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CORE EQUIVALENCE THEOREMS FOR INFINITE CONVEX GAMES

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Abstract

We show that the core of a continuous convex game on a measurable space of players is a von Neumann-Morgenstern stable set. We also extend the definition of the Mas-Colell bargaining set to games with a measurable space of players, and show that for continuous convex games the core may be strictly included in the bargaining set but it coincides with the set of all countably additive payoff measures in the bargaining set. We provide examples which show that the continuity assumption is essential to our results.

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§1- Introduction

Convex coalitional games were introduced in Shapley (1971). They include in particular any convex function of a measure, and occur in many applications. For example, Sorenson, Tschirhart and Whinston (1978) showed that the coalitional game modeling a producer and a set of potential consumers under decreasing costs is convex. The airport game (see Section XI.4 in Owen (1982)) and the bankruptcy game (see Aumann and Maschler (1985)) are also convex. Demange (1987) gave several examples of convex games which arise from public good models. The core of a convex game with a finite set of players was studied in Shapley (1971) and other solution concepts were investigated in Maschler, Peleg and Shapley (1972). Where different approaches lead to the same solution, this reinforces the appeal of the solution. In this work we study the equivalence between the core, von Neumann-Morgenstern stable sets, and the Mas-Colell bargaining set in convex coalitional games over a measurable space of players.

Stable sets for cooperative games were introduced by von Neumann and Morgenstern in their seminal book (see von Neumann and Morgenstern (1944)). Shapley (1971) showed that the core of a finite convex game is a von Neumann-Moregenstern stable set. The result was extended to cooperative games without side payments in Peleg (1986). Stable sets for coalitional games with a finite set of players have been studied intensively (for a comprehensive survey see Lucas (1992)). There are a few works concerning stable sets of games with an infinite set of players. Davis (1962) showed that for symmetric simple games with a continuum of players, results analogous to those of Bott (1953) can be obtained. Hart (1974) dealt with stable sets of market games with a continuum of players. Einy et al. (1995) analyzed stable sets of some non-atomic games, and showed that the core of a non-atomic glove market game which is defined as the minimum of a finite number of non-atomic probability measures is a stable set. Such a game is usually not convex. The stability of the core in games with a countable set of players was studied in Einy and Shitovitz (1995). The core of games with an infinite set of players was investigated in many works (for a comprehensive survey see Kannai (1992)). In this work we show that the core of a continuous convex game with a measurable space of players is its unique von Neumann-Morgenstern stable set.

The first definition of a bargaining set for cooperative games was given by Aumann and Maschler (1964). Recently, several new concepts of bargaining sets have been introduced (see Mas-Colell (1989), Dutta et al. (1989), Greenberg (1990, 1992); for a comprehensive survey see Maschler (1992)). All these sets contain the core of the game. However, there are important cases in which some of these sets coincide with the core. It is known that for convex coalitional games with a finite set of players these sets coincide with the core (see Maschler, Peleg and Shapley (1972) for the Aumann-Maschler bargaining set, Dutta et al. (1989) for the Mas-Colell and the consistent bargaining sets and Greenberg (1992) for the stable bargaining set). Einy and Wettstein (1995) studied the equivalence between bargaining sets and the core in simple games. The Mas-Colell bargaining set was introduced in Mas-Colell (1989), where it was proved that in an atomless pure exchange economy it coincides with the set of competitive equilibria (and hence, by Aumann's equivalence theorem (Aumann (1964)), it also coincides with the core). Shitovitz (1989) showed that for a large class of both finite and mixed market games the Mas-Colell bargaining set coincides with the core. In this work we extend the definition of the Mas-Colell bargaining set to coalitional games with transferable utility which have a measurable space of players, and prove that for continuous convex games the core coincides with the set of countably additive measures in the bargaining set. We give an example which shows that the continuity assumption is essential. We also give an example which shows that the bargaining set of an infinite continuous convex game may contain non-countably additive measures, and thus strictly include the core.

The class of games to which our results apply is very general and includes, in particular, games with a finite set of players, games with countably many players, non-atomic games, and mixed games. The known proofs of the corresponding results for finite games do not seem to admit an extension to games with a measurable space of players. Our approach is different, and thus in particular provides new proofs in the finite case. Our proofs employ Delbaen's (1974) characterization of convex games, Schmeidler's (1986) characterization of convex games in terms of the Choquet integral, and a general minmax theorem due to Sion (1958).

The paper is organized as follows. In section 2 we define the basic notions which are relevant to our work. In section 3 we state and prove a lemma which constitutes the main part of the proofs of our main results. In section 4 we show that the core of a continuous convex game on a measurable space of players is its unique von Neumann-Morgenstern stable set. In section 5 we extend the definition of the Mas-Colell bargaining set to games with a measurable space of players, and prove that for continuous convex games the core coincides with the set of countably additive measures in the bargaining set.

§2 - Preliminaries

Let (T,Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. We refer to the members of T as *players* and to those of Σ as *coalitions*. A *coalitional game*, or simply a *game* on (T,Σ), is a function $v : \Sigma \rightarrow \Re_+$ with $v(\emptyset) = 0$. A coalition C is a *carrier* of v if $v(S) = v(S \cap C)$ for all $S \in \Sigma$. A coalition S is *null* in v if $T \setminus S$ is a carrier of v. A game v on (T,Σ) is *continuous at* $S \in \Sigma$ if for all non-decreasing sequences $\{S_n\}_{n=1}^{\infty}$ of coalitions such that $\bigcup_{n=1}^{\infty} S_n = S$, and all non-increasing sequences $\{S_n\}_{n=1}^{\infty}$ of coalitions such that $\bigcap_{n=1}^{\infty} S_n = S$, we have $v(S_n) \rightarrow v(S)$. The game v is *continuous* if it is continuous at each S in Σ.

A *payoff measure* in a game v is a bounded finitely additive measure $\xi: \Sigma \to \Re$ (not necessarily nonnegative) which satisfies $\xi(T) \leq v(T)$. The *core* of a game v, denoted by $Core(v)$, is the set of all payoff measures ξ such that $\xi(S) \geq v(S)$ for all $S \in \Sigma$.

As observed by Schmeidler (see the first part of the proof of Theorem 3.2 in Schmeidler (1972)), if v is continuous at T, then every member of $Core(v)$ is countably additive.

A game v is *convex* if for every $A, B \in \Sigma$ we have

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B).$$

It is well known that the core of a convex game is non-empty (see Shapley (1971) for finite games and Schmeidler (1972) for games with a measurable space of players). We note that Proposition 3.15 in Schmeidler (1972) implies that a convex game which is continuous at the grand coalition is continuous at every coalition.

We denote by $ba = ba(T,\Sigma)$ the Banach space of all bounded finitely additive measures on (T,Σ) with the variation norm. If μ is a countably additive measure on (T,Σ) we denote by $ba(\mu) = ba(T,\Sigma,\mu)$ the subspace of ba which consists of all bounded finitely additive measures on (T,Σ) which vanish on the μ-measure zero sets in Σ. The subspace of ba which consists of all bounded countably additive measures on (T,Σ) is denoted by $ca = ca(T,\Sigma)$. If μ is a measure in ca then $ca(\mu) = ca(T,\Sigma,\mu)$ denotes the set of all members of ca which are absolutely continuous with respect to μ. Finally, if A is a subset of an ordered vector space we denote by A_+ the set of all nonnegative members of A.

The following fact is a consequence of Theorem 3.10 in Schmeidler (1972), and is recorded here for later use.

Proposition 2.1

Let v be a continuous convex game on (T,Σ). Then there exists a measure $\mu \in ca_+$ such that a coalition $S \in \Sigma$. is null in v iff $\mu(S) = 0$. Moreover, $Core(v) \subset ca_+(\mu)$.

§3 - The Main Lemma

In this section we state and prove a lemma concerning continuous convex games which constitutes the main part of the proofs of our equivalence theorems.

Lemma 3.1

Let v be a continuous convex game on (T,Σ). Assume that $\xi \in ca$ satisfies $\xi(S) < v(S)$ for some $S \in \Sigma$. Then there exist $A \in \Sigma$ and $\eta \in Core(v)$ such that:

(3.1) $v(A) - \xi(A) = max\{v(C) - \xi(C) \mid C \in \Sigma\}$.

(3.2) $\eta(A) = v(A) > \xi(A)$ and $\eta(B) \geq \xi(B)$ for all $B \in \Sigma$ with $B \subset A$.

Proof

We need the following notation: If $\xi \in ca$ and f is a ξ-integrable function then the integral $\int_T f d\xi$ will be denoted by $\xi(f)$.

Let $\mu \in ca_+$ be a measure as guaranteed in Proposition 2.1. Let $\boldsymbol{B}$ be the unit ball of $L_\infty(\mu) = L_\infty(T,\Sigma,\mu)$. The proof proceeds in several steps.

Step 1: We extend v to a function $\bar{v}$ defined on $\boldsymbol{B}_+$.

For each $f \in \boldsymbol{B}_+$ let $\bar{v}(f) = min\{\zeta(f) \mid \zeta \in Core(v)\}$ (the minimum exists because $Core(v)$ is a weak*-compact non-empty subset of $ba(\mu)$, which is the norm-dual of $L_\infty(\mu)$). Since v is convex, by Proposition 3 in Schmeidler (1986) (see also Theorem 2.2 in Gilboa and Schmeidler (1995)), for each $f \in \boldsymbol{B}_+$ we have

$$\bar{v}(f) = \int_0^1 v(\{t \in T \mid f(t) \geq x\})dx \ ,$$

where the integral is a Riemann integral and is known as the Choquet integral of f with

v. In particular for each $S \in \Sigma$ we have $\bar{v}(\chi_S) = v(S)$.

In Steps 2-5 we treat the special case when $\xi \in ca_+(\mu)$.

Step 2: We prove that $\bar{v} - \xi$ attains its maximum on $\boldsymbol{B}_+$.

As $Core(v) \subset ca_+(\mu)$, by the Radon-Nikodym theorem for each $\zeta \in Core(v)$ the function $\zeta(f) = \int_T f d\zeta$ is continuous on $\boldsymbol{B}_+$ with respect to the weak*-topology which is induced by $L_\infty(\mu)$ on $\boldsymbol{B}_+$. Therefore $\bar{v}$ is weak*-upper semicontinuous on $\boldsymbol{B}_+$, as it is the minimum of weak*-continuous functions on $\boldsymbol{B}_+$. As $\xi \in ca_+(\mu)$, $\bar{v} - \xi$ is a weak*- upper semicontinuous function on $\boldsymbol{B}_+$. Now by Alaoglu's theorem $\boldsymbol{B}_+$ is compact in the weak*- topology. Therefore $\bar{v} - \xi$ attains its maximum on $\boldsymbol{B}_+$.

Step 3: We prove that the maximum in Step 2 is attained at a coalition A .

Let $M = \max_{f \in \boldsymbol{B}_+}(\bar{v} - \xi)(f)$. Let $f^* \in \boldsymbol{B}_+$ be such that $M = (\bar{v} - \xi)(f^*)$. For each $0 \leq x \leq 1$ let $T_x = \{t \in T \mid f^*(t) \geq x\}$. Then $M = \int_0^1 (v - \xi)(T_x)dx$ and thus $\int_0^1 [M - (v - \xi)(T_x)]dx = 0$. As $M \geq (v - \xi)(T_x)$ for each $0 \leq x \leq 1$, there is $0 \leq x_0 \leq 1$ such that $M = (v - \xi)(T_{x_0})$. Let $A = T_{x_0}$. Then $v(A) - \xi(A) \geq \bar{v}(f) - \xi(f)$ for all $f \in \boldsymbol{B}_+$.

It is clear that the coalition A found in Step 3 satisfies (3.1). Moreover, since $v(S) > \xi(S)$, we have $v(A) > \xi(A)$. It remains to find $\eta \in Core(v)$ which satisfies (3.2).

Step 4: We prove that for each $f \in \boldsymbol{B}_+$ with $f \leq \chi_A$ there exists $\eta \in Core(v)$ with $\eta(A) = v(A)$ such that $\eta(f) \geq \xi(f)$.

Let $f \in \boldsymbol{B}_+$ be such that $f \leq \chi_A$. Denote $g = \chi_A - f$, and for each $0 \leq x \leq 1$ let $A_x = \{t \in A \mid g(t) \geq x\}$. It is clear that if $x, y \in [0,1]$ and $x \leq y$ then $A_x \supset A_y$. Thus $\{A_x\}_{0 \leq x \leq 1}$ is a chain in Σ. As v is convex, by Corollary 3 in Delbaen (1974), there exists $\eta \in Core(v)$ such that $\eta(A_x) = v(A_x)$ for each $0 \leq x \leq 1$. As $A_0 = A$, we have $\eta(A) = v(A)$. Now for each $x > 0$ we have $\{t \in T \mid g(t) \geq x\} = A_x$, and so

$$\eta(g) = \int_0^1 \eta(A_x)dx = \int_0^1 v(A_x)dx = \bar{v}(g).$$

Thus $\eta(A)-\xi(A)=v(A)-\xi(A)\geq \bar{v}(g)-\xi(g)=\eta(g)-\xi(g)$, and therefore $\eta(A)-\eta(g)\geq \xi(A)-\xi(g)$. Since $g=\chi_A-f$, we have $\eta(f)\geq \xi(f)$.

Step 5: We prove that the order of the quantifiers in Step 4 can be reversed, that is, there exists $\eta \in Core(v)$ with $\eta(A)=v(A)$ that works for all $f \in \boldsymbol{B}_+$ with $f \leq \chi_A$.

Denote $\boldsymbol{B}_+(A)=\{f \in \boldsymbol{B}_+ \mid f \leq \chi_A\}$ and $C(A)=\{\eta \in Core(v) \mid \eta(A)=v(A)\}$. Then the sets $\boldsymbol{B}_+(A)$ and $C(A)$ are weak*-compact and convex in $L_\infty(\mu)$ and $ba(\mu)$ respectively.

Define a real-valued function H on $C(A)\times \boldsymbol{B}_+(A)$ by $H(\eta,f)=\eta(f)-\xi(f)$. Then H is affine and continuous in each of its variables separately. Thus the sets $C(A)$, $\boldsymbol{B}_+(A)$ and the function H satisfy the assumptions of Sion's minmax theorem (see Sion (1958)), and therefore

(3.3) $$\min_{f\in \boldsymbol{B}_+(A)}\ \max_{\eta\in C(A)} H(\eta,f)=\max_{\eta\in C(A)}\ \min_{f\in \boldsymbol{B}_+(A)} H(\eta,f).$$

Now by Step 4, $\min_f \max_\eta H(\eta,f)\geq 0$ and thus by (3.3), $\max_\eta \min_f H(\eta,f)\geq 0$. Therefore there exists $\eta \in C(A)$ such that $H(\eta,f)\geq 0$ for all $f \in \boldsymbol{B}_+(A)$. It is clear that η satisfies (3.2).

Step 6: We now show that if ξ is in ca_+ but is not absolutely continuous with respect to μ, then there exist A and η such that (3.1) and (3.2) are satisfied.

By the Lebesgue decomposition theorem, there exist two measures, ξ_a and ξ_s, in ca_+ such that $\xi=\xi_a+\xi_s$, where ξ_a is absolutely continuous with respect to μ and the measures ξ_s and μ are mutually singular. As $\xi(S)<v(S)$, $\xi_a(S)<v(S)$. As $\xi_a \in ca_+(\mu)$, by what we have already shown, there exist $A_o \in \Sigma$ and $\eta \in Core(v)$ such that $v(A_o)-\xi_a(A_o)=max\{v(C)-\xi_a(C) \mid C\in\Sigma\}$, $\eta(A_o)=v(A_o)>\xi_a(A_o)$, and $\eta(B)\geq\xi_a(B)$ for all $B\in\Sigma$ with $B\subset A_o$. Let C_o be a carrier of μ such that $\xi_s(C_o)=0$, and let $A=A_o\cap C_o$. Since C_o is a carrier of v, for each $C\in\Sigma$ we have

$$v(A)-\xi(A)=v(A)-\xi_a(A)=v(A_o)-\xi_a(A)\geq v(A_o)-\xi_a(A_o)\geq v(C)-\xi_a(C)\geq v(C)-\xi(C).$$

Hence $v(A)-\xi(A)=max\{v(C)-\xi(C) \mid C\in\Sigma\}$. So (3.1) is satisfied by A. Also, $\eta(A)\leq\eta(A_o)=v(A_o)=v(A)$, and as $\eta\in Core(v)$, we have $\eta(A)=v(A)$. Finally, if $B\in\Sigma$ and $B\subset A$ then $\eta(B)\geq\xi_a(B)=\xi(B)$. Thus (3.2) is satisfied by A and η.

Step 7: We now assume that $\xi\in ca$ is a signed measure, and show that there exist A and η such that (3.1) and (3.2) are satisfied.

By the Jordan decomposition theorem $\xi=\xi_+ +\xi_-$, where ξ_+ and ξ_- are the positive and the negative parts of ξ respectively. Let $w=v-\xi_-$. Then w is convex, continuous and $\xi_+(S)<w(S)$. Therefore we can apply what we have already shown for the game w and the measure ξ_+, and this yields the existence of $A\in\Sigma$ and $\eta\in Core(v)$ such that (3.1) and (3.2) are satisfied. Q.E.D.

§4 - Stability of the Core of Convex Games

The main purpose of this section is to prove that the core of a continuous convex game on a measurable space of players is a von Neumann-Morgenstern stable set. This generalizes the result of Shapley (1971) who showed that the core of a convex game with a finite set of players is a von Neumann-Morgenstern stable set.

A game v on (T,Σ) is *superadditive* if $v(A\cup B)\geq v(A)+v(B)$ whenever A and B are disjoint coalitions in Σ. When dealing with von Neumann-Morgenstern stable sets, it is natural to restrict attention to superadditive games, and we shall do so in this section. Clearly, a convex game is superadditive.

For a superadditive game v on (T,Σ) and any coalition $S\in\Sigma$, we define

$$\sigma_v(S)=inf\sum_{i=1}^{\infty}v(S_i)$$

where the infimum is taken over all countable partitions $S_1,S_2,...$ of S such that $S_i\in\Sigma$ for all i. It is easy to verify, using superadditivity, that $\sigma_v\in ca_+$. Clearly, $\sigma_v(S)\leq v(S)$ for all $S\in\Sigma$. Intuitively, $\sigma_v(S)$ is the amount that the members of S are guaranteed to obtain in v without cooperation. This permits to extend the notion of individual rationality from finite games. We say that a member ξ of *ba* is *individually rational* with respect to the game v if $\xi(S) \geq \sigma_v(S)$ for each $S \in \Sigma$. The set of all individually rational payoff measures in a superadditive game v on (T,Σ) is denoted by $I(v)$, i.e.,

$$I(v) = \{\xi \in ba_+ \mid \xi \text{ is individually rational and } \xi(T) \leq v(T)\}.$$

Let v be a game on (T,Σ). A coalition A in Σ is a *dummy coalition* in v if $\sigma_v(A) = v(A)$ and for each coalition B such that $A \cap B = \emptyset$ we have $v(A \cup B) = v(A) + v(B)$. Observe that a null coalition is also a dummy coalition, but not vice versa.

We now define a dominance relation on $I(v)$. Let $\xi, \eta \in I(v)$ and $A \in \Sigma$ be a non-dummy coalition. Then η *dominates* ξ *via A*, denoted by $\eta \succ_A \xi$, if $\eta(A) \leq v(A)$ and $\eta(B) > \xi(B)$ for each $B \in \Sigma$ such that $B \subset A$ and B is non-dummy. We say that η *dominates* ξ, denoted by $\eta \succ \xi$, if there exists a non-dummy coalition $A \in \Sigma$ such that $\eta \succ_A \xi$.

We note that if v is a superadditive game on (T,Σ), and there is a measure μ on (T,Σ) such that $\mu(S) > 0$ for each coalition S in Σ which is non-dummy in v (by Proposition 2.1, for a continuous convex game v such a μ always exists), then $Core(v)$ consists of those $\xi \in I(v)$ for which there is no $\eta \in I(v)$ such that $\eta \succ \xi$.

We come now to the definition of a von Neumann-Morgenstern stable set:

Let v be a superadditive game on (T,Σ). A set $V \subset I(v)$ is a *von Neumann-Morgenstern stable set* (or simply a *stable set*) of the game v if:

(4.1) *V is internally stable*, i.e., if $\xi \in V$ then there is no $\eta \in V$ such that $\eta \succ \xi$.

(4.2) *V is externally stable*, i.e., if $\xi \in I(v) \setminus V$ then there is $\eta \in V$ such that $\eta \succ \xi$.

The main result of this section is:

Theorem A

Let v be a continuous convex game on (T,Σ). Then the core of v is its unique von Neumann-Morgenstern stable set.

Proof

It is sufficient to show that $Core(v)$ is externally stable. Let $\mu \in ca_+$ be a measure as guaranteed in Proposition 2.1. Let $\xi \in I(v) \setminus Core(v)$. Let $S \in \Sigma$ be such that $\xi(S) < v(S)$.

We first assume that $\xi \in ca_+$. Let $\varepsilon > 0$ be such that $\xi(S) + \varepsilon\mu(S) < v(S)$. Then by Lemma 3.1 applied to $\xi + \varepsilon\mu \in ca$, there exist $A \in \Sigma$ and $\eta \in Core(v)$ such that $\eta(A) = v(A) > \xi(A) + \varepsilon\mu(A)$ and $\eta(B) \geq \xi(B) + \varepsilon\mu(B)$ for all $B \in \Sigma$ with $B \subset A$. Since $\sigma_v(A) \leq \xi(A) < v(A)$, A is not a dummy in v. Now if $B \subset A$ is a non-dummy coalition in v then $\mu(B) > 0$, and therefore $\eta(B) > \xi(B)$. Hence, $\eta \succ_A \xi$.

We now assume that ξ is in ba_+ but is not countably additive. By Theorem 1.23 in Yosida and Hewitt (1952), ξ can be decomposed uniquely into a sum of a nonnegative countably additive measure ξ^c and a nonnegative purely finitely additive measure ξ^p. As $\mu \in ca_+$, by Theorem 1.22 in Yosida and Hewitt (1952), there exists an increasing sequence of sets $C_n \in \Sigma$ such that $\xi^p(C_n) = 0$ for all n, and $\lim_{n\to\infty} \mu(T \setminus C_n) = 0$. Let $C = \bigcup_{n=1}^{\infty} C_n$. Then $\mu(T \setminus C) = 0$. Therefore C is a carrier of v. For each n let $S_n = S \cap C_n$. As v is continuous, $\lim_{n\to\infty} v(S_n) = v(S \cap C) = v(S)$. Since $\xi(S) < v(S)$, there exists a natural number k such that $\xi(S_k) < v(S_k)$. Let $D = S_k$ and $\Sigma_D = \{A \in \Sigma \mid A \subset D\}$. Let v_D be the restriction of v to Σ_D. It is clear that v_D is a continuous convex game on (D, Σ_D). As ξ^p vanishes on Σ_D, ξ coincides with ξ^c on Σ_D. Let ξ^c_D be the restriction of ξ^c to Σ_D. Then $\xi^c_D \in ca_+(D, \Sigma_D)$ and $\xi^c_D(D) = \xi(D) < v(D)$. Now we can apply what we have already shown to the game v_D and the measure ξ^c_D, in order to obtain the existence of a non-dummy coalition $A \in \Sigma_D$ and a measure $\zeta \in Core(v_D)$ such that $\zeta(A) = v(A)$ and $\zeta(B) > \xi(B)$ for all

$B \in \Sigma$ with $B \subset A$ and $\mu(B) > 0$. Since v is convex and $\zeta \in Core(v_D)$, by Proposition 3.8 in Einy and Shitovitz (1995), ζ can be extended to a measure η on (T,Σ) such that $\eta \in Core(v)$. Then $\eta \succ_A \xi$ in v and the proof is completed. Q.E.D.

Einy and Shitovitz (1995) showed that the core of a continuous convex game with a countable set of players on a field of coalitions which contains all the finite sets (not necessarily a σ-field) is a von Neumann-Morgenstern stable set. Their result is not implied by Theorem A, but it also does not imply Theorem A in the case of a countable set of players.

Let T be the set of natural numbers and Σ be the set of all subsets of T. As it was done in Example 3.5 of Einy and Shitovitz (1995), one can show that the convex game

$$v(S) = \begin{cases} 1 & \text{if } T \setminus S \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

does not have a von Neumann-Morgenstern stable set. Therefore the continuity assumption in Theorem A is essential.

§5 - Equivalence of the Core and the Mas-Colell Bargaining set in Convex Games

In this section we extend the definition of the bargaining set in Mas-Colell (1989) to coalitional games on a measurable space of players and prove that for continuous convex games the core coincides with the set of all countably additive payoff measures in the bargaining set. We also give an example which shows that the continuity assumption is essential, and an example which shows that the bargaining set may strictly include the core even in a continuous convex game.

Let v be a game on (T,Σ) and $\xi \in ba$ be a payoff measure in v. An *objection* to ξ is a pair (A,η) such that $A \in \Sigma$ and $\eta \in ba$ satisfies $\eta(A) \leq v(A)$, $\eta(A) > \xi(A)$ and $\eta(B) \geq \xi(B)$ for all $B \in \Sigma$ with $B \subset A$. A *counter objection* to the objection (A,η) is a pair (C,ζ) such that:

(5.1) $C \in \Sigma$, $\zeta \in ba$ and $\zeta(C) \leq v(C)$.

(5.2) If $B \in \Sigma$ satisfies $B \subset A \cap C$ then $\zeta(B) \geq \eta(B)$, and if $D \in \Sigma$ satisfies $D \subset C \setminus A$ then $\zeta(D) \geq \xi(D)$.

(5.3) $\zeta(C) > \eta(A \cap C) + \xi(C \setminus A)$.

An objection to a payoff measure is called *justified* if there is no counter objection to it. The *Mas- Colell bargaining set* of a game v, denoted by $MB(v)$, is the set of all payoff measures in v which have no justified objection.

It is known that for finite convex games the bargaining set coincides with the core (see Proposition 3.3 in Dutta et al. (1989)).

The following example shows that for infinite games the bargaining set may strictly include the core even when the game is continuous and convex.

Example 5.1

Let T be the set of natural numbers and Σ be the set of all subsets of T. Define a game v on (T,Σ) by $v(S) = \sum_{i \in S} 2^{-i}$ for each $S \in \Sigma$. It is clear that v is continuous and convex. In fact, v is a countably additive measure, and hence $Core(v) = \{v\}$. Let $F = \{S \in \Sigma \mid T \setminus S \text{ is finite}\}$. Then F is a filter in Σ. Let F_0 be a maximal filter which contains F. Define a measure ξ on Σ by $\xi(S) = 1$ if $S \in F_0$ and $\xi(S) = 0$ otherwise. We show that $\xi \in MB(v)$. Let (A,η) be an objection to ξ. Then $A \neq T$. Let $i \in T \setminus A$. Since $\xi(\{i\}) = 0$ and $v(\{i\}) > 0$, $(\{i\}, v)$ is a counter objection to (A,η), and thus $\xi \in MB(v)$.

We come now to the main result of this section.

Theorem B

Let v be a continuous convex game on (T,Σ). Then

$$Core(v) = MB(v) \cap ca.$$

The following corollary is an immediate consequence of Theorem B.

Corollary B

Let T be finite, and let v be a convex game on (T,Σ). Then

$$Core(v) = MB(v) \ .$$

Note that if T is finite and Σ is the set of all subsets of T then Corollary B follows from Proposition 3.3 in Dutta et al. (1989).

Proof of Theorem B

From the definition of $MB(v)$ it is clear that $Core(v) \subset MB(v)$. As v is continuous $Core(v) \subset ca$ (see Section 2). Therefore $Core(v) \subset MB(v) \cap ca$. We will show that $MB(v) \cap ca \subset Core(v)$. Assume, on the contrary, that there is $\xi \in MB(v) \cap ca$ such that $\xi \notin Core(v)$. As $\xi(T) \leq v(T)$, there is $S \in \Sigma$ such that $\xi(S) < v(S)$. Therefore by Lemma 3.1, there exist $A \in \Sigma$ and $\eta \in Core(v)$ such that (3.1) and (3.2) are satisfied. Clearly, (A,η) is an objection to ξ in the game v. We show that (A,η) is a justified objection and this will contradict the fact that $\xi \in MB(v)$. Let C be any coalition in Σ. Then by the convexity of v we have

$$v(C) - \xi(C) \leq v(A \cup C) - \xi(A \cup C) + v(A \cap C) - \xi(A \cap C) - v(A) + \xi(A).$$

By (3.1), $v(A \cup C) - \xi(A \cup C) \leq v(A) - \xi(A)$. Therefore $v(C) - \xi(C) \leq v(A \cap C) - \xi(A \cap C)$. As $\eta \in Core(v)$, we have $v(A \cap C) \leq \eta(A \cap C)$, and thus $v(C) \leq \eta(A \cap C) + \xi(C \setminus A)$. Hence there is no counter objection of C to (A,η), and as C was arbitrary, (A,η) is a justified objection to ξ. Q.E.D.

We now give an example which shows that the continuity assumption in Theorem B is essential.

Example 5.2

Let T be the set of natural numbers and Σ the set of all subsets of T. Define a game v on (T,Σ) by

$$v(S) = \begin{cases} 1 & \text{if } T \setminus S \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

For each $S \in \Sigma$ let $\xi(S) = \sum_{i \in S} 2^{-i}$. Then $\xi \in ca$ and $\xi \notin Core(v)$. We will show that $\xi \in MB(v)$. Let (A,η) be any objection to ξ. Then $T \setminus A$ is finite. Let $i \in A$. Then $\eta(\{i\}) > 0$. Now it is easy to construct a counter objection of the form $(A \setminus \{i\},\zeta)$ to (A,η), and thus $\xi \in MB(v)$.

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