}$$

$$T_2 = \{i \in T_S | \mu(S_i \bigcap I^{n+1}_{2i}) > 0\}$$

Note that $T_S = T_1 \bigcup T_2$. We distinguish two different cases.

First, if $T_S = T_1 = T_2$, then $\tilde{g}(t) \succ^{n+1}_t f(t)$ for all $t \in S$. It is also verified that

$$\int_S \tilde{g}(t) d\mu(t) = \int_S g(t) d\mu(t) \leq \int_S \omega^n(t) d\mu(t) = \int_S \omega^{n+1}(t) d\mu(t)$$

So the coalition S blocks f via $\tilde{g}$ in the economy $\mathcal{E}_c^{n+1}$. This is in contradiction with $f \in \mathcal{S}^{n+1}\text{-}Core(\mathcal{E}_c^{n+1})$.

Secondly, let us suppose $T_1 \neq T_2$. For each $i \notin T_1$ (resp. for each $i \notin T_2$) let us consider $S_{2i-1} \subset I_{2i-1}^{n+1}$ (resp. $S_{2i} \subset I_{2i}^{n+1}$), with $\mu(S_{2i-1}) = \mu(S_i)$ (resp. $\mu(S_{2i}) = \mu(S_i)$). As $g_i \succ_t f_i$ for all $t \in S_i$, then by continuity of the preferences we have that there exists

a bounded function $r_i : S_i \to \mathbb{R}_+^\ell$, such that $g_i - r_i(t) \succ_t f_i$ for all $t \in S_i$. Let $\hat{r}_i = \int_{S_i} r_i(t) d\mu(t)$. Let us consider now $\hat{g} : S \to \mathbb{R}_+^\ell$, defined as follows

$$\hat{g}(t) = \begin{cases} g_i - r_i(t) & \text{if } t \in S_i, \text{ with } i \notin T_1 \text{ or } i \notin T_2 \\ g_i + \dfrac{1}{\mu(S_{2i-1})}\hat{r}_i & \text{if } t \in S_{2i-1}, \text{ with } i \notin T_1 \\ g_i + \dfrac{1}{\mu(S_{2i})}\hat{r}_i & \text{if } t \in S_{2i}, \text{ with } i \notin T_2 \\ \tilde{g}(t) & \text{in other case} \end{cases}$$

By construction, we have that $\int_S \hat{g}(t) d\mu(t) \leq \int_S \omega^{n+1}(t) d\mu(t)$. In fact,

$$\begin{aligned} \int_S \hat{g}(t)d\mu(t) = {} & \sum_{i\notin T_1} \mu(S_i) g_i - \sum_{i \notin T_1} \int_{S_i} r_i(t) d\mu(t) + \sum_{i\notin T_2} \mu(S_i) g_i - \sum_{i\notin T_2} \int_{S_i} r_i(t) d\mu(t) + \\ & \sum_{i\notin T_1} \mu(S_{2i-1}) g_i - \sum_{i\notin T_1} \hat{r}_i + \sum_{i\notin T_2} \mu(S_{2i}) g_i - \sum_{i\notin T_2} \hat{r}_i + \\ & \sum_{i\notin T_1} \mu\left((I_{2i-1}^{n+1} \setminus S_{2i-1})\right) g_i + \sum_{i\notin T_2} \mu\left((I_{2i}^{n+1} \setminus S_{2i})\right) g_i + \\ & \sum_{i\notin T_1} \mu\left((I_{2i}^{n+1} \setminus S_i)\right) g_i + \sum_{i\notin T_2} \mu\left((I_{2i-1}^{n+1} \setminus S_i)\right) g_i + \sum_{i \in T_1 \cap T_2} \mu(I_i^n) g_i \\ = {} & \sum_{i \in \bigcap T_2} \mu(I_i^n) g_i + \sum_{i\notin T_1} \mu\left(I_{2i-1}^{n+1} \cup I_{2i}^{n+1}\right) g_i + \sum_{i\notin T_2} \mu\left(I_{2i-1}^{n+1} \cup I_{2i}^{n+1}\right) g_i \end{aligned}$$

So, it is verified that $\int_S \hat{g}(t)d\mu(t) = \sum_{i\in T_S} \mu(I_i^n) g_i = \int_S \tilde{g}(t) d\mu(t)$.

By continuity and monotony of the preferences, we obtain that $\hat{g}(t) \succ_t^{n+1} f(t)$ for all $t \in S$. Therefore, $f \notin \mathcal{S}^{n+1}\text{-}Core(\mathcal{E}_c^{n+1})$.

Q.E.D.

Theorem 4.2 *Let $\overline{n}$ be a positive integer number, and let f be a feasible allocation in the economy $\mathcal{E}_c$, such that $f(t) = f_i$ for all $t \in I_i^{\overline{n}}$, $i = 1, \ldots, 2^{\overline{n}}$.*

If $f \in \mathcal{S}$-$Core(\mathcal{E}_c)$, then $f \in \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$, for all $n \geq \overline{n}$. In particular, if $f \in Core(\mathcal{E}_c)$, then $f \in \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$, for all $n \geq \overline{n}$.

Proof. Let us suppose that $f \notin \mathcal{S}^n$-$Core(\mathcal{E}_c^n)$, for some $n \geq \overline{n}$. Then, there exist $S \in \mathcal{S}^n$ and $g : S \to \mathbb{R}_+^\ell$, such that $\int_S g(t)d\mu(t) \leq \int_S \omega^n(t)d\mu(t)$ and $g(t) \succ_t^n f(t)$ for all $t \in S$. Let $T_S \subset \{1, \ldots, 2^n\}$, such that $S = \bigcup_{i \in T_S} I_i^n$. We have that $g(t) \succ_i^n f_i$ for all $t \in I_i^n$, with $i \in T_S$. For each $i \in T_S$, let $g_i = \dfrac{1}{\mu(I_i^n)} \int_S g(t)d\mu(t)$.

Let us consider $\tilde{g} : S \to \mathbb{R}_+^\ell$, given by $\tilde{g}(t) = g_i$, if $t \in I_i^n$. By construction, we obtain that $\int_S \tilde{g}(t)d\mu(t) = \int_S g(t)d\mu(t)$. By lemma 4.1, it is verified that $g_i \succ_i^n f_i$, for all $i \in T_S$. By the definition of unanimous preference $\succeq_i^n$, this is equivalent to the fact that for each $i \in T_S$ there exists $S_i^n \subset I_i^n$, with $\mu(S_i^n) > 0$, such that $g_i \succeq_t f_i$, for almost all $t \in I_i^n$ and $g_i \succ_t f_i$, for all $t \in S_i^n$. Therefore, the allocation f is blocked by the coalition S in the economy $\mathcal{E}_c$.

Q.E.D.

4.3 Some counterexamples

Example 4.1. Our first example is similar to the example 3.1 and shows that the converse of theorem 4.1 is not true.

By Cantor's nested intervals theorem we can take α as below. Let $\alpha \in I$, such that for infinitely many $n = n_k$ and for all $i = i(n)$, with $\alpha \in I_i^n$, it is verified that $a_i^n < \alpha < b_i^n$, being

$$
\begin{aligned}
a_i^n &= \tfrac{2}{3}\left(\tfrac{i-1}{2^n}\right) + \tfrac{1}{3}\left(\tfrac{i}{2^n}\right) \\
a_i^n &= \tfrac{1}{3}\left(\tfrac{i-1}{2^n}\right) + \tfrac{2}{3}\left(\tfrac{i}{2^n}\right)
\end{aligned}
$$

Note that this implies that $\mu(\{t \in I_i^n | t < \alpha\}) < 2\mu(\{t \in I_i^n | t > \alpha\})$ and besides $\mu(\{t \in I_i^n | t > \alpha\}) < 2\mu(\{t \in I_i^n | t < \alpha\})$.

Let us consider the economy $\mathcal{E}_c$ with the commodity space $\mathbb{R}^2$. Each agent $t \in [0,1]$ is characterized by his initial endowment $\omega(t)$ and his preference relation represented by the utility function $U(t,(x,y))$, defined as follows

$$
\omega(t) = \begin{cases} (1,0) & \text{if } t < \alpha \\ (0,1) & \text{if } \alpha < t \end{cases}
$$

$$
U(t,(x,y)) = \begin{cases} 2x + y & \text{if } t < \alpha \\ x + 2y & \text{if } \alpha < t \end{cases}
$$

As in example 3.1, it is easy to prove that the allocation f, given by $f(t) = \omega(t)$, belongs to $Core(\mathcal{E}_c)$. However, the subsequence (n_k) chosen before, verifies that $f^{n_k} \notin S^{n_k}\text{-}Core(\mathcal{E}_c^{n_k})$ whatever n_k may be. In fact, for each n_k the coalition $S_{n_k} = I_i^{n_k}$ blocks the allocation f^{n_k} via g_{n_k} in the economy $\mathcal{E}_c^{n_k}$, being $g_{n_k}(t) = f(t)$. To see this, observe that if $t \in I_i^{n_k}$, then

$$U(t, f^n(t)) = \begin{cases} 2\left(\alpha - \frac{i-1}{2^n}\right)2^n + \left(\frac{i}{2^n} - \alpha\right)2^n & \text{if } \ t < \alpha \\ \left(\alpha - \frac{i-1}{2^n}\right)2^n + 2\left(\frac{i}{2^n} - \alpha\right)2^n & \text{if } \ t < \alpha \end{cases}$$

Therefore, if $t \in S_{n_k}$, then $f(t)$ is unanimously prefered to $f^{n_k}(t)$, for all n_k, because $U(t, f^n(t)) < U(t, f(t))$, for all $t \in S_{n_k}$, due to the fact that

$$\max\left\{\alpha - \frac{i-1}{2^n}, \frac{i}{2^n} - \alpha\right\} < \frac{2}{3}2^{-n}$$

Example 4.2. In example 4.1, we can get that $f^n \notin S^n\text{-}Core(\mathcal{E}_c^n)$ for all even number n, or $f^n \notin S^n\text{-}Core(\mathcal{E}_c^n)$ for all odd number n, but not both of them simultaneously. Now, as we did in example 3.2, we define an economy $\mathcal{E}_c$, such that there exists $f \in Core(\mathcal{E}_c)$, but $f^n \notin S^n\text{-}Core(\mathcal{E}_c^n)$ for all n.

Let us consider the economy $\mathcal{E}_c$ with the commodity space $\mathbb{R}^2$. Each agent $t \in [0,1]$ is characterized by his initial endowment $\omega(t)$ and his preference relation represented by the utility function $U(t,(x,y))$, defined as follows

$$\omega(t) = \begin{cases} (1,0) & \text{if } \ t < \alpha \ \text{ or } \ t > \beta \\ (0,1) & \text{if } \ \alpha < t < \beta \end{cases}$$

$$U(t,(x,y)) = \begin{cases} 2x + y & \text{if } \ t < \alpha \ \text{ or } \ t > \beta \\ x + 2y & \text{if } \ \alpha < t < \beta \end{cases}$$

where α is chosen as in example 4.1 for all odd n, and β satisfies for all even n the following both inequalities:

$$\mu\left(\{t \in I_i^n | t < \beta\}\right) < 2\mu\left(\{t \in I_i^n | t > \beta\}\right)$$

$$\mu\left(\{t \in I_i^n | t > \beta\}\right) < 2\mu\left(\{t \in I_i^n | t < \beta\}\right)$$

As in example 4.1, it is clear that the allocation f, given by $f(t) = \omega(t)$, belongs to $Core(\mathcal{E}_c)$. However, if n is an odd number (resp. n is an even number), then the coalition $S_n = I_i^n$, with $\alpha \in I_i^n$ (resp. with $\beta \in I_i^n$) blocks the allocation f^n via g_n in the economy $\mathcal{E}_c^n$, being $g_n(t) = f(t)$.

Example 4.3. As we have noticed earlier, the continuity of f in theorem 4.1 can not be dropped. To prove this, let us consider a continuum economy $\mathcal{E}_c$ with two commodities, and let K be a Cantor subset of I, with $\mu(K) > 0$. For each agent $t \in I$, let $U(t,\cdot) = (1+x)^{1+\beta(t)}(1+y)^{1-\beta(t)}$, where $\beta : I \to \mathbb{R}$ is a continuos function defined by

$$\beta(t) = \begin{cases} 0 & \text{if } t \in K \\ (t-a_k)(t-b_k)\sin\left(\frac{1}{(t-a_k)(t-b_k)}\right) & \text{if } t \in (a_k, b_k) \end{cases}$$

being $I \setminus K = \bigcup_{k=1}^{\infty}(a_k, b_k)$.

Let $A = \{t \in I | \beta(t) > 0\}$ and $B = \{t \in I | \beta(t) < 0\}$. Let $K = K_1 \bigcup K_2$, with $\mu(K_1) = \mu(K_2)$, and $K_1 \bigcap K_2 = \emptyset$.

For each agent $t \in I$ let $\omega(t)$ be his initial endowment, given by

$$\omega(t) = \begin{cases} (1+\gamma, 1-\gamma) & \text{if } t \in K_1 \\ (1-\gamma, 1+\gamma) & \text{if } t \in K_2 \\ (1,0) & \text{if } t \in A \\ (0,1) & \text{if } t \in B \end{cases}$$

It is easy to prove that if $K \subset S$, then the coalition S blocks the allocation $f = \omega$ via g, defined by $g(t) = (1,1)$ if $t \in K$, and $g(t) = f(t)$ if $t \in S \setminus K$.

However, $f^n \in Core(\mathcal{E}_c^n)$ for all n. To prove this, let us assume that there exist a coalition $\hat{S}$ and an allocation g, such that $f^n(t) = (f_1^n(t), f_2^n(t)) \prec_t^n g(t) = (g_1(t), g_2(t))$. Then, it is verified $g_1(t) + g_2(t) > f_1^n(t) + f_2^n(t)$, for all $t \in \hat{S}$, which is a contradiction.

To prove the last inequality, it is enough to notice that any I_i^n satisfies one of the three following facts: *(a)* $I_i^n \subset A$; *(b)* $I_i^n \subset B$; *(c)* $I_i^n \bigcap A \neq \emptyset$ and $I_i^n \bigcap B \neq \emptyset$. Let us consider i such that $\mu(\hat{S} \bigcap I_i^n) > 0$. If *(a)* holds, then the agents of type i prefer commodity 1 better than commodity 2, but they have not any of it, so they verify the inequality. If *(b)* holds, then for a similar reason, the inequality is also verified for all $t \in I_i^n$. Finally, if *(c)* holds, the inequality is verified for all $t \in I_i^n$, because all the agents $t \in A$ (resp. $t \in B$) prefer commodity 1 (resp. 2) better than 2 (resp. 1).

Example 4.4. Example 4.3 shows that, in theorem 4.1, the continuity of f can not be dropped. Now we give an example which shows that neither the continuity of $\mathcal{U}$ can be deleted.

For this, let A, B be disjoint subsets of I, such that $\mu(I_i^n \bigcap A) > 0$ and $\mu(I_i^n \bigcap B) > 0$, for all n and i. For example, we can take A and B as follows

$$A = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{2^n} A_i^n, \quad B = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{2^n} B_i^n$$

such that A_i^n, B_i^n are non negligible Cantor subsets of I_i^n, verifying that

$$\left(\bigcup_{n=1}^{N}\bigcup_{i=1}^{2^n} A_i^n\right)\bigcap\left(B=\bigcup_{n=1}^{N}\bigcup_{i=1}^{2^n} B_i^n\right)=\emptyset, \quad \text{for all } N$$

Let us consider an economy $\mathcal{E}_c$ with two commodities. Each agent $t \in I$ is characterized by his endoment $\omega(t) = (1,1)$, and his utility function, given by

$$U(t,(x,y)) = \begin{cases} 2x+y & \text{if } t \in A \\ x+2y & \text{if } t \in B \\ U(t,(x,y)) & \text{if } t \in I \setminus (A \bigcup B) \end{cases}$$

It is clear that the coalition $S = A \bigcup B$ blocks the allocation $f = \omega$. However, $f^n \in Core(\mathcal{E}_c^n)$, for all n. That is so because if $g(t) = (g_1(t), g_2(t))$ is unanimously prefered to $(1,1)$ by all $t \in S$, with $\mu(S) > 0$, then $g_1(t) + g_2(t) > 2$ for all $t \in S$.

Example 4.5. In section 3, we have considered a discrete approach to continuum economies, introducing the average preference. In that case, as a consequence of corollary 3.2 and the results in Kannai (1970), we have obtained that, if $f \in Core(\mathcal{E}_c)$, then for each $\varepsilon > 0$ it is verified that $f^n \in \varepsilon\text{-}Core(\mathcal{E}_c^n)$ for all $n \geq n_0$.

Now, we give an example which proves that this result does not hold if we consider the unanimous preference as discrete approach. For this, let us see that the fact that an allocation f belongs to $Core(\mathcal{E}_c)$ does not imply that for each $\varepsilon > 0$ it is verified that f^n belongs to $\varepsilon\text{-}Core(\mathcal{E}_c^n)$ for all n large enough.

Let us consider a continuum economy $\mathcal{E}_c$ with a single commodity. Each agent $t \in I = [0,1]$ has as initial endowment $\omega(t) \in \mathbb{R}_+$, such that $\int_I \omega(t)d\mu(t) = 1$ and $\int_0^{\frac{1}{2}} \omega(t)d\mu(t) = 2^{-1} + \alpha$, $0 < \alpha \leq 2^{-1}$. The preference relation of the agent $t \in I$ is represented by the utility function $U(t,\cdot)$, defined by

$$U(t,x) = \begin{cases} x & \text{if } x \leq 2^{-1} + t \\ 2^{-1} + t & \text{if } x > 2^{-1} + t \end{cases}$$

It is easy to prove that the allocation f, given by $f(t) = 2^{-1} + t$ belongs to the core of the economy $\mathcal{E}_c$. That is so because any agent $t \in I$ is satiated with the quantity $f(t)$. However, if $\varepsilon < \alpha$, then $f^n \notin \varepsilon\text{-}Core(\mathcal{E}_c^n)$ for all n. In fact, we claim that whatever n may be, we have that f^n is blocked by the coalition $S = [0, 2^{-1})$ via $g(t) = 1$ for every $t \in S$. To prove our claim, let t_i^n be the half point of the subinterval I_i^n. Then for all $I_i^n \subset S$, it is verified that $g(t) = 1$ is strictly prefered to $f^n(t) = 2^{-1} + t_i^n$ for all agent $t \in \left(t_i^n, \frac{i}{2^n}\right)$, and both are indifferent for all the agents $t \in \left[\frac{i-1}{2^n}, t_i^n\right]$.

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